NATIONAL UNIVERSITY OF SINGAPORE

MA2101 — Linear Algebra II

(Semester 2 : AY2015/2016)

Final Examination — 28 April 2016

Time allowed : 2 hours

INSTRUCTIONS TO STUDENTS

(1) Please write only your matriculation/student number. Do not write your name.

(2) This examination paper contains **FIVE (5)** questions printed on **THREE (3)** pages (including this cover page).

(3) Attempt **ALL** questions. Each question is worth **TWENTY (20)** points. When a question consists of multiple parts, each part carries the same weightage of points.

(4) This is a CLOSED BOOK examination; but one piece of help sheet (A4-size, hand-written on either or both sides) is allowed.

(5) The use of any form of computing or communication device (e.g. calculator, mobile phone, laptop, etc.) is strictly prohibited during this examination.

(6) Write your solutions legibly in the answer booklet provided. Please start your solution to each question on a new page.

Notation

$\mathbb{M}_2(\mathbb{R})$ denotes the $\mathbb{R}$-vector space of 2×2-matrices over the field $\mathbb{R}$ of real numbers.

$\mathbb{M}_3(\mathbb{C})$ denotes the $\mathbb{C}$-vector space of 3×3-matrices over the field $\mathbb{C}$ of complex numbers.

1. Let $A \in \mathbb{M}_2(\mathbb{R})$ be the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$, and consider the map

$$T : \mathbb{M}_2(\mathbb{R}) \to \mathbb{M}_2(\mathbb{R}) \qquad \text{given by} \qquad T(X) \;=\; A\,X - X\,A.$$

(a) Show that T is a linear map.

(b) (i) Determine the matrix $[T]_{\mathcal{B}}$ of T with respect to the standard ordered basis

$$\mathcal{B} \;=\; \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \qquad \text{of } \mathbb{M}_2(\mathbb{R})\,.$$

(ii) Determine the characteristic polynomial $\det(\,T - x \cdot \mathrm{Id}\,) \in \mathbb{R}[x]$ of T as a polynomial in the variable x.

2. Let $X, Y, Z \in \mathbb{M}_2(\mathbb{R})$ be three arbitrarily given matrices, and consider the linear map

$$T \;:\; \mathbb{R}^3 \to \mathbb{M}_2(\mathbb{R}) \qquad \text{given by} \qquad T\begin{pmatrix} a \\ b \\ c \end{pmatrix} \;=\; a\,X + b\,Y + c\,Z.$$

(a) Show that T is injective if and only if X, Y, Z are linearly independent in $\mathbb{M}_2(\mathbb{R})$.

(b) Suppose $X = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, $Y = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(i) Determine the rank of T and the nullity of T.

(ii) Determine the matrix $[T]^{\mathcal{B}}_{\mathcal{A}}$ of the linear map T with respect to the ordered bases

$$\mathcal{A} \;=\; \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \qquad \text{of } \mathbb{R}^3$$

and $$\mathcal{B} \;=\; \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \qquad \text{of } \mathbb{M}_2(\mathbb{R}).$$

3. (a) Let V be a finite dimensional $\mathbb{C}$-vector space, and let $T : V \to V$ be a linear map. Suppose $v_1, \ldots, v_k \in V$ are $k \geqslant 1$ eigenvectors of T, of corresponding eigenvalues $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ which are pairwise distinct. Show that $v_1, \ldots, v_k$ are linearly independent.

(b) Let $T : \mathbb{M}_3(\mathbb{C}) \to \mathbb{M}_3(\mathbb{C})$ be the linear map given by

$$T(X) \;=\; X^t \qquad \text{(matrix transpose)}.$$

(i) Determine an ordered basis $\mathcal{B}$ of $\mathbb{M}_3(\mathbb{C})$ such that the matrix $[T]_{\mathcal{B}}$ of T with respect to $\mathcal{B}$ is diagonal.

(ii) Compute the determinant $\det(T)$ of T.

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4. Let V be a vector space over $\mathbb{C}$ of dimension 3. Let $\{v_1, v_2, v_3\}$ be a basis for V, and let p_1, p_2, p_3 be any three vectors in V.

(a) Show that there are at most three distinct values of $\lambda \in \mathbb{C}$ such that

$$\{ v_1{+}\lambda\, p_1\, ,\ v_2{+}\lambda\, p_2\, ,\ v_3{+}\lambda\, p_3 \} \qquad \text{fails to be a basis for } V.$$

(b) Determine the exact values of $\lambda \in \mathbb{C}$ such that $\{ v_1{+}\lambda\, p_1\, ,\ v_2{+}\lambda\, p_2\, ,\ v_3{+}\lambda\, p_3 \}$ fails to be a basis for V, when $V = \mathbb{C}^3$ and

$$v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix},\ v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix},\ v_3 = \begin{pmatrix}0\\0\\1\end{pmatrix},\quad p_1 = \begin{pmatrix}1\\1\\0\end{pmatrix},\ p_2 = \begin{pmatrix}1\\0\\1\end{pmatrix},\ p_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

5. Let $V = \mathbb{M}_3(\mathbb{C})$, and consider the map

$$\langle\ ,\ \rangle\ :\ V \times V \to \mathbb{C} \qquad \text{given by} \qquad \langle X, Y \rangle \;=\; \mathrm{Tr}(\,Y^* X\,),$$

where $Y^* = \overline{Y^t}$ denotes the conjugate-transpose (adjoint) matrix of Y.

(a) Show that $\langle\ ,\ \rangle$ is an inner-product on V.

(b) Determine an orthonormal basis for the subspace of V spanned by

$$A \;=\; \begin{pmatrix}1 & i & 0\\ 0 & 0 & -i\\ 0 & 0 & -1\end{pmatrix} \qquad \text{and} \qquad B \;=\; \begin{pmatrix}1 & 0 & 0\\ i & 0 & 0\\ 0 & -i & -1\end{pmatrix}.$$

END OF PAPER