0 Review

0.1 Introduction

Definition 0.1 (Zero-coupon Bond)**.** A contract to deliver \$1 on a future date T is known as a **zero-coupon bond**.
The price of the bond at time $t < T$ is denoted by $P(t, T)$, where T is the maturity of the bond.

Definition 0.2 (Money Market Account)**.** The **money market account** is an asset created by the following procedure, called continuously compounded:

- The initial amount equal to \$1 is invested at time $t = 0$ in the bond with the shortest available maturity(i.e. the next infinitesimal instant).
- The position is rolled over to the bond with the next shortest maturity once the first bond expires.

The price of the money market at time t is denoted by M_t.

Theorem 0.1. We have

$$\mathrm{d}\, M_t = r(t) M_t \,\mathrm{d}\, t$$

to describe the value of money market account, where $r(t)$ is a parameter known as the interest rate.
Solving, we have

$$M_t = \exp\left(\int_0^t r(u)\,\mathrm{d}\, u\right)$$

Remark: The relationship between M_t and $P(t, T)$ is non-trivial if $r(t)$ is stochastic. However,

Theorem 0.2. Suppose interest rate r is constant. Then,

$$P(t, T) = \frac{M_t}{M_T} = e^r(T - t)$$

It is obvious that $P(t, T) < 1$ as long as $r > 0$.

Definition 0.3 (Value of Asset)**.** The **value** of an asset is the amount of dollars that an investor will pay to own that asset.
The term **stock price** refers to the value (per unit) of the stock under consideration.

Definition 0.4 (Position)**.** A **position** in an asset is the equantity of an asset owned or owed by an investor.

- **long position**: the investor *owns* the asset.
- **short position**: the investor *sells* the asset that he does not own.

Definition 0.5 (Portfolio)**.** A **portfolio** is a combination of various positions in financial assets.
At any time t, the **value of the portfolio** Π_t is just eh sum of the values of all the positions held in the portfolio at that particular time t:

$$\Pi_t = a_t^{(1)} A_t^{(1)} + \cdots + a_t^{(n)} A_t^{(n)}$$

where $a_t^{(i)}$ is the position where $A_t^{(i)}$ is the price of the asset at time t.
Since the price of assets is not determined by us, therefore, Π_t can be written as the vector

$$(a_t^{(1)}), \dots, a_t^{(n)})$$

Definition 0.6 (Self-financing)**.** A portfolio $\Pi_t = (a_t^{(1)}), \dots$ is **self-financing** over the time interval $[0, T]$ if there is *no* exogeneous infusion or withdrawal of money, *except* possibly at the initiation time 0 or maturity date T:

$$\mathrm{d}\,\Pi_t = a_t^{(1)} dA_t^{(1)} + \cdots + a_t^{(n)} dA_t^{(n)}$$

Roughly speaking, the differential equation says tha thte change in portfolio is completely due to the change in the underlying asset prices and nothing else.

Definition 0.7 (Direction of Cash Flow)**.** Suppose I am an investor

- If cash flow is positive, it measn that someone pays me.
- If cash flow is negative, it means that I pay someone.

Cash flow depends on

1. the value of the underlying portfolio, and
2. whether or not the portfolio is entered into or liquidated.

0.2 Financial Market

Definition 0.8 (Arbitrage)**.** An **arbitrage opportunity** is the existence of a self-financing portfolio $\Pi_t, 0 \le t \le T$, having the following properties:

1. $\Pi_0 = 0$
2. $\Pi_T \ge 0$ for **all possible** outcomes
3. There is a positive probability that $\Pi_T > 0$.

