**NATIONAL UNIVERSITY OF SINGAPORE**

**MA2108 - MATHEMATICAL ANALYSIS I**

(Semester 2 : AY2016/2017)

Time allowed : 2 hours

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**INSTRUCTIONS TO STUDENTS**

1. Please write your matriculation/student number only. Do not write your name.
2. This examination paper contains **SIX (6)** questions and comprises **FIVE (5)** printed pages.
3. Answer **ALL** questions.
4. Please start each question on a new page.
5. This is a CLOSED BOOK (with help sheet) examination.
6. You are allowed to use one A4-sized help sheet.
7. You may use non-programmable calculators. However, you should lay out systematically the various steps in the calculations.

**Convention:** Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Q}$ denotes the set of rational numbers, and $\mathbb{R}$ denotes the set of real numbers. For any non-negative real number $a$ and $n \in \mathbb{N}$, $a^{\frac{1}{n}}$ denotes the non-negative $n$-th root of $a$, and $\sqrt{a}$ denotes the non-negative square root of $a$.

**Question 1 [20 marks]**

(a) For each of the following sequences, either find the limit or show that the limit does not exist. Justify your answers.

(i) $\left(n - \dfrac{2n^4}{\sqrt{4n^6 + 3n^5 + 1}}\right)$.

(ii) $\left(\left(\dfrac{3^n + 5}{3^n + 3}\right)^{3^{n+1}}\right)$.

(iii) $\left(\left(3^{4n} + \left(4 + \dfrac{1}{n}\right)^{3n}\right)^{\frac{1}{n}}\right)$.

(b) Let $f : (2, \infty) \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \sqrt{4x - 8} & \text{if } x \in (2, \infty) \cap \mathbb{Q}, \\ x - 1 & \text{if } x \in (2, \infty) \cap (\mathbb{R} \setminus \mathbb{Q}). \end{cases}$$

Determine the points, if any, at which $f$ is continuous. Justify your answer.

**Question 2 [17 marks]**

(a) Determine the convergence or divergence of each of the following series. Justify your answers.

(i) $\displaystyle\sum_{n=1}^{\infty} n^2\left(6+\frac{1}{n}\right)^n\left(1+\frac{1}{3n^2}\right)^{-6n^3}$.

(ii) $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt{n^4+5n+1}-n^2\right)$.

(b) Determine the absolute convergence, conditional convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{\sqrt{n+1-\sqrt{n}}}$. Justify your answer.

**Question 3 [16 marks]**

(a) Consider the sequence $(a_n)$ given by

$$a_1=-5, \quad \text{and} \quad a_{n+1}=\frac{9a_n}{3-a_n} \quad \text{for all } n\in\mathbb{N}.$$

Is it true that $(a_n)$ converges? Find also its limit if $(a_n)$ converges. Justify your answers.

[Remark: The approximate values of the first few terms of $(a_n)$ are as follows:

$$a_1=-5,\ a_2=-5.63,\ a_3=-5.87,\ a_4=-5.96,\ a_5=-5.99,\cdots.]$$

(b) Let $f:(0,\infty)\to\mathbb{R}$ be a function such that $f$ is continuous on $(0,\infty)$, and

$$f(x)>0 \quad \text{for all } x\in(0,\infty).$$

Let $x_1$, $x_2$, $x_3\in(0,\infty)$. Is it true that there exists $c\in(0,\infty)$ such that

$$f(c)=\frac{4}{\dfrac{1}{f(x_1)}+\dfrac{2}{f(x_2)}+\dfrac{1}{f(x_3)}}?$$

Justify your answer.

**Question 4 [18 marks]**

(a) Let $(a_n)$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove that the series $\sum_{n=1}^{\infty}(a_n - 2a_{n+1} + a_{n+2})$ converges.

(b) Let $f : [1,2) \to \mathbb{R}$ be a function such that $f$ is uniformly continuous on $[1,2)$, and $f(x) > 0$ for all $x \in [1,2)$. Consider the functions $g : [1,2) \to \mathbb{R}$ and $h : [1,2) \to \mathbb{R}$ given by

$$g(x) = \big(f(x)\big)^2 \quad \text{and} \quad h(x) = \frac{1}{f(x)} \quad \text{for all } x \in [1,2).$$

(i) Show that $g$ is uniformly continuous on $[1,2)$.

(ii) Is it true that $h$ must be uniformly continuous on $[1,2)$? Justify your answer.

**Question 5 [15 marks]**

(a) Use the $\epsilon - \delta$ definition of limit to show that

$$\lim_{x\to 2} \frac{3x^2}{x-1} = 12.$$

(b) Let $(a_n)$ be a sequence of real numbers such that

$$\lim_{n\to\infty} (3a_{n+1} - a_n) = 1.$$

Is it true that $(a_n)$ converges? Justify your answer.

**Question 6 [14 marks]**

(a) Consider the limit

$$\lim_{x \to 4} \tan\left(\left[\frac{x}{4}\right] + \left[\frac{4}{x}\right]\right).$$

Either find the limit or show that the limit does not exist. Justify your answer. Here $[a]$ denotes the greatest integer less than or equal to $a$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f$ is continuous on $\mathbb{R}$, and

$$|f(x) - f(y)| \geq |x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

Is it true that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(g(x)) = x \quad \text{for all } x \in \mathbb{R}?$$

Justify your answer.

**End of Paper**