# PYP Answer - MA2216 AY1617Sem2

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1. (a) We can get the cdf of $X$ by integrating $f_X(x)$:

$$F_X(x) = \begin{cases} 0 & x \leq -1 \\ \frac{1}{4}(1+x)^2 & -1 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

Since $Y = X^2$, we see that $F_Y(y) = P(Y < y) = P(X^2 < y)$. Therefore,

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ P(-\sqrt{y} < x < \sqrt{y}) = \frac{1}{4}(1+\sqrt{y})^2 - \frac{1}{4}(1-\sqrt{y})^2 = \sqrt{y} & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

By differentiating the cdf of $Y$, we have

$$f_Y(y) = \begin{cases} \frac{1}{2}y^{-\frac{1}{2}} & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(b) As $y$ vanishes in region other than $[0,1]$, we can evaluate $\mathrm{E}(Y)$ and $\mathrm{var}(Y)$ in $[0,1]$ only.

$$\mathrm{E}(Y) = \int_0^1 y f_Y(y)\mathrm{d}y = \int_0^1 \frac{1}{2}y^{\frac{1}{2}}\mathrm{d}y = \left[\frac{1}{3}y^{\frac{3}{2}}\right]_0^1 = \frac{1}{3}$$

And similarly,

$$\mathrm{var}(Y) = \int_0^1 (y^2 - \mathrm{E}(Y)^2) f_Y(y)\mathrm{d}y = \int_0^1 (y^2 - \frac{1}{9})\frac{1}{2}y^{-\frac{1}{2}}\mathrm{d}y = \frac{4}{45}$$

2. (a) Since $Y \sim \text{Poisson}(n)$, we can express $Y = \sum_{i=1}^n Y_i$, where $Y_i \sim \text{Poisson}(1)$ for $i = 1, \ldots, n$. We note $Y_i$ has mean $\mu = 1$ and variance $\sigma^2 = 1$.
When $n$ is large, apply Central Limit Theorem on $Y_i, i = 1, \ldots, n$, we have

$$\frac{\sum_{i=1}^n Y_i - n(1)}{\sqrt{1}\sqrt{n}} = \frac{Y-n}{\sqrt{n}} \sim Z(0,1)$$

which implies $P(Y < y) \approx \Phi(\frac{y-n}{\sqrt{n}})$.
By continuity correction,

$$P(Y = n) = P(Y < n + \frac{1}{2}) - P(Y < n - \frac{1}{2}) = \Phi(\frac{1}{2\sqrt{n}}) - \Phi(\frac{-1}{2\sqrt{n}}) \quad (\#)$$

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(b) On one hand, $P(Y=n)=\frac{e^{-n}n^n}{n!}$, since $Y\sim \text{Poisson}(n)$.
On the other hand, $P(Y=n)$ can be evaluated using (#). When $n$ is large, $\frac{1}{2\sqrt{n}}$ is small, so the RHS of (#) becomes the area of rectangle on the pdf of $Z(0,1)$ distribution with base length $(\frac{1}{2\sqrt{n}}-(-\frac{1}{2\sqrt{n}}))=\frac{1}{\sqrt{n}}$ and height given by the pdf $\frac{1}{\sqrt{2\pi}}$.Therefore, we have

$$\frac{\left(\frac{n}{e}\right)^n}{n!}=\frac{1}{\sqrt{2\pi n}}$$

Rearranging the above equation implies the result.

3. (a) $\mathrm{E}(X_1)=0, \mathrm{var}(X_1)=1, \mathrm{cov}(X_1,X_2)=\rho$.

(b) We calculate $\mathrm{E}(Z)$ by conditioning on $X_1$:

$$\begin{aligned}\mathrm{E}(Z)&=\mathrm{E}(\mathrm{E}[Z|X_1])\\&=\mathrm{E}(Z|X_1<X_2)P(X_1<X_2)+\mathrm{E}(Z|X_1\geq X_2)P(X_1\geq X_2)\\&=\mathrm{E}(X_2)P(X_1<X_2)+\mathrm{E}(X_1)P(X_1\geq X_2)\end{aligned}$$

Next, calculate $\mathrm{E}(Y)$ by conditioning also on $X_1$:

$$\begin{aligned}\mathrm{E}(Y)&=\mathrm{E}(\mathrm{E}[Y|X_1])\\&=\mathrm{E}(Y|X_1<X_2)P(X_1<X_2)+\mathrm{E}(Y|X_1\geq X_2)P(X_1\geq X_2)\\&=\mathrm{E}(X_1)P(X_1<X_2)+\mathrm{E}(X_2)P(X_1\geq X_2)\end{aligned}$$

Adding the two equations above, we have

$$\begin{aligned}\mathrm{E}(Y)+\mathrm{E}(Z)&=\mathrm{E}(X_1)(P(X_1<X_2)+P(X_1\geq X_2))+\mathrm{E}(X_2)(P(X_1<X_2)+P(X_1\geq X_2))\\&=\mathrm{E}(X_1)+\mathrm{E}(X_2)\end{aligned}$$

(c) We recognise that $X_1$ follows a normal distribution with mean 0.
By Chebyshev's inequality,

$$P(|X_1-0|\geq 3)\leq\frac{1^2}{3^2}=\frac{1}{9}$$

Therefore, $P(X_1\geq 3)\leq\frac{1}{18}$, by symmetry of normal distribution.

