Mathematics 551

INTRODUCTION TO TOPOLOGY

The photocopied material presented in this course consists of a list of axioms, theorems, questions, and definitions, and this course will be mainly devoted to a development of proofs of these theorems by members of the class. Each student should attempt to develop such proofs without consultation with any other person and without any reference to material in the literature. Such proofs will then be presented orally in class, and some written proofs may be required. This procedure is intended to emphasize a very interesting logical development of the topology of a line, and to enable members of the class to develop ability and gain experience in proving theorems.

This material will be supplemented with discussions in class, and it is not intended that it have any priority over other questions and theorems that are stated in class discussion. Each student should try to find other questions related to this material. It is suggested that each student prepare written proofs of all the theorems and include them in a notebook.

Linear Point Set Theory

Undefined Notions. The word point and the expression the point x precedes the point y will not be defined. This undefined expression will be written $x < y$.

AXIOM 1. S is a collection of points such that

(a) If x and y are different points, then either $x < y$ or $y < x$.

(b) If the point x precedes the point y, then $x \neq y$.

(c) If x, y, and z are points such that $x < y$ and $y < z$, then $x < z$.

Theorem 1. If x and y are points such that $x < y$, then $y \not< x$.

Definition. If c is a point of a point set M such that no point of M precedes c, then c is called a first point of M. Define last point similarly.

Theorem 2. No point set has two first (last) points.

Theorem 3. Every finite point set has a first point and a last point.

Theorem 4. If M is a finite point set consisting of n points, then there exist points $a_1, a_2, \ldots, a_n$ of M such that $a_1 < a_2 < a_3 < \cdots < a_n$.

Definition. If $x < z$ and $z < y$, then z is said to be between x and y. The set of all points between x and y is called a region and will be referred to as the region xy.

AXIOM 2. S has no first point and no last point.

Theorem 5. If x is a point, then there is a region containing x.

Question. Is Theorem 5 equivalent to Axiom 2?

Definition. A set K is said to be a subset of a set M if every element of K is an element of M.

Definition. A point p is said to be a limit point of a point set M if every region containing p contains a point of M distinct from p.

Theorem 6. If the point set H is a subset of the point set K and p is a limit point of H, then p is a limit point of K.

Notation. The intersection of R_1 and R_2 is denoted by $R_1 \cdot R_2$.

Theorem 7. If the point x is common to the regions R_1 and R_2, then some region lies in $R_1 \cdot R_2$ and contains x.

Theorem 8. If the point p is a limit point of the sum of two point sets H and K, then p is a limit point of at least one of the sets H and K.

Theorem 9. Extend Theorem 8 to any finite number of point sets.

Definition. Two point sets are said to be <u>mutually exclusive</u> if they have no point in common.

If G is a collection of point sets such that each pair of them is mutually exclusive, then the sets of G are said to be <u>mutually exclusive</u>.

Theorem 10. If x and y are two points, then there exist two mutually exclusive regions, one containing x and the other containing y. (Hausdorff property)

Theorem 11. No finite point set has a limit point.

Theorem 12. If p is a limit point of the point set M, then every region containing p contains infinitely many points of M.

Definition. A sequence of points $p_1, p_2, p_3, \ldots$ is said to <u>converge</u> to a point p if for every region R containing p there is a positive integer k such that if $n > k$, p_n lies in R.

Theorem 13. No sequence of points converges to each of two points.

Theorem 14. If a is a sequence of points converging to a point p, then every subsequence of a converges to p.

Theorem 15. If $p_1, p_2, p_3, \ldots$ is a sequence of distinct points converging to the point p, then p is the only limit point of the set $p_1 + p_2 + p_3 + \ldots$

Theorem 16. If $p_1, p_2, p_3, \ldots$ is a sequence of points converging to a point p, then some region contains the set $p + p_1 + p_2 + p_3 + \ldots$

Definition. A point set is said to be <u>closed</u> if it contains all of its limit points.

Notation. If H is a point set, then $\overline{H}$ denotes H together with all of its limit points. The set $\overline{H}$ is called the closure of H.

Theorem 17. If H is a point set, then $\overline{H}$ is closed.

Definition. A point set H is said to be <u>open</u> if for every point x of H there is a region containing x and lying in H.

Theorem 18. If x is a point, then the set of all points which precede (follow) x is an open set.

Theorem 19. Every region is an open set.

Definition. If x and y are two points, the set consisting of x and y, together with all points between x and y, is called the <u>interval</u> xy.

Theorem 20. Every interval is closed.

Definition. A set K is said to be a proper subset of a set M if K is a subset of M and does not contain M.

Theorem 21. If the open set M is a proper subset of S, then $S - M$ is closed.

Theorem 22. If the closed set M is a proper subset of S, then $S - M$ is open.

