Revision notes - MA4264

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1 Static Game of Complete Information

1.1 Pure Strategies

Definition 1.1 (Normal Form Representation)**.** The normal-form representation of an n-player game specifies the players'

- **Strategy space** $S_1, \ldots, S_n$, and
- their **payoff functions** $u_1, \ldots, u_n$, where $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$.

We denote this game by $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$.
Let $(s_1, \ldots, s_n)$ be a combination of strategies, one for each player. Then $u_i(s_1, \ldots, s_n)$ is the payoff to player i if for each $j = 1, \ldots, n$, player j chooses strategy s_j.

Definition 1.2 (Strictly Dominated)**.** In a normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, let $s_i', s_i'' \in S_i$. Strategy s_i' is strictly dominated by strategy s_i'' if

$$u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

i.e., for each feasible combination of the other players' strategies, player i's payoff from playing s_i' is **strictly** less than the payoff from playing s_i''.

Since rational players do not play strictly dominated strategies, we can eliminate these strictly dominated strategies iteratively, so as to reduce the dimension of $S_i, i = 1, \ldots, n$, without removing the best response.

Definition 1.3 (Best response)**.** In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, the **best response** for player i to a combination of other player's strategies $s_{-i} \in S_{-i}$ is

$$R_i(s_{-i})L = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$$

i.e., $R_i(s_{-i})$ is the *set of best responses* by player i to the other player's strategies s_{-i}.

Remark: $R_i(s_{-i}) \subset S_i$ can be an empty set, a singleton, or a finite or infinite set.

Definition 1.4 (Nash Equilibrium)**.** In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, the strategies $(s_i^*, \ldots, s_n^*)$ is called a **Nash Equilibrium** if

$$s_i^* \in R_i(s_{-i}^*) \quad \forall i = 1, \ldots, n$$

equivalently,

$$u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*) \quad \forall i = 1, \ldots, n$$

In other words, no player has incentive to deviate from Nash Equilibrium.

To find a Nash Equilibrium in 2 player game, we can use graph. Let $G(R_i)$ denote the graph of R_i defined by

$$G(R_i) = \{(s_i, s_{-i}) \mid s_i \in R_i(s_{-i}), s_{-i} \in S_{-i}\}$$

Then $(s_i^*, \ldots, s_n^*) \in \cap_{i=1}^n G(R_i)$ if and only if it is in a Nash Equilibrium.
Specifically, in a 2-person game, we can compute the graph $R_1(s_2)$ and $R_2(s_1)$ and find the intersection.
If the game can be represented via a bimatrix, we can use the underline to denote the other player's best payoff to current player's strategy; do this for the 2 players and the cell with both underlined will be the best strategy.

Theorem 1.1 (Relation between Nash Equilibrium and IESDS)**.** If the strategies $(s_1^*, \ldots, s_n^*)$ are a Nash equilibrium in an n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots$ then each s_i^* cannot be eliminated in iterated elimination of strictly dominated strategies.

This implies: $\{$Nash Equilibria$\} \subseteq \{$Outcomes of IESDS$\}$.

Theorem 1.2. In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$ where $S_1, \ldots, S_n$ are *finite* sets, if IESDS eliminates all but the strategy $(s_1^*, \ldots, s_n^*)$, then these strategies are unique Nash equilibrium of the game.

In general, to compute Nash Equilibrium, find out expression $\pi_i(s_i, s_j)$ (usually in the form of piecewise functions), and take maximum to get a equation $s_i(s_j)$. Similarly, compute π_j and get a equation $s_j(s_i)$. Find all intersections.

1.2 Mixed Strategies

Definition 1.5 (Mixed Strategy)**.** In the normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$. Suppose $S_i = \{s_{i1}, \ldots, s_{iK}\}$. Then

- Each strategy s_{ik} in $S_i$is called a pure strategy for player i.
- A mixed strategy for player i is a probability distribution $p_i(p_{i1}, \ldots, p_{iK})$, where $\sum_{k=1}^K p_{ik} = 1$ and $p_{ik} \geq 0$.