Definition 0.9 (Equivalent Definition of Arbitrage)**.** An arbitrage opportunity is the existence of a self-financing portfolio $\Pi_t, 0 \le t \le T$, having the following properties:

1. $\Pi_T - \Pi_0 e^{rT} \ge 0$ for **all possible** outcomes

2. There is a positive probability that $\Pi_T - \Pi_0 e^{rT} > 0$

Theorem 0.3 (Consequence of No Arbitrage)**.** Suppose there are two self-financing portfolios Π_t^A and Π_t^B over the time interval $[t, T]$ such that $\Pi_T^A \geq \Pi_T^B$. Then in the absence of arbitrage, we must have

$$\Pi_t^A \geq \Pi_t^B$$

Theorem 0.4 (Law of One Price)**.** Law of one price is a consequence of the above theorem.
Suppose there are two self-financing portfolios Π_t^A and Π_t^B over the time interval $[t, T]$ such that $\Pi_T^A = \Pi_T^B$. Then, in the absence of arbitrage, we must have

$$\Pi_t^A = \Pi_t^B$$

Theorem 0.5. All *risk-free* portfolios must earn the same return, i.e., *riskless interest rate*. Suppose Π_t is teh valueo f a riskfree portfolio, and $\mathrm{d}\,\Pi_t$ is the price increment during a small period of time interval $[t, t + \mathrm{d}\,t]$. Then

$$\mathrm{d}\,\Pi_t = r\Pi_t\,\mathrm{d}\,t$$

where r is the riskless interest rate.

0.3 Forward Contracts & Options

Definition 0.10 (Forward Contract)**.** A **forward contract** is a contract that delivers one unit of the underlying asset on a known future date T for a certain price K agreed today.
Here,

- K is the **delivery price**
- T is called the **delivery date**
- the buyer of the contract is in the long position
- the seller of the contract is in the short position
- the delivery price K is the amount the long sides pays the short side in exchange of one unit of the **udnerlying asset whose value** is S_T on the delivery date T.

Definition 0.11 (Forward Price)**.** The **forward price** at time t is the delivery price of a forward contract which costs nothing to enter into at time t. We denote the forward price at time t by

$$F(S_t, t, T)$$

Remark: The forward price $F(S, t, T)$ is *not* the value of corresponding forward contract.

Definition 0.12 (Payoff, Profit)**.** The **payoff** to a position is the value of the position at the maturity date T.
The **profit** to a position is the payoff to the position at maturity dat T, subtracted by the time-T value of the initial investment in the position:

$$\Pi_T - \Pi_t e^{r(T-t)}$$

Theorem 0.6 (Payoff of Forward Contract)**.** It is obvious from the definition that

- the payoff to a long forward contract is $S_T - K$;
- the payoff to a short forward contract is $K - S_T$

Suppose it costs nothing to enter into a forward contract, then by using the forward price definition,

- the payoff and the profit to a long forward contract are the same:
$$S_T - F(S, t, T)$$
- the payoff and profit to a short forward contract are the same:
$$F(S, t, T) - S_T$$

Theorem 0.7 (Forward Price)**.** Suppose the underlying stock S does not pay dividends. Then the forward price $F(S, t, T)$ of stock at time t is given by

$$F(S, t, T) = Se^{r(T-t)}$$

where S is the price of the stock at time t.

Definition 0.13 (Call Option)**.** A **call option** is an agreeme where the buyer has the *right*, but not the obligation to buy the underlying asset, for a certain price K agreed at the initiation of the contract. Here, K is called the stike price, whereby T is used to denote maturity, which is the darte by which option must be exercised or it becomes worthless.
For now, we consider only European call option, where exercise of the contract occurs only at maturity T.
The payoff to a long European call option with strike price K and maturity T is

$$(S_T - K)^+ = \max\{S_T - K, 0\}$$

where the payoff to a short European call option with same strike price and maturity is $-(S_T - K)^+$.

Definition 0.14 (Put Option)**.** A **put option** is an agreeme where the buyer has the right to sell an asset, but not the obligation to sell, for a certain price K agreed at the initiation of the contract.
The payoff to a long European put option with strike price K and expiration T is

$$(K - S_T)^+ = \max\{K - S_T, 0\}$$

whereas the payoff to a short European put option with same strike price and expiration T is $-(K - S_T)^+$.

Definition 0.15 (Moneyness)**.** Options are often described by their degree of moneyness. At any time t, an option is said to be

- **in-the-money** if payoff at time $t > 0$.
- **at-the-money** if payoff$=0$, i.e., $S_t = K$.
- **out-of-the-money** if payoff<0.

