Revision notes - MA1102R

Ma Hongqiang

January 15, 2017

Contents

1 **Limits** 2

2 **Continuous Functions** 4

3 **Derivatives** 6

4 **Applications of Differentiation** 7

5 **Integrals** 11

6 **Inverse Functions and Techniques of Integration** 13

7 **Application of Integral** 17

8 **Ordinary Differential Equations** 18

1 Limits

1.1 Precise Definition of Limits

Definition 1.1 (Limit).
Let f be a function defined on an open interval containing a, except possibly at a.
The **limit** of $f(x)$ when x approaches a, equals L if
for every $\epsilon > 0$ **there is** a $\delta_\epsilon > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta_\epsilon$$

Definition 1.2 (Left-hand Limit).
We write $\lim\limits_{x \to a^-} f(x) = L$ if
for every $\epsilon > 0$ **there is** a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < a - x < \delta$$

Definition 1.3 (Right-hand Limit).
We write $\lim\limits_{x \to a^-} f(x) = L$ if
for every $\epsilon > 0$ **there is** a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < x - a < \delta$$

Definition 1.4 (Infinite Limit).
We write $\lim\limits_{x \to a^-} f(x) = \infty$ if **for every** $M > 0$ **there is** a $\delta > 0$ such that

$$f(x) > M \quad \text{whenever} \quad 0 < |x - a| < \delta$$

Definition 1.5 (Negative Infinite Limit).
We write $\lim\limits_{x \to a^-} f(x) = -\infty$ if **for every** $M < 0$ **there is** a $\delta > 0$ such that

$$f(x) < M \quad \text{whenever} \quad 0 < |x - a| < \delta$$

1.2 Law of Limits

Theorem 1.1 (Scalar Multiplication Law).
If $\lim\limits_{x \to a} f(x) = L$, then $\lim\limits_{x \to a} (c(f(x))) = cL$, where c is a constant.

Theorem 1.2 (Sum Law).
If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then $\lim\limits_{x \to a} (f(x) + g(x)) = L + M$.

Theorem 1.3 (Product Law).
If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then $\lim\limits_{x \to a} f(x)g(x) = LM$.

Theorem 1.4 (Quotient Law).
If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided that $M \neq 0$.

Theorem 1.5.
Suppose $f(x) = g(x)$ for all x near a, except possibly at a.
If $\lim\limits_{x \to a} f(x) = L$, then $\lim\limits_{x \to a} g(x)$ exists and equals L.

Theorem 1.6 (Inequality on Limits).
Suppose $f(x) \leq g(x)$ for all x near a, except possibly at a.
If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $L \leq M$.

Theorem 1.7 (Squeeze Theorem).
Let f, g, h be functions such that $f(x) \leq g(x) \leq h(x)$ for all x near a, and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$
Then $\lim_{x \to a} g(x)$ exists and equals L.

2 Continuous Functions

2.1 Continuity at a Point

Definition 2.1 (Continuity at Point)**.**
A function f is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a)$$

The definition of continuity requires the following:

1. f is **defined** at a,
2. $\lim\limits_{x \to a} f(x)$ exists,
3. $\lim\limits_{x \to a} f(x) = f(a)$.

Definition 2.2 (One-sided continuity)**.**
A function f is said to **be continuous from the right** at a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is said to **be continuous from the left** at a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Theorem 2.1.
*f is **continuous at** a if and only if f is **continuous from the left at** a and **continuous from the left at** a.*

2.2 Continuity on an Interval

Definition 2.3 (Continuity on Interval)**.**
f is **continuous on an interval** if it is **continuus at every number** in the interval.

f is continuous on open interval (a, b)

$$\Leftrightarrow f \text{ is continuous at every } x \in (a, b)$$

f is continuous on closed interval $[a, b]$

$$\Leftrightarrow \begin{cases} f \text{ is continuous at every } x \in (a, b) \\ f \text{ is continuous from the right at } a \\ f \text{ is continuous from the left at } b \end{cases}$$

2.3 Properties of Continuous Function

Suppose f and g are continuous at a, then

- cf is continuous at a,
- $f + g$ is continuous at a,
- fg is continuous at a,
- f/g is continuous at a, provided that $g(a) \neq 0$.

