PYP Answer - MA2108 AY1617Sem2

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1. (a) i. Use squeeze theorem

$$n - \frac{2n^4}{\sqrt{4n^6+3n^5+1}} = \frac{\left(n - \frac{2n^4}{\sqrt{4n^6+3n^5+1}}\right)\left(n + \frac{2n^4}{\sqrt{4n^6+3n^5+1}}\right)}{n + \frac{2n^4}{\sqrt{4n^6+3n^5+1}}}$$
$$= \frac{3n^7+n^2}{(4n^6+3n^5+1)\left(n+\frac{2n^4}{\sqrt{4n^6+3n^5+1}}\right)} > \frac{3n^7}{8n^7}$$

Also,

$$\frac{3n^7+n^2}{(4n^6+3n^5+1)\left(n+\frac{2n^4}{\sqrt{4n^6+3n^5+1}}\right)} < \frac{3n^7+\frac{9}{4}n^6+\frac{3}{4}n}{(4n^6+3n^5+1)\left(n+\frac{2n^4}{\sqrt{4n^6+3n^5+1}}\right)} = \frac{\frac{3}{4}n}{n+\sqrt{\frac{4n^8}{4n^6+o(n^6)}}}$$

Applying squeeze, we see that

$$\lim_{n\to\infty} \frac{\frac{3}{4}n}{n+\sqrt{\frac{4n^8}{4n^6+o(n^6)}}} = \frac{3}{8} \quad \text{and} \quad \lim_{n\to\infty}\frac{3n^7}{8n^7} = \frac{3}{8}$$

Therefore, the required limit exists and equals $\frac{3}{8}$.

ii. We recognise that the sequence enumerated by $\left(\frac{3^n+5}{3^n+3}\right)^{3^{n+1}}$ is the subsequence of the sequence enumerated by $\left(\frac{n+5}{n+3}\right)^{3n} = \left(1+\frac{2}{n+3}\right)^{3n}$, whose limit is equal to $\left(1+\frac{2}{n}\right)^{3(n-3)}$, and equal to its subsequence $\left(1+\frac{2}{2n}\right)^{3(2n-3)}$. We then evaluate this subsequence's limit.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{6n-9} = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{6n}}{\left(1+\frac{1}{n}\right)^9} = \frac{e^6}{1} = e^6$$

iii. Use squeeze theorem, for all $n \geq 10$,

$$3^4 = (3^{4n})^{\frac{1}{n}} < \left(3^{4n} + \left(4+\frac{1}{n}\right)^{3n}\right)^{\frac{1}{n}} < (2\cdot 3^{4n})^{\frac{1}{n}} = 2^{\frac{1}{n}}\cdot 81$$

And as $\lim_{n\to\infty} 2^{\frac{1}{n}} = 1$, we have the limit exist and equal 81.

(b) Let $a \in \mathbb{R}$. Take a rational sequence (x_n) and an irrational sequence (y_n) such that $x_n \to a$ and $y_n \to a$. Then

$$f(x_n) = \sqrt{4x_n - 8} \to \sqrt{4a - 8} \quad \text{and} \quad f(y_n) = y_n - 1 \to a - 1$$

If f is continuous at $x = a$, then

$$\sqrt{4a - 8} = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = a - 1$$

so that $a = 3$. It follows that if $a \neq 3$, then f is not continuous at $x = a$.
Next we prove that f is continuous at $x = 3$, i.e., $\lim_{x\to 3} f(x) = f(3) = 2$.
Let $\varepsilon > 0$. We choose $\delta = \min\{1, \frac{\varepsilon}{2}\}$. Then, if $|x - 3| < \delta$, we have

$$|f(x) - 2| = \begin{cases} |\sqrt{4x - 8} - 2| = \frac{|4x-12|}{\sqrt{4x-8}+2} < \frac{4|x-3|}{2} < 2 \cdot \frac{\varepsilon}{2} = \varepsilon & \text{if } x \text{ is rational} \\ |x - 3| < \frac{\varepsilon}{2} < \varepsilon & \text{if } x \text{ is irrational} \end{cases}$$

In other words, $|x - 3| < \delta \Rightarrow |f(x) - f(3)| < \varepsilon$, so f is continuous at $x = 3$.