Theorem 0.8 (Put Call Parity)**.** We have the following relationship between call c and put p price, over the underlying asset at time t. Here K is the strike price of the options and F is the forward price at time t.

$$c - p + (K - F)e^{-r(T-t)} = 0$$

0.4 Binomial Model

Definition 0.16 (One-period Binomial Model)**.** Suppose the non-dividend paying stock price per share today is S_0. We assume that, at the end of the one period, the stock price is either S_0u or S_0d where d and u are positive real numbers such that $d < u$.
We call u the **up factor** and d the **down factor**.
Consider a derivative on the stock with time T to maturity. Let V_0 be the price of derivative at time 0.
We can price V_0 by constructing $\Pi_0 = V_0 - \phi S_0$, and make it riskless, i.e. $\Pi_T = V_u - \phi S_0 u = V_d - \phi S_0 d$, by picking a suitable ϕ. Since Π_0 is riskless, its payoff should be the same as any other riskless payoff, e.g. money market account,i.e., $\Pi_T = (V_0 - \phi S_0)e^{rT}$.
Solving, we have

$$V_0 = e^{-rT}(pV_u + (1-p)V_d)$$

where $p = \frac{e^{rT}-d}{u-d}$.

We can interpret p and $1-p$ as probabilities distribution on S_T, so that we can write

$$V_0 = e^{-rT}E^{\mathbb{Q}}[V_T]$$

The expectation of S_T under $\mathbb{Q}$ is

$$E^{\mathbb{Q}}[S_T] = S_0e^{rT}$$

which matches our riskless argument.

Theorem 0.9 (Restriction on u and d)**.** In the one-period binomial model where the one period is $[0,T]$ and the corresponding up-factor and down-factor of a non-dividend paying stock are u and d respectively with $d < u$, we have

$$d < e^{rT} < u$$

Definition 0.17 (Multi-period Binomial Model)**.** At any time $j\Delta t$, there are $j+1$ possible stock prices:

$$S_0d^j, S_0d^{j-1}u, \ldots, S_0u^j$$

Without loss of generality, we can assume that $ud = 1$.
If V_j^k is the price of the derivative at time $j\Delta t$ when the underlying stock price is $S_0d^{j-k}u^k$, i.e. there are k period out of j that the price goes up.
We then have

$$V_0 = e^{-rn\Delta t}\sum_{i=0}^{n}\binom{n}{i}p^i(1-p)^{n-i}V_n^i$$

where V_0 is the price of the European style derivative given by the n period binomial model.

1 Brownian Motion

1.1 Brownian Motion

Definition 1.1 (Standard Brownian Motion)**.** A **standard Brownian Motion** is a stochastic process $W_t, t \geq 0$ with the following defining characteristics:

(W1) $W_0 = 0$.

(W2) With probability 1 (almost surely), the function $t \to W_t$ is continuous in t.

(W3) For every $0 \leq t_1 < t_2$, $W_{t_2} - W_{t_1}$ is normally distributed with mean 0 and variance $t_2 - t_1$.

(W4) $W_{t_3} - W_{t_2}$ is independent of $W_{t_1} - W_{t_0}$ for any $0 \leq t_0 \leq t_1 \leq t_2 \leq t_3$, i.e. non-overlapping increments are independently distributed.

Property (W3) implies that for any Δt, $W_{t+\Delta t} - W_t \sim N(0, \Delta t)$. This implies that $|W_t| \leq 1.96\sqrt{t}$ with 95% probability.
Property (W4) implies that $\mathrm{Cov}(W_t, W_s) = E[W_tW_s] = \min\{t, s\}$.[1]

Theorem 1.1 (Binomial Approximation to Brownian Motion)**.** Let $\varepsilon_1, \ldots$ be a sequence of independent, identically distributed random variables with mean 0 and variance 1. For each $n \geq 1$, define a continuous time stochastic process $W_t^{(n)}$ by

$$W_t^{(n)} = \frac{1}{\sqrt{n}}\sum_{1\leq i\leq[nt]}\varepsilon_i$$

$W_t^{(n)}$ approaches a standard Brownian motion $N(0,t)$ as $n \to \infty$.

[1] We write $W_t = (W_t - W_s + W_s)$ if $t > s$.

