NATIONAL UNIVERSITY OF SINGAPORE

MA2101 - Linear Algebra II

May 2017

Time allowed : 2 hours

INSTRUCTIONS TO STUDENTS

1. Please write your matriculation number only. Do not write your name.
2. This examination paper contains **EIGHT** questions and comprises **FIVE** printed pages.
3. Students are required to answer **ALL** questions.
4. Please start each question on a **NEW** page.
5. This is a CLOSED BOOK (with helpsheet) examination.
6. You may use scientific calculators. However, you should lay out systematically the various steps in the calculations.

Question 1 [12 marks]

Let $A = (a_{ij}) \in M_2(\mathbf{C})$ be a complex matrix of size 2×2 and let

$$P := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

such that

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Let $y_i = y_i(x)$ $(i = 1, 2)$ be differentiable functions in x. Solve the following system of differential equations:

$$Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = AY = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Hint. You may assume the solution of the differential equation

$$z'(x) + p(x)\, z = q(x)$$

is given by

$$z = \frac{1}{\mu}\Big(\int \mu\, q(x)\, dx + C\Big)$$

where

$$\mu := e^{\int p(x)\, dx}.$$

Question 2 [12 marks]

Let $A \in M_n(\mathbf{C})$ be a complex matrix of size $n \times n$. Let $A^* = \overline{(A^t)}$ be the conjugate transpose of A.

(a) Prove the existence of a unitary matrix U such that $U^{-1}(A\,A^*)U$ is equal to a diagonal matrix D.

(b) If $AA^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, find U and D in (a).

Question 3 [12 marks]

Let $A, B \in M_n(\mathbf{C})$ be complex matrices of the same size $n \times n$. Suppose that B is obtained from A by elementary row operations $Op_1, \dots, Op_r$ corresponding to elementary matrices $E_1, \dots, E_r$. Namely, $B = E_r \cdots E_1 A$. It is known that each E_i is invertible.

(a) State the definition of $rank(A)$.

(b) Show that a relation

$$\sum_{j=1}^{n} c_j \mathbf{b}_j = \mathbf{0}$$

holds among columns $\mathbf{b}_j$ of B if and only if the same relation

$$\sum_{j=1}^{n} c_j \mathbf{a}_j = \mathbf{0}$$

holds among the corresponding columns $\mathbf{a}_j$ of A.

(c) Prove that $rank(A) = rank(B)$.

Question 4 [12 marks]

Let V be a vector space over a field F. Let V_1 be a subset of V. Let $T : V \to V$ be a map. Set $T^n := T \circ \cdots \circ T$, the composition of n-copies of the same T, and

$$R_n := \operatorname{Im} T^n = \{T^n(\mathbf{v}) \mid \mathbf{v} \in V\} \subseteq V.$$

(a) State the definition for V_1 to be a vector subspace of V.

(b) State the definition for T to be a linear transformation.

In the following, assume that T is a linear transformation.

(c) Show that R_n is a vector subspace of V for all $n \geq 1$.

(d) Show that $R_{m+1} \subseteq R_m$ for all $m \geq 1$.

(e) Show that $R_s = R_{s+1} = R_{s+2} = \cdots$ for some $s \geq 1$, when $\dim V$ is finite.

(f) If $\dim V = \infty$, can you still say that $R_s = R_{s+1}$ for some $s \geq 1$?

Justify your answer by proving it, or giving a **concrete** counterexample.

Question 5 [12 marks]

Let $A \in M_4(\mathbf{C})$ be a complex matrix of size 4×4. Suppose that

$$(A^2 - I_4)^2 = 0.$$

Suppose further that A is diagonalizable over $\mathbf{C}$.

(a) State the definitions of the minimal polynomial $m_A(x)$ of A and the characteristic polynomial $p_A(x)$ of A.

(b) Find all possible minimal polynomials $m_A(x)$ of A. Justify your answers.

(c) Find all possible characteristic polynomials $p_A(x)$ of A. Justify your answers.

(d) Show that A is invertible.

(e) Find a polynomial $g(x)$ in $\mathbf{C}[x]$ such that $A^{-1} = g(A)$.

Question 6 [12 marks]

Let $(V, \langle\,\rangle)$ be a complex inner product space. Let $T : V \to V$ be a linear operator.

(a) Let $\mathbf{u}_0, \mathbf{v}_0 \in V$. Show that if $\langle \mathbf{u}_0, \mathbf{v} \rangle = 0$ for all $\mathbf{v}$ in V then $\mathbf{u}_0 = \mathbf{0}$, and that if $\langle \mathbf{u}, \mathbf{v}_0 \rangle = 0$ for all $\mathbf{u}$ in V then $\mathbf{v}_0 = \mathbf{0}$.

(b) Prove the existence of a linear operator $S : V \to V$ such that

$$\langle \mathbf{u}, T(\mathbf{v}) \rangle = \langle S(\mathbf{u}), \mathbf{v} \rangle, \quad (\forall\, \mathbf{u},\, \mathbf{v} \in V).$$

(c) Prove that the S in (b) above is unique.

Question 7 [14 marks]

Let $A \in M_n(\mathbf{C})$ be a complex matrix of size $n \times n$. Let $A^* = \overline{(A^t)}$ be the conjugate transpose of A.

(a) State the definition for A to be a unitary matrix.

In the following, assume A is unitary. Equip the complex column n-space

$$V := \mathbf{C}_c^n$$

with the standard inner product $\langle , \rangle$ and norm $\| \|$.

(b) Prove that

$$(A - \alpha I_n)(A^* - \overline{\alpha} I_n) = (A^* - \overline{\alpha} I_n)(A - \alpha I_n)$$

for every complex scalar α.

(c) Is it true that $\|AX\| = \|A^*X\|$ for all $X \in V$?

Justify your answer by proving it, or giving a **concrete** counterexample.

(d) If $AY = \lambda Y$ for some $Y \in V$ and scalar $\lambda \in \mathbf{C}$, is it true that $A^*Y = \lambda Y$?

Justify your answer by proving it, or giving a **concrete** counterexample.

(e) If Z is an eigenvector of A^*, is it true that Z is also an eigenvector of A?

Justify your answer by proving it, or giving a **concrete** counterexample.

Question 8 [14 marks]

Let $n \geq 1$. Let $A \in M_n(\mathbf{C})$ be a complex matrix of size $n \times n$.

(a) State the definition for A to be a positive definite matrix.

(b) State the definition for a complex vector space V to be an inner product space.

(**Warning**: You are not supposed to assume V is a column space).

(c) Let

$$W := \mathbf{C}_c^n$$

be the complex column n-space. Show that the function H on W below

$$\begin{aligned} H : W \times W &\to \mathbf{C} \\ (X, Y) &\mapsto \langle X, Y \rangle := (AX)^t\, \overline{Y} = X^t A^t \overline{Y} \end{aligned}$$

defines an inner product on W if and only if A is positive definite.

END OF PAPER