PYP Answer - MA2101 AY1516Sem2

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1. (a) We shall show that T respects addition and scalar multiplication.
Observe, for $X, Y \in \mathbb{M}_2(\mathbb{R})$ and $c \in \mathbb{R}$,

$$\begin{aligned} T(X+Y) &= A(X+Y) - (X+Y)A = AX + AY - XA - YA \\ &= (AX - XA) + (AY - YA) = T(X) + T(Y) \end{aligned}$$

and

$$T(cX) = A(cX) = cAX = c(AX) = cT(X)$$

(b) i. Denote the vectors in $\mathcal{B}$ in the same sequence as that in the question, as b_1, b_2, b_3, b_4.
We have

$$[T]_{\mathcal{B}} = ([T(b_1)]_{\mathcal{B}}, [T(b_2)]_{\mathcal{B}}, [T(b_3)]_{\mathcal{B}}, [T(b_4)]_{\mathcal{B}})$$

Here, for instance,

$$T(b_1) = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0b_1 + 0b_2 + 0b_3 + 0b_4$$

Therefore, $[T(b_1)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Using the same process, we have

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

ii.

$$\begin{aligned} \det(T - xI) &= \det([T]_{\mathcal{B}} - xI) \\ &= \det \begin{pmatrix} -x & 0 & 0 & 0 \\ 0 & -3-x & 0 & 0 \\ 0 & 0 & 3-x & 0 \\ 0 & 0 & 0 & -x \end{pmatrix} \\ &= x^2(x+3)(x-3) \end{aligned}$$

2. (a)

$$\begin{aligned}
& T \text{ is injective} \\
\Leftrightarrow\ & \operatorname{Ker}(T) = \{0\}. \\
\Leftrightarrow\ & T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \text{ has only solution } \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \\
\Leftrightarrow\ & aX + bY + cZ = 0 \text{ only has solution } (a,b,c) = (0,0,0) \\
\Leftrightarrow\ & X, Y, Z \text{ linearly independent}
\end{aligned}$$

(b) i. The rank of T equals to the rank of $[T]_{\mathcal{A}}^{\mathcal{B}}$ in (ii), so it has rank 2. Use rank-nullity theorem, nullity $= \dim \mathbb{R}^3 - \text{rank} = 3 - 2 = 1$.

ii. Let $\mathcal{A} = (a_1, a_2, a_3)$ and $\mathcal{B} = (b_1, b_2, b_3, b_4)$.
We have

$$[T]_{\mathcal{A}}^{\mathcal{B}} = ([T(a_1)]_B, [T(a_2)]_B, [T(a_3)]_B)$$

For instance,

$$T(a_1) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + (-1)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = b_2 + (-1)b_4$$

Therefore, $[T(a_1)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$.
Using the same process, we have

$$[T]_{\mathcal{A}}^{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

3. (a) Since V is finite dimensional, we apply inducion on k. The claim is trivial when $k = 1$.
Now suppose that the claim holds when $k = r$. We proceed to show that the claim is true when $k = r + 1$.
From assumption, we have

$$\sum_{i=1}^{r} \alpha_i v_i = 0$$

has only trivial solutions. Then for the case $k = r + 1$. Suppose $v_1, \ldots, v_r, v_{r+1}$ is not linearly independent, then, after relabelling

$$v_{r+1} = \sum_{i=1}^{r} \alpha_i v_i$$

, where $\alpha_i \neq 0$ for some $1 \leq i \leq r$.
Apply T on both sides yields

$$\lambda_{r+1} w_{r+1} = \sum_{i=1}^{r} \alpha_i \lambda_i w_i \qquad (1)$$

whereas multiply λ_{r+1} on both sides yields

$$\lambda_{r+1} w_{r+1} = \sum_{i=1}^{r} \alpha_i \lambda_{r+1} w_i \qquad (2)$$

$(2) - (1)$, we have

$$0 = \sum_{i=1}^{r} \alpha_i (\lambda_i - \lambda_{r+1}) w_i$$

Since by assumption, $w_i, i = 1, \ldots, r$ are independent, we have

$$\alpha_i(\lambda_i - \lambda_{r+1}) = 0 \text{ for all } i = 1, \ldots, n$$

and for those i, where $\alpha_i \neq 0$, we have

$$\lambda_i = \lambda_{r+1}$$

which is a contradiction about the pairwise distinctiveness of λ_i.Therefore, the claim must be true for $k = r+1$, and the claim for any k is true.