We define the expected payoff for player 1 to play mixed strategy $p_1 := (p_{11}, \ldots, p_{1J}$ is

$$v_1(p_1, p_2) = \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j} p_{2k} u_1(s_{1j}, s_{2k})$$

Definition 1.6 (Nash Equilibrium)**.** In the two-player normal-form game $G = \{S_1, S_2; u_1, u_2\}$, the mixed strategies (p_1^*, p_2^*) are a **Nash equilibrium** if each player's mixed strategy is a best response to the other player's mixed strategy, i.e.,

$$v_1(p_1^*, p_2^*) \geq v_1(p_1, p_2^*)$$

and

$$v_2(p_1^*, p_2^*) \geq v_2(p_1^*, p_2)$$

for all probability distribution p_1, p_2 on S_1, S_2.

Since it is completely known to us the value of $u_{1,2}(s_{1j}, s_{2k})$, the mixed strategy Nash Equilibrium only concerns solving the probability distribution. In a simplified setting where each player has only 2 strategies, let $p_1 := (r, 1-r)$ and $p_2 = (q, 1-q)$, then

$$v_1(p_1, p_2) = rv_1(s_{11}, p_2) + (1-r)v_1(s_{12}, p_2)$$

As you can see, r here is dependent on q. So we can solve $r^*(q)$ by maximising the above equation. Specifically, we can have

$$r^*(q) = \begin{cases} 1, & \text{if } v_1(s_{11}, p_2) > v_1(s_{12}, p_2) \\ 0, & \text{if } v_1(s_{11}, p_2) < v_1(s_{12}, p_2) \\ [0,1], & \text{if } v_1(s_{11}, p_2) = v_1(s_{12}, p_2) \end{cases}$$

And this is also true for $q^*(r)$. Then find intersections.

Theorem 1.3 (Strategies eliminated by IESDS)**.** If a pure strategy $s_{kj} \in S_{kj}$ is eliminated by IESDS, then this strategy will be played with zero probability $p_{kj} = 0$, in any mixed strategy Nash Equilibrium. If there are only 2 strategies left for each player, then we can use the approach discussed before.

Theorem 1.4 (Existence Theorem on Nash Equilibrium)**.** In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, if n is *finite* and S_i is *finite* for every i, then there eixsts at least one Nash equilibrium, possibly involving mixed strategies.

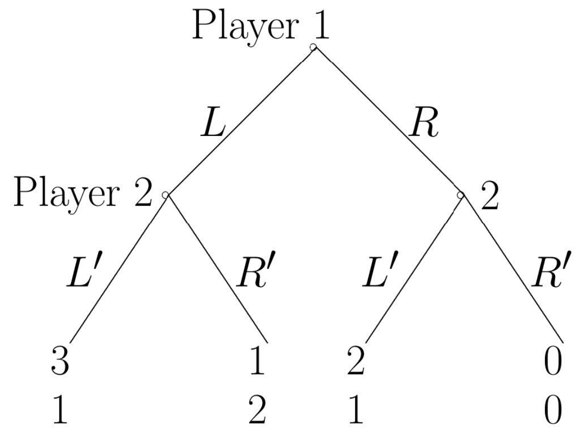
2 Dynamic Games on Complete Information

Definition 2.1 (Dynamic Game of Complete and Perfect Information)**.** A dynamic game of complete and perfect information is a game where

- Players move in sequence,
- (Perfect Information) All previous moves are observed before next move is chosen,
- (Complete Information): Payoffs are common knowledge

Such games can be represented by a game tree.
In this chapter, we will see games with perfect, and imperfect information in sequence.

Definition 2.2 (Backward Induction)**.** The steps are as follow:



1. At the second stage, player 2 observes the action chosen by player 1 at the first stage, say a_1, and then chooses an action by solving

$$\arg\max_{a_2 \in A_2} u_2(a_1, a_2)$$

   Assume this optimization problem has a unique solution, denoted by $R_2(a_1)$. This will be the best response.

