1 Review

Definition 1.1 (Limit of Sequence)**.** For a sequence $(x_n)_{n\in\mathbb{N}}$, we say $\lim_{n\to\infty} x_n = a$ if and only if

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0, |x_n - a| \leq \epsilon$$

Similarly, we define $\lim_{n\to\infty} x_n = \infty$ if and only if

$$\forall M > 0, \exists n_0 \text{ depending on } M, \text{s.t. } \forall n \geq n_0, x_n \geq M$$

Definition 1.2 (Limit Point(Subsequential Limit in `MA2108` notes))**.** A number $a \in [-\infty, \infty]$ is called a **limit point** of the sequence $(x_n)_{n\in\mathbb{N}}$, if there exists an increasing sequence of indices $n_1 < n_2 < n_3 < \cdots$ such that $\lim_{i\to\infty} x_{n_i} = a$.

Theorem 1.1. $\lim_{n\to\infty} x_n$ does not exist in $[-\infty, \infty]$ if and only if $(x_n)_{n\in\mathbb{N}}$ has more than 1 limit point in $[-\infty, \infty]$.

Definition 1.3 (Supremum and Infimum)**.** Let $A \subset [-\infty, \infty]$. The **supremum** of A, denoted by $\sup A$, is defined to be the **least upper bound** of A.
Essentially, $p = \sup A$ if and only if

1. $x \leq p \forall x \in A$
2. if $x \leq u \forall x \in A$ for some $u \in [-\infty, \infty]$, then $p \leq u$.

The infimum is defined in a similar fashion. For detailed definition, check MA2108 revision notes.

Definition 1.4 (Limit Supremum and Infimum)**.** Given a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$,

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} (\sup_{m\geq n} x_m)$$

and

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} (\inf_{m\geq n} x_m)$$

Theorem 1.2. $\limsup_{n\to\infty} x_n$ is a limit point and the **largest limit point** of the sequence $(x_n)_{n\in\mathbb{N}}$. $\liminf_{n\to\infty} x_n$ is the smallest limit point.

Theorem 1.3. $\lim_{n\to\infty} x_n$ exists in $[-\infty, \infty]$ if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$.

Definition 1.5 (Continuity)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **continuous** at x if $\lim_{y\to x} f(y)$ exists and equals $f(x)$.
Equivalently,

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \sup_{y\in[x-\delta,x+\delta]} |f(y) - f(x)| \leq \epsilon$$

2 Derivative

Definition 2.1 (Derivative)**.** Let $I \subseteq \mathbb{R}$ be an interval, and let $c \in I$. A function $f : I \to \mathbb{R}$ is differentiable at c if

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = L$$

for some $L \in \mathbb{R}$.
Equivalently, we need

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that} \forall x \in I, 0 < |x - c| < \delta \Rightarrow |\frac{f(x) - f(c)}{x - c} - L| \leq \epsilon$$

Here, L is called the **derivative** of f at c, denoted by $f'(c)$, or $\frac{\mathrm{d}f}{\mathrm{d}x}|_{x=c}$.

If f is differentiable at every $x \in S \subseteq I$, we say f is differentiable on S.

Definition 2.2 (Equivalent Definition of Derivative). f is differentiable at c, if $f(x)$ can be approximated by the line $l(x) := f(c) + f'(c)(x - c)$ near $x = c$, i.e.,

$$\forall \epsilon > 0, \exists \delta > 0, \text{such that } \forall x \in [c - \delta, c + \delta], |f(x) - l(x)| \leq \epsilon |x - c|$$

Theorem 2.1 (Differentiability infers Continuity). If $f : I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c.

Theorem 2.2 (Derivative Rules). Suppose that $f, g : I \to \mathbb{R}$ are differentiable at $c \in I$, then

- (Linearity) For any $a, b \in \mathbb{R}$, $af + bg$ is differentiable at c, and

$$(af + bg)'(c) = af'(c) + bg'(c)$$

- (Product Rule) fg is differentiable at c and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

- (Quotient Rule) If $g(c) \neq 0$, then f/g is differentiable at c and

$$(\frac{f}{g})'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}$$

Theorem 2.3 (Caratheodory's Representation Lemma). Let $f : I \to \mathbb{R}$ and let $c \in I$. The following conditions are equivalent:

1. f is differentiable at c.
2. There exists a function $\phi : I \to \mathbb{R}$ such that ϕ is continuous at c and

$$f(x) - f(c) = \phi(x)(x - c) \quad \forall x \in I$$

   In this case, $\phi(c) = f'(c)$.