Polynomials, **rational functions**, **root functions**, **trigonometric functions** are continuous over domain.

Theorem 2.2 (Composite of Continuous Functions)**.**
If f is **continuous** at b and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(b)$$

Or equivalently,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

If g is continuous at a and f is continuous at $g(a)$, then the **composite** $f \circ g$ is continuous at a.

2.3.1 Remarks on substitution

Suppose $y = g(x)$ such that $\lim_{x \to a} g(x) = b$. If

1. f is **continuous** at b; **or**
2. $g(x) \neq b$ for all x near a, and $\lim_{y \to b} f(y)$ exists;

Then $\lim_{x \to a} f(g(x)) = \lim_{y \to b} f(y)$.
In particular, assuming that $\lim_{y \to b} f(y)$ exists, then (2) holds if g is a **one-to-one** function

2.4 Intermediate Value Theorem

Theorem 2.3 (Intermediate Value Theorem(Simple))**.**
Let f be a function **continuous on** a finite closed interval $[a, b]$.
If $f(a) < 0$ and $f(b) > 0$, or if $f(a) > 0$ and $f(b) < 0$, then
there exists a number $c \in (a, b)$ such that $f(c) = 0$.

Theorem 2.4 (Intermediate Value Theorem)**.**
Let f be a function **continuous on** $[a, b]$ with $f(a) \neq f(b)$.
Let N be a number between $f(a)$ and $f(b)$, then **there exists** $c \in (a, b)$ such that $f(c) = N$.

3 Derivatives

Definition 3.1 (Derivative)**.**
The **derivative** of a function f at a number a is

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

f is differentiable at a if $f'(a)$ exists.
$f'(a)$ is the slope of $y = f(x)$ at $x = a$.
An equivalent definition is:

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Theorem 3.1 (Differentiability implies Continuity)**.**
If f is **differentiable** at a then f is **continuous** at a.

Theorem 3.2 (Differentiation Formulas)**.**

- $(cf)' = cf'$
- $(f \pm g)' = f' \pm g'$
- $(fg)' = f'g + fg'$
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, if $g(x) \neq 0$.

Theorem 3.3 (Some useful results)**.**

- $(x^n)' = nx^{n-1}$ for all $n \in \mathbb{Q}$
- Lemma A: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$
- Lemma B: $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \sec^2 x$
- $(\cot x)' = -\csc^2 x$
- $(\sec x)' = \sec x \tan x$
- $(\csc x)' = -\csc x \cot x$

Theorem 3.4 (Chain Rule)**.**
Let f and g be differentiable functions.
Then $F = f \circ g$ is differentiable and

$$F' = (f' \circ g)(g')$$

4 Applications of Differentiation

Definition 4.1 (Absolute Maximum and Minimum)**.**
Let f be a function, and D be its domain.
f has an **global** (or **absolute**) **maximum** at $c \in D$

$$\Leftrightarrow f(c) \geq f(x) \text{ for all } x \in D.$$

f has an **global** (or **absolute**) **minimum** at $c \in D$

$$\Leftrightarrow f(c) \leq f(x) \text{ for all } x \in D.$$

The **absolute maximum** and **absolute minimum** are called the **(absolute) extreme values**.

Definition 4.2 (Local Maximum and Minimum)**.**
Let f be a function with domain D.
f has a **local maximum** at $c \in D$

$$\Leftrightarrow f(c) \geq f(x) \text{ for all } x \text{ near } c \text{ (i.e., for all } x \text{ in an open interval containing } c\text{)}$$

f has a **local minimum** at $c \in D$

$$\Leftrightarrow f(c) \leq f(x) \text{ for all } x \text{ near } c \text{ (i.e., for all } x \text{ in an open interval containing } c\text{)}$$

Theorem 4.1 (Extreme Value Theorem)**.**
If f is **continuous** on a **finite closed** interval $[a, b]$, then f attains **extreme values** on $[a, b]$.
Precisely, f attains an

- **absolute maximum** value $f(c)$ at some $c \in [a, b]$,
- **absolute minimum** value $f(d)$ at some $d \in [a, b]$.

Theorem 4.2 (Finding Extreme Values)**.**
Let f be a continuous function on closed interval $[a, b]$.