Notation. If G is a collection of point sets, then G^* denotes the sum of the sets of G.

Theorem 23. If G is a finite collection of closed sets, then G^* is closed.

Theorem 24. If G is a collection of open sets, then G^* is open.

Definition. The intersection of the elements of a collection G of point sets is the set of all points that belong to every element of G.

Theorem 25. If G is a collection of closed sets having a common point, then the intersection of the sets of G is closed.

Theorem 26. If G is a finite collection of open sets having a common point, then the intersection of the sets of G is open.

Definition. Two sets are said to be mutually separated if they have no point in common and neither of them contains a limit point of the other.

Definition. A point set is said to be connected if it is not the sum of two mutually separated point sets.

Theorem 27. If H and K are two mutually separated point sets and M is a connected subset of $H + K$, then M is a subset of one of the sets H and K.

Theorem 28. Every point is a connected set.

Theorem 29. If a finite point set M contains more than one point, then M is not connected.

Theorem 30. If G is a collection of connected point sets having a common point, then G^* is connected.

Theorem 31. If M is a connected point set, then $\overline{M}$ is connected.

Definition. A point set is said to be nondegenerate if it contains more than one point.

Theorem 32. If M is a nondegenerate connected point set, then every point of M is a limit point of M.

Theorem 33. If x and y are two points of a connected point set M, then the region xy lies in M.

Theorem 34. If the point set M is not connected, then M is the sum of two mutually separated sets H and K such that every point of H precedes every point of K. (Dedekind cut)

AXIOM 3. S is connected.

Theorem 35. If H and K are two point sets such that $H + K = S$ and every point of H precedes every point of K, then either (1) H has a last point or (2) K has a first point. Furthermore, (1) and (2) are mutually exclusive.

Theorem 36. Every interval is connected.

Theorem 37. Every region is connected.

Theorem 38. No proper subset of S is both open and closed.

Question. Is each of the above four theorems equivalent to Axiom 3?

Theorem 39. If x and y are two points, there is a point between them.

Theorem 40. Every point of a region R is a limit point of R.

Theorem 41. Each end point of a region R is a limit point of R.

Theorem 42. No region has a first (last) point.

Theorem 43. S contains infinitely many points and every point is a limit point of S.

Definition. A sequence of points is said to be <u>bounded</u> if some region contains every point of this sequence.

Definition. A sequence of points $p_1, p_2, p_3, \ldots$ is said to be <u>increasing</u> if for each positive integer n, $p_n < p_{n+1}$.

Theorem 44. Every bounded increasing sequence of points converges.

Theorem 45. If a is a bounded sequence of points, then some subsequence of a converges. (Sequential compactness)

Definition. A point set is said to be <u>bounded</u> if it lies in some region.

Theorem 46. Every bounded infinite point set has a limit point. (Bolzano-Weierstrass)

Theorem 47. Every closed and bounded point set contains a first point and a last point.

Theorem 48. If $M_1, M_2, M_3, \ldots$ is a sequence of closed and bounded point sets such that for each n, M_n contains M_{n+1}, then the sets $M_1, M_2, M_3, \ldots$ have a point in common. Furthermore, the set of all points common to these sets is closed.

Definition. A collection G of point sets is said to <u>cover</u> a point set M if every point of M lies in at least one set of G.

Theorem 49. If ab is a region and G is a collection of regions covering $\overline{ab}$, then some finite subcollection of G covers $\overline{ab}$.

Theorem 50. Strengthen Theorem 49 by letting G be a collection of open sets.

Theorem 51. If G is a collection of open sets covering the closed and bounded point set M, then some finite subcollection of G covers M. (Heine-Borel-Lebesgue)

Definition. A point set M is said to be <u>perfect</u> if M is closed and every point of M is a limit point of M.

Theorem 52. No countable point set is perfect. (Baire)

Theorem 53. Every region is uncountable.

Theorem 54. Every uncountable point set has a limit point.

Definition. A point set M is said to be <u>nowhere dense in</u> S if every region contains a region which does not intersect M.

Definition. If H and K are point sets, H is <u>nowhere dense in</u> K if every region intersecting K contains a region which intersects K but not H.

Theorem 55. If the point set H is nowhere dense in the point set K, then every subset of H is nowhere dense in K.

Theorem 56. If each of the point sets H and K is nowhere dense in the point set M, then $H + K$ is nowhere dense in M.

Theorem 57. No region is the sum of a countable number of point sets such that each of them is nowhere dense in S. (Baire)

Theorem 58. No closed point set M is the sum of a countable number of closed point set such that if x is any one of them, then every point of X is a limit point of $M - X$. (Baire)

Theorem 59. No closed point set M is the sum of a countable number of point sets such that each of them is nowhere dense in S. (Baire)

Definition. A point set H is said to be <u>everywhere dense</u> in a point set K if $\overline{H}$ contains K.