2. Player 1 will then solve $\max_{a_1 \in A_1} u_1(a_1, R_2(a_1))$. Assume it has a unique solution a_1^*, we call $(a_1^*, R_2(a_1^*))$ the backwards-induction outcome of the game.

2.1 Two Stage Games of Complete, Imperfect Information

Definition 2.3 (Subgame Perfect Outcome)**.** Let players 1 and 2 simultaneously choose actions a_1 and a_2 from the feasible set A_1, A_2.
Let players 3 and 4 observe the outcome of the first stage (a_1, a_2) and then simultaneously choose action a_3, a_4 from the feasible sets A_3, A_4 respectively.
Payoffs are $u_i(a_1, a_2, a_3, a_4)$ for $i = 1, 2, 3, 4$.
For each given (a_1, a_2), player 3 and 4 try to find Nash equilibrium in stage 2. Assume the second-stage game has a unique Nash Equilibrium $(a_3(a_1, a_2), a_4(a_1, a_2))$, then player 1 and player 2 play a simultaneous-move game with payoffs $u_i(a_1, a_2, a_3(a_1, a_2), a_4(a_1, a_2))$. Suppose (a_1^*, a_2^*) is the Nash equilibrium of the simultaneous-move game, then

$$(a_1^*, a_2^*, a_3(a_1^*, a_2^*), a_4(a_1^*, a_2^*))$$

is the **subgame-perfect** outcome of the 2-stage game.

Definition 2.4 (Extensive Form Representation)**.** The **extensive form** representation of a game specifies

- The players in teh game
- – When each player has the move

- What each player can do at each move
- What each player knows at each of his or her move
- The payoff received by each player for each combinations of moves that could be chosen by the players

Definition 2.5 (Information Set)**.** An **information set** for a player is a collection of decision nodes satisfying:

- The player needs to move at every node in the information set
- When the play of the game reached a node in the information set, the player with the move does not know which node in the set has been reached.

The second point impliees the player must have the **same set** of feasible actions at each decision node in an information set.

A game is said to have **imperfect information** if some of its information sets are *non-singletons*.
In an extensive-form game, a collection of decision nodes, which constitutes an information set, is connected by a dotted line.

Definition 2.6 (Strategy)**.** A **strategy** for a player is a **complete plan of actions**.
It specifies a feasible action for the player in every contingency in which the player might be called on to act.

Definition 2.7 (Payoffs)**.** In the extensive-form representation, payoffs are given for **each sequence of actions**, namely

$$u_i(a_1, \ldots, a_m), \quad i = 1, \ldots, n$$

where $a_1, \ldots, a_m$ are a sequence of actions.
Let $s = (s_1, \ldots, s_n)$ be a combination of strategies of n players and $(a_1(s), \ldots, a_m(s))$ be the sequence of actions specified by $s(s_1, \ldots, s_n)$. Then the payoff received by playing $s = (s_1, \ldots, s_n)$ is

$$\tilde{u}(s) = u(a_1(s), \ldots, a_m(s))$$

where s on LHS is strategy while the parameters in the RHS are actions taken.

Definition 2.8 (Normal Form and Nash Equilibrium)**.** *The normal form of dynamic game specifies payoffs for each combination of* ***strategies****. Nash Equilibrium is obtained from teh normal-form representation.*

Remark: The Nash Equilibrium for dynamic games concerns about players' respective best **strategies**.
In general, we are interested in finding the Nash Equilibrium (s_1, s_2) where $s_1 \in A_1$ and $s_2 = f : A_1 \to A_2$. This means

- Player 1 is interested in finding $\arg\max_{s_1} \tilde{u}_1(a_1 = s_1, s_2^*)$
- Player 2 is interested in finding $\arg\max_{s_2} \tilde{u}_2(a_1, s_2)$ for each $a_1 \in A_1$

Here, although R_2 gives an arg max of whatever player 1 plays, player 2 may *not* follow this strategy.

Theorem 2.1. (a_1^*, R_2) is a Nash equilibrium.