1. Compute the values at **endpoints**: $f(a)$, $f(b)$.
2. Find **local max** and **local min** of f on (a, b).
3. Compare the values obtained above to seek out the **extreme values**.

Theorem 4.3 (Fermat's Theorem)**.**
Suppose f has a **local maximum** or **local minimum** at c.

$$\text{If } f'(c) \text{ exists, then } f'(c) = 0$$

Definition 4.3 (Critical Number)**.**
Let f be a function with domain D.
Then $c \in D$ is called a **Critical Number** of f if $f'(c)$ does not exist, or $f'(c)$ exists and equals 0.

Theorem 4.4 (Closed Interval Method)**.**
Condition: f is a **continuous** function on interval $[a, b]$.

1. Find the values of f at **endpoints**: $x = a$, $x = b$,
2. Find the values of f at **critical numbers** of f in (a, b)
3. Compare the values of $f(x)$ evaluated in 1 and 2.

Theorem 4.5 (Rolle's Theorem)**.**
Condition: f is a function such that

- f is **continuous** on $[a, b]$,
- f is **differentiable** on (a, b),
- $f(a) = f(b)$

Then there is a number $c \in (a, b)$ such that $f'(c) = 0$.

Theorem 4.6 (Mean Value Theorem)**.**
Condition: f is a function such that

- f is **continuous** on $[a, b]$,
- f is **differentiable** on (a, b),

Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 4.7 (Cauchy's Mean Value Theorem)**.**
Condition: f, g are functions **continuous** on $[a, b]$, **differentiable** on (a, b), and $g'(x) \neq 0$ for any $x \in (a, b)$.
Then there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Theorem 4.8 (An application of Mean Value Theorem)**.**
Let f be a function such that

- f is **continuous** on $[a, b]$
- $f'(x) = 0$ for any $x \in (a, b)$

Then f is **constant** on $[a, b]$.
From this theorem, let f and g be continuous on $[a, b]$.
If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + C$ on $[a, b]$ for a constant C.

Theorem 4.9 (Increasing and Decreasing Test)**.**
Condition: f is **continuous** on $[a, b]$, **differentiable** on (a, b).
If $f'(x) > 0$ for any $x \in (a, b)$, then f is **increasing** on $[a, b]$.
If $f'(x) < 0$ for any $x \in (a, b)$, then f is **decreasing** on $[a, b]$.

Theorem 4.10 (Weak Converse of Increasing Test)**.**
Let f be **differentiable** on an open interval I.

- f is **increasing** $\Leftrightarrow f' \geq 0$ on I.
- f is **decreasing** $\Leftrightarrow f' \leq 0$ on I.

Theorem 4.11 (First Derivative Test)**.**
Condition: f is a **continuous** function, and c is a **critical number** of f. Suppose f is **differentiable** near c, except possibly at c.

- If f' changes from **positive** to **negative** at c, then f has a **local maximum** at c.
- If f' changes from **negative** to **positive** at c, then f has a **local minimum** at c.
- If f' does not change sign at c, then f has **neither local max/min** at c.

Definition 4.4 (Concavity)**.**
The graph of f is **concave up** on open interval I if $f(x) > f'(y)(x-y)+f(y)$ for any $x \neq y$ in I.
The graph of f is **concave down** on open interval I if $f(x) < f'(y)(x-y)+f(y)$ for any $x \neq y$ in I.

Theorem 4.12 (Relations of Concavity and f')**.**
Condition: f is **differentiable** on an open interval I.

- The graph is **concave up** $\Leftrightarrow f'$ is **increasing**.
- The graph is **concave down** $\Leftrightarrow f'$ is **decreasing**.

Theorem 4.13 (Concavity Test)**.**
Condition: f is **twice differentiable** on an open interval I.

- If $f'' > 0$ on I, by increasing test, f' is increasing, then the graph of f is **concave up**.
- If $f'' < 0$ on I, by decreasing test, f' is decreasing, then the graph of f is **concave down**.

Remark: Concavity, however, only requires f be differentiable.

Theorem 4.14 (Second Derivative Test)**.**
Condition: f'' exists at **critical numbers** and f'' is non-zero at **critical numbers**.

- $f'(c) = 0$ and $f''(c) > 0 \Rightarrow f$ has a **local minimum** at c.

