PYP Answer - MA2101 AY1617Sem2

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1. Let $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Z := P^{-1}Y$. We have $Y = PZ$ and $Y' = PZ'$.

$$Y' = AY \Rightarrow PZ' = APZ$$
$$Z' = P^{-1}APZ = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} Z$$

Therefore,

$$\begin{cases} z_1' = z_1 & (1) \\ z_2' = z_1 - z_2 & (2) \end{cases}$$

Solving (1), we have

$$z_1 = Ae^x, \text{where } A \text{ is an aribitrary constant}$$

Substituting z_1 to (2), we have

$$z_2' + z_2 = Ae^x$$

Solving z_2 using hint, we have

$$z_2 = Be^{-x} + \frac{A}{2}e^x, \text{where } B \text{ is an arbitrary constnant}$$

Therefore,

$$Y = \begin{pmatrix} \frac{A}{2}e^x + Be^{-x} \\ \frac{3A}{2}e^x + Be^{-x} \end{pmatrix}$$

2. (a) Note, that $(AA^*)^* = A^{**}A^* = AA^*$, so AA^* is self-adjoint.
By Principle Axis Theorem, there exists a unitary matrix U, such that

$$D = U^*(AA^*)U$$

(b) Note the ith column $(i = 1, 2)$ of U consists of eigenvectors of $AA^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the corresponding ith column of D consists of the eigenvalues at the diagonal. Therefore, solving

$$p(x) = |xI - AA^*| = x^2 - 1 = 0 \Rightarrow x = \pm 1$$

We have two distinct eigenvalues $1, -1$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
Solving

$$\begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} v = 0$$

where $\lambda = 1, -1$, we have the two eigenvectors $e_{\lambda=1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $e_{\lambda=-1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
Therefore, $U' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.After normalisation.

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

3. (a) In any matrix, column and row spaces have the same dimension

$$\dim_F \mathrm{Col}(A) = \dim_F \mathrm{R}(A) := \mathrm{rank}(A)$$

which is called the **rank** of A.

(b) ($\Leftarrow$) We first note that for any column j, $\mathbf{b}_j = E_r \cdots E_1 \mathbf{a}_j$.

$$\begin{aligned} \sum_{j=1}^{n} c_j \mathbf{a}_j &= 0 \\ \Rightarrow E_r \cdots E_1 \sum_{j=1}^{n} c_j \mathbf{a}_j &= E_r \cdots E_1 \\ \Rightarrow \sum_{j=1}^{n} c_j E_r \cdots E_1 \mathbf{a}_j &= 0 \\ \Rightarrow \qquad \sum_{j=1}^{n} c_j \mathbf{b}_j &= 0 \end{aligned}$$

($\Rightarrow$) We note that all elementary matrices are invertible. So,

$$\begin{aligned} \sum_{i=1}^{n} c_j \mathbf{b}_j &= 0 \\ \Rightarrow \sum_{j=1}^{n} c_j E_r \cdots E_1 \mathbf{a}_j &= 0 \\ \Rightarrow E_r \cdots E_i \sum_{j=1}^{n} c_j \mathbf{a}_j &= 0 \\ \Rightarrow \qquad \sum_{j=1}^{n} c_j \mathbf{a}_j &= E_1^{-1} \cdots E_n^{-1} 0 = 0 \end{aligned}$$

(c) $\text{rank}(B) \leq \text{rank}(A)$ is clear. For $\text{rank}(B) \geq \text{rank}(A)$, suppose that $\text{rank}(A) = k$. Then by definition of rank, there exists some columns $i \in I$ from A ($|I| = k$), where $\sum_{i \in I} c_i \mathbf{a}_i = 0$ has only trivial solution.
According to (b), B will also have the corresponding property: the corresponding columns $i \in I$ of B ($|I| = k$), admit that $\sum_{i \in I} c_i \mathbf{b}_i = 0$ has only trivial solution. Hence, $\text{rank}(B) \geq k$. Therefore, $\text{rank}(B) = \text{rank}(A)$.

4. (a) V_1 is a vector subspace of V if:

(CA) (**C**losed under vector **A**ddition)

$$\mathbf{v}_1, \mathbf{v}_2 \in V_1 \Rightarrow \mathbf{v}_1 + \mathbf{v}_2 \in V_1$$

(CS) (**C**losed under **S**calar Multiplication)

$$\mathbf{v} \in V_1 \Rightarrow a\mathbf{v} \in V_1$$

(b) T is a linear transformation since from V to V if for all $v, v_1, v_2 \in V$:

$$\begin{aligned} T(\mathbf{v}_1 + \mathbf{v}_2) &= T(\mathbf{v}_1) + T(\mathbf{v}_2) \\ T(a\mathbf{v}) &= aT(\mathbf{v}) \end{aligned}$$

(c) Let v_1, v_2 be two vectors from R_n, and $c \in F$. There is $u_1, u_2 \in V$ such that $v_i = T^n(u_i), i = 1, 2$. Note that, as T is a linear transformation, the composition T^n is also a linear transformation, by which we have

(CA)

$$\begin{aligned} v_1 + v_2 &= T^n(u_1) + T^n(u_2) \\ &= T^n(u_1 + u_2) \in R_n \end{aligned}$$

(CS)

$$\begin{aligned} cv_1 &= cT^n(u_1) \\ &= T^n(cu_1) \in R_n \end{aligned}$$

(d) For all $v \in R_{m+1}$, $v = T^{n+1}(u)$ for some $u \in V$. Let $w = T(u) \in V$, so $v = T^n(w)$ implies $v \in R_m$. Hence, $R_{m+1} \subseteq R_m$.