However, apart from a_1^*, there exists other Nash Equilibriums where player 1 not necessarily playing a_1^*.

Definition 2.9 (Subgame Perfect Nash Equilibrium)**.** A **subgame** in an extensive-form game

- begins at a decision node n that is a singleton information set (but is not the game's first decision node)
- includes all the decision and terminal nodes following node n in teh game tree (but no nodes that do not follow n)
- does not cut any information sets (i.e., if a decision node n' follows n in the game tree, then all other nodes in the information set containing n' must also follow n, and so must be included in teh subgame)

A Nash Equilibrium is **subgame-perfect** if the players' strategies constitute a Nash Equilibrium in every subgame.

It can be shown that any finite dynamic game of complete information has a subgame-perfect Nash Equilibrium (which can be in mixed strategies).

2.2 Infinitely Repeated Games

Let π_t be the payoff in stage t. Given a discount factor $\delta \in (0, 1)$, the **present value** of sequence of payoff $\{\pi_1, \pi_2, \ldots\}$ is

$$\pi_1 + \delta\pi_2 + \cdots = \sum_{t=1}^{\infty} \delta^{t-1}\pi_t$$

Here the period 1 is un-discounted.

Definition 2.10 (Infinitely Repeated Games)**.** In the first stage, the player play the stage game G, and receive payoff $\pi_{1,1} and \pi_{2,1}$.
The game is repeated infinitely. In the tth stage, the players observe the actions chosen in the preceding $(t-1)$ stages, and then play G to receive $(\pi_{1,t}, \pi_{2,t})$
The payoff of infinitely repeated game is the **presetn value** of sequence of payoffs:

$$(\sum_{t=1}^{\infty} \delta^{t-1}\pi_{1,t}, \sum_{t=1}^{\infty} \delta^{t-1}\pi_{2,t})$$

Playing the stage game G does not mean having to play an equilibrium of G.
Denote by A_{it} the action space of player i in stage t. We have $A_t := A_{1t} \times A_{2t}$.
A strategy by player i is of the form $\{a_{i1}, a_{i2}, \ldots\}$ where $a_{it} : A_1 \times \cdots \times A_{t-1} \to A_{it}$.
The payoff received at stage t is $\pi_{it} = u_i(a_{it}, a_{jt})$.

Here, the non-cooperative strategy in Infinite Prisoner Dilemma is a Nash Equilibrium, whereas *trigger strategy* is a Nash Equilibrium if and only if $\delta \geq \frac{1}{4}$. A trigger strategy chooses cooperation until being betrayed.

Theorem 2.2. Trigger strategy Nash Equilibrium($\delta \geq 1/4$) is subgame perfect.

3 Static games of incomplete information

Games of **incomplete information** are also called **Bayesian games**. In a game of *incomplete* information, at least one player is uncertain about another player's payoff function. If there are uncertainty, we will maximise the *expected* profit.
In general, let player i's possible payoff functions by $u_i(a_1, \ldots, a_n; t_i)$ where $t_i \in T_i$ is the **type** of player i.
Let t_{-i} be the types of other players and T_{-i} the set of all the t_{-i}.
Player i knows his own type, but *only* knows probability distribution $P_i(t_{-i} \mid t_i)$ on T_{-i}, a belief about other player's types, given i's knowledge of his own t_i.

Definition 3.1 (Normal-form Representation)**.** The **normal-form** representation of an n-player static Bayesian game specifies

- Players' action spaces $A_1, \ldots, A_n$
- Player's type spaces $T_1, \ldots, T_n$
- their beliefs $P_1, \ldots, P_n$ where $P_i : T_i \to P(T_{-i} \mid T_i)$
- their payoff functions $u_1, \ldots, u_n$ where $u_n : \prod_i A_i \times t_i$.

