NATIONAL UNIVERSITY OF SINGAPORE

MA2101 - Linear Algebra II

(Semester 2 : AY2013/2014)

Time allowed : 2 hours

INSTRUCTIONS TO STUDENTS

1. Please write your matriculation number only. Do not write your name.
2. This examination paper contains **SEVEN** questions and comprises **FOUR** printed pages.
3. Students are required to answer **ALL** questions.
4. Students should write the answers for each question on a **NEW** page.
5. This is a CLOSED BOOK (with helpsheet) examination.
6. You may use scientific calculators. However, you should lay out systematically the various steps in the calculations.

Question 1 [14 marks]

Let $f : V \to W$ be a linear transformation between two vector spaces over a field F, which are not necessarily finite-dimensional. Let

$$T = \{\mathbf{t}_\alpha \mid \alpha \in I\} \subseteq W$$

be a subset (of W) which is not necessarily finite. Take $\mathbf{s}_\alpha \in V$ such that

$$f(\mathbf{s}_\alpha) = \mathbf{t}_\alpha \quad (\alpha \in I).$$

Set $S := \{\mathbf{s}_\alpha \mid \alpha \in I\}$.

(a) If T is a linearly independent subset of W, show that $\dim \mathrm{Span}(S) = \dim \mathrm{Span}(T)$.

(b) If T is linearly dependent, is it true that $\dim \mathrm{Span}(S) = \dim \mathrm{Span}(T)$? Justify your answer by either proving it or disproving it by a *concrete* counterexample.

Question 2 [14 marks]

Let $T : V_1 \to V_2$ be a linear transformation between two finite-dimensional vector spaces over a field F. Let B_i be a basis of V_i for $i = 1, 2$. Let

$$A := [T]_{B_1, B_2}$$

be the representation matrix of T relative to the bases B_i of V_i. Prove that T is an isomorphism if and only if A is an invertible (square) matrix over F.

Question 3 [14 marks]

Let $A_i \in M_n(F)$ be two similar matrices over a field F: $A_1 \sim A_2$. Prove the following:

(i) The characteristic polynomials are identical: $p_{A_1}(x) = p_{A_2}(x)$.

(ii) The traces are the same: $\mathrm{Tr}(A_1) = \mathrm{Tr}(A_2)$.

(iii) The determinants are the same: $|A_1| = |A_2|$.

(iv) The minimal polynomials are identical: $m_{A_1}(x) = m_{A_2}(x)$.

(v) A_1 is diagonalizable over F if and only if so is A_2.

(vi) A_1 has a Jordan canonical form over F if and only if so does A_2.

Question 4 [14 marks]

Let $A \in M_n(\mathbf{R})$ be an $n \times n$ real matrix.

(i) Prove that there is an orthogonal matrix P such that $P^{-1}(A\,A^t)P$ is equal to a diagonal matrix D.

(ii) If $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, find P and D in (i).

Question 5 [14 marks]

Let $A \in M_5(\mathbf{R})$ be a 5×5 real matrix such that $(A^2 - I)(A^2 - 4I) = 0$.

(i) Prove that A is diagonalizable over $\mathbf{R}$.

(ii) Prove that the determinant $|A|$ is a real number satisfying $-32 \le |A| \le 32$.

(iii) Prove that A is invertible and find a polynomial $g(x)$ such that $A^{-1} = g(A)$.

Question 6 [14 marks]

Let V be a finite-dimensional vector space over a field F where $1 + 1 + 1 \ne 0_F$. Let $T_i : V \to V \quad (i = 1, 2)$ be two linear operators. Assume that

$$T_1 \circ T_1 = T_1, \quad T_2 \circ T_2 = 3\,T_2, \quad T_1 \circ T_2 = 0\,I_V = T_2 \circ T_1$$

where $0\,I_V$ is the zero map. It is known that

$$T := T_1 + T_2$$

is a linear operator on V.

(i) If $\mathbf{v} \in R(T_1)$ (the range of T_1), show that $T(\mathbf{v}) = \mathbf{v}$.

(ii) If $\mathbf{v} \in R(T_2)$ (the range of T_2), show that $T(\mathbf{v}) = 3\mathbf{v}$.

(iii) Show that $\dim V = \dim \mathrm{Ker(T)} + \dim \mathrm{R(T)}$.

(iv) Show that $T \circ T = T + 2\,T_2$.

(v) Show that the sum $\mathrm{Ker(T)} + \mathrm{R(T)}$ is a direct sum of subspaces $\mathrm{Ker(T)}$ and $R(T)$.

(vi) Show that $V = \mathrm{Ker(T)} \oplus \mathrm{R(T)}$.

Question 7 [16 marks]

For each of the 8 statements below, if it is true write T (or True) as the answer; if it is false write F (or False) as the answer. Answer them **in order.** *No* proof is required.

(i) If $f : V \to W$ is a surjective linear transformation, then the cardinality $|f^{-1}(\mathbf{w}_1)| = |f^{-1}(\mathbf{w}_2)|$ for all $\mathbf{w}_i \in W$. Here $f^{-1}(\mathbf{w}) := \{\mathbf{v} \in V \mid f(\mathbf{v}) = \mathbf{w}\}$.

(ii) If T is an orthogonal linear operator on a real inner product space V then its representation matrix $[T]_B$ relative to any basis B of V is an orthogonal matrix.

(iii) If U_i $(i = 1, 2)$ are both unitary complex matrices then so is $U_1 U_2^{-1}$.

(iv) If a square matrix is diagonalizable over a field F then its eigenvalues are all distinct.

(v) If matrix A is invertible and diagonalizable over field F then so is A^n for all integers n.

(vi) Every real matrix in $M_n(\mathbf{R})$ has a Jordan canonical form.

(vii) If A is a self-adjoint complex matrix then so is its transpose A^t.

(viii) If C is a complex square matrix then $C^* C$ is a positive definite matrix.

END OF PAPER