Revision notes - MA3110

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Contents

1 Review

Definition 1.1 (Limit of Sequence)**.**
For a sequence $(x_n)_{n\in\mathbb{N}}$, we say $\lim_{n\to\infty} x_n = a$ if and only if

$$\forall\epsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0, |x_n - a| \leq \epsilon$$

Similarly, we define $\lim_{n\to\infty} x_n = \infty$ if and only if

$$\forall M > 0, \exists n_0 \text{ depending on } M, \text{s.t. } \forall n \geq n_0, x_n \geq M$$

Definition 1.2 (Limit Point(Subsequential Limit in `MA2108` notes))**.**
A number $a \in [-\infty, \infty]$ is called a **limit point** of the sequence $(x_n)_{n\in\mathbb{N}}$, if there exists an increasing sequence of indices $n_1 < n_2 < n_3 < \cdots$ such that $\lim_{i\to\infty} x_{n_i} = a$.

Theorem 1.1.
$\lim_{n\to\infty} x_n$ does not exist in $[-\infty, \infty]$ if and only if $(x_n)_{n\in\mathbb{N}}$ has more than 1 limit point in $[-\infty, \infty]$.

Definition 1.3 (Supremum and Infimum)**.**
Let $A \subset [-\infty, \infty]$. The **supremum** of A, denoted by $\sup A$, is defined to be the **least upper bound** of A.
Essentially, $p = \sup A$ if and only if

1. $x \leq p \forall x \in A$
2. if $x \leq u \forall x \in A$ for some $u \in [-\infty, \infty]$, then $p \leq u$.

The infimum is defined in a similar fashion. For detailed definition, check MA2108 revision notes.

Definition 1.4 (Limit Supremum and Infimum)**.**
Given a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$,

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} (\sup_{m\geq n} x_m)$$

and

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} (\inf_{m\geq n} x_m)$$

Theorem 1.2.
$\limsup_{n\to\infty} x_n$ is a limit point and the **largest limit point** of the sequence $(x_n)_{n\in\mathbb{N}}$. $\liminf_{n\to\infty} x_n$ is the smallest limit point.

Theorem 1.3.
$\lim_{n\to\infty} x_n$ exists in $[-\infty, \infty]$ if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$.

Definition 1.5 (Continuity)**.**
A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **continuous** at x if $\lim_{y\to x} f(y)$ exists and equals $f(x)$.
Equivalently,

$$\forall\epsilon > 0, \exists\delta > 0 \text{ such that } \sup_{y\in[x-\delta, x+\delta]} |f(y) - f(x)| \leq \epsilon$$

2 Derivative

Definition 2.1 (Derivative).
Let $I \subseteq \mathbb{R}$ be an interval, and let $c \in I$. A function $f : I \to \mathbb{R}$ is differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = L$$

for some $L \in \mathbb{R}$.
Equivalently, we need

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that} \forall x \in I, 0 < |x - c| < \delta \Rightarrow |\frac{f(x) - f(c)}{x - c} - L| \leq \epsilon$$

Here, L is called the **derivative** of f at c, denoted by $f'(c)$, or $\frac{\mathrm{d}f}{\mathrm{d}x}|_{x=c}$.
If f is differentiable at every $x \in S \subseteq I$, we say f is differentiable on S.

Definition 2.2 (Equivalent Definition of Derivative).
f is differentiable at c, if $f(x)$ can be approximated by the line $l(x) := f(c) + f'(c)(x - c)$ near $x = c$, i.e.,

$$\forall \epsilon > 0, \exists \delta > 0, \text{such that } \forall x \in [c - \delta, c + \delta], |f(x) - l(x)| \leq \epsilon |x - c|$$

Theorem 2.1 (Differentiability infers Continuity).
If $f : I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c.

Theorem 2.2 (Derivative Rules).
Suppose that $f, g : I \to \mathbb{R}$ are differentiable at $c \in I$, then

- (Linearity) For any $a, b \in \mathbb{R}$, $af + bg$ is differentiable at c, and
$$(af + bg)'(c) = af'(c) + bg'(c)$$
- (Product Rule) fg is differentiable at c and
$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$
- (Quotient Rule) If $g(c) \neq 0$, then f/g is differentiable at c and
$$(\frac{f}{g})'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}$$

Theorem 2.3 (Caratheodory's Representation Lemma).
Let $f : I \to \mathbb{R}$ and let $c \in I$. The following conditions are equivalent:

1. f is differentiable at c.
2. There exists a function $\phi : I \to \mathbb{R}$ such that ϕ is continuous at c and
$$f(x) - f(c) = \phi(x)(x - c) \quad \forall x \in I$$
In this case, $\phi(c) = f'(c)$.

Theorem 2.4 (Chain Rule).
Let I, J be intervals in $\mathbb{R}$. Let $g : I \to J$ and $f : J \to \mathbb{R}$. Suppose g is differentiable at $c \in I$ and f is differentiable at $g(c) \in J$, then $f \circ g$ is differentiable at c, with

$$(f \circ g)'(c) = f'(g(c))g'(c)$$

3 Mean Value Theorem

Theorem 3.1 (Derivative of an Inverse Function)**.**
Let I be an interval, and $f : I \to \mathbb{R}$ be continuous and stricly monotone on I. Let $J := f(I)$ be the **range** of f, and $g : J \to I$ be the inverse of f.
If f is differentiable at $c \in I$ and $f'(c) \neq 0$, then g is differentiable at $f(c) \in J$, and

$$g'(f(c)) = \frac{1}{f'(c)}$$

Theorem 3.2 (Mean Value Theorem)**.**
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

A special case of Mean Value Theorem is Rolle's Theorem.