Theorem 60. If the point set H is everywhere dense in the point set K, then every region intersecting K contains a point of H.

Theorem 61. If the point set H is everywhere dense in the point set K, then H fails to be nowhere dense in K.

Theorem 62. If the point set H is nowhere dense in the closed point set M, then $M - H$ is everywhere dense in M.

Theorem 63. If G is a countable collection of open subsets of the point set M, each of which is everywhere dense in M, then the intersection of the elements of G is everywhere dense in M.

Definition. If the point set M contains a countable set which is everywhere dense in M, then M is said to be <u>separable</u>.

AXIOM 4. S is separable.

Theorem 64. There do not exist uncountable many mutually exclusive regions.

Question. Is Theorem 64 equivalent to Axiom 4? (Souslin question—undecidable)

Theorem 65. There exists an increasing unbounded sequence.

Theorem 66. If p is a point, there exists an increasing sequence converging to p.

Theorem 67. If p is a limit point of the point set M, there exists a sequence of distinct points in M converging to p.

Theorem 68. There exists a countable collection F of regions such that if R is a region and p is a point of R, then some region of F contains p and lies in R. (countable basis—second countability—perfectly separable)

Question. Is Theorem 68 equivalent to Axiom 4?

Theorem 69. If G is a collection of regions covering the point set M, then some countable subcollection of G covers M. (Lindelöf)

Theorem 70. Every uncountable point set contains a limit point of itself.

Theorem 71. Every point set is separable.

Theorem 72. There does not exist a collection of mutually exclusive intervals which covers S. (Baire)

Transformations

Definition. A <u>transformation</u> f of a point set H onto a point set K is a collection of ordered pairs of points such that:

(1) Each ordered pair in f has a point of H as its first element and a point of K as its second element.
(2) Every point of H is the first element of some ordered pair in f and every point of K is the second element of some ordered pair in f.
(3) No two pairs in f have the same first element.

For the following definitions and notation, let f be a transformation of a point set H onto a point set K.

Definition. If K is a subset of K', then f is said to be a transformation of H into K'.

Notation. If x is a point of H, then $f(x)$ denotes the second element of the ordered pair of f that has x as a first element.

If H' is a subset of H, then $f(H')$ denotes the set of all points $f(x)$ such that x is a point of H'.

If y is a point of K, then $f^{-1}(y)$ denotes the set of all points x in H such that $f(x) = y$.

If K' is a subset of K, then $f^{-1}(K')$ denotes the set of all points in H such that $f(x)$ is a point of K'.

Definition. The transformation f is said to be <u>one-to-one</u> if no two ordered pairs in f have the same second element.

The transformation f is said to be <u>continuous</u> if for any sequence $x_1, x_2, x_3, \dots$ of points in H converging to a point x in H, the sequence $f(x_1), f(x_2), f(x_3), \dots$ converges to $f(x)$.

Notation. If the transformation f is one-to-one, then f^{-1} denotes the collection of all ordered pairs $(y, f^{-1}(y))$ such that y is a point of K.

Definition. If the transformation f is one-to-one and both f and f^{-1} are continuous, then f is said to be a <u>homeomorphism</u> of H onto K, and H and K are said to be <u>homeomorphic</u>.

Theorem 73. If f is a continuous transformation of an interval H into itself, then there is a point x of H such that $f(x) = x$. (H is in a space satisfying Axioms 1, 2, 3, 4.)

Theorem 74. If f is a continuous transformation from a point set H into a point set K, then:

(1) If H is closed and bounded, then K is closed and bounded.

(2) If H is connected, then K is connected.

(3) If H is an interval, then K is either an interval or a point.

Definition. A transformation f of a space S_1 onto a space S_2 is said to preserve order if for any two points x and y in S_1 such that if $x < y$, then $f(x) < f(y)$.

Define reverse order similarly.

Theorem 75. If S_1 and S_2 are spaces satisfying Axioms 1, 2, 3, and 4 and f is an order preserving (reversing) transformation of S_1 onto S_2, then f is a homeomorphism.

Theorem 76. If S_1 and S_2 are spaces satisfying Axioms 1, 2, 3, and 4 and f is a homeomorphism of S_1 and S_2, then f either preserves order or reverses order.

Theorem 77. Every two spaces satisfying Axioms 1, 2, 3, and 4 are homeomorphic.

AXIOM 2′. S is nondegenerate and has a first point and a last point.

Note: It should be observed here that with the previous definition of a region, the first (last) point of S does not lie in a region. In addition to the sets previously defined as regions, consider the following sets as regions: For each point p different from the first (last) point of S, let the set of all points that precede (follow) p be a region.

Theorem 78. Every two spaces satisfying Axioms 1, 2′, 3, and 4 are homeomorphic.