We denote the game as $G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; P_1, \ldots, P_n; u_1, \ldots, u_n\}$.
The timing is as follows:

- Nature draws a type vector $t = (t_1, \ldots, t_n)$
- Nature reveals t_i to player i only.
- Players simultaneously choose action $a_i \in A_i$
- Payoff $u_i(a_1, \ldots, a_n; t_i)$are received

Suppose we have a prior probability distribution $P(t) : T \to \mathbb{R}$, then the belief $P_i(t_{-i} \mid t_i)$ can be computed by Bayes' rule:

$$P(t_{-i} \mid t_i) = \frac{P(t_{-i}, t_i)}{P(t_i)} = \frac{P(t_{-i}, t_i)}{\sum_{t_{-i;} \in T_{-i}} P(t'_{-i}, t_i)}$$

Definition 3.2 (Strategy)**.** In the static Bayesian Game G, a **strategy** for player i is a function $s_i : T_i \to A_i$. For given type t_i, $s_i(t_i)$ gives the action. Player i's strategy space S_i is set of all functions from T_i to A_i.

Definition 3.3 (Bayesian Nash Equilibrium)**.** In static Bayesian game G, the strategies $s^* = (s_1^*, \ldots, s_n^*)$ are a pure-strategy **Bayesian Nash Equilibrium** if, for each player i and each of i's type t_i, $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} E_{t_i} u_i(a_i, s_{-i}^*(t_{-i}); t_i)$$

where

$$E_{t_i} u_i(a_i, s_{-i}^*(t_{-i}); t_i) = \sum_{t_{-i} \in T_{-i}} P_i(t_{-i} \mid t_i) u_i(a_i, s_{-i}^*(t_{-i}); t_i)$$

Theorem 3.1. In a general, finite action space static Bayesian game, a Bayesian Nash equilibrium exists, perhaps in mixed strategies.

The steps we take to find Nash Equilibrium is

- For each player,
  - For each type of games itself has
    - Draw tables of payoff, one each for each type of games opponent has
  - Collate the tables into an expected payoff table of itself playing this type
- Collate the tables generated to the best response $R_i : S_i \to S_j$.
- Use a table to find out Nash Equilibrium, with rows and columns populated by all possible strategies. This is because, R may map to a set.

Definition 3.4 ("Nature Select a Game" Form)**.** Nature selects a game from a set of possible games for each player to play. Some player may know some information about which games/subset of games he is playing.
In general, such static Bayesian games are defined by $\{G, P, ($
where

- G a set of games, where $g \in G$ specifies players' action spaces A_i and payoffs $u_i(a; g)$ for $a = (a_1, \ldots, a_n)$, $1, \ldots, n$.
- P a probability distribution of G

- T_i player i's type space, where each $t_i \in T_i$ is a subset of G, and type space T_i a partition of G.
- The players' types are determined by nature. If nature selects $g \in G$, the player will have type t_j where $g \in t_j$.
- Define $\sigma_i : G \to T_i$, $\sigma_i(g) = t_i$ if $g \in t_i$. So it maps the games to type.
- $(\sigma_1, \ldots, \sigma_n)$ are common knowledge.

The games proceeds as such

- Nature selects a game
- Each player learns its own type, but not others'.
- Player act given its type.

To find equilibria, consider a strategy profile $s_{-i} : T_{-i} \to A_{-i}$ of other $n-1$ players,

- Let $u_i(a_i, s_{-i}; t_i)$ be the expected payoff received by type t_i of player i if i players a_i.
- (s_i^*, s_{-i}^*) is a Bayesian equilibrium if for each $t_i \in T_i$

$$u_i(s_i^*(t_i), s_{-i}^*; t_i) = \max_{a_i \in A_i} u_i(a_i, s_{-i}^*; t_i)$$

Theorem 3.2. For any type $t_i \in T_i$, and strategy profile s_{-i},

$$u_i(a_i, s_{-i}; t_i) = \frac{1}{P(t_i)} \sum_{g \in t_i} P(g) u_i(a_i; s_{-i}(\sigma_{-i}(g)); g)$$

Here, we take the following steps to find equilibria:

- For each player,
  - For each type, compute the expected payoff
  - Find best response

The important tables are the payoff table of each game as well as the formula.