Theorem 1.2 (Infinitesimal Brownian Increment)**.** We can approximate $\mathrm{d}\,W_t$ is a very small time interval Δt:

$$\Delta W_t := W_{t+\Delta t} - W_t$$

and then we have

$$\Delta W_t = \phi\sqrt{\Delta t}\text{i.e., } \Delta W_t \sim N(0, \Delta t)$$

where ϕ has a standard normal distribution.

Definition 1.2 (Generalised Wiener Process)**.** A **generalised Wiener Process** for a variable X can be defined in terms of $\mathrm{d}\,W_t$ as

$$\mathrm{d}\,X_t = a\,\mathrm{d}\,t + b\,\mathrm{d}\,W_t$$

where a and b are constants. The parameter a is called the **drift rate** and b^2 is called the **variance rate** of the process.
In a small time interval Δt, the change ΔX_t is given by

$$\Delta X_t = a\Delta t + b\Delta W_t$$

Therefore, ΔX_t has a normal distribution with mean $a\Delta t$ and variance $b^2\Delta t$.

Here, we can safely write $X_t = X_0 + at + bW_t$.

1.2 Quadratic Variation

Definition 1.3 (Quadratic Variation)**.** *Any sequence of values $0 = t_0 < t_1 < \cdots < t_n = T$ is called a partition $\Pi = \Pi(t_0, \ldots, t_n)$ of a fixed interval $[0, T]$. The discrete quadratic variation of a standard Brownian motion W relative to the partition Π is defined as*

$$Q(W, \Pi) = \sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}})^2$$

For any partition Π, define

$$\|\Pi\| = \max_{1\leq i\leq n}|t_i - t_{i-1}|$$

Theorem 1.3 (nth Moment of Standard Normal Z)**.** The nth moment of a random variable X is defined to be $E[X^n]$. If $\phi \sim N(0, 1)$, then

$$E[\phi^n] = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{(2k)!}{2^k k!} & \text{if } n = 2k \end{cases}$$

In particular, we have $E(\phi^2) = 1$ and $E(\phi^4) = 3$.

Theorem 1.4. Consider an arbitrary sequence of partitions Π_n, where $n = 1, 2, \ldots$. Suppose $\lim_{n\to\infty}\|\Pi_n\| = 0$, then

$$\lim_{n\to\infty} E(Q(W, \Pi_n) - T)^2] = 0$$

That is the standard Brownian motion has quadratic variation which is equal to T, in the mean square limit.

Definition 1.4. Define the integral $\int_0^T(\mathrm{d}\,W)^2$ by

$$\lim_{n\to infty} E[\sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}})^2 - \int_0^T(\mathrm{d}\,W_t)^2]^2 = 0$$

However, from quadrativ variation theorem, we have

$$\lim_{n\to infty} E[\sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}})^2 - \int_0^T \mathrm{d}\,t]^2 = 0$$

Therefore,

$$\int_0^T(\mathrm{d}\,W_t)^2 = \int_0^T \mathrm{d}\,t$$

In fact, we can write

$$(\mathrm{d}\,W_t)^2 = \mathrm{d}\,t$$

which gives

$$(\Delta W)^2 \approx \Delta t$$

in discrete time approximation.

1.3 Itô's lemma

Definition 1.5 (Itô's Process)**.** The Itô's Process $\mathrm{d}\,X_t$ is defined as

$$\mathrm{d}\,X_t = a(X_t, t)\,\mathrm{d}\,t + b(X_t, t)\,\mathrm{d}\,W_t$$

Theorem 1.5 (Itô's Lemma)**.** Let $V(X_t, t)$ be a smooth function of t and of the Itô's process X_t:

$$\mathrm{d}\,X_t = a\,\mathrm{d}\,t + b\,\mathrm{d}\,W_t$$

for some $a = a(X_t, t)$ and $b = b(X_t, t)$. Then we have

$$\mathrm{d}\,V(X_t, t) = (a\frac{\partial V}{\partial X} + \frac{\partial V}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial X^2})\,\mathrm{d}\,t + b\frac{\partial V}{\partial X}\,\mathrm{d}\,W_t$$