(e) By (d), if $\dim V$ is finite, then

$$\dim V \geq \dim R_1 \geq \dim R_2 \geq \cdots$$

is a nonincreasing sequence of nonnegative integers, and hence must be eventually constant, i,e,

$$\dim R_s = \dim R_{s+1} = \dim R_{s+2} = \cdots$$

for some $s \geq 1$. Now note that

$$R_s \supseteq R_{s+1} \supseteq R_{s+2} \supseteq \cdots$$

so for all $n \geq 1$, R_{s+n} is a subspace of R_s with the same dimension as R_s. Hence $R_{s+n} = R_s$.

(f) No. Let V be the vector space of infinite sequence of real numbers $(x_0, x_1, x_2 \ldots)$, under componentwise addition and scalar multiplication. The linear transformation is the right shift operator $T : (x_0, x_1, x_2, \ldots) \mapsto (0, x_0, x_1, \ldots)$.
Consider $v = T^s(1, 0, 0, 0, \ldots) \in R_s$ is not in R_{s+1}, since the sth coordinate of v is 1, but the 0th, 1st, $\ldots$, sth coordinates of any vector in R_{s+1} are 0. Hence, the claim is false.

5. (a) Minimal polynomial of matrix A,$m_A(x)$, is the monic polynomial of minimal degree such that $m_A(A) = 0$.
Characteristic polynomial of matrix A, $p_A(x)$, is defined as $p_A(x) = \det(xI_n - A)$.

(b) First note that $(A - I)^2(A + I)^2 = 0$. So $(x - 1)^2(x + 1)^2$ kills p_A. Since A is diagonalisable over $\mathbb{C}$, $m(x)$ should have only simple zero(s). Therefore, possible $m(x)$ are

$$\begin{aligned} m(x) &= x - 1 \\ m(x) &= x + 1 \\ m(x) &= (x - 1)(x + 1) \end{aligned}$$

Examples of A satisfying these three cases are $I_4, -I_4$ and $\text{diag}[1, 1, 1, -1]$ respectively.

(c) p_A has the same zero set as m_A, so possible $p_A(x)$, which must of degree 4 are

$$p(x) = (x - 1)^i(x + 1)^{4-i} \quad i = 0, 1, 2, 3, 4$$

The diagonal matrix with i 1's and $4 - i$ -1's will satisfy the above $p(x)$.

(d) The determinant of A is non-zero.

$$\begin{aligned} |\det(A)| &= |\textbf{constant term of} p_A(x)| \\ &= 1 \quad \text{from (c)} \end{aligned}$$

(e) $(A^2 - I)^2 = 0 \Rightarrow A^4 - 2A^2 + I = 0$.
Therefore,

$$\begin{aligned} I &= 2A^2 - A^4 \\ A^{-1} &= 2A - A^3 \end{aligned}$$

So we can take, $g(x) = 2x - x^3$

6. (a) Note that $\langle u_0, u_0 \rangle = 0$ implies $u_0 = 0$ by positivity. Similarly, $\langle v_0, v_0 \rangle = 0$ implies $v_0 = 0$.

(b) Choose an orthonormal basis B with respect to the inner product. Then $\langle x, y \rangle = [x]_B^*[y]_B$ for any $x, y \in V$. Now

$$\begin{aligned} \langle u, T(v) \rangle &= [u]_B^*[T]_B[v]_B \\ &= ([T]_B[u]_B)^*[v]_B \\ &= \langle S(u), v \rangle \end{aligned}$$

(c) We prove by contradiction. Suppose $\langle u, T(v)\rangle = \langle S(u), v\rangle = \langle S'(u), v\rangle$ where $S \neq S'$. Rearranging,

$$\langle S(u) - S'(u), v\rangle = 0 \forall v \in V$$

which implies, from (a), that

$$S(u) - S'(u) = 0 \forall u \in V$$

and that implies $S = S'$, contradiction. Therefore, the claim (c) must be true.

7. (a) A complex matrix A in $\mathbb{M}_n(\mathbb{C})$ is **unitary** if

$$AA^* = I_n$$

(b)

$$\begin{aligned} \text{LHS} &= AA^* - \alpha A^* - \overline{\alpha} A + II \\ &= A^*A - \overline{\alpha} A - \alpha A^* + (-\overline{\alpha})(-\alpha) I \\ &= \text{RHS} \end{aligned}$$

(c) We first note that A and A^* are both unitary. Therefore.

$$\|AX\| = \langle AX, AX\rangle = \langle X, X\rangle = \langle A^*X, A^*X\rangle = \|A^*X\|$$

(d) No. Let $A = iI_2, \lambda = i$ and $Y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then we have $A^* = -iI$, but LHS$=$ $-iY \neq iY =$RHS.