Theorem 2.4 (Chain Rule). Let I, J be intervals in $\mathbb{R}$. Let $g : I \to J$ and $f : J \to \mathbb{R}$. Suppose g is differentiable at $c \in I$ and f is differentiable at $g(c) \in J$, then $f \circ g$ is differentiable at c, with

$$(f \circ g)'(c) = f'(g(c))g'(c)$$

3 Mean Value Theorem

Theorem 3.1 (Derivative of an Inverse Function). Let I be an interval, and $f : I \to \mathbb{R}$ be continuous and stricly monotone on I. Let $J := f(I)$ be the **range** of f, and $g : J \to I$ be the inverse of f.
If f is differentiable at $c \in I$ and $f'(c) \neq 0$, then g is differentiable at $f(c) \in J$, and

$$g'(f(c)) = \frac{1}{f'(c)}$$

Theorem 3.2 (Mean Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

A special case of Mean Value Theorem is Rolle's Theorem.

Theorem 3.3 (Rolle's Theorem). When $f(a) = f(b)$ in the Mean Value Theorem, we obtain the existence of a $c \in (a, b)$ with

$$f'(c) = 0$$

Definition 3.1 (Relative Extremum). Let $f : I \to \mathbb{R}$ for some subset $I \subseteq \mathbb{R}$ and let $c \in I$. Then

1. f has a **relative maximum** at c, if for some $\delta > 0$,

$$f(c) \geq f(x) \forall x \in I \cap (c-\delta, c+\delta)$$

2. **Relative minimum** of f on I are defined analogously.

Relative Extremum refers to either relative maximum or relative minimum.

Theorem 3.4 (Interior Extremum Theorem)**.** Let $f : I \to \mathbb{R}$, and let $c \in I$ be an interior point of I, i,e, $(c-\delta, c+\delta) \subseteq I$ for some $\delta > 0$.
If f is differentiable at c adn has a relative extremum at c, then $f'(c) = 0$.

Theorem 3.5. Let $f : I \to \mathbb{R}$ and assume that $f'(c)$ exists for some $c \in I$.

1. If $f'(c) > 0$, then for some $\delta > 0$, we have

$$f(x) < f(c) \quad \forall x \in I \cap (c-\delta, c)$$

   and

$$f(x) > f(c) \quad \forall x \in I \cap (c, c+\delta)$$

2. If $f'(c) < 0$, then the directions of the two inequalities above are reversed.

Theorem 3.6 (Cauchy's Mean Value Theorem)**.** Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) = g(a))f'(c)$$

4 Application of Mean Value Theorem

Theorem 4.1 (Monotonicity Properties)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then f is increasing(resp. decreasing) on $[a, b]$ if and only if $f'(x) \geq 0$(resp. $f'(x) \leq 0$) for all $x \in (a, b)$.

If strict monotonicity is concerned, we will have $f'(x) > 0 \Rightarrow f(x) < f(y)$ for all $x < y$, but **not** the other direction.

Theorem 4.2 (Uniqueness of Anti-derivative Modulo Shift)**.** Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).
Suppose that f and g have the same derivative, i.e., $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists a constant $C \in \mathbb{R}$ such that

$$f(x) = g(x) + C \quad \forall x \in [a, b]$$

Theorem 4.3 (Intermediate Value Theorem for Derivatives)**.** Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Suppose that $f'(a) < f'(b)$, then for any $r \in (f'(a), f'(b))$, there exists some $c \in (a, b)$ with $f'(c) = r$.

Theorem 4.4 (First Derivative Test)**.** Let f be continuous on (a, b). Let $c \in (a, b)$. Assume that $f'(x)$ exists for all $x \in (a, b) \setminus \{c\}$. Then

1. If $f'(x) \geq 0$ for all $x \in (a, c)$ and $f'(x) \leq 0$ for all $x \in (c, b)$, then f has a relative maximum at c.

2. If $f'(x) \leq 0$ for all $x \in (a, c)$ and $f'(x) \geq 0$ for all $x \in (c, b)$, then f has a relative minimum at c.