Another version of the Ito's Lemma where we do not have explicit form of $\mathrm{d}\,X$ is

$$\mathrm{d}\,V_t = \mathrm{d}\,V(X_t, t) = \frac{\partial V}{\partial t}\,\mathrm{d}\,t + \frac{\partial V}{\partial X}\,\mathrm{d}\,X_t + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}(\mathrm{d}\,X_t)^2$$

Definition 1.6 (Ito Integral)**.** In order to have differential form of the SDE $\mathrm{d}\,X_t = a(X_t, t)\,\mathrm{d}\,t + b(X_t, t)\,\mathrm{d}\,W_t$, we require $a(X_t, t)$ and $b(X_t, t)$ to be non-anticipative, which means that its value at t can only be available at time t. With the above assumption, we can define

$$\int_0^T a(X_t, t)\,\mathrm{d}\,t = \lim_{n\to\infty}\sum_{i=1}^{n} a(X_{t_{i-1}}, t_{i-1})(t_i - t_{i-1})$$

and Ito integral

$$\int_0^T b(X_t, t)\,\mathrm{d}\,W_t$$

is the mean-square limit of the sum $\sum_{i=1}^{n} b(X_{t_{i-1}}, t_{i-1})(W_{t_i} - W_{t_{i-1}})$.

2 Black-Scholes Model

In Black-Scholes Model, we assume the following 2 conditions:

1. The money market(riskless asset) M_t is given by
$$\mathrm{d}\, M_t = rM_t \,\mathrm{d}\, t$$
2. The stock price follows the Geometric Brownian motion:
$$\mathrm{d}\, S_t = \mu S_t \,\mathrm{d}\, t + \sigma S_t \,\mathrm{d}\, W_t$$
where μ and σ are constants.
The explicit formula is $S_t = \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t)$.

We can derive Black Scholes PED from either delta hedging, where we take $\Pi_t = V_t - \phi_t S_t,\ 0 \le t \le T$, such that ϕ_t is chosen to make Π_t self-financing and riskless.
Self financing condition gives

$$\mathrm{d}\,\Pi_t = \mathrm{d}\, V_t - \phi_t \,\mathrm{d}\, S_t$$

Also, since ϕ_t is chosen so that Π_t is riskless, we also need

$$\mathrm{d}\,\Pi_t = r\Pi_t \,\mathrm{d}\, t$$

Here, $\mathrm{d}\, V_t$ can be calculated via Ito's lemma and $\mathrm{d}\, S_t$ is given in assumption. Solving, we will have

$$\phi_t = \frac{\partial V}{\partial S}$$

and

$$\frac{\partial V}{\partial t} + r\frac{\partial V}{\partial S}S_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} = rV$$

Here, we call $\frac{\partial V}{\partial S}$ **delta** of the derivative.
We can derive the Black Scholes PDE by *replication* also, by considering an asset $\Pi_t = a_t S_t + b_t M_t$, which satisfies $\Pi_t = V_t$ for all $t \le T$. The equality throughout T gives $\mathrm{d}\,\Pi_t = \mathrm{d}\, V_t$.
Similarly, self financing condition gives

$$\mathrm{d}\,\Pi_t = a_t \,\mathrm{d}\, S_t + v_t \,\mathrm{d}\, M_t$$

where $\mathrm{d}\, S_t$ and $\mathrm{d}\, M_t$ are readily available.
Also, we can conpute $\mathrm{d}\, V_t$ via Ito's Lemma:

$$\mathrm{d}\, V_t = (\frac{\partial V}{\partial S}\mu S_t + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2)\,\mathrm{d}\, t + \frac{\partial V}{\partial S}\sigma S_t \,\mathrm{d}\, W_t$$

Then we compare coefficients of $\mathrm{d}\, W_t$, arriving at

$$a_t = \frac{\partial V}{\partial S}$$

and therefore b_t can be wriiten as

$$b_t = \frac{1}{M_t}(V_t - \frac{\partial V}{\partial S}S_t)$$

whereas comparing $\mathrm{d}\, t$ and make necessary computation, we can arrive at

$$\frac{\partial V}{\partial t} + r\frac{\partial V}{\partial S}S_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

Remark:

1. The drift parameter μ of the stock never enters into the PDE.
2. To uniquely determine the solution, we must prescribe
   - Boundary conditions
   - Initial, or final conditions

Theorem 2.1 (Solution to Black Scholes PDE)**.** For a European call option with a call price $c(S_t, t)$, we can make the following observation:

1. Final condition: $c(S_T, T) = \max\{S_T - K, 0\}$
2. Boundary condition 1: $S_0 = 0 \Rightarrow c(0,t) = 0$ for all $0 \le t \le T$.
3. Boundary condition 2: With $S_t \gg K$, we have $c(S_t, t)$ S_t for all $0 \le t \le T$.