Theorem 3.3 (Rolle's Theorem)**.**
When $f(a) = f(b)$ in the Mean Value Theorem, we obtain the existence of a $c \in (a, b)$ with

$$f'(c) = 0$$

Definition 3.1 (Relative Extremum)**.**
Let $f : I \to \mathbb{R}$ for some subset $I \subseteq \mathbb{R}$ and let $c \in I$. Then

1. f has a **relative maximum** at c, if for some $\delta > 0$,

$$f(c) \geq f(x) \forall x \in I \cap (c - \delta, c + \delta)$$

2. **Relative minimum** of f on I are defined analogously.

Relative Extremum refers to either relative maximum or relative minimum.

Theorem 3.4 (Interior Extremum Theorem)**.**
Let $f : I \to \mathbb{R}$, and let $c \in I$ be an interior point of I, i,e, $(c - \delta, c + \delta) \subseteq I$ for some $\delta > 0$. If f is differentiable at c adn has a relative extremum at c, then $f'(c) = 0$.

Theorem 3.5.
Let $f : I \to \mathbb{R}$ and assume that $f'(c)$ exists for some $c \in I$.

1. If $f'(c) > 0$, then for some $\delta > 0$, we have

$$f(x) < f(c) \quad \forall x \in I \cap (c - \delta, c)$$

   and

$$f(x) > f(c) \quad \forall x \in I \cap (c, c + \delta)$$

2. If $f'(c) < 0$, then the directions of the two inequalities above are reversed.

Theorem 3.6 (Cauchy's Mean Value Theorem)**.**
Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) = g(a))f'(c)$$

4 Application of Mean Value Theorem

Theorem 4.1 (Monotonicity Properties)**.**
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then f is increasing(resp. decreasing) on $[a, b]$ if and only if $f'(x) \geq 0$(resp. $f'(x) \leq 0$) for all $x \in (a, b)$.

If strict monotonicity is concerned, we will have $f'(x) > 0 \Rightarrow f(x) < f(y)$ for all $x < y$, but **not** the other direction.

Theorem 4.2 (Uniqueness of Anti-derivative Modulo Shift)**.**
Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).
Suppose that f and g have the same derivative, i.e., $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists a constant $C \in \mathbb{R}$ such that

$$f(x) = g(x) + C \quad \forall x \in [a, b]$$

Theorem 4.3 (Intermediate Value Theorem for Derivatives)**.**
Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Suppose that $f'(a) < f'(b)$, then for any $r \in (f'(a), f'(b))$, there exists some $c \in (a, b)$ with $f'(c) = r$.

Theorem 4.4 (First Derivative Test)**.**
Let f be continuous on (a, b). Let $c \in (a, b)$. Assume that $f'(x)$ exists for all $x \in (a, b) \setminus \{c\}$. Then

1. If $f'(x) \geq 0$ for all $x \in (a, c)$ and $f'(x) \leq 0$ for all $x \in (c, b)$, then f has a relative maximum at c.

2. If $f'(x) \leq 0$ for all $x \in (a, c)$ and $f'(x) \geq 0$ for all $x \in (c, b)$, then f has a relative minimum at c.

Theorem 4.5 (Second Derivative Test)**.**
Let f be differentiable on $[a, b]$ with derivative f'. Suppose $f'(c) = 0$ at some $c \in (a, b)$, and f' is differentiable at c with derivative $f''(c)$. Then,

1. If $f''(c) > 0$, then f has a relative minimum at c.

2. If $f''(c) < 0$, then f has a relative maximum at c.

5 L'Hospital's Rule

Theorem 5.1 (L'Hospital's Rule)**.**
Let $-\infty \leq a < b \leq \infty$. Let f and g be differentiable on (a, b). Assume that $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a, b)$.

(I) If $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$, and $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$ for some $L \in [-\infty, \infty]$, then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$$

(II) If $\lim_{x\to a^+} g(x) = \infty$ and $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$ for some $L \in [-\infty, \infty]$, then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$$

Remark: L'Hospital's rule also holds if we replace $x \to a^+$ above by $x \to b^-$. We can also replace b be $a + \delta$ for some $\delta > 0$. Also, note that we make no assumption on f in (II).

Theorem 5.2 (Taylor Expansion)**.**
Let f be n times differentiable on $[a, x]$, with $f^{(i)}$ denoting the ith derivative of f. Suppose that $f^{(n+1)}(x)$ exists on (a, x). Then there exists $c \in (a, x)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

6 More on Taylor

Theorem 6.1 (Taylor Theorem)**.**
Let f be n times differentiable on $[a, x]$ with $f^{(i)}$ denoting the ith derivative of f.
Suppose that $f^{(n+1)}$ exists on (a, x). Then there exists $c \in (a, x)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Theorem 6.2 (Higher Order Derivative Tests)**.**
Let $f : [a, b] \to \mathbb{R}$. Suppose that $f^{(1)}(x_0) = f^{(2)}(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ for some $x_0 \in (a, b)$.
Assume also that $f^{(n)}$ exists at x_0 with $f^{(n)}(x_0) \neq 0$. Then

1. If n is even
   (a) and $f^{(n)}(x_0) > 0$, then x_0 is a relative minimum of f.
   (b) and $f^{(n)}(x_0) < 0$, then x_0 is a relative maximum of f.
2. If n is odd, then x_0 is neither a relative maximum nor a relative minimum of f.

7 Riemann Integral

Definition 7.1 (Partition)**.**
Let $[a, b]$ be a bounded closed interval. A **partition** P of $[a, b]$ is a finite collection of ordered points:

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

The norm of P, denoted by $\|P\| := \max_{1 \le i \le n}\{x_i - x_{i-1}\}$.