- $f'(c) = 0$ and $f''(c) < 0 \Rightarrow f$ has a **local maximum** at c.

Definition 4.5 (Inflection Point).
A point P on the curve of $y = f(x)$ is called an **inflection point** if

- f is **continuous** at P, and
- the **concavity** of the curve changes at P.

Theorem 4.15 (Properties of Inflection Point).
Suppose f has an inflection point at c, if f is **twice differentiable** at c, then $f''(c) = 0$.

Theorem 4.16 (Taylor's Theorem).
Suppose f is $n + 1$ times differentiable, then,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} + R_n$$

$$\text{where } R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \text{ for a } c \text{ between } x \text{ and } a.$$

Theorem 4.17 (l'Hopital's Rule).
Conditions:

- $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$ or ∞
- f and g are **differentiable** near a, except at a
- $g'(x) \neq 0$ near a.

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, provided that the limit on the right hand side exists or equal $\pm\infty$

5 Integrals

Definition 5.1 (Definite Integral)**.**
Condition: f is a **continuous** function on $[a, b]$.
The **definite integral** of f from a to b:

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\frac{b-a}{n})\frac{1}{n}$$

Theorem 5.1 (Properties of Definite Integral)**.**

- $\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$
- $\int_a^b c\mathrm{d}x = (b-a)c$
- $\int_a^b cf(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x$
- $\int_a^b (f(x) + g(x))\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x$
- $\int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$
- Let f be a continuous function on $[a, b]$,$(a < b)$. Suppose $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\mathrm{d}x \geq 0$.
- Let f and g be continuous and $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)\mathrm{d}x \geq \int_a^b g(x)\mathrm{d}x$
- Let f be continuous and $m \leq f(x) \leq M$ on $[a, b]$. $m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a)$

Theorem 5.2 (Fundamental Theorem of Calculus (I))**.**
Let f be a **continuous** function on $[a, b]$. Define $g(x) = \int_a^x f(t)\mathrm{d}t$. Then

- g is **continuous** on $[a, b]$
- g is **differentiable** on (a, b), and $g'(x) = f(x)$ on (a, b).

Theorem 5.3 (Fundamental Theorem of Calculus (II))**.**
Let f be a **continuous** function on $[a, b]$. If F is **continuous** on $[a, b]$ and $F' = f$ and $[a, b]$, then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

Theorem 5.4 (Substitution Rule)**.**
Let $u = g(x)$ be a differentiable function whose range is an interval. If f is continuous and g' is continuous, then

$$\int f(g(x))g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u$$

$$\int_a^b f(g(x))g'(x)\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\mathrm{d}u$$

Definition 5.2 (Improper Integral)**.**
Let f be **continuous** on $[a, b)$, and discontinuous at b.

$$\int_a^b f(x)\mathrm{d}x = \lim_{t\to b^-} \int_a^t f(x)\mathrm{d}x$$

Theorem 5.5 (Properties of Improper Integral)**.**
Suppose f has a discontinuity at $c \in (a, b)$.

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x \text{ if both are convergent}$$

Definition 5.3 (Improper Integral over Infinite Intervals)**.**
If $\int_a^t f(x)\mathrm{d}x$ exists for every $t \geq a$, the improper integral of f from a to ∞ is

$$\int_a^\infty f(x)\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\mathrm{d}x$$

6 Inverse Functions and Techniques of Integration

Definition 6.1 (One to One Function)**.**
f is said to be **one-to-one** if

$$\text{for any } a, b \in D, a \neq b \Rightarrow f(a) \neq f(b)$$

Or equivalently,

$$\text{for any } a, b \in D, f(a) = f(b) \Rightarrow a = b$$

Definition 6.2 (Inverse Function)**.**
Let f be a **one-to-one** function with domain A and range B.
Its **inverse function** f^{-1} is the function with

- **domain** B and **range** A, and
- $f^{-1}(y) = x \Leftrightarrow y = f(x)$ for any $x \in A, y \in B$.

Theorem 6.1 (Calculus of Inverse Function)**.**
Let f be a one-to-one continuous function on an open interval I, then the inverse function f^{-1} is also continuous.