With these observation, we have, for European call

$$c_t = S_t N(d_+) - Ke^{-r\tau}N(d_-)$$

For European put:

$$p_t = Ke^{-r\tau}N(-d_-) - S_t N(-d_+)$$

where

$$d_\pm = \frac{\ln(S_t/K) + (r \pm \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

and

$$\tau = T - t$$

Theorem 2.2 (Black Scholes PDE with Presence of Dividends)**.** With presence of dividends,

$$\mathrm{d}\,\Pi_t = a_t \,\mathrm{d}\, S_t + b_t \,\mathrm{d}\, M_t + a_t q S_t \,\mathrm{d}\, t$$

The black scholes PDE becomes

$$\frac{\partial V}{\partial t} + (r-q)\frac{\partial V}{\partial S}S_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

2.1 Preliminaries on Martingale Pricing

Definition 2.1 (Equivalent Probability Measure)**.** Suppose there are two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on space $(\Omega, \mathcal{F})$. We say that $\mathbb{P}$ and $\mathbb{Q}$ are **equivalent**, denoted by $\mathbb{P} \sim \mathbb{Q}$ if

$$\mathbb{P}(A) > 0 \Leftrightarrow \mathbb{Q}(A) > 0 \text{ for all } A \in \mathcal{F}$$

Essentially, two equivalent measures agree on **all** certain and impossible events.
We hope to derive an equivalent probability measure $\mathbb{Q}$ from an existing one $\mathbb{P}$. To do this, we require a **positive** random variable L with property $\mathrm{E}^{\mathbb{P}}[L] = 1$. These

two conditions are two defining characteristic of a Radon-Nikodym Derivative L.
Define $\mathbb{Q}$ by

$$\mathbb{Q}(A) = \mathrm{E}^{\mathbb{P}}[L \cdot \mathbf{1}_A]$$

where $\not{\Bbbk}_A$ is the indicator random variable for the event A.

Theorem 2.3 (Radon Nikodym)**.** Consider two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega, \mathcal{F})$. The following are equivalent:

1. $\mathbb{P} \sim \mathbb{Q}$
2. There eixsts a positive random variable L such that for every event $A \in \mathcal{F}$
   - $\mathbb{Q}(A) = \mathrm{E}^{\mathbb{P}}[L \cdot \mathbf{1}_A]$
   - $\mathbb{P}(A) = \mathrm{E}^{\mathbb{Q}}[\frac{1}{L} \cdot \mathbf{1}_A]$.

Here $L = \frac{\mathrm{d}\,\mathbb{Q}}{\mathrm{d}\,\mathbb{P}}$, as derived from theorem, is called the Radon Nikodym derivative of $\mathbb{Q}$ wrt $\mathbb{P}$.

Theorem 2.4. Let X be a random variable. With above notations, we have

$$\mathrm{E}^{\mathbb{Q}}[X] = \mathrm{E}^{\mathbb{P}}[L \cdot X]$$

and

$$\mathrm{E}^{\mathbb{P}}[X] = \mathrm{E}^{\mathbb{Q}}[\frac{1}{L} \cdot X]$$

Definition 2.2 (Filtration)**.** Let $\{\mathcal{F}_t\}, 0 \leq t \leq T$ be a filtration, where $\mathcal{F}_t$ is information available to us at time t.
Trivially, we have $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$.

Suppose we now want to move from $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$ to $(\Omega, \{\mathcal{F}_t\}, \mathbb{Q})$, we require a random variable L_T satisfying the following:

- L_T is $\mathcal{F}_T$ measurable, which means that L_T will be known at time T.
- L_T is positive.
- $\mathrm{E}^{\mathbb{P}}[L_T] = 1$.

