NATIONAL UNIVERSITY OF SINGAPORE

MA2101 Linear Algebra II

SEMESTER 1: AY 2014/2015

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation number only. Do not write your name.
2. This examination paper contains a total of **EIGHT (8)** questions and comprises **THREE (3)** printed pages.
3. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.
4. Use a separate page for each question.
5. This is a **CLOSED BOOK** examination.
6. Each candidate is allowed to use two pieces of A4-size, handwritten help sheets.
7. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [15 marks]

Let V be a vector space over $\mathbb{C}$, and $T \in \mathcal{L}(V)$ such that $T^2 = 1_V$. Define

$$V_1 = \{\boldsymbol{v} \in V \mid T(\boldsymbol{v}) = \boldsymbol{v}\} \quad \text{and} \quad V_2 = \{\boldsymbol{v} \in V \mid T(\boldsymbol{v}) = -\boldsymbol{v}\}.$$

(i) Prove that V_1 and V_2 are subspaces of V.

(ii) Prove that $V = V_1 \oplus V_2$.

Question 2 [15 marks]

Let V be a vector space and $T \in \mathcal{L}(V)$.

(i) Prove that $\mathrm{Im}(T^k) \subseteq \mathrm{Im}(T^{k-1})$ for every positive integer k.

(ii) Prove that if $\mathrm{Im}(T^k) = \mathrm{Im}(T^{k-1})$ for a positive integer k, then $\mathrm{Im}(T^{k+1}) = \mathrm{Im}(T^k)$.

(iii) If $V = \mathcal{P}(\mathbb{F})$, find $T \in \mathcal{L}(V)$ such that $\mathrm{Im}(T^k) \subsetneqq \mathrm{Im}(T^{k-1})$ for all positive integer k.

Question 3 [15 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$ and define

$$T(\boldsymbol{X}) = \boldsymbol{A}\boldsymbol{X} - \boldsymbol{X}\boldsymbol{A}, \quad \boldsymbol{X} \in \mathcal{M}_{2\times 2}(\mathbb{R}).$$

(i) Find $[T]_{\mathcal{B}}$, where $\mathcal{B} = \{\boldsymbol{E}_{11}, \boldsymbol{E}_{12}, \boldsymbol{E}_{21}, \boldsymbol{E}_{22}\}$.

(ii) Find $c_T(x)$ and $m_T(x)$, and determine if T is diagonalizable.

Question 4 [10 marks]

It is given that $\langle\ ,\ \rangle$ defined by

$$\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)\,dx$$

is an inner product on $\mathcal{P}(\mathbb{R})$. Find an orthogonal basis for $V = \mathrm{span}_{\mathbb{R}}\{x^2, x^3, x^4\}$.

Question 5 [15 marks]

Consider the linear system

$$\begin{cases} x_1 + ix_2 - x_3 = 0 \\ x_1 + x_3 = 6 \\ 2x_1 + ix_2 = 3 \\ ix_2 - 2x_3 = 3. \end{cases}$$

(i) Find all the least squares solutions.

(ii) Find the optimal least squares solution.

Question 6 [10 marks]

Let V be a finite-dimensional vector space over $\mathbb{C}$, and $T \in \mathcal{L}(V)$. Suppose that T has exactly two eigenvalues 1 and -1, such that

(i) the algebraic multiplicities of -1 and 1 are 5 and 7 respectively, and

(ii) the geometric multiplicities of -1 and 1 are 2 and 3 respectively.

Find all the possible Jordan canonical forms of T, and write down the corresponding minimal polynomials. (Write the Jordan canonical form as $\boldsymbol{J}_{k_1}(\lambda_1) \oplus \cdots \oplus \boldsymbol{J}_{k_s}(\lambda_s)$).

Question 7 [10 marks]

Let V be a finite-dimensional vector space, and $S, T \in \mathcal{L}(V)$. Prove that

(i) $\mathrm{rank}(S+T) \leq \mathrm{rank}(S) + \mathrm{rank}(T)$.

(ii) $\mathrm{nullity}(S \circ T) \leq \mathrm{nullity}(S) + \mathrm{nullity}(T)$.

Question 8 [10 marks]

Let V be a finite-dimensional inner product space, and $T \in \mathcal{L}(V)$ invertible. Prove that there exists a unitary operator U and a positive operator P on V such that $T = U \circ P$.

End of Paper