NATIONAL UNIVERSITY OF SINGAPORE

MA2101 - Linear Algebra II

December 2015

Time allowed : 2 hours

INSTRUCTIONS TO STUDENTS

1. Please write your matriculation number only. Do not write your name.
2. This examination paper contains **EIGHT** questions and comprises **FOUR** printed pages.
3. Students are required to answer **ALL** questions.
4. Please start each question on a **NEW** page.
5. This is a CLOSED BOOK (with helpsheet) examination.
6. You may use scientific calculators. However, you should lay out systematically the various steps in the calculations.

Question 1 [12 marks]

Let $A \in M_2(\mathbf{R})$ be the following symmetric real matrix

$$A = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}.$$

Find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Question 2 [12 marks]

Let $A = (a_{ij}) \in M_2(\mathbf{R})$ be a real matrix and let

$$P := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

such that

$$P^{-1}AP = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Let $y_i = y_i(x)$ $(i = 1, 2)$ be differentiable functions in x. Solve the following system of differential equations:

$$Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = AY = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Note. For the differential equation $z'(x) + p(x)\, z = q(x)$ you may assume, without proof, that its general solution is given as $z(x) = \frac{1}{\mu}(\int \mu\, q(x)\, dx + C)$ with $\mu := e^{\int p(x)\, dx}$.

Question 3 [12 marks]

Let U and V be vector spaces over a scalar field F, let $T : U \to V$ be a surjective linear transformation and let W be a vector subspace of V.

(i) Show that the pre-image

$$T^{-1}(W) := \{\mathbf{u} \in U \mid T(\mathbf{u}) \in W\}$$

of W is a vector subspace of U.

(ii) Show that

$$\dim T^{-1}(W) + \dim V = \dim W + \dim U.$$

Warning: $\dim U$ or $\dim V$ might be infinite.

Question 4 [12 marks]

Let $Q \in M_3(\mathbf{R})$ be an orthogonal real matrix of order 3. Let

$$p_Q(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$$

be the characteristic polynomial of Q, where $\lambda_i \in \mathbf{C}$.

(i) Show that $\lambda_i^2 = 1$ for at least one of the λ_i.

(ii) Is it true that $\lambda_i^2 = 1$ for all i? If your answer is 'yes', prove it; if your answer is 'no', provide a **concrete** counterexample.

Question 5 [12 marks]

Let $(V, \langle , \rangle)$ be a real inner product space. Let $T : V \to V$ be a linear operator and T^* the adjoint of T. Let W be a T^*-invariant vector subspace of V, and let

$$W^\perp := \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0, \forall\, \mathbf{w} \in W\rangle$$

be the orthogonal complement of W.

(i) Show that $W^\perp$ is vector subspace of V.

(ii) Is $W^\perp$ a T-invariant subspace of V? If your answer is 'yes', prove it; if your answer is 'no', provide a **concrete** counterexample.

(iii) Is $W^\perp$ a T^*-invariant subspace of V? If your answer is 'yes', prove it; if your answer is 'no', provide a **concrete** counterexample.

Question 6 [12 marks]

Let $A \in M_n(\mathbf{C})$ be a complex matrix of order $n \geq 9$ and let

$$f(x) := (x - 1)^2(x - 2)^3(x - 3)^4.$$

Suppose that A is self-adjoint and $f(A) = 0$. Find all possible minimal polynomials $m_A(x)$ of A.

Question 7 [14 marks]

Let $T : V \to V$ be a linear operator. For positive integer n, let $T^n := T \circ \cdots \circ T$ be the composition of n-copies of the same T and set

$$K_n := \mathrm{Ker}(T^n).$$

(i) Show that $K_m \subseteq K_{m+1}$ for all $m \geq 1$.

(ii) Show that

$$K_r = K_{r+1} = K_{r+2} = \cdots$$

for some $r \geq 1$, when V is finite-dimensional.

(iii) If V is infinite-dimensional, can one still say that $K_r = K_{r+1}$ for some $r \geq 1$? If your answer is 'yes', prove it; if your answer is 'no', provide a **concrete** counterexample.

Question 8 [14 marks]

Let $A \in M_n(\mathbf{C})$ be a matrix of order $n \geq 2$. Let

$$p_A(x) = (x - \lambda_1) \cdots (x - \lambda_n)$$

be the characteristic polynomial of A such that all λ_i are positive real numbers.

(a) When A is a real matrix, is A then a positive definite matrix? If your answer is 'yes', prove it; if your answer is 'no', provide a **concrete** counterexample.

(b) Suppose that A is a normal matrix. Prove that one can write:

(bi) $A = G^4$ for some self-adjoint matrix G, and

(bii) $A = H^* H$ for some invertible matrix H.

END OF PAPER