PYP Answer - MA2101 AY1415Sem2

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September 26, 2017

1. (a)

$$p_A(x) = |xI - A| = \det \begin{pmatrix} x-2 & -1 & 3 \\ & x+1 & 0 \\ & & x+1 \end{pmatrix}$$
$$= (x-2)(x+1)^2$$

   (b) $m)A(x)$ and $p_A(x)$ have the same zero sets, therefore $m_A(x) = (x-2)(x+1)^i, i \geq 1$. First we try $i = 1$.

$$m_A(A) = (A - 2I)(A + I) = 0 \qquad (\#)$$

   Therefore, $(x-2)(x+1)$ kills A, so the minimum polynomial is $m_A(x) = (x-2)(x+1)$.

   (c) Simple $m_A(x)$ has only simple zeros in $\mathbf{R}$, A is diagonalisable in $\mathbf{R}$.

   (d) From (#), we have $A^2 - A - 2I = 0$. Therefore, A^2 is dependent on A and I_3. Recursively, we will have A^i dependent on A and I_3, where $2 \leq i \leq 9$. Next $c_1 A + c_2 I = 0$ clearly has trivial solultion $(c_1, c_2) = (0, 0)$. Therefore, A and I_3 are linearly independent. Hence, the basis of W is

$$\{I_3, A\}$$

2. Let $Y = PZ$.
   By writing $Y' = AY$ in terms of Z' and Z, we have

$$PZ' = APZ \Rightarrow Z' = P^{-1}APZ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} Z$$

   which yields

$$\begin{cases} z_1' = z_1 & \Rightarrow z_1 = C_1 e^x \\ z_2' = 2z_2 + z_3 & (*) \\ z_3' = 2z_3 & \Rightarrow z_3 = C_2 e^x \end{cases}$$

This leaves us with the unknown z_2. From $(*)$, $z_2' = 2z_2 + C_2e^x$.Using hint,

$$\mu = C_3e^{-2x}$$

and therefore

$$z_2 = -C_2e^x + C_4e^{2x}$$

By back substituting z, we yield

$$\begin{cases} y_1 = -C_2e^x + C_4e^{2x} \\ y_2 = C_4e^{2x} \\ y_3 = (C_1 - C_2)e^x + C_4e^{2x} \end{cases}$$

3. (a) We construct V_1 as such. Suppose $T(V)$ is spanned by B. For each basis vector $\mathbf{b}_i$ in W, we pick $\mathbf{v}_i$ such that $T(\mathbf{v}_i) = \mathbf{b}_i$. The vector subspace V_1 is spanned by the union of all such $\mathbf{v}_i$.

   i. We first show T_1 is a linear transformation. We check T_1 respects addition and scalar multiplication. Let $\mathbf{u}, \mathbf{v} \in V_1$ and $c \in F$.

$$\begin{aligned} T_1(\mathbf{u} + \mathbf{v}) &= T(\mathbf{u} + \mathbf{v}) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \\ &= T_1(\mathbf{u}) + T_1(\mathbf{v}) \end{aligned}$$

and

$$\begin{aligned} T_1(c\mathbf{u}) &= T(c\mathbf{u}) \\ &= cT(\mathbf{u}) \\ &= cT_1(\mathbf{u}) \end{aligned}$$

Therefore T_1 is a linear transformation.
Next we show that T_1 is injective. Suppose $T_1(\mathbf{v}) = T_1(\mathbf{u}) \Rightarrow T_1(\mathbf{v}) - T_1(\mathbf{u}) = T_1(\mathbf{v} - \mathbf{u}) = 0$, where

$$\mathbf{v} = \sum_{i \in I} c_i\mathbf{v}_i \quad \mathbf{u} = \sum_{i \in I} d_i\mathbf{v}_i$$

then

$$\begin{aligned} T_1(\mathbf{v} - \mathbf{u}) &= T_1(\sum_{i \in I} c_i\mathbf{v}_i - \sum_{i \in I} d_i\mathbf{v}_i) \\ &= \sum_{i \in I}(c_i - d_i)T_1(\mathbf{v}_i) \\ &= \sum_{i \in I}(c_i - d_i)\mathbf{b}_i = 0 \end{aligned}$$