2. (a) i. We recognise this series is eventually non-negative, so apply simplified root test:

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left(n^2 \left(6 + \frac{1}{n}\right)^n \left(1 + \frac{1}{3n^2}\right)^{-6n^3} \right)^{\frac{1}{n}} \\ &= \lim_{n\to\infty} n^{\frac{2}{n}} \lim_{n\to\infty} \left(6 + \frac{1}{n}\right) \lim_{n\to\infty} \left(1 + \frac{1}{3n^2}\right)^{-6n^2} \\ &= 1 \cdot 6 \cdot \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-2n} \\ &= 6 \cdot e^{-2} < 1 \end{aligned}$$

Therefore, it converges.

ii. As $\sqrt{n^4 + 5n + 1} - n^2 = \frac{(\sqrt{n^4+5n+1}-n^2)(\sqrt{n^4+5n+1}+n^2)}{\sqrt{n^4+5n+1}+n^2} = \frac{5n+1}{\sqrt{n^4+5n+1}+n^2}$. Also note that

$$\frac{5n + 1}{\sqrt{n^4 + 5n + 1} + n^2} > \frac{5n}{2\sqrt{n^4 + 5n + 1}}$$

Apply limit comparison test on series enumerated by $\frac{5n}{2\sqrt{n^4+5n+1}}$ and $\frac{1}{n}$, we have

$$\rho = \lim_{n\to\infty} \frac{\frac{5n}{2\sqrt{n^4+5n+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{5}{2} \sqrt{\frac{n^4}{n^4 + 5n + 1}} = \frac{5}{2} > 0$$

Therefore, as series enumerated by $\frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{5n}{2\sqrt{n^4+5n+1}}$ diverges, and by comparison test, the series in the question diverges.

(b) $\sum_{i=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n+1-\sqrt{n}}} = \sum_{i=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1-\sqrt{n}}}$. By alternating series test,

- $\frac{1}{\sqrt{n+1-\sqrt{n}}} \geq 0$ for all n

- $\frac{1}{\sqrt{n+1-\sqrt{n}}}$ is decreasing. It is obvious as the next term has a denominator $\sqrt{n+2-\sqrt{n+1}}$, since both denominator are positive, we can compare their squares: $(n+1-\sqrt{n})-(n+2-\sqrt{n+1}) = -1+(\sqrt{n+1}-\sqrt{n}) < 0$, therefore, the next term's denominator is always larger, which makes the next term smaller than any previous term.
- $\lim_{n\to\infty} \frac{1}{\sqrt{n+1-\sqrt{n}}} = 0$.

This series converges conditionally. We then show that this series does not converges absolutely. i.e., $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1-\sqrt{n}}}$ does not converge. We see this by comparing this series against $\sum_{n=1}^{\infty} \frac{1}{n}$. It is clear that for $n > 2$, $\frac{1}{n} < \frac{1}{\sqrt{n+1-\sqrt{n}}}$, and by comparison test, since $\sum_{n=1}^{\infty}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1-\sqrt{n}}}$ also diverges.

3. (a) Yes, (a_n) converges. We claim that (a_n) is monotone decreasing and bounded below by -6. From monotone convergence theorem, (a_n) converges. Next we prove the two claim above.
First (a_n) is bounded below by -6. We write

$$a_{n+1} = \frac{9a_n}{3-a_n} = -9 + \frac{27}{3-a_n}$$

Since $a_1 = -5 > -6$, $a_2 > -9 + \frac{27}{3-(-6)} = -6$. And by induction, we can show that $a_n > -6$ for all n.
Next we show that (a_n) is monotone decreasing. We note the obvious fact that $a_n < 0$ for all n. Then, observe

$$a_{n+1} - a_n = \frac{9a_n}{3-a_n} - a_n = \frac{a_n(a_n+6)}{3-a_n}$$

Note, that $a_n < 0$, $a_n + 6 > 0$ and $3 - a_n > 0$, so $a_{n+1} - a_n < 0$. This suggests that for all n, $a_{n+1} < a_n$. This prove our claim.

(b) Let $M := \max\{f(x_1), f(x_2), f(x_3)\}$ and $m = \min\{f(x_1), f(x_2), f(x_3)\}$. We see that the right hand side

$$m = \frac{4}{\frac{1}{m}+\frac{2}{m}+\frac{1}{m}} \leq \frac{4}{\frac{1}{f(x_1)}+\frac{2}{f(x_2)}+\frac{1}{f(x_3)}} \leq \frac{4}{\frac{1}{M}+\frac{2}{M}+\frac{1}{M}} = M$$

And by intermediate value theorem, we have our c.