(d) We have, from the joint density,

$$\begin{aligned}\mathbb{P}\left\{\min(X_1,X_2)\leq y\right\}&=1-\mathbb{P}\left\{X_1>y,X_2>y\right\}\\&=1-\int_y^\infty\int_y^\infty f_{X_1,X_2}(s,t)dsdt\\&=1-\int_y^\infty f_{X_2}(t)\int_y^\infty f_{X_1|X_2}(s,t)dsdt\end{aligned}$$

where

$$f_{X_2}(t)=\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}},\qquad f_{X_1|X_2}(s,t)=\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{(s-\rho t)^2}{2(1-\rho^2)}}$$

and so

$$\mathbb{P}\left\{\min(X_1, X_2) \leq y\right\} = 1 - \int_y^\infty \varphi(t)\left(1 - \Phi\left(\frac{y - \rho t}{\sqrt{1-\rho^2}}\right)\right) dt.$$

$$= 1 - \int_y^\infty \varphi(t)\Phi\left(\frac{\rho t - y}{\sqrt{1-\rho^2}}\right) dt$$

To get the density we differentiate with respect to $y$ giving

$$\begin{aligned} f_Y(y) &= -\frac{\partial}{\partial y}\int_y^\infty \varphi(t)\Phi\left(\frac{\rho t - y}{\sqrt{1-\rho^2}}\right) dt \\ &= \varphi(y)\Phi\left(\frac{\rho y - y}{\sqrt{1-\rho^2}}\right) + \int_y^\infty \varphi(t)\frac{1}{\sqrt{1-\rho^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\rho t - y)^2}{2(1-\rho^2)}} dt \qquad (1) \end{aligned}$$

Completing the square of the last term in the above equation we have

$$\begin{aligned} \varphi(t)\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{(\rho t-y)^2}{2(1-\rho^2)}} &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{1}{2(1-\rho^2)}\left((1-\rho^2)t^2+(\rho t-y)^2\right)} \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{1}{2(1-\rho^2)}\left(t^2-2t\rho y+\rho^2y^2+(1-\rho^2)y^2\right)} \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{(t-\rho y)^2}{2(1-\rho^2)}}. \end{aligned}$$

So putting this back in to (1) we get

$$\begin{aligned} f_Y(y) &= \varphi(y)\Phi\left(\frac{\rho y - y}{\sqrt{1-\rho^2}}\right) + \varphi(y)\int_y^\infty \frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{(t-\rho y)^2}{2(1-\rho^2)}} dt \\ &= \varphi(y)\Phi\left(\frac{\rho y - y}{\sqrt{1-\rho^2}}\right) + \varphi(y)\left(1 - \Phi\left(\frac{y-\rho y}{\sqrt{1-\rho^2}}\right)\right) \\ &= 2\varphi(y)\Phi\left(\frac{\rho y - y}{\sqrt{1-\rho^2}}\right). \end{aligned}$$

4. (a) We note that

$$P(X = x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}$$

and

$$P(X = x - 1) = \frac{\binom{m}{x-1}\binom{N-m}{n-x+1}}{\binom{N}{n}}$$

So

$$\frac{P(X=x)}{P(X=x-1)} = \frac{\frac{m!}{x!(m-x)!}\frac{(N-m)!}{(N-m-n+x)!(n-x)!}}{\frac{m!}{(x-1)!(m-x+1)!}\frac{(N-m)!}{(N-m-n+x-1)!(n-x+1)!}}$$
$$= \frac{m-x+1}{x} \cdot \frac{n-x+1}{N-m-n+x} = (\#)$$

Similarly,

$$\frac{P(Y=x)}{P(Y=x-1)} = \frac{\binom{n}{x}\left(\frac{m}{n}\right)^x\left(\frac{N-m}{N}\right)^{n-x}}{\binom{n}{x-1}\left(\frac{m}{n}\right)^{x-1}\left(\frac{N-m}{N}\right)^{n-x+1}}$$
$$= \frac{n-x+1}{x}\frac{m}{N-m} = (*)$$

To show the equality of limit, write

$$\lim_{m,N\to\infty}(\#) = \frac{n-x+1}{x}\frac{1-\frac{x-1}{m}}{\frac{N}{m}-1-\frac{n-x}{m}}$$
$$= \frac{n-x+1}{x}\frac{1}{p-1}$$
$$= \frac{n-x+1}{x}\frac{m}{N-m} = (*)$$

(b) Let the number of cocainer packets be $m$. Then the scenario will have the following possibilities to occur

$$P(X=4)\times P(X=0) \approx \binom{4}{4}\left(\frac{m}{496}\right)^4\binom{2}{0}\left(\frac{492-(m-4)}{492}\right)$$

By differentiating the expression to solve for maximum, we have $m=\frac{992}{3}$, which gives the number 0.0233 if substituted back.

(c) So $20-6=14$ more packets is sufficient.