**NATIONAL UNIVERSITY OF SINGAPORE**

**MA2101 - Linear Algebra II**

April 2015

Time allowed : 2 hours

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**INSTRUCTIONS TO STUDENTS**

1. Please write your matriculation number only. Do not write your name.
2. This examination paper contains **SEVEN** questions and comprises **FIVE** printed pages.
3. Students are required to answer **ALL** questions.
4. Please start each question on a **NEW** page.
5. This is a CLOSED BOOK (with helpsheet) examination.
6. You may use scientific calculators. However, you should lay out systematically the various steps in the calculations.

**Question 1** [14 marks]

Let $A \in M_3(\mathbf{R})$ be the following real matrix

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(i) Find the characteristic polynomial $p_A(x)$ of $A$.

(ii) Determine the minimal polynomial $m_A(x)$ of $A$. Justify your answer.

(iii) Is $A$ diagonalizable over $\mathbf{R}$? Justify your answer.

(iv) Let $W$ be the subspace of $M_3(\mathbf{R})$ spanned by $\{I_3, A, A^2, A^3, A^4, A^5, A^6, A^7, A^8, A^9\}$. Find a basis of $W$. Justify your answer.

**Question 2** [14 marks]

Let $A = (a_{ij}) \in M_3(\mathbf{R})$ be a real matrix and let

$$P := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

such that

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Let $y_i = y_i(x)$ $(1 \le i \le 3)$ be differentiable functions in $x$. Solve the following system of differential equations:

$$Y' = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = AY = A \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

**Note.** For the differential equation $z'(x) + p(x)\,z = q(x)$ you may assume, without proof, that its general solution is given as $z(x) = \frac{1}{\mu}(\int \mu\, q(x)\,dx + C)$ with $\mu := e^{\int p(x)\,dx}$.

**Question 3** [14 marks]

Let $T : V \to W$ be a linear transformation between two vector spaces over a field $F$.

(a) Show that there is a vector subspace $V_1$ of $V$ such that the restriction map

$$\begin{aligned} T_1 = T_{|V_1} : V_1 &\to W \\ \mathbf{v}_1 &\mapsto T(\mathbf{v}_1) \end{aligned}$$

satisfies **(ai)** and **(aii)** below.

(ai) $T_1$ is an injective linear transformation.

(aii) $\mathrm{Im}(T_1) = \mathrm{Im}(T)$.

For the $T_1 : V_1 \to W$ constructed in **(a)** above, prove the following:

(b) The intersection $V_1 \cap \mathrm{Ker}(T) = \{\mathbf{0}\}$.

(c) $V$ is the direct sum of $V_1$ and $\mathrm{Ker}(T)$:

$$V = V_1 \oplus \mathrm{Ker}(T).$$

**Note**. You **cannot** assume that $V$ or $W$ is finite-dimensional. You may assume that every vector space has a basis.

**Question 4** [14 marks]

Let $V$ be a vector space over a field $F$ and let $T_i : V \to V$ $(i = 1, 2)$ be two linear operators such that

$$T_1 \circ T_2 = T_2 \circ T_1.$$

Suppose that $\lambda_1 \in F$ is an eigenvalue of $T_1$ such that the eigenspace $V_{\lambda_1}(T_1) = \mathrm{Ker}(T_1 - \lambda_1 I_V)$ is 1-dimensional and is spanned by a vector $\mathbf{v}_1$.

(a) Show that $\mathbf{v}_1$ is also an eigenvector of $T_2$ so that $T_2(\mathbf{v}_1) = \lambda_2 \mathbf{v}_1$ for some $\lambda_2 \in F$.

(b) Let

$$S_1 := I_V + T_1 + 2T_1^2 + 3T_1^3 + 4T_1^4 + 5T_1^5, \quad S_2 := I_V + T_2 + 2T_2^2 + 3T_2^3 + 4T_2^4 + 5T_2^5.$$

Show that there is a common eigenvector $\mathbf{v}_2 \in V$ of $S_1$ and $S_2$ such that

$$S_1(\mathbf{v}_2) = \mu_1 \mathbf{v}_2, \quad S_2(\mathbf{v}_2) = \mu_2 \mathbf{v}_2$$

for some $\mu_1$, $\mu_2 \in F$.

(c) Determine $\mu_1$ and $\mu_2$ as functions of $\lambda_1$, $\lambda_2$.

**Note.** $T^n = T \circ \cdots \circ T$ denotes the composition of $n$ copies of the same $T$.

**Question 5** [14 marks]

Let $(V, \langle , \rangle)$ be a **complex** inner product space.