- If f is **differentiable** at $a \in I$, and $f'(a) \neq 0$,
- then f^{-1} is **differentiable** at $b = f(a)$,

  and $(f^{-1})'(b) = \frac{1}{f'(a)}$

6.1 Inverse Trigonometric Functions

Inverse Function	Domain	Range
$\arcsin x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos x$	$[-1, 1]$	$[0, \pi]$
$\arctan x$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$\operatorname{arccot} x$	$\mathbb{R}$	$(0, \pi)$
$\operatorname{arcsec} x$	$\mathbb{R} \setminus (-1, 1)$	$[0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$
$\operatorname{arccsc} x$	$\mathbb{R} \setminus (-1, 1)$	$(0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$

6.2 Derivative of Inverse Trigonometric Functions

- $\frac{\mathrm{d}}{\mathrm{d}x} \arcsin x = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
- $\frac{\mathrm{d}}{\mathrm{d}x} \arccos x = \frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$
- $\frac{\mathrm{d}}{\mathrm{d}x} \arctan x = \frac{1}{1+x^2}, x \in \mathbb{R}$
- $\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{arccot} x = \frac{-1}{1+x^2}, x \in \mathbb{R}$

- $\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{arcsec} x = \frac{1}{x\sqrt{x^2-1}}, |x| > 1$
- $\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{arccsc} x = \frac{-1}{x\sqrt{x^2-1}}, |x| > 1$

  And following are the identities.

- $\arcsin x + \arccos x = \frac{\pi}{2}$
- $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$
- $\operatorname{arcsec} x + \operatorname{arccsc} x = \begin{cases} \frac{\pi}{2}, & \text{if } x \geq 1 \\ \frac{5\pi}{2}, & \text{if } x \leq -1 \end{cases}$

6.3 Natural Logarithmic Function

Definition 6.3 (Natural Logarithmic Function($\ln x$))**.**

$$\ln x = \int_1^x \frac{1}{t} \mathrm{d}t$$

Theorem 6.2 (Properties of Natural Logarithmic Function)**.**

- $\ln 1 = 0, \ln e = 1$
- $\ln x$ is increasing on $\mathbb{R}^+$
- $\lim\limits_{x\to\infty} \ln x = \infty, \lim\limits_{x\to\infty} \ln x = -\infty$
- Let $a > 0$ and $x > 0$. $\ln ax = \ln a + \ln x$
- Let $x > 0$. $\ln x^r = r \ln x$.
- $\int \frac{1}{x} \mathrm{d}x = \ln |x| + C$

Theorem 6.3 (Logarithmic Differentiation)**.**
Given $y = f_1(x)f_2(x)\cdots f_n(x)$, we have

$$\ln |y| = \ln |f_1(x)| + \ln |f_2(x)| + \cdots + \ln |f_n(x)|$$

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f_1'(x)}{f_1(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left[\frac{f_1'(x)}{f_1(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}\right] f_1(x)f_2(x)\cdots f_n(x)$$

6.4 Exponential Function

Definition 6.4 (Exponential Function).
Exponential function (exp x), is defined as

$$e^x = \ln^{-1}(x), \text{ for all } x \in \mathbb{R}$$

Theorem 6.4 (Properties of Exponential Function).

- $\lim\limits_{x\to\infty} e^x = \infty, \lim\limits_{x\to-\infty} e^x = 0$
- $\lim\limits_{x\to\infty} \frac{e^x}{x^n} = \infty, n \in \mathbb{Z}^+$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Definition 6.5 ((Generalised) Exponential Function).
For $a > 0$ and $x \in \mathbb{R}$, define

$$a^x = \exp(x \ln a) = e^{x \ln a}$$

Theorem 6.5 (More Properties of Exponential Function).

1. $a^u a^v = a^{u+v}$
2. $a^{-u} = \frac{1}{a^u}$
3. $(a^u)^v = a^{uv}$
4. $(a^x)' = a^x \ln a$
5. $\int x^r \mathrm{d}x = \begin{cases} \frac{x^{r+1}}{r+1} + C \text{ if } r \neq -1 \\ \ln|x| + C \text{if } r = -1 \end{cases}$
6. $e = \lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = \lim\limits_{y\to\infty}(1+\frac{1}{y})^y$

6.5 Hyperbolic Trigonometric Functions

Definition 6.6 (Hyperbolic Trigonometric Functions).