Define a stochastic process L_t as follows:

$$L_t = \mathrm{E}^{\mathbb{P}}[L_T \mid \mathcal{F}_t], 0 \leq t \leq T$$

The above process is called **Radon-Nikodym** derivative/likelihood process.

Theorem 2.5. Suppose $X_t, 0 \leq t \leq T$, is an adapted process on $\Omega, \mathcal{F}_t$. We have

$$\mathrm{E}^{\mathbb{Q}}[X_t] = \mathrm{E}^{\mathbb{P}}[L_t \cdot X_t], 0 \leq t \leq T$$

Theorem 2.6 (Bayes' Formula)**.** For $0 \leq s \leq t \leq T$, we have

$$\mathrm{E}^{\mathbb{Q}}[X_t \mid \mathcal{F}_s] = \frac{1}{L_s} \mathrm{E}^{\mathbb{P}}[L_t \cdot X_t \mid \mathcal{F}_s]$$

3 Martingale & Girsanov

Definition 3.1 (Martingale)**.** Let $(\Omega, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space. Consider an adapted stochastic process $I_t, 0 \leq t \leq T$.
We say that I_t is a $\mathbb{P}$-**martingale** if

$$\mathrm{E}^{\mathbb{P}}[I_t \mid \mathcal{F}_s] = I_s \text{ for all } 0 \leq s \leq t \leq T$$

Heuristically, we have

$$\mathrm{E}_t[\mathrm{d}\, I_t] = 0$$

Theorem 3.1. Suppose X_t is an adapted process, and W_t a Brownian motion under $\mathbb{P}$. Define

$$I_t = \int_0^t X_u \,\mathrm{d}\, W_u$$

or equivalently,

$$\mathrm{d}\, I_t = X_t \,\mathrm{d}\, W_t$$

Then I_t is a $\mathbb{P}$-martingale.

Theorem 3.2 (Martingale Representation Theorem)**.** Suppose I_t is a $\mathbb{P}$-martingale, and W_t a Brownian motion under $\mathbb{P}$. Then there is an adapted process X_t under $\mathbb{P}$, such that

$$I_t = I_0 + \int_0^t X_u \,\mathrm{d}\, W_u$$

i.e.,

$$\mathrm{d}\, I_t = X_t \,\mathrm{d}\, W_t$$

Theorem 3.3 (Girsanov)**.** Suppose W_t is a Brownian motion under measure $\mathbb{P}$, and θ a constant. Define

$$\tilde{W}_t = W_t + \theta t$$

i.e.,

$$\mathrm{d}\, \tilde{W}_t = \mathrm{d}\, W_t + \theta \,\mathrm{d}\, t$$

Then there exists a measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that $\tilde{W}_t$ is a $\mathbb{Q}$-Brownian motion.
Moreover, the probability $\mathbb{Q}$ is defined by

$$L_T = \frac{\mathrm{d}\,\mathbb{Q}}{\mathrm{d}\,\mathbb{P}} = e^{-\frac{1}{2}\theta^2 T - \theta W_T}$$

The Radon-Nikodym process L_t in the Girsanov Theorem is given by

$$L_t = \mathrm{E}^{\mathbb{P}}[L_T \mid \mathcal{F}_t] = e^{-\frac{1}{2}\theta^2 t - \theta W_t}$$

By Ito's Lemma, L_t has the following equivalent differential form:

$$\mathrm{d}\, L_t = -\theta L_t \,\mathrm{d}\, W_t$$

Theorem 3.4 (Martinalizing Discounted Stock Price)**.** In $\mathbb{P}$, $\mathrm{d}\, S_t = \mu S_t \,\mathrm{d}\, t + \sigma S_t \,\mathrm{d}\, W_t$. We denote $\frac{S_t}{M_t}$ the **discounted stock price**, where M_t is the money market account. By Ito's Lemma,

$$\mathrm{d}(\frac{S_t}{M_t} = \sigma \frac{S_t}{M_t}(\frac{\mu - r}{\sigma}\,\mathrm{d}\, t + \mathrm{d}\, W_t)$$