Partition can be then used to construct upper and lower bounds for any sensible definition of $\int_a^b f(x)\mathrm{d}x$:
Let P be a partition of $[a, b]$ defined above. Let $f : [a, b] \to \mathbb{R}$. Define

$$m_i := \inf_{x_{i-1} \le x \le x_i} f(x) \text{ and } M_i := \sup_{x_{i-1} \le x \le x_i} f(x)$$

Then,

Definition 7.2 (Upper Sum and Lower Sum)**.**
The upper sum and lower sum of f, with respect to P is defined by

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \text{ and } L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

It is clear, geometrically that any sensible definition of $\int_a^b f(x)\mathrm{d}x$ should satisfy

$$L(f, P) \le \int_a^b f(x)\mathrm{d}x \le U(f, P) \text{ for any } P$$

However, in this way, the definition of integral will be dependent on P. We hope to get rid of P.

Theorem 7.1.
Let $f : [a, b] \to \mathbb{R}$. For any partition P of $[a, b]$, we have

$$L(f, P) \le U(f, P)$$

Definition 7.3 (Refinement of Partition)**.**
Let P and Q be two partitions of $[a, b]$. We say Q is a refinement of P, or Q is a finer partition than P, if $P \subset Q$.

Essentially, some subintervals of P-partition are further divided into smaller subintervals under Q.

Theorem 7.2.
Let $f : [a, b] \to \mathbb{R}$. Let Q be a finer parition of $[a, b]$ than P, then

$$L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P)$$

Definition 7.4 (Upper and Lower Integrals)**.**
Let $f : [a, b] \to \mathbb{R}$. The upper and lower integrals are defined by

$$U(f) := \inf_P U(f, P)$$

$$L(f) := \sup_P L(f, P)$$

where inf and sup are taken over all partitions P of $[a, b]$.

Theorem 7.3. *$L(f) \leq U(f)$.*

Theorem 7.4 (Riemann Integral)**.**
Let $f : [a, b] \to \mathbb{R}$. We say that f is Riemann integrable on $[a, b]$ if $L(f) = \inf_P L(f, P) = \sup_P L(f, P) = U(f)$. In this case, we define

$$\int_a^b f(x)\mathrm{d}x := L(f) = U(f)$$

We also define $\int_b^a f := -\int_a^b f$.

8 Integrability

The Criteria 1 is by definition.

Theorem 8.1.
Let $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}$. If we can find a sequence of partitions P_n of $[a, b]$ such that $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} Y(f, P_n) =: I \in \mathbb{R}$, then f is Riemann integrable on $[a, b]$ with $\int_a^b f = I$.

Theorem 8.2 (Riemann Integrability Criterion)**.**
Let $f : [a, b] \to \mathbb{R}$. f is Riemann integrable on $[a, b]$ *if and only if* for all $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) \leq \epsilon$$

Theorem 8.3 (Bounded Monotone Function)**.**
Let $f : [a, b] \to \mathbb{R}$ be **bounded and monotone**. Then f is Riemann integrable on $[a, b]$.

Theorem 8.4 (Bounded Continuous Function)**.**
Let $f : [a, b] \to \mathbb{R}$ be **continuous** on $[a, b]$. Then f is Riemann integrable on $[a, b]$.

9 Integral Properties

Theorem 9.1 (Properties of the Riemann Integral)**.**
Let f and g be Riemann integrable on $[a, b]$.

1. For each $c \in \mathbb{R}$, cf is integrable with $\int_a^b cf = c \int_a^b f$.
2. $f + g$ is integrable with $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.
3. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.
4. $|f|$ is integrable, and $|\int_a^b f| \leq \int_a^b |f|$.
5. $f \cdot g$ is integrable.

Theorem 9.2 (Piecewise Integration)**.**
Let $f : [a, b] \to \mathbb{R}$ and let $c \in (a, b)$.

1. If f is integrable on $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$ with

$$\int_a^b f = \int_a^c f + \int_c^b f$$

2. If f is integrable on $[a, b]$, then f is integrable on $[a, c]$ and $[c, b]$.

Remark: By induction, the theorem above can extend to the case when $[a, b]$ is partitioned into a finite number of intervals.

10 Riemann Sum

Definition 10.1 (Riemann Sum)**.**
Let $P = \{x_0 = a < \cdots < x_n = b\}$ and $f : [a, b] \to \mathbb{R}$. Let $\xi := (\xi_1, \ldots, \xi_n)$ with $\xi \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$. Then

$$S(f, P, \xi) := \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

is called the **Riemann Sum** of f wrt P and ξ.

Theorem 10.1 (Convergence of Riemann Sums)**.**
Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then uniformly in teh choice of sample point ξ,

$$\lim_{\|P\| \to 0} S(f, P, \xi) = \int_a^b f$$

More precisely,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{s.t.} \forall P \text{ with } \|P\| \leq \delta \text{ and } \forall \xi, |S(f, P, \xi) - \int_a^b f| \leq \epsilon$$

11 Fundamental Theorem of Calculus

Theorem 11.1.
Let f be integrable on $[a, b]$. Let $F(x) := \int_a^x f$ for all $x \in [a, b]$, with $F(a) := 0$. Then F is **uniformly continuous** on $[a, b]$.

Theorem 11.2 (Fundamental Theorem of Calculus(I))**.**
Let f be integrable on $[a, b]$. Let $F(x) := \int_a^x f$ for $x \in [a, b]$, with $F(a) := 0$. If f is continuous at $x_0 \in [a, b]$, then $F'(x_0) = f(x_0)$.