Theorem 3.3 (Weakly Dominating Leading Bayesian NE)**.** Let $(s_1^*, \ldots, s_n^*)$ be strategies in a static Bayesian game. If for any $t_i \in T_i, a_i \in A_i$ and $a_{-i} \in A_{-i}$,

$$u_i(s_i^*(t_i), a_{-i}; t_i) \geq u_i(a_i, a_{-i}; t_i)$$

(i.e., $s_i^*(t_i)$ weakly dominates every $a_i \in A_i$), then $(s_1^*, \ldots, s_n^*)$ is a Bayesian Nash Equilibrium.

4 Dynamic Games of Incomplete Information

4.1 Perfect Bayesian Equilibrium

Definition 4.1 (Perfect Bayesian Equilibrium)**.** A **perfect Bayesian equilibrium** consists of **strategies** *and* **beliefs** satisfying the following requirements:

- Given their beliefs, the players' strategies must be **sequentially rational**.
  That is, at each information set, the action taken by the player with the move (and the player's subsequent strategy) must be optimal, given the player's belief at that information set and the other players' subsequent strategies, (where a "subsequent strategy" is a complete plan of action covering every contingency that might arise after the given information set has been reached).
- **Consistency**: At every information set, beliefs are determined by Bayes' rule and the players' strategies, whenever possible.

Remark: The strategy profile in any perfect Bayesian equilibrium is a Nash Equilibrium.

Definition 4.2 (Pure Strategy Perfect Bayesian Equilibrium)**.** A **pure strategy** perfect Bayesian Equilibrium in a signalling game is a pair of strategies and a belief satisfying sequential rationality and consistency.

Definition 4.3 (Sender Receiver Game)**.** In sender receiver game, the nature will choose the type of the game. In each type, there are some actions that the sender can take, and the receiver will act on the actions of the sender but not the type(which is unknown). The payoffs are perfect information to both players.

For sender-receiver game, the following steps are used to determine perfect bayesian equilibrium.

- For each case, determine the Nash Equilibrium(sequen rational). This can be done by backward induction, or via normal-form representation.
- Determine the belief from the case
- Check the optimality according to the belief(consistent

Definition 4.4 (Worker-Firm)**.** The nature draws the ability of the worker $\{\eta_H, \eta_L\} \in \eta$, and the worker will execute the strategy $S_W : \eta \to \{e_c, e_s\} \in e$, the educational level. The firm will execute the strategy $S_F : e \to w$, the wage.
In general, The payoff of the worker is $w - c(\eta_., e_.)$, whereas the payoff of the firm is $-[y(\eta_., e_.) - w]^2$, whereas c and y are some functions.

Theorem 4.1. Let $c(\eta, e) = c_1(\eta) + c_2(e)$. Assume the worker's payoffs from playing e_c and e_s are different, then there does not exist separating perfect Bayesian equilibrium.

The possible Bayesian Equilibria are

- $\{(e_c, e_c), (w^*(e_c, p), w**(e_s, q)), p = 1/2, q \in [0, \bar{q}]\}$
- $\{(e_s, e_s), (w^*(e_c, p), w**(e_s, q)), p \in [0, \bar{p}], q = 1/2\}$

where $w^*(e_i, r) = ry(\eta_H, e_i) + (1-r)y(\eta_L, e_i)$, $\bar{q}$ the maximum that $\frac{1}{2}B(\eta_H, e_c) + \frac{1}{2}B(\eta_L, e_c) \geq qB(\eta_H, e_s) + (1-q)B(\eta_L, e_s)$ holds; $\bar{p}$ the maximum that $pB(\eta_H, e_c) + (1-p)B(\eta_L, e_c) \leq \frac{1}{2}B(\eta_H, e_s) + \frac{1}{2}B(\eta_L, e_s)$ holds, and $B(\eta., e.) = y(\eta., e.) - c_2(e.)$.