(b) i. We first find the eigenvalues. Let λ be one eigenvalue, so $Tv = \lambda v$. We have $v^t = \lambda v$, and $v = \lambda v^t$ by taking transpose. It yields,

$$v^t = \lambda^2 v^t$$

which implies the eigenvalues are only $\lambda = \pm 1$.
Solving $Av = \lambda v$ for each λ, we will have

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} -1 &&&&&&&& \\ & -1 &&&&&&& \\ && -1 &&&&&& \\ &&& 1 &&&&& \\ &&&& 1 &&&& \\ &&&&& 1 &&& \\ &&&&&& 1 && \\ &&&&&&& 1 & \\ &&&&&&&& 1 \end{pmatrix}$$

ii.

$$\det(T) = \det(D) = (-1)^3 \times 1^6 = -1$$

4. (a) Denote $B = \{v_1, v_2, v_3\}$. Clearly, $\lambda neq 0$. Let $k = \frac{1}{\lambda}$.
The claim is then equivalent to: There are at most three distinct value of $k \in \mathbb{C}$ such that $\{kv_1 + p_1, kv_2 + p_2, kv_3 + p_3\}$ fails to be a basis of V.
Let us represent p_i in terms of basis B.

$$\begin{aligned} p_1 &= a_1 v_1 + a_2 v_2 + a_3 v_3 \\ p_2 &= b_1 v_1 + b_2 v_2 + b_3 v_3 \\ p_3 &= c_1 v_1 + c_2 v_2 + c_3 v_3 \end{aligned}$$

Then the claim is equivalent to: There are at most three distinct values of $k \in \mathbb{C}$ such that $\{(k+a_1)v_1 + a_2 v_2 + a_3 v_3, b_1 v_1 + (k+b_2)v_2 + b_3 v_3, c_1 v_1 + c_2 v_2 + (k+c_3)v_3\}$ fails to be a basis for V.
The above claim is equivalent to: There are at most three distinct values of $k \in \mathbb{C}$ such that

$$\det \begin{pmatrix} k+a_1 & a_2 & a_3 \\ b_1 & k+b_2 & b_3 \\ c_1 & c_2 & k+c_3 \end{pmatrix} = 0 \quad (\#)$$

The determinant is a polynomial of degree 3 in k, and by Fundamental Theorem of Algebra, it has most three roots. This proves our claim.

(b) Note that

$$\begin{aligned} p_1 &= v_1 + v_2 \\ p_2 &= v_1 + v_3 \\ p_3 &= v_2 + v_3 \end{aligned}$$

So,

$$\begin{matrix} a_1 = 1 & a_2 = 1 & a_3 = 0 \\ b_1 = 1 & b_2 = 0 & b_3 = 1 \\ c_1 = 0 & c_2 = 1 & c_3 = 1 \end{matrix}$$

And substituting in the (#) equation,

$$\det\begin{pmatrix} k+1 & 1 & 0 \\ 1 & k & 1 \\ 0 & 1 & k+1 \end{pmatrix} = 0$$

we have $k = -2, -1, 1$. So $\lambda = -\frac{1}{2}, -1, 1$.

5. (a) We shall show that $\operatorname{Tr}(Y^*X)$ satisfies sesquilinearity, symmetry and positivity.

- Sesquilinearity

$$\begin{aligned} &\langle a_1X_1 + a_2X_2, Y\rangle \\ =& \operatorname{Tr}(Y^*(a_1X_1 + a_2X_2)) \\ =& a_1 \operatorname{Tr}(Y^*X_1) + a_2 \operatorname{Tr}(Y^*X_2) \\ =& a_1\langle X_1, Y\rangle + a_2\langle X_2, Y\rangle \end{aligned}$$

$$\begin{aligned} &\langle X, b_1Y_1 + b_2Y_2\rangle \\ =& \operatorname{Tr}(((b_1Y_1 + b_2Y_2)^*)X) \\ =& \operatorname{Tr}((\overline{b_1}Y_1^* + \overline{b_2}Y_2^*)X) \\ =& \overline{b_1} \operatorname{Tr}(Y_1^*X) + \overline{b_2} \operatorname{Tr}(Y_2^*X) \\ =& \overline{b_1}\langle X_1, Y\rangle + \overline{b_2}\langle X_2, Y\rangle \end{aligned}$$

- Symmetry

$$\begin{aligned} &\overline{\langle Y, X\rangle} \\ =& \overline{\operatorname{Tr}(X^*Y)} \\ =& \operatorname{Tr}(\overline{X^*Y}) \\ =& \operatorname{Tr}(X^t\overline{Y}) \\ =& \operatorname{Tr}((X^t\overline{Y})^t) \\ =& \operatorname{Tr}(Y^*X) \\ =& \langle X, Y\rangle \end{aligned}$$

- Positivity

$$\begin{aligned} \langle X, X\rangle &= \operatorname{Tr}(X^*X) \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} X_{kl}\overline{X_{kl}} > 0 \end{aligned}$$

with equality if and only if $X_{kl} = 0$ for all k, l i.e., X is the $\mathbf{0}$ matrix.

(b) Applying Gram Schmidt Process employed with the above inner-product.

$$\begin{aligned} v_1 &= A \\ v_2 &= B - \frac{\langle B, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ i & 0 & 0 \\ 0 & -i & -1 \end{pmatrix} - \frac{2}{4} v_1 \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}i & 0 \\ i & 0 & \frac{1}{2}i \\ 0 & -i & -\frac{1}{2} \end{pmatrix} \end{aligned}$$

Normalise v_1 and v_2, we have

$$b_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{4}} \frac{1}{2} \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$b_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}i & 0 \\ i & 0 & \frac{1}{2}i \\ 0 & -i & -\frac{1}{2} \end{pmatrix}$$