(a) Show that if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ then $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

(b) When $\mathbf{v} \neq \mathbf{0}$, find a scalar $\alpha$ such that $\mathbf{z} := \mathbf{u} - \alpha\mathbf{v}$ satisfies $\langle \mathbf{z}, \mathbf{v} \rangle = 0$.

(c) Prove the Cauchy-Schwarz inequality (the **complex** vector space version):

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|.$$

for all $\mathbf{u}$ and $\mathbf{v}$ in $V$.
(**Warning**: $V$ is a **complex** vector space and may not be defined over $\mathbf{R}$.)

(d) Show that if $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$ or $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$, then $|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\| \, \|\mathbf{v}\|$.

(e) Show that if $|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\| \, \|\mathbf{v}\|$, then either $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$ or $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$.

**Question 6** [14 marks]

Label the following statements as true (T) or false (F). *No* proof is required.

(i) A linear transformation $T : V \to W$ is an isomorphism if the representation matrix $A = [T]_{B_V, B_W}$ relative to bases $B_V$ of $V$ and $B_W$ of $W$ satisfies $AB = I$ (the identity matrix) for some matrix $B$.

(ii) If the characteristic polynomial $p_A(x)$ of a real matrix $A \in M_3(\mathbf{R})$ has only one real zero $\alpha$ and $\alpha$ is a simple zero of $p_A(x)$, then $A$ is diagonalizable over the complex field $\mathbf{C}$.

(iii) If $T : V \to V$ is an orthogonal linear operator and $B$ is a canonical basis for $T$, then the representation matrix $[T]_B$ relative to $B$ is an orthogonal matrix.

(iv) If a real matrix $A \in M_2(\mathbf{R})$ satisfies $A^3 - A = 0$ then $A$ is diagonalizable over $\mathbf{R}$.

(v) If $\mathbf{v}$ is an eigenvector of a complex matrix $A \in M_2(\mathbf{C})$, then the conjugate $\overline{\mathbf{v}}$ is an eigenvector of the adjoint matrix $A^*$.

(vi) If $A$ is a complex matrix with $A^*$ its adjoint, then every eigenvalue of $AA^*$ is a real number.

(vii) If a symmetric complex matrix $A \in M_2(\mathbf{C})$ has 2 distinct positive eigenvalues, then $A$ is a positive definite matrix.

**Question 7** [16 marks]

Let $V$ be an $n$-dimensional complex vector space and let $T : V \to V$ a linear operator. Let $\lambda_i$ $(1 \le i \le k)$ exhaust all distinct eigenvalues of $T$ and let

$$p(x) = p_T(x) = \prod_{i=1}^{k}(x-\lambda_i)^{n_i}, \quad m(x) = m_T(x) = \prod_{i=1}^{k}(x-\lambda_i)^{m_i}$$

be the characteristic polynomial and minimal polynomial of $T$. Set

$$V_i := \mathrm{Ker}(T - \lambda_i I_V)^{n_i}$$

which is known to be a vector subspace of $V$.

(a) Show that the vector subspace $V_i$ of $V$ is $T$-invariant.

(b) Show that $\mathrm{Ker}(T - \lambda_i I_V)^{m_i} = V_i$ for all $i$.

(c) Show that the sum $\sum_{i=1}^{k} V_i$ which is known to be a vector subspace of $V$, is the direct sum of $V_i$:

$$\sum_{i=1}^{k} V_i = \oplus_{i=1}^{k} V_i.$$

(d) Let $q_i(x) = m_T(x)/(x-\lambda_i)^{m_i} = \prod_{j \ne i}(x-\lambda_j)^{m_j}$ and consider the linear transformation:

$$\begin{aligned} q_i(T) : V &\to V \\ \mathbf{v} &\mapsto q_i(T)(\mathbf{v}). \end{aligned}$$

Show that $\mathrm{Im}\, q_i(T) \subseteq V_i$.

(e) Show that $V$ is the direct sum of $V_i$:

$$V = \oplus_{i=1}^{k} V_i.$$

(f) Show that the characteristic polynomial and minimal polynomial of the restriction map

$$\begin{aligned} T_i = T_{|V_i} : V_i &\to V_i \\ \mathbf{v} &\mapsto T(\mathbf{v}) \end{aligned}$$

are given as follows:

$$p_{T_i}(x) = (x-\lambda_i)^{n_i}, \quad m_{T_i}(x) = (x-\lambda_i)^{m_i}.$$

**Note.** Given $s$ complex polynomials $f_i(x)$ $(1 \le i \le s)$ with no common zero, you may assume, without proof, that $\sum_{i=1}^{s} f_i(x)u_i(x) = 1$ for some polynomials $u_i(x)$.

END OF PAPER