5 Cooperative Games

Definition 5.1 (Domination, Pareto Optimal)**.** Let (u, v) and (u', v') be two payoff pairs. We say (u, v) **dominates** (u', v') if

$$u \geq u', \quad v \geq v'$$

Payoff pairs which are not dominated by any other pair are said to be **Pareto optimal**.

Definition 5.2 (Two-person bargaining game)**.** The pair $\Gamma = (H, d)$ is a **two-person bargaining game** if

- $H \subset \mathbb{R}^2$, the set of payoff pair, is compact(closed and bounded) and convex.
- $d \in H$, a threat point
- and H contains at least one element u, such that $u \gg d$.

Definition 5.3 (Nash Bargaining Solution)**.** The Nash bargaining solution is a mapping $f : W \to \mathbb{R}^2$ that associates a unique element $f(H, d) = (f_1(H, d), f_2(H, d))$ with the game $(H, d) \in W$, satisfying the following axioms:

1. Feasibility: $f(H, d) \in H$
2. Individual rationality: $f(H, d) \gg d$ for all $(H, d) \in W$.
3. $f(H, d)$ is Pareto Optimal.
4. Invariance under linear transformations:
   Let $a_1, a_2 > 0, b_1, b_2 \in \mathbb{R}$ and $(H, d), (H', d') \in W$ where $d'_i = a_i d_i + b_i, i = 1, 2$, and $H' = \{x \in \mathbb{R}^2 \mid x_i = a_i y_i + b_i, u = 1, 2, y \in H\}$. Then $f_i(H'_i, d'_i) = a_i f_i(H, d) + b_i, i = 1, 2$.
5. Symmetry: If $(H, d) \in W$ satisfies $d_1 = d_2$ and $(x_1, x_2) \in H$ implies $(x_2, x_1) \in H$, then $f_1(H, d) = f_2(H, d)$.
6. Independence of irrelevant alternatives:
   If $(H, d), (H', d') \in W, d = d', H \subset H'$ and $f(H', d') \in H$, then $f(H, d) = f(H', d')$.

Theorem 5.1 (Solving Nash Bargaining Game)**.** A game $(H, d) \in W$ has a unique Nash solutino $u^* = f(H, d)$ satisfying conditions 1 to 6. The solution u^* satisfies condition 1 to 6 if and only if

$$(u_1^* - d_1)(u_2^* - d_2) > (u_1 - d_1)(u_2 - d_2)$$

for all $u \in H$, $u \gg d$ and $u \neq u^*$.

Definition 5.4 (n=person Cooperative Games)**.** For an n-person game with the set of players $N = \{1, 2, \ldots, n\}$

- any nonempty subset of N is called a **coalition**
- For each coalition S, the characteristic function v of the game gives the amount $v(S)$ that the coalition S can be sure of receiving.
- The game is denoted by $\Gamma = (N, v)$.

Assumption: The characteristic function v satisfies

1. $v(\varnothing) = 0$
2. For any **disjoint** coalitions, K and L contained in N,
$$v(K \cup L) \geq v(K) + v(L)$$

Definition 5.5 (Imputation)**.** An **imputation** in the game (N, v) is a payoff vector $x = (x_1, \ldots, x_n)$ satisfying

1. $\sum_{i=1}^n x_i = v(N)$ (group rational)
2. $x_i \geq v(\{i\})$, for all $i \in N$ (individually rational)

Let $I(N, v)$ denote the set of all imputations of the game (N, v).

Definition 5.6 (Dominate)**.** Let $x, y \in I(N, v)$ and let S be a coalition. We say x dominates y via S, i.e. $x \succ_S y$, if

1. $x_i > y_i$ for all $i \in S$
2. $\sum_{i \in S} x_i \leq v(S)$

We say x **dominates** y, i.e. $x \succ y$ if there is *any* coalition S such that $x \succ_S y$.

A dominated imputation is unstable, and we hope to solve to get a undominated imputation.

Definition 5.7 (Core)**.** The set of all undominated imputations for a game (N, v) is called the **core**, denoted by $C(N, v)$.