- $\sinh x = \frac{e^x - e^{-x}}{2}$
- $\cosh x = \frac{e^x + e^{-x}}{2}$

  Remark:

  $\cosh^2 x - \sinh^2 x = 1$

  $(\sinh x)' = \cosh x, (\cosh x)' = \sinh x$
- $\tanh x = \frac{\sinh x}{\cosh x}$

Theorem 6.6 (Inverse Trigonometric Functions)**.**
Let $y = \sinh^{-1} x$. $\frac{\mathrm{d}}{\mathrm{d}x} \sinh^{-1} x = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+x^2}}$
Hence, $\sinh^{-1} x = \ln(x + \sqrt{1 + x^2}), x \in \mathbb{R}$.
Similarly,
$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), x \geq 1$;
$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x}\right), -1 < x < 1$

6.6 More Techniques on Integration

Theorem 6.7 (Substitution Rule(2nd version))**.**
Let f be a continuous function, and $x = g(t)$ be a differentiable function. If g' is continuous, then

$$\int f(x)\mathrm{d}x = \int f(g(t))g'(t)\mathrm{d}t$$

6.6.1 Useful Substituents

Scenario	Substituent
$\frac{1}{x(1+x^4)}$	$x = \frac{1}{t}$
$1 + x^2$	$x = \tan t$
$\sqrt{1 - x^2}$	$x = \sin t$
$\sqrt{x^2 - 1}$	$x = \sec t$
$\sin x, \cos x, \cdots$	$t = \tan \frac{x}{2}$
$\frac{f(x)}{g(x)}$, f, g polynomial	Partial fraction

Theorem 6.8 (Integration by Parts)**.**

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

7 Application of Integral

Theorem 7.1 (Finding Net Area)**.**
Let f be a **continuous** function on $[a, b]$. Then the **net area** under $y = f(x)$ from a to b is

$$A = \int_a^b f(x)\mathrm{d}x$$

Theorem 7.2 (Finding Volume (Washer Method))**.**
Suppose the solid is obtained by rotating the region between the curve $y = f(x)$ and the x-axis from a to b about the x-axis.

$$V = \int_a^b \pi(f(x))^2\mathrm{d}x$$

Theorem 7.3 (Finding Volume (Cylindrical Shell Method))**.**
Let f be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$ where $0 \leq a < b$. Then the volume of the solid obtained by rotating the region between the curve $y = f(x)$ and the x-axis from a to b about the y-axis is

$$V = \int_a^b 2\pi x f(x)\mathrm{d}x$$

Theorem 7.4 (Finding Arc Length)**.**
The **arc length** of smooth curve $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\mathrm{d}x$$

Theorem 7.5 (Finding Surface Area of Revolution)**.**
Let f be a smooth function such that $f(x) \geq 0$ on $[a, b]$. Then the area of the surface obtained by rotating the curve $y = f(x), a \leq x \leq b$ about the x-axis is

$$A = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\mathrm{d}x$$

8 Ordinary Differential Equations

8.1 Simplest Ordinary Differential Equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \Rightarrow y = \int f(x)\mathrm{d}x + C$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(y) \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{g(y)} \Rightarrow x = \int \frac{1}{g(y)}\mathrm{d}y$$

8.2 Seperation of Variables

Consider $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$.
In **differential forms**, $\frac{1}{g(y)}\mathrm{d}y = f(x)\mathrm{d}x$

$$\int \frac{1}{g(y)}\mathrm{d}y = \int f(x)\mathrm{d}x$$

Remark: Try substitute $z = \frac{y}{x}$ when encountering homogeneous equations of degree zero.

8.3 First Order Linear Equations

Definition 8.1 (First Order Linear Equations)**.**
First order linear equations are in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$$

Theorem 8.1 (Solving First Order Linear Equations)**.**
Consider $\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$.

- Evaluate $P(x) = \int p(x)\mathrm{d}x$
- Let $v(x) = e^{P(x)}$
- $y = \frac{1}{v(x)} \int v(x)q(x)\mathrm{d}x$

Theorem 8.2 (Bernoulli's Equation)**.**
Consider $\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n$.
Substitute $z = y^{1-n}$. We have $\frac{\mathrm{d}z}{\mathrm{d}x} + (1-n)p(x)z = (1-n)q(x)$.