Therefore, we define

$$\mathrm{d}\, \tilde{W}_t = \frac{\mu - r}{\sigma}\,\mathrm{d}\, t + \mathrm{d}\, W_t$$

to arrive at

$$\mathrm{d}(\frac{S_t}{M_t}) = \sigma \frac{S_t}{M_t}\,\mathrm{d}\, \tilde{W}_t$$

which makes $\frac{S_t}{M_t}$ a $\mathbb{Q}$-martingale for a equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$.
Girsanov Theorem ensures $\tilde{W}_t$ is a Brownian motion in $\mathbb{Q}$, given by the Radon-Nikodym process

$$\mathrm{d}\, L_t = -\theta L_t \,\mathrm{d}\, W_t$$

where $\theta = \frac{\mu - r}{\sigma}$ is the Sharpe ratio.
The dynamic of S_t in $\mathbb{Q}$ is

$$\mathrm{d}\, S_t = r S_t \,\mathrm{d}\, t + \sigma S_t \,\mathrm{d}\, \tilde{W}_t$$

so

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma \tilde{W}_t}$$

3.1 Risk Neutral Valuation

We often denote $\mathbb{Q}$ as the risk neutral measure.

Definition 3.2 (European Contingent Claim)**.** A **European contingent claim** or a T-claim is a financial instrument consisting of a payment V_T at maturity date T. Here V_T is non-negative random variable.

Theorem 3.5 (Risk-Neutral Valuation Formula)**.**

$$\frac{V_t}{M_t} = \mathrm{E}_t^{\mathbb{Q}}[\frac{V_T}{M_T}]$$

Below is an outline of derivation:

1. Define $U_t := \mathrm{E}_t^{\mathbb{Q}}[\frac{V_T}{M_T}]$. Note U_t is a $\mathbb{Q}$ martingale.
2. Martingale Representation Theorem suggests there exists some adapted process η_t such that
$$\mathrm{d}\, U_t = \eta_t \,\mathrm{d}\, \tilde{W}_t$$
3. We already have $\mathrm{d}(\frac{S_t}{M_t}) = \sigma \frac{S_t}{M_t}\,\mathrm{d}\, \tilde{W}_t$. Therefore, $\mathrm{d}\, U_t = \phi_t \,\mathrm{d}(\frac{S_t}{M_t})$ where $\phi_t = \frac{\eta_t M_t}{\sigma S_t}$.
4. We define $\Pi_t = \phi_t S_t + \gamma_t M_t$ where $\gamma_t = U_t - \phi_t \frac{S_t}{M_t}$. Therefore, $\Pi_t = U_t M_t$ and $\Pi_T = U_T M_T = V_T$. We claim $U_t M_t$ is arbitrage-free price of derivative.
5. Since Π is self financing, we can show $\mathrm{d}\, \Pi_t = \mathrm{d}(U_t M_t) = \mathrm{d}(U_t e^{rt})$. Applying Ito's Lemma, $\mathrm{d}\, \Pi_t = r U_t M_t \,\mathrm{d}\, t + e^{rt}\phi_t \,\mathrm{d}(\frac{S_t}{M_t})$. Applying Ito's Lemma one more time and we can arrive at $\Pi_t = \phi_t \,\mathrm{d}\, S_t + r_t \,\mathrm{d}\, M_t$, which is the definition of self-financing condition.
6. Then it follows $U_t M_t = V_t$, and theorem follows.

Using this theorem, we can calculate the derivative price at time t from its final payoff at time T, by taking expectation

$$V_t = \mathrm{E}^{\mathbb{Q}}[e^{-r(T-t)} V_T \mid \mathcal{F}_t]$$

where one can write $V_T = V_t \cdot f(T-t)$ to get rid of conditional expectation, and take integral eventually after finding out the upper/lower bound of the integral.

4 Useful Properties

Theorem 4.1 (Lognormal Distribution)**.** *Suppose X follows lognormal distribution(μ, σ^2), then the mean is* $\exp(\mu + \sigma^2/2)$ *and variance is* $(\exp(\sigma^2) - 1)\exp(2\mu + \sigma^2)$.