More generally, if $\lim_{h \to 0^+} f(x+h) = \alpha$ and $\lim_{h \to 0^-} f(x+h) = \beta$, then

$$\lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h} = \alpha \text{ and } \lim_{h \to 0^-} \frac{F(x+h) - F(x)}{h} = \beta$$

Theorem 11.3 (Fundamental Theorem of Calculus II)**.**
Let f be differentiable on $[a, x]$, and assume that f' is integrable on $[a, x]$. Then

$$\int_a^x f' = f(x) - f(a)$$

Theorem 11.4 (Integration by Parts)**.**
Let $f, g : [a, b] \to \mathbb{R}$ have integrable derivatives f', g' on $[a, b]$. Then

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g$$

Theorem 11.5 (Integration by Substitution)**.**
Let $\phi : [a, b] \to I$, where I is an inteval. Suppose there is an integrable derivative ϕ' on $[a, b]$. Let $f : I \to \mathbb{R}$ be continuous on I. Then

$$\int_a^b f(\phi(t))\phi'(t)\mathrm{d}t = \int_{\phi(a)}^{\phi(b)} f(x)\mathrm{d}x$$

12 Taylor And Improper Integral

Theorem 12.1 (Integral Version of MVT)**.**
Let f be continuous on $[a,b]$. Then $\exists c \in (a,b)$ such that $\int_a^b = f(c)(b-a)$.

Theorem 12.2 (Generalized Integral Version of MVT)**.**
Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Let $g : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and assume that g has a *constant* sign on $[a,b]$. Then $\exists c \in (a,b)$ such that $\int_a^b fg = f(c)\int_a^b g$.

Theorem 12.3 (Taylor Expansion in Integral Form)**.**
Let $f : [a,b] \to \mathbb{R}$. Given $x \in (a,b)$, assume that $f^{(1)}, \ldots, f^{(n+1)}$ exists on $[a,x]$ and $f^{(n+1)}$ integrable on $[a,x]$. Then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n \mathrm{d}t$$

Definition 12.1 (Singularities)**.**
$b \in [-\infty, \infty]$ is a singularity of f if either $b = \pm\infty$ or f is unbounded in every neighbourhood of b, which can be formulated as one of the following equivalent claims:

- $\limsup_{x\to b} |f(x)| = \infty$
- $\forall \delta > 0, \sup_{x\in[b-\delta,b+\delta]} |f(x)| = \infty$
- $\forall \delta > 0, \forall N > 0, \exists x \in [b-\delta, b+\delta]$ such that $|f(x)| > N$.
- $\exists x_1, \ldots$ with $\lim_{x\to\infty} x_n = b$ such that $\lim_{n\to\infty} |f(x_n)| = \infty$.

Definition 12.2 (Improper Integral)**.**
Let b be a singularity of f and assume $\int_a^c f$ exists for all $c \in [a,b)$. Then the improper integral $\int_a^b f$ is defined by

$$\int_a^b f := \lim_{c\to b^-} \int_a^c f$$

if the limit exists.
Similarly, if a is a singularity of f, then

$$\int_a^b f := \lim_{c\to a^+} \int_c^b f$$

if the limit exists.
If $c \in (a,b)$ is the only singularity of f on $[a,b]$, then

$$\int_a^b f := \int_a^c f + \int_c^b f$$

if both improper integral limit exists.

Definition 12.3 (Cauchy Mean Value Theorem).
Suppose $c \in (a, b)$ is the only singularity of f on $[a, b]$. Then

$$\lim_{\varepsilon \to 0} \left(\int_a^{c-\varepsilon} f + \int_{c+\varepsilon}^{b} f \right)$$

is the Cauchy Principle Value of $\int_a^b f$ if the limit exists.
Similarly, if $a = -\infty$ and $b = \infty$ are only singularities of f, then Cauchy Principle Value is defined as

$$\lim_{t \to \infty} \int_{-t}^{t} f$$

Remark: Cauchy Principle Value may exists even improper integral does not exists.

13 Pointwise and Uniform Convergence

In this section, we study the convergence of a sequence of functions.

Definition 13.1 (Pointwise Convergence)**.**
Let $E \subset \mathbb{R}$. Let $f_n : E \to \mathbb{R}, n \in \mathbb{N}$, be a sequence of functions. We say that f_n converges pointwise to a limiting function $f : E \to \mathbb{R}$, if $\forall x \in E$, we have

$$f(x) = \lim_{n\to\infty} f_n(x)$$

In other words,

$$\forall x \in E, \forall \epsilon > 0, \exists N_{x,\epsilon} \in \mathbb{N}\text{such that}\forall n > N_{x,\epsilon}, |f_n(x) - f(x)| \leq \epsilon$$

In this case, we say f_n converges to f, i.e., $f_n \to f$, pointwise on E.

Remark: Suppose f_n pointwise converges to f,

- f_n continuous does *not* imply f continuous.
- f_n may have different integral value to f.
- f_n differentiable at x does not imply $f_n'(x) \to f'(x)$ at any x.

Definition 13.2 (Uniform Convergence)**.**
A sequence of functions $f_n : E \to \mathbb{R}$ is said to be converge **uniformly** on E to a limiting function $f : E \to \mathbb{R}$ if

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \text{ such that } \forall n \geq N_\epsilon \text{ and } \forall x \in E, |f_n(x) - f(x)| \leq \epsilon$$

or equivalently,

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \text{ such that } \forall n \geq N_\epsilon, \sup |f_n(x) - f(x)| \leq \epsilon$$

We say that f_n converges to f, $f_n \to f$, uniformly on E.

Definition 13.3 (Sup Norm)**.**
We define sup-norm of a function f to be

$$\|f\| := \sup_{x\in E} |f(x)|$$

Sup-norm $\|\cdot\|$ has the following properties:

- $\|f\| = 0$ if and only if $f = 0$
- for $c > 0$, $\|cf\| = c\|f\|$
- $\|f + g\| \leq \|f\| + \|g\|$

The condition for uniform convergence can be written as

$$\lim_{n\to\infty} \|f_n - f\| = 0$$

Theorem 13.1. if $f_n \to f$ uniformly on E, then $f_n \to f$ pointwise on E.