Theorem 5.2. The core of a game (N, v) is the set of all n-vectors x, satisfying

- $\sum_{i \in S} x_i \geq v(S)$ for all $S \subset N$.
- $\sum_{i \in N} x_i = v(N)$

The core of the game (N, v) is nonempty *if and only if* there exists x satisfying the above theorem, *if and only if*

$$v(B) \geq \min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} x_i$$
$$\text{such that } \sum_{i \in S} x_i \geq v(S), \quad \forall S \subset N, S \neq N$$

Definition 5.8 (Constant Sum Game)**.** A game (N, v) is said to be **constant sum** if for all $S \subset N$,

$$v(S) + v(N - S) = v(N)$$

Definition 5.9 (Essential Game)**.** A game (N, v) is **essential** if

$$v(B) > \sum_{i \in N} v(\{i\})$$

It is **inessential** otherwise.

Theorem 5.3. If (N, v) is inessential, then for any coalition S, $v(S) = \sum_{i \in S} v(\{i\})/$

Theorem 5.4. If (N, v) is inessential, then it is constant sum.

Theorem 5.5. If a game (N, v) is constant sum, then its core is either empty or a singleton

$$\{(v(\{1\}), \ldots, v(\{n\}))\}$$

Theorem 5.6. If (N, v) is inessential, then $C(N, v) = \{(v(\{1\}), \ldots, v(\{n\}))\}$.

Theorem 5.7. If game (N, v) is both essential and constant-sum, then its core is empty.

Definition 5.10 (Strategically Equivalent)**.** Two games v, u are **strategically equivalent** if there exist constants $a > 0$ and $c_1, \ldots, c_n$ such that for every coalition S

$$u(S) = av(S) + \sum_{i \in S} c_i$$

Theorem 5.8. Suppose that u and v are strategically equivalent. Then

1. u is essential(resp. inessential) *if and only if* v is essential(resp. inessential).
2. x is an imputation for v *if and only if* $ax + c$ is an imputation for u, where $c = (c_1, \ldots, c_n)$.
3. $x \succ_S y$ with respect to v *if and only if* $ax + c \succ_S ay + c$ with respect to u.
4. *$x \in C(N, v)$ if and only if $ax + c \in C(N, u)$.*

Definition 5.11 ($(0,1)$-reduced form)**.** A characteristic function v is in $(0,1)$**-reduced form** if

- $v(\{i\}) = 0$ for all $i \in N$.
- $v(N) = 1$.

Theorem 5.9. Any essential game (N, v) is strategically equivalent to a game (N, i) in $(0,1)$-reduced form.

Definition 5.12 (Shapley Value)**.** Given n-person game (N, v), the Shapley value is a n-vector, denoted by $\phi(v)$, satisfying a set of axioms.
The ith component of $\phi(v)$ can be uniquely determined by

$$\begin{aligned}\phi_i(v) &= \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)] \\ &= \frac{1}{n} \sum_{s=0}^{n-1} \frac{1}{\binom{n-1}{s}} \sum_{S \subset N \setminus \{i\}, |S| = s} [v(S \cup \{i\} - v(S))]\end{aligned}$$

We can understand $\phi_i(v)$ is the expected contribution of i to coalition. This is because the probability of finding coalition S already there when player i arrives is

$$\gamma(s) = \frac{s!(n-s-1)!}{n!}$$

Theorem 5.10. The Shapley value has the following desirable properties, assuming $v(\varnothing) = 0$.

- Individual rationality: $\phi_i(v) \geq v(\{i\})$ for every $i \in N$.
- Efficiency: The total gain is distributed: $\sum_{i \in N} \phi_i(v) = v(N)$.
- Symmetry: If player i and j are such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for every coalition S not containing i and j, then $\phi_i(v) = \phi_j(v)$.
- Additivity: If v and w are characteristic functions, then $\phi(v + w) = \phi(v) + \phi(w)$.
- Dummy: If player i is such that $v(S \cup \{i\}) = v(S)$ for every S not containing i, then $\phi_i(v) = 0$.