(e) Yes. Observe that $(A - \lambda I)Y = 0$, and therefore $\langle (A-\lambda I)Y, (A-\lambda I)Y\rangle = 0$. So,

$$\begin{aligned} 0 &= \langle Y, (A^* - \overline{\lambda} I)(A - \lambda I)Y\rangle \\ &= \langle Y, (A - \lambda I)(A^* \overline{\lambda} I)Y\rangle \\ &= \langle (A^* - \overline{\lambda} I)Y, (A^* - \overline{\lambda} I)Y\rangle \\ &= \left\|(A^* - \overline{\lambda} I)Y\right\|^2 \end{aligned}$$

So $(A^* - \overline{\lambda} I)Y = 0$, i.e. $A^*Y = \overline{\lambda} Y$. Note that eigenvalues of a complex matrix will either be real, or in conjugate form. So we have the set of eigenvalues of A equals that of A^*.

8. (a) A is **positive definite** if A is self-adjoint and

$$(AX)^t \overline{X} = X^t A^t \overline{X} > 0$$

for all $X \in \mathbb{C}^n$.

(b) V is an inner product space if it has an inner product, ie. a function $\langle -, -\rangle$ satisfying the following

i. For all

$$\mathbf{x}_i, \mathbf{y}_j, \mathbf{x}, \mathbf{y} \in V, a_i, b_i \in \mathbb{R}$$

we have

$$\langle a_1\mathbf{x}_1 + a_2\mathbf{x}_2, \mathbf{y}\rangle = a_1\langle \mathbf{x}_1, \mathbf{y}\rangle + a_2\langle \mathbf{x}_2, \mathbf{y}\rangle$$
$$\langle \mathbf{x}, b_1\mathbf{y}_1 + b_2\mathbf{y}_2\rangle = \bar{b}_1\langle \mathbf{x}, \mathbf{y}_1\rangle + \bar{b}_2\langle \mathbf{x}, \mathbf{y}_2\rangle$$

ii. H is symmetric, i.e., for all $\mathbf{x}, \mathbf{y} \in V$, we have

$$\langle \mathbf{x}, \mathbf{y}\rangle = \overline{\langle \mathbf{y}, \mathbf{x}\rangle}$$

iii. Positivity:
For all $\mathbf{0} \neq \mathbf{x} \in V$, we have

$$\langle \mathbf{x}, \mathbf{x}\rangle > 0$$

(c) ($\Rightarrow$) We show that a positive definite A will satisfy the above three condition of inner product.

i.

$$\begin{aligned}
\langle a_1X_1 + a_2X_2, Y\rangle &= (a_1AX_1 + a_2AX_2)^t\overline{Y} \\
&= (a_1X_1^t + a_2X_2^t)A^t\overline{Y} \\
&= a_1X_1^tA^t\overline{Y} + a_2X_2A^t\overline{Y} \\
&= a_1\langle X_1, Y\rangle + a_2\langle X_2, Y\rangle
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle X, b_1Y_1 + b_2Y_2\rangle &= X^tA^t\overline{b_1Y_1 + b_2Y_2} \\
&= X^tA^t\overline{b_1Y_1} + X^tA^t\overline{b_2Y_2} \\
&= \overline{b_1}X^tA^t\overline{Y_1} + \overline{b_2}XA^t\overline{Y_2} \\
&= \overline{b_1}\langle X, Y_1\rangle + \overline{b_2}\langle X, Y_2\rangle
\end{aligned}$$

ii.

$$\begin{aligned}
\langle X, Y\rangle &= X^tA^t\overline{Y} \in F \\
&= (X^tA^t\overline{Y})^t \\
&= \overline{Y}^t\overline{A}^t\overline{\overline{X}} \\
&= \overline{Y^tA^t\overline{X}} \\
&= \overline{\langle Y, X\rangle}
\end{aligned}$$

iii.

$$\langle X, X\rangle = X^tA^t\overline{X} > 0$$

by positive definiteness of A.

($\Leftarrow$) Since $\langle -,-\rangle$ is an inner product, we have

$$\langle X,X\rangle = X^tA^t\overline{X} \geq 0$$

for all X in W.
Also, consider the symmetric property

$$\begin{aligned}\overline{\langle X,Y\rangle} &= \langle Y,X\rangle\\ \overline{(AX)^t\overline{Y}} &= (AY)^tX\\ \overline{X^tA^t\overline{Y}} &= Y^tA^tX = (Y^tA^tX)^t\\ X^*A^*Y &= X^*AY \text{ for all } X,Y\end{aligned}$$

which implies $A^* = A$. Thus, A is self adjoint.
Combining with the result obtained from positivity, we prove the claim.