14 Cauchy Criterion

In this section, we introduce another way to show uniform onvergence, apart from definition or sup-norm.

Theorem 14.1 (Cauchy Sequence).
$\lim_{x\to\infty} a_n$ exists in $\mathbb{R}$ *if and only if* $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e.,

$$\forall\epsilon > 0, \exists N(\epsilon), \text{ such that } \forall m, n \geq N(\epsilon), |a_m - a_n| \leq \epsilon$$

Cauchy sequence is useful in checking convergence if it is difficult to guess the limit.

Theorem 14.2 (Cauchy's Criterion for Pointwise/Uniform Convergence).
Let $f_n : E \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions.

- There exists an $f : E \to \mathbb{R}$ such that $f_n \to f$ pointwise on E, if and only if for all $x \in E$, $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e.,

$$\forall x \in E, \forall\epsilon > 0, \exists N_{x,\epsilon} \in \mathbb{N}, \text{ such that } \forall m, n \geq N_{x,\epsilon}, |f_m(x) - f_n(x)| \leq \epsilon$$

- There exists an $f : E \to \mathbb{R}$ such that $f_n \to f$ uniformly on E, if and only if (f_n) is a Cauchy sequence with respect to supnorm $\|\cdot\|$, i,e,

$$\forall\epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \text{ such that } \forall m, n \geq N_\epsilon, \|f_m - f_n\| \leq \epsilon$$

Theorem 14.3 (Uniform Equivalent to Pointwise on Finite Set).
If E finite set and $f_n, f : E \to \mathbb{R}$, then $f_n \to f$ pointwise is equivalent to $f_n \to f$ uniformly on E.

15 Property of Uniform Convergence

Theorem 15.1 (Preservation of Continuity)**.**
Let $f_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$ be a sequence of functions which converges **uniformly** on $[a,b]$ to a limit $f : [a,b] \to \mathbb{R}$. If $(f_n)_{n\in\mathbb{N}}$ are all **continuous** at a given $x_0 \in [a,b]$ then f is also **continuous** to x_0.
If $(f_n)_{n\in\mathbb{N}}$ are all continuous on $[a,b]$, then f is also continuous on $[a,b]$.

Theorem 15.2 (Convergence of Integrals)**.**
Let $f_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions converging **uniformly** on $[a,b]$ to a limit $f : [a,b] \to \mathbb{R}$. Suppose that each f_n is **integrable** on $[a,b]$, then f is **integrable** on $[a,b]$, and for any $x_0 \in [a,b]$, $F_n(x) := \int_{x_0}^{x} f_n$ converges **uniformly** to $F(x) := \int_{x_0}^{x} f$ on $[a,b]$. In particular, $\int_a^b f_n \to \int_a^b f$.

Theorem 15.3 (Interchanging Limits with differential operator)**.**
Let $f_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions. Suppose that

- For some $x_0 \in [a,b]$, we have $\lim_{n\to\infty} f_n(x_0) = L \in \mathbb{R}$ exists.
- f_n' exists on $[a,b]$ and $f_n' \to g$ uniformly on $[a,b]$ for some $g : [a,b] \to \mathbb{R}$.
- Each f_n' integrable.

Then $f_n \to f$ **uniformly** on $[a,b]$ for some $f : [a,b] \to \mathbb{R}$ with $f(x_0) = L$, and f' exists on $[a,b]$ with $f' = g$ on $[a,b]$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}x} f_n(x)$$

Remark: In fact, condition (3) can be removed.

16 Functional Series

Definition 16.1 (Infinite Series of Functions).
Let $f_n : E \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions. We call $\sum_{n=1}^{\infty} f_n$ an **infinite series** of functions, which is to be interpreted as follow:

1. We call $S_n : E \to \mathbb{R}$, defined by $S_n := \sum_{i=1}^{n} f_i$, the nth partial sum of $\sum_{n=1}^{\infty} f_n$.

2. If S_n converges pointwise (resp. uniformly) to a function S defined on a subset $E_0 \subset E$, then we say that the series $\sum_{n=1}^{\infty} f_n$ converges pointwise (resp. uniformly) on E_0, and we define $\sum_{n=1}^{\infty} f(x) := S(x)$ for all $x \in E_0$.
   If for some $x \in E$, $\lim_{n\to\infty} S_n(x)$ does not exist, then $\sum_{n=1}^{\infty} f_n$ is undefined at x.

Theorem 16.1 (Cauchy's Criterion for Series of Functions).

1. $\sum_{n=1}^{\infty} f_n$ converges pointwise on E if and only if

$$\forall \epsilon > 0, \forall x \in E, exitsN_{\epsilon,x} \in \mathbb{N} \text{ such that } \forall m \geq n \geq N_{\epsilon,x}, |\sum_{i=n}^{m} f_i(x)| < \epsilon$$

   i.e., $\forall x \in E$, $S_n(x) = \sum_{i=1}^{n} f_i(x)$ is a Cauchy sequence of reals.

2. $\sum_{n=1}^{\infty} f_n$ converges uniformly on E if and only if

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \text{ such that } \forall m \geq n \geq N_\epsilon, \|\sum_{i=n}^{m} f_i\| \leq \epsilon$$

   i.e., $S_n(x) = \sum_{i=1}^{n} f_i(x)$ is a Cauchy sequence of functions with respec tto the sup-norm $\|\cdot\|$.

Theorem 16.2 (A necessary condition for Uniform Convergence).
If $\sum_{n=1}^{\infty} f_n$ converges **uniformly** on E¡ then

$$\|f_n\| = \sup_{x\in E} |f_n(x)| \to 0 \text{ as } n \to \infty$$

Theorem 16.3 (Weierstrass M-test).
Let $f_n : E \to \mathbb{R}$ for $n \geq 1$. Let $M_n := \|f_n\|$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on E. Furthermore, $\sum_{n=1}^{\infty} |f_n|$ also converges uniformly on E.