4. (a) We note that the required sum, by telescoping, equals

$$\lim_{k\to\infty} \sum_{n=1}^{k} (a_n - 2a_{n+1} + a_{n+2}) = \lim_{k\to\infty} a_1 - a_2 - a_{n+1} + a_{n+2} = a_1 - a_2$$

Therefore, the series converges

(b) i. Let $x_1, x_2 \in [1,2)$. By continuous extension theorem, f is continuous on $[1,2]$. Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x_2) - f(x_1)| < \frac{\varepsilon}{2M}$ where $M = \max_{1 \le x < 2} f(x)$ ($0 < M < \infty$, the lower bound is specified by the question and upper bound given by extreme value theorem) for all $|x_2 - x_1| < \delta$. Next we show g is uniformly continuous on $[1,2)$. For the same δ , such that $|x_1 - x_2| < \delta$,

$$|g(x_1) - g(x_2)| = |f(x_1)^2 - f(x_2)^2| = |f(x_1) + f(x_2)||f(x_1) - f(x_2)| < 2M \cdot \frac{\varepsilon}{2M} = \varepsilon$$

Therefore, g is uniformly continuous on $[1,2)$.

ii. No. Let (x_n) (u_n) be two series such that $x_n = 2 - \frac{1}{n}$ and $u_n = 2 - \frac{2}{n}$. It is clear that $x_n - u_n \to 0$. However, $h(x_n) - h(u_n) = \frac{n}{2} \to \infty$.Therefore, h is not uniformly continuous on $[1,2)$.

5. (a) For any $\varepsilon > 0$, choose $\delta = \min\{\frac{1}{2}, \sqrt{\frac{\varepsilon}{6}}\}$ such that $|x - 2| < \delta$, which gives $\frac{1}{2} < x - 1 < \frac{3}{2}$

$$\left| \frac{3x^2}{x-1} - 12 \right| = \left| \frac{3(x-2)^2}{x-1} \right| < 2 \cdot 3\delta^2 \le \varepsilon$$

and the result follows.

(b) From the limit, we have, for any $\varepsilon > 0$, there exists $M > 0$ such that $n > M$ gives

$$|3a_{n+1} - a_n - 1| < \varepsilon$$

which is equivalent to

$$\frac{a_n}{3} + \frac{1-\varepsilon}{3} < a_{n+1} < \frac{a_n}{3} + \frac{1+\varepsilon}{3}$$

Recursively apply this inequality until $n = M + 1$, we have

$$\frac{a_{M+1}}{3^{n-M-1}} + f_1(\varepsilon) < a_{n+1} < \frac{a_{M+1}}{3^{n-M-1}} + f_2(\varepsilon)$$

Taking limit on the inequality and let $\varepsilon \to 0$, we will have, by squeeze, that $a_{n+1} \to 0$, which implies (a_n) converges.

6. (a) We note that $\lim_{x\to 4^+} \left[\frac{x}{4}\right] = 1$ and $\lim_{x\to 4^-} \left[\frac{x}{4}\right] = 0$. Also, $\lim_{x\to 4^+} \left[\frac{4}{x}\right] = 0$ and $\lim_{x\to 4^-} \left[\frac{4}{x}\right] = 1$. Then

$$\lim_{x\to 4^+} \tan\left(\left[\frac{x}{4}\right] + \left[\frac{4}{x}\right]\right) = \lim_{x\to 4^+} \tan 1 = \tan 1 = \lim_{x\to 4^-} \tan 1 = \lim_{x\to 4^-} \tan\left(\left[\frac{x}{4}\right] + \left[\frac{4}{x}\right]\right)$$

Therefore, the limit is $\tan 1$.

(b) Without loss of generality, let $x > y$. Then we have, $|f(x) - f(y)| \ge x - y$ for all $x, y \in \mathbb{R}$. We claim that f is monotone increasing or monotone decreasing.
We prove this claim by contradiction. Consider the case that f increases on some interval (a, x_M) and decreases on (x_M, b), then f takes value $(f(a), f(x_M))$ on

the first interval and $(f(b), f(x_M))$ on the second. We can always choose a $\lambda_1 \in (a, x_M)$ and $\lambda_2 \in (x_M, b)$ such that $\max\{f(a), f(b)\} < f(\lambda_1) = f(\lambda_2) < f(x_M)$, but it contradicts the assumption. Similarly, the other possibility will also lead to contradiction.
We further note that f cannot be constant for any interval, as it will lead to the contradiction also. Since f is strictly monotone, there exists an inverse $f^{-1} := g$ such that $f(g(x)) = x$.