Theorem 4.5 (Second Derivative Test)**.** Let f be differentiable on $[a, b]$ with derivative f'. Suppose $f'(c) = 0$ at some $c \in (a, b)$, and f' is differentiable at c with derivative $f''(c)$. Then,

1. If $f''(c) > 0$, then f has a relative minimum at c.

2. If $f''(c) < 0$, then f has a relative maximum at c.

5 L'Hospital's Rule

Theorem 5.1 (L'Hospital's Rule)**.** Let $-\infty \le a < b \le \infty$. Let f and g be differentiable on (a, b). Assume that $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a, b)$.

(I) If $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$, and $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$ for some $L \in [-\infty, \infty]$, then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$$

(II) If $\lim_{x\to a^+} g(x) = \infty$ and $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$ for some $L \in [-\infty, \infty]$, then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$$

Remark: L'Hospital's rule also holds if we replace $x \to a^+$ above by $x \to b^-$. We can also replace b be $a + \delta$ for some $\delta > 0$. Also, note that we make no assumption on f in (II).

Theorem 5.2 (Taylor Expansion)**.** Let f be n times differentiable on $[a, x]$, with $f^{(i)}$ denoting the ith derivative of f.
Suppose that $f^{(n+1)}(x)$ exists on (a, x). Then there exists $c \in (a, x)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

6 More on Taylor

Theorem 6.1 (Taylor Theorem)**.** Let f be n times differentiable on $[a, x]$ with $f^{(i)}$ denoting the ith derivative of f. Suppose that $f^{(n+1)}$ exists on (a, x). Then there exists $c \in (a, x)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Theorem 6.2 (Higher Order Derivative Tests)**.** Let $f : [a, b] \to \mathbb{R}$. Suppose that $f^{(1)}(x_0) = f^{(2)}(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ for some $x_0 \in (a, b)$.
Assume also that $f^{(n)}$ exists at x_0 with $f^{(n)}(x_0) \neq 0$. Then

1. If n is even
   (a) and $f^{(n)}(x_0) > 0$, then x_0 is a relative minimum of f.
   (b) and $f^{(n)}(x_0) < 0$, then x_0 is a relative maximum of f.
2. If n is odd, then x_0 is neither a relative maximum nor a relative minimum of f.

7 Riemann Integral

Definition 7.1 (Partition)**.** Let $[a, b]$ be a bounded closed interval. A **partition** P of $[a, b]$ is a finite collection of ordered points:

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

The norm of P, denoted by $\|P\| := \max_{1\le i\le n}\{x_i - x_{i-1}\}$.

Partition can be then used to construct upper and lower bounds for any sensible definition of $\int_a^b f(x)\mathrm{d}x$:
Let P be a partition of $[a,b]$ defined above. Let $f:[a,b]\to\mathbb{R}$. Define

$$m_i := \inf_{x_{i-1}\le x\le x_i} f(x) \text{ and } M_i := \sup_{x_{i-1}\le x\le x_i} f(x)$$

Then,

Definition 7.2 (Upper Sum and Lower Sum)**.** The upper sum and lower sum of f, with respect to P is defined by

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \text{ and } L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

It is clear, geometrically that any sensible definition of $\int_a^b f(x)\mathrm{d}x$ should satisfy

$$L(f,P) \le \int_a^b f(x)\mathrm{d}x \le U(f,P) \text{ for any } P$$

However, in this way, the definition of integral will be dependent on P. We hope to get rid of P.

Theorem 7.1. Let $f:[a,b]\to\mathbb{R}$. For any partition P of $[a,b]$, we have

$$L(f,P) \le U(f,P)$$

Definition 7.3 (Refinement of Partition)**.** Let P and Q be two partitions of $[a,b]$. We say Q is a refinement of P, or Q is a finer partition than P, if $P\subset Q$.

Essentially, some subintervals of P-partition are further divided into smaller subintervals under Q.

Theorem 7.2. Let $f:[a,b]\to\mathbb{R}$. Let Q be a finer parition of $[a,b]$ than P, then

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$$

Definition 7.4 (Upper and Lower Integrals)**.** Let $f:[a,b]\to\mathbb{R}$. The upper and lower integrals are defined by

$$U(f) := \inf_P U(f,P)$$

$$L(f) := \sup_P L(f,P)$$

where inf and sup are taken over all partitions P of $[a,b]$.