Theorem 16.4 (Continuity of an Infinite Series of Continuous Functions).
If $\sum_n f_n$ converges uniformly to a function S on $[a, b]$, and each f_n is **continuous** at $x_0 \in [a, b]$, then S is also **continuous** at x_0.

Theorem 16.5 (Interchanging Series with Integral).
If $\sum_n f_n$ converges uniformly to a function S on $[a, b]$, and each f_n is integrable on $[a, b]$, then S is integrable on $[a, b]$, and for every $x \in [a, b]$,

$$\int_a^x S(t)\mathrm{d}t = \int_a^x \sum_{n=1}^{\infty} f_n(t)\mathrm{d}t = \sum_{n=1}^{\infty} \int_a^x f_n(t)\mathrm{d}t$$

where the convergence on the right-hand-side is uniform on $[a, b]$.

Theorem 16.6 (Interchanging Series with Derivative)**.**
Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions on $[a, b]$ such that

1. $sum_{n=1}^{\infty} f_n(x_0)$ converges for some $x_0 \in [a, b]$
2. $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[a, b]$

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$ to a differentiable function $S : [a, b] \to \mathbb{R}$, with

$$S'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(\sum_{n=1}^{\infty} f_n(x)) = \sum_{n=1}^{\infty} f_n'(x)$$

This theorem provides an alternative to Weierstrass M-test for proving uniform convergence of a series of functions.

17 Test of Convergence of Functional Series

Definition 17.1 (Uniform Boundedness)**.**
A sequence of functions $(f_n)_{n\in\mathbb{N}} : E \to \mathbb{R}$ is called **uniformly bounded** on E if there exists a $K \in (0, \infty)$ such that $|f_n(x)| \leq K$ for all $x \in E$ and $n \in \mathbb{N}$, i.e.,

$$\sup_{n\in\mathbb{N}} \|f_n\| \leq K$$

Theorem 17.1 (Dirichlet's Test)**.**
Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be two sequences of functions on E. Then $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ is **uniformly convergent** on E if the following conditions are satisfied:

1. The partial sums of $\sum_n g_n$, $G_n := \sum_{i=1}^{n} g_i$, are uniformly bounded(i.e., $sup_{n\in\mathbb{N}}\|G_n\| \leq \infty$)
2. $\lim_{n\to\infty} \|f_n\| = 0$, i.e., f_n converges to the constant function 0 uniformly on E.
3. For each $x \in E$, the sequence of real numbers $(f_n(x))_{n\in\mathbb{N}}$ is **monotone**.

We have the following corollary from the above theorem:

Theorem 17.2 (Dirichlet's Test for Reals)**.**
Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of real numbers. Then $\sum_{n=1}^{\infty} a_n b_n$ converges if

1. $B_n := \sum_{i=1}^{n} b_i, n \in \mathbb{N}$, are bounded(i.e., $sup_{n\in\mathbb{N}}|B_n| = K < \infty$)
2. $\lim_{n\to\infty} a_n = 0$.
3. a_n is **monotone** in n.

Theorem 17.3 (Alternating Series Test)**.**
Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions on E such that

1. $f_n \to 0$ uniformly on E, i.e., $\lim_{n\to\infty} \|f_n\| = 0$
2. $\forall x \in E$, $f_n(x)$ is **monotone** in n.

Then the series $\sum_{n=1}^{\infty} (-1)^n f_n(x)$ is **uniformly convergent** on E.

Theorem 17.4 (Abel's Test)**.**
Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be two sequences of functions on E. Then $\sum_{i=1}^{\infty} f_n g_n$ is uniformly convergent if

1. $\sum_{n=1}^{\infty} g_n$ converges uniformly on E.
2. $(f_n)_{n\in\mathbb{N}}$ are **uniformly bounded** on E, i.e., $\sup_{n\in\mathbb{N}} \|f_n\| = K < \infty$.
3. For each $x \in E$, $f_n(x)$ is monotone in n.

Theorem 17.5 (Abel's Test For Reals)**.**
Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of real numbers on E. Then $\sum_{i=1}^{\infty} a_n b_n$ is uniformly convergent if

1. $\sum_{n=1}^{\infty} b_n$ converges.
2. $\sup_{n\in\mathbb{N}} |a_n| = K < \infty$.
3. a_n is monotone in n.

18 Dini Theorem and Power Series

Dini's Theorem is another test of uniform convergence.

Definition 18.1 (Monotone Sequence of Functions)**.**
A sequence of functions $f_n : E \to \mathbb{R}, n \in \mathbb{N}$, is called **monotone** is it is either **increasing** $(\forall x \in E, f_1(x) \leq f_2(x) \leq \cdots)$ or decreasing.

Theorem 18.1 (Dini's Theorem)**.**
Let $(f_n)_{n\in\mathbb{N}}$ and f be defined on $[a, b]$. Suppose that

1. $f_n \to f$ pointwise on $[a, b]$.
2. $(f_n)_{n\in\mathbb{N}}$ and f all continuous on $[a, b]$.
3. $(f_n)_{n\in\mathbb{N}}$ is monotone

Then $f_n \to f$ uniformly on $[a, b]$.

Definition 18.2 (Power Series)**.**
A series of functions of the form

$$f(x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where $(a_n)_{n\geq 0}$ are constants, is called a **power series** in $(x - x_0)$.

Definition 18.3 (Absolute Uniform Convergence)**.**
A series of functions $\sum_{n=1}^{\infty} f_n$ converges **absolutely-uniformly on** E if $\sum_{n=1}^{\infty} |f_n|$ converges uniformly on E.

Theorem 18.2. Absolute uniform convergence of $\sum_n f_n$ implies uniform convergence.