And since $\mathbf{b}_i$ are linearly indepedent, we have $c_i - d_i = 0$ for all $i \in I$, and therefore $\mathbf{v} = \mathbf{u}$. This proves the injective part.

ii. By our choice of V_1, we have

$$\forall w = T(u), \exists v \in V_1, T_1(v) = w$$

Therefore, $\mathrm{Im}(T_1) \supseteq \mathrm{Im}(T)$. For the other direction, suppose $\mathbf{w} \in \mathrm{Im}(T_1)$, then $\mathbf{w} = T(u)$ for some $u \in V$, and therefore $\mathrm{Im}(T_1) \subseteq \mathrm{Im}(T)$.

(b) Suppose the contrary is true, then we have $0 \neq \mathbf{v} \in V_1 \cap \mathrm{Ker}(T)$. Then $T_1(\mathbf{v}) = 0 = T_1(0)$, which contradicts the injective property of T_1. Therefore the claim is true.

(c) We can always docompose any vector $\mathbf{v} \in V$ in such way:

$$V \ni v = v_1 + u$$

where $v_1 \in V_1$ and $u \in V \setminus V_1$. Furthermore, we can choose v_1 such that $T(v) = T_1(v_1)$, as T_1 and T always have the same image. Then, for any v, we have

$$T(v) = T_1(v_1) + T(u) \Rightarrow 0 = T(u)$$

Therefore, $u \in \mathrm{Ker}(T)$. This proves the claim.

4. (a) We have $T_1\mathbf{v}_1 = \lambda_1\mathbf{v}_1$. Applying T_2 on both slide, and using the commutativity of T_2 and T_1 we have

$$T_2T_1\mathbf{v}_1 = T_2(\lambda_1\mathbf{v}_1) \Rightarrow T_1(T_2\mathbf{v}_1) = \lambda_1(T_2\mathbf{v}_1)$$

Therefore, $T_2\mathbf{v}_1 \in V_{\lambda_1}(T_1)$, so

$$T_2\mathbf{v}_1 = \lambda_2\mathbf{v}_1$$

for some $\lambda_2 \in F$.

(b) Choose $\mathbf{v}_2 \in V_{\lambda_1}(T_1)$, then we have

$$\begin{aligned} S_1\mathbf{v}_2 &= (1 + \lambda_1 + 2\lambda_1^2 + \cdots + 5\lambda_1^5)\mathbf{v}_2 := \mu_1\mathbf{v}_2 \text{ and} \\ S_2\mathbf{v}_2 &= (1 + \lambda_2 + 2\lambda_2^2 + \cdots + 5\lambda_2^5)\mathbf{v}_2 := \mu_2\mathbf{v}_2 \end{aligned}$$

(c) See (b).

5. (a)

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|^2 &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + 0 + \overline{0} + \langle \mathbf{y}, \mathbf{y} \rangle \\ &= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \end{aligned}$$

(b) We want to have $\langle \mathbf{z}, \mathbf{v} \rangle = 0$, that is

$$\langle \mathbf{u} - \alpha\mathbf{v}, \mathbf{v} \rangle = 0 \Rightarrow \langle \mathbf{u}, \mathbf{u} \rangle - \alpha\langle \mathbf{v}, \mathbf{v} \rangle = 0$$

Solving which, we have

$$\alpha = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$$

(c) By (b), we have, for all $\mathbf{u}$, $\mathbf{u} = \mathbf{z} + \alpha\mathbf{v}$, where $\langle \mathbf{z}, \mathbf{v} \rangle = 0$. Then

$$\begin{aligned} \|\mathbf{u}\|^2 &= \langle \mathbf{z} + \alpha\mathbf{v}, \mathbf{z} + \alpha\mathbf{v} \rangle = \|\mathbf{z}\|^2 + \|\alpha\mathbf{v}\|^2 = \|\mathbf{z}\|^2 + \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{|\langle \mathbf{v}, \mathbf{v} \rangle|^2} \|\mathbf{v}\|^2 \\ &= \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\|\mathbf{v}\|^2} + \|\mathbf{z}\|^2 \geq \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\|\mathbf{v}\|^2} \end{aligned}$$

This proves the claim.