Theorem 7.3. $L(f) \le U(f)$.

Theorem 7.4 (Riemann Integral)**.** Let $f:[a,b]\to\mathbb{R}$. We say that f is Riemann integrable on $[a,b]$ if $L(f) = \inf_P L(f,P) = \sup_P L(f,P) = U(f)$. In this case, we define

$$\int_a^b f(x)\mathrm{d}x := L(f) = U(f)$$

We also define $\int_b^a f := -\int_a^b f$.

8 Integrability

The Criteria 1 is by definition.

Theorem 8.1. Let $(x_n)_{n\in\mathbb{N}}\in\mathbb{R}$. If we can find a sequence of partitions P_n of $[a,b]$ such that $\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} Y(f,P_n) =: I\in\mathbb{R}$, then f is Riemann integrable on $[a,b]$ with $\int_a^b f = I$.

Theorem 8.2 (Riemann Integrability Criterion)**.** Let $f:[a,b]\to\mathbb{R}$. f is Riemann integrable on $[a,b]$ *if and only if* for all $\epsilon>0$, there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) \le \epsilon$$

Theorem 8.3 (Bounded Monotone Function)**.** Let $f:[a,b]\to\mathbb{R}$ be **bounded and monotone**. Then f is Riemann integrable on $[a,b]$.

Theorem 8.4 (Bounded Continuous Function)**.** Let $f:[a,b]\to\mathbb{R}$ be **continuous** on $[a,b]$. Then f is Riemann integrable on $[a,b]$.

9 Integral Properties

Theorem 9.1 (Properties of the Riemann Integral)**.** Let f and g be Riemann integrable on $[a, b]$.

1. For each $c \in \mathbb{R}$, cf is integrable with $\int_a^b cf = c \int_a^b f$.
2. $f + g$ is integrable with $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.
3. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$.
4. $|f|$ is integrable, and $|\int_a^b f| \leq \int_a^b |f|$.
5. $f \cdot g$ is integrable.

Theorem 9.2 (Piecewise Integration)**.** Let $f : [a, b] \to \mathbb{R}$ and let $c \in (a, b)$.

1. If f is integrable on $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$ with

$$\int_a^b f = \int_a^c f + \int_c^b f$$

2. If f is integrable on $[a, b]$, then f is integrable on $[a, c]$ and $[c, b]$.

Remark: By induction, the theorem above can extend to the case when $[a, b]$ is partitioned into a finite number of intervals.

10 Riemann Sum

Definition 10.1 (Riemann Sum)**.** Let $P = \{x_0 = a < \cdots < x_n = b\}$ and $f : [a, b] \to \mathbb{R}$. Let $\xi := (\xi_1, \ldots, \xi_n)$ with $\xi \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$. Then

$$S(f, P, \xi) := \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

is called the **Riemann Sum** of f wrt P and ξ.

Theorem 10.1 (Convergence of Riemann Sums)**.** Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then uniformly in teh choice of sample point ξ,

$$\lim_{\|P\| \to 0} S(f, P, \xi) = \int_a^b f$$

More precisely,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{s.t.} \forall P \text{ with } \|P\| \leq \delta \text{ and } \forall \xi, |S(f, P, \xi) - \int_a^b f| \leq \epsilon$$

11 Fundamental Theorem of Calculus

Theorem 11.1. Let f be integrable on $[a, b]$. Let $F(x) := \int_a^x f$ for all $x \in [a, b]$, with $F(a) := 0$. Then F is **uniformly continuous** on $[a, b]$.

Theorem 11.2 (Fundamental Theorem of Calculus(I))**.** Let f be integrable on $[a, b]$. Let $F(x) := \int_a^x f$ for $x \in [a, b]$, with $F(a) := 0$. If f is continuous at $x_0 \in [a, b]$, then $F'(x_0) = f(x_0)$.