Theorem 18.3 (Radius of Convergence)**.**
given a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$. Let

$$R := \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}} \in [0, \infty]$$

with $R := 0$(resp. ∞) if $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \infty$(resp. 0). Then

1. If $R \in (0, \infty)$, then for any fixed $r \in [0, R)$, then power series converges **absolutely uniformly** on $[x_0 - r, x_0 + r]$ and diverges for any x with $|x - x_0| > R$.
2. If $R = 0$, then the series converges at $x = x_0$ only.
3. If $R = \infty$, then for any fixed $r > 0$, the series converges **absolutely uniformly** on $[x_0 - r, x_0 + r]$. In particular, it converges absolutely at each $x \in \mathbb{R}$.

Theorem 18.4.
If $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, a_n \neq 0$, and $\rho := \lim_{n\to\infty} \frac{|a_{n+1}}{a_n}$ exists in $[0, \infty]$, then $R = \frac{1}{\rho}$.

19 Power Series Properties

Recall, the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is given by

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}} \in [0,\infty]$$

The power series converges pointwise on $(x_0 - R, x_0 + R)$, converges uniformly on any **bounded closed subinterval** of $(x_0 - R, x_0 + R)$, and diverges on $(-\infty, x_0 - R) \cup (x_0 + R, \infty)$. The convergence/divergence at $x_0 \pm R$ depends o nthe concrete problem at hand. Also, the radius of convergence can be given by

$$R = \frac{1}{\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}}$$

if the limit exists, since $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ is the geometric growth rate of $|a_n|$.

Definition 19.1 (Domain of a Power Series)**.**
The domain of a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is defined to be the set

$$\{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges}\}$$

Theorem 19.1 (Derivatives of Power Series)**.**
If $f(x) := \sum_{n=0}^{\infty} a_n(x-x_0)^n$ has radius of convergence $R > 0$, then f is **infinitely differentiable** on $(x_0 - R, x_0 + R)$, with

$$f'(x) = \sum_{n=1} \infty n a_n (x-x_0)^{n-1} \quad \forall |x - x_0| < R$$

and for all $k \in \mathbb{N}$,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} a_n n(n-1)\cdots(n-k+1)(x-x_0)^{n-k} \quad \forall |x-x_0| < R$$

The radius of convergence of these power series all equal R.

Theorem 19.2 (Facts)**.**

- *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two sequences such that $\lim_{n\to\infty} u_n := U \in (0,\infty)$ exists. Then*

$$\limsup_{n\to\infty} u_n v_n = U \limsup_{n\to\infty} v_n$$

- $\limsup_{n\to\infty} |a_n|^{\frac{1}{n-1}} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$.

20 Properties of Power Series

Theorem 20.1 (Power Series as Taylor Series)**.**
Let $f(x) := \sum_{n=0}^{\infty} a_n(x - x_0)^n$ be convergent on $(x_0 - r, x_0 + r)$ for some $r > 0$. Then

$$a_k = \frac{f^{(k)}(x_0)}{k!} \quad \forall k \in \{0\} \cup \mathbb{N}$$

Essentially, if f can be represented as a power series in powers of $x - x_0$ in a neighbourhood of x_0, then this power series is in fact the Taylor series for f expanded around x_0.

Theorem 20.2 (Uniqueness of Power Series)**.**
If

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} b_n(x - x_0)^n$$

on $(x_0 - r, x_0 + r)$ for some $r > 0$, then $a_n = b_n$ for all $n \in \{0\} \cup \mathbb{N}$.

Theorem 20.3 (Integrating a Power Series)**.**
Let $f(x) := \sum_{n=0}^{\infty} a_n(x - x_0)^n$ be convergent on $(x_0 - r, x_0 + r)$ for some $r > 0$. Then for all $x \in (x_0 - r, x_0 + r)$, f is integrable on $[x_0, x]$ and

$$F(x) := \int_{x_0}^{x} f(t)\mathrm{d}t = \sum_{n=0}^{\infty} a_n(t - x_0)^n \mathrm{d}t = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}$$

In particular, f is integrable on any $[a, b] \subset (x_0 - r, x_0 + r)$, with

$$\int_{a}^{b} f(x)\mathrm{d}x = F(b) - F(a) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(b - x_0)^{n+1} - \sum_{n=0}^{\infty} \frac{a_n}{n+1}(a - x_0)^{n+1}$$

Abel theorem gives us teh ability to check for continuity of a power series at the boundary of its interval of convergence.

Theorem 20.4 (Abel's Theorem)**.**
Let $f(x) := \sum_{n=0}^{\infty} a_n(x - x_0)^n$, which converges on $(x_0 - R, x_0 + R)$, and assume $R > 0$ to be the radius of convergence.

1. If the power series converges at $x = x_0 + R$, i.e., $\sum_{n=0}^{\infty} a_n R^n$ converges, then

$$\lim_{x \to (x_0 + R)^-} f(x) = \sum_{n=0}^{\infty} a_n R^n$$

2. If the power series converges at $x = x_0 - R$, i.e., $\sum_{n=0}^{\infty} a_n (-R)^n$ converges, then

$$\lim_{x \to (x_0 - R)^+} f(x) = \sum_{n=0}^{\infty} a_n (-R)^n$$

In other words, if the power series converges at boundary points, then the power series must be continuous at that point(one-sided).

21 Taylor Series

Definition 21.1 (Taylor, Maclaurin Series)**.**
Suppose that f is infinitely differentiable on $(x_0 - r, x_0 + r)$ for some $r > 0$. Then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the **Taylor series** of f about x_0. When $x_0 = 0$, the series becomes

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

which is called the Maclaurin series of f.