(d) Suppose $\mathbf{u} = k\mathbf{v}$, where $k \in \mathbf{C}$. then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = |k\langle \mathbf{v}, \mathbf{v} \rangle| = |k| \, \|\mathbf{v}\|^2 = \|k\mathbf{v}\| \, \|\mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\|$$

In a similar way, we can prove the second part of the claim.

(e) From (c), we see the equality holds if and only if $\|\mathbf{z}\| = 0$, which happends if and only if $\mathbf{u} = \alpha\mathbf{v}$.

6. (a) False. V and W may not have the same dimension.
   (b) True. $m(x)$ has only simple zeroes.
   (c) True. By principle axis theorem.
   (d) True. Since $m(x)$ has only simple zeroes.
   (e) False.
   (f) True.
   (g) True.

7. (a) Consider any $x \in V_i$. Choose a positive integer p such that $(T - \lambda I)^p(x) = 0$. Then

$$(T - \lambda I)^p T(x) = T(T - \lambda I)^p(x) = T(0) = 0$$

(b) Direction ($\subseteq$) is clear. For ($\supseteq$), we claim that V_i and V_j are disjoint. Then for $x \in V_i$, $m_T(T)(x) = \prod_{j \neq i}(T - \lambda_j)^{m_j}(T - \lambda_i)^{m_i}(x) = 0$. Suppose $x \notin \mathrm{Ker}(T - \lambda_i I)$, then $m_T(T)(x) = \prod_{j \neq i}(T - \lambda_j)^{m_j} y \neq 0$, Therefore, we have the inclusion.
Next we prove that all generalised eigenspaces V_i are disjoint. Suppose the contrary. Then there is $x \in \beta_i \cap \beta_j$, where β_i, β_j are basis of $V_i, V_j, i \neq j$. We will observe that $T - \lambda_i I$ is one-to-one on V_j, and therefore, $(T - \lambda_i I)^p(x) \neq 0$ for any positive integer p. But this contradicts the fact that $x \in V_i$, and the result follows.
The observation is a direct result of (a). Let $x \in V_i$ and $(T - \lambda_j I)(x) = 0$. By way of contradiction, suppose that $x \neq 0$. Let p be the smallest integer for which $(T - \lambda_i I)^p(x) = 0$, and let $y = (T - \lambda_i I)^{p-1}(x)$. Then

$$(T - \lambda_i I)(y) = (T - \lambda_i I)^p(x) = 0$$

and hence y is in the eigenspace of λ_i. Furthermore,

$$(T - \lambda_j I)(y) = (T - \lambda_j I)(T - \lambda_i I)^{p-1}(x) = (T - \lambda_i I)^{p-1}(T - \lambda_j I)(x) = 0$$

so that y is also in the eigenspace of λ_j. But eigenspace of different eigenvector intersects only trivially. Thus, $y = 9$, contradiction. So $x = 0$, and the restriction of $T - \lambda_j$ to V_i is one-to-one.

(c) We have proven this in (b), by showing V_i and V_j intersect trivially when $i \neq j$.

(d) This is shown in (b). Suppose not, $m(T)$ can no longer kill.

(e) First, we observe $q_i(x)$ are coprime, then using hint, we have $\sum q_i(x)u_i(x) = 1$. Then let $x = T$, we will have $V = \sum_{i=1}^{n} V_i$. Together with (d), we show the claim.

(f) We consider T in the basis of $B = (B_1, B_2, \ldots, B_k)$, where B_i is the basis of V_i. Then a suitable choice of basis, which makes $[T]_B$ diagonal, will suggests that $[T_i]_B$ has n_i diagonal entry, and a power of m_i is required to kill. Therefore we have this final result.