More generally, if $\lim_{h \to 0^+} f(x + h) = \alpha$ and $\lim_{h \to 0^-} f(x + h) = \beta$, then

$$\lim_{h \to 0^+} \frac{F(x + h) - F(x)}{h} = \alpha \text{ and } \lim_{h \to 0^-} \frac{F(x + h) - F(x)}{h} = \beta$$

Theorem 11.3 (Fundamental Theorem of Calculus II)**.** Let f be differentiable on $[a, x]$, and assume that f' is integrable on $[a, x]$. Then

$$\int_a^x f' = f(x) - f(a)$$

Theorem 11.4 (Integration by Parts)**.** Let $f, g : [a, b] \to \mathbb{R}$ have integrable derivatives f', g' on $[a, b]$. Then

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g$$

Theorem 11.5 (Integration by Substitution)**.** Let $\phi : [a, b] \to I$, where I is an inteval. Suppose there is an integrable derivative ϕ' on $[a, b]$. Let $f : I \to \mathbb{R}$ be continuous on I. Then

$$\int_a^b f(\phi(t))\phi'(t)\mathrm{d}t = \int_{\phi(a)}^{\phi(b)} f(x)\mathrm{d}x$$

12 Taylor And Improper Integral

Theorem 12.1 (Integral Version of MVT)**.** Let f be continuous on $[a, b]$. Then $\exists c \in (a, b)$ such that $\int_a^b = f(c)(b-a)$.

Theorem 12.2 (Generalized Integral Version of MVT)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Let $g : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and assume that g has a *constant* sign on $[a, b]$. Then $\exists c \in (a, b)$ such that $\int_a^b fg = f(c)\int_a^b g$.

Theorem 12.3 (Taylor Expansion in Integral Form)**.** Let $f : [a, b] \to \mathbb{R}$. Given $x \in (a, b)$, assume that $f^{(1)}, \ldots, f^{(n+1)}$ exists on $[a, x]$ and $f^{(n+1)}$ integrable on $[a, x]$. Then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n \mathrm{d}t$$

Definition 12.1 (Singularities)**.** $b \in [-\infty, \infty]$ is a singularity of f if either $b = \pm\infty$ or f is unbounded in every neighbourhood of b, which can be formulated as one of the following equivalent claims:

- $\limsup_{x\to b} |f(x)| = \infty$
- $\forall\delta > 0, \sup_{x\in[b-\delta,b+\delta]} |f(x)| = \infty$
- $\forall\delta > 0, \forall N > 0, \exists x \in [b-\delta, b+\delta]$ such that $|f(x)| > N$.
- $\exists x_1, \ldots$ with $\lim_{x\to\infty} x_n = b$ such that $\lim_{n\to\infty} |f(x_n)| = \infty$.

Definition 12.2 (Improper Integral)**.** Let b be a singularity of f and assume $\int_a^c f$ exists for all $c \in [a, b)$. Then the improper integral $\int_a^b f$ is defined by

$$\int_a^b f := \lim_{c\to b^-}\int_a^c f$$

if the limit exists.
Similarly, if a is a singularity of f, then

$$\int_a^b f := \lim_{c\to a^+}\int_c^b f$$

if the limit exists.
If $c \in (a, b)$ is the only singularity of f on $[a, b]$, then

$$\int_a^b f := \int_a^c f + \int_c^b f$$

if both improper integral limit exists.

Definition 12.3 (Cauchy Mean Value Theorem)**.** Suppose $c \in (a, b)$ is the only singularity of f on $[a, b]$. Then

$$\lim_{\varepsilon \to 0}\left(\int_a^{c-\varepsilon} f + \int_{c+\varepsilon}^b f\right)$$

is the Cauchy Principle Value of $\int_a^b f$ if the limit exists.
Similarly, if $a = -\infty$ and $b = \infty$ are only singularities of f, then Cauchy Principle Value is defined as

$$\lim_{t \to \infty} \int_{-t}^t f$$

Remark: Cauchy Principle Value may exists even improper integral does not exists.

13 Result from Tutorial

Theorem 13.1. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show if $\lim_{x \to a} f'(x) = A$, then $f'(a)$ exists and equals A.

Theorem 13.2. Suppose f'' exists and bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Then $f'(x) \to 9$ as $x \to \infty$.

Theorem 13.3. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. Suppose g differs from f at a finite number of points, then g is integrable also on $[a, b]$ and $\int g = \int f$.

Theorem 13.4. If $f : [a, b] \to \mathbb{R}$ integrable on $[a, b]$ and $\phi : \mathbb{R} \to \mathbb{R}$ continuous. Then $\phi \circ f$ is integrable on $[a, b]$.