In general, $f(x) \neq \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ on interval of convergence of Taylor series, unless f is defined via a power series:

Theorem 21.1 (Power Series as Taylor Series)**.**
If $f(x) := \sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges on $(x_0 - r, x_0 + r)$ for some $r > 0$, then $\sum_n a_n(x - x_0)^n$ is the Taylor series of f about x_0, i.e., $a_n = \frac{f^{(n)}(x_0)}{n!}$.

In genreall, we want to investage when function f equals its Taylor Series. We first observe the following proven theorem:

Theorem 21.2 (Taylor Expansion with Remainder)**.**
Suppose $f^{(n+1)}$ exists on $I := (x_0 - r, x_0 + r)$. Then $\forall x \in I, \exists c_n$ between x_0 and x, which depends on n, x, x_0 such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c_n)}{(n+1)!}(x - x_0)^{n+1}$$

It follows immediately that

Theorem 21.3 (Equality Between a Function and Its Taylor Expansion)**.**
Suppose that f is *infinitely differentiable* on $I = (x_0 - r, x_0 + r)$, then for each $x \in I$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

if and only if

$$\lim_{n\to\infty} R_n(x) := \lim_{n\to\infty} \frac{f^{(n+1)}(c_n)}{(n+1)!}(x - x_0)^{n+1} = 0$$

The Taylor series of f about x_0 converges uniformly to f on $[a, b] \subset I$ if and only if $R_n \to 0$ uniformly on $[a.b]$.

22 Power Series Arithmetic

Let $f(x) := \sum_{n=0} 6\infty a_n x^n$ and $g(x) := \sum_{n=0}^{\infty} b_n x^n$ be two power seires. Formal multiplication gives

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n := \sum_{k=0}^{n} a_k b_{n-k}$.

Definition 22.1 (Cauchy Product)**.**
The Cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is defined to be $\sum_{n=0}^{\infty} c_n$. Then $\sum_{n=0}^{\infty} c_n x^n$ is teh Cauchy product of $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$.

Theorem 22.1 (Merten)**.**
If $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$, and if either $\sum a_n$ or $\sum b_n$ converges **absolutely**, then Cauchy product $\sum_{n=0}^{\infty} c_n = AB$.

Remark: $\sum_n c_n$ may not converge absolutely.

Theorem 22.2. If both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely, then the Cauchy product $\sum_{n=0}^{\infty} c_n$ converge absolutely.

Theorem 22.3 (Arithmetic Operations)**.**
Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \ \ |x - x_0| < R_1$$

and

$$g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n, \ \ |x - x_0| < R_2$$

then for any $\alpha, \beta \in \mathbb{R}$,

$$\alpha f(x) + \beta g(x) = \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x - x_0)^n, \ \ |x - x_0| \leq R_1 \wedge R_2$$

and

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \text{ with } c_n = \sum_{i=0}^{n} a_i b_{n-i}, \ \ |x - x_0| \leq R_1 \wedge R_2$$

23 Open and Closed Sets, Compactness

Definition 23.1 (Neighbourhood)**.**
A neighbourhood of a point $x \in \mathbb{R}$ is any set V that contains a ϵ-neighbourhood $V_\epsilon(x) := (x-\epsilon, x+\epsilon)$ of x for some $\epsilon > 0$.

Definition 23.2 (Open/Closed Subset)**.**
A subset G of $\mathbb{R}$ is called open if for each $x \in G$ there exists a neighbourhood V of x such that $V \subset G$.
A subset F of $\mathbb{R}$ is called closed in $\mathbb{R}$ if the complement $\mathbb{R} \setminus F$ is open in $\mathbb{R}$.

Remark: We can have a set that is neigher open nor closed, say $[0,1)$. Note, the empty set $\varnothing$ is open in $\mathbb{R}$.

Theorem 23.1 (Properties of Open Set)**.**

- THe union of an arbitrary collection of open subsets in $\mathbb{R}$ is open.
- The intersection of any finite collection of open sets in $\mathbb{R}$ is open.

Theorem 23.2 (Properties of Closed Set)**.**

- The intersecion of an arbitrary collection of closed subsets in $\mathbb{R}$ is closed.
- The union of any finite collection of closed sets in $\mathbb{R}$ is closed.

Theorem 23.3 (Characterisation of Open Sets)**.**
A subset of $\mathbb{R}$ is open if and only if it is the union of countably many disjoint open intervals in $\mathbb{R}$.

Theorem 23.4 (Characterisation of Closed Sets)**.**
Let $F \subset \mathbb{R}$, then the following assertions are equivalent:

1. F is closed subset of $\mathbb{R}$.
2. If $X = (x_n)$ is any convergent sequence of elements in F, then $\lim X$ belongs to F.

Theorem 23.5. A subset of $\mathbb{R}$ is closed if and only if it contains all of its limit points.

Definition 23.3 (Open Cover)**.**
Let A be a subset of $\mathbb{R}$. An **open cover** of A is a collection $\mathcal{G} = \{G_\alpha\}$ of open sets in $\mathbb{R}$ whose union contains A, that is

$$A \subseteq \cup_\alpha G_\alpha$$

If $\mathcal{G}'$ is a subcollection of sets from $\mathcal{G}$ such that the union of the sets in $\mathcal{G}'$ also contains A, then $\mathcal{G}'$ is called a **subcover** of $\mathcal{G}$. If $\mathcal{G}'$ consists of finitely many sets, then we call $\mathcal{G}'$ a finite subcover of $\mathcal{G}$.

Definition 23.4 (Compact Set)**.**
A subset K of $\mathbb{R}$ is said to be **compact** if every open cover of K has a finite subcover.

Theorem 23.6 (Heine-Borel)**.**
A subset K of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Theorem 23.7. A subset K of $\mathbb{R}$ is compact if and only if every sequence in K has a subsequence that converges to a point in K.