Revision notes - MA4264

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Contents

1 Static Game of Complete Information

1.1 Pure Strategies

Definition 1.1 (Normal Form Representation).
The normal-form representation of an n-player game specifies the players'

- **Strategy space** $S_1, \ldots, S_n$, and
- their **payoff functions** $u_1, \ldots, u_n$, where $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$.

We denote this game by $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$.
Let $(s_1, \ldots, s_n)$ be a combination of strategies, one for each player. Then $u_i(s_1, \ldots, s_n)$ is the payoff to player i if for each $j = 1, \ldots, n$, player j chooses strategy s_j.

Definition 1.2 (Strictly Dominated).
In a normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, let $s_i', s_i'' \in S_i$. Strategy s_i' is strictly dominated by strategy s_i'' if

$$u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

i.e., for each feasible combination of the other players' strategies, player i's payoff from playing s_i' is **strictly** less than the payoff from playing s_i''.

Since rational players do not play strictly dominated strategies, we can eliminate these strictly dominated strategies iteratively, so as to reduce the dimension of $S_i, i = 1, \ldots, n$, without removing the best response.

Definition 1.3 (Best response).
In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, the **best response** for player i to a combination of other player's strategies $s_{-i} \in S_{-i}$ is

$$R_i(s_{-i})L = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$$

i.e., $R_i(s_{-i})$ is the *set of best responses* by player i to the other player's strategies s_{-i}.

Remark: $R_i(s_{-i}) \subset S_i$ can be an empty set, a singleton, or a finite or infinite set.

Definition 1.4 (Nash Equilibrium).
In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, the strategies $(s_i^*, \ldots, s_n^*)$ is called a **Nash Equilibrium** if

$$s_i^* \in R_i(s_{-i}^*) \quad \forall i = 1, \ldots, n$$

equivalently,

$$u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*) \quad \forall i = 1, \ldots, n$$

In other words, no player has incentive to deviate from Nash Equilibrium.

To find a Nash Equilibrium in 2 player game, we can use graph. Let $G(R_i)$ denote the graph of R_i defined by

$$G(R_i) = \{(s_i, s_{-i}) \mid s_i \in R_i(s_{-i}), s_{-i} \in S_{-i}\}$$

Then $(s_i^*, \ldots, s_n^*) \in \cap_{i=1}^n G(R_i)$ if and only if it is in a Nash Equilibrium.
Specifically, in a 2-person game, we can compute the graph $R_1(s_2)$ and $R_2(s_1)$ and find the intersection.
If the game can be represented via a bimatrix, we can use the underline to denote the other player's best payoff to current player's strategy; do this for the 2 players and the cell with both underlined will be the best strategy.

Theorem 1.1 (Relation between Nash Equilibrium and IESDS).
If the strategies $(s_1^*, \ldots, s_n^*)$ are a Nash equilibrium in an n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, then each s_i^* cannot be eliminated in iterated elimination of strictly dominated strategies.

This implies: $\{\text{Nash Equilibria}\} \subseteq \{\text{Outcomes of IESDS}\}$.

Theorem 1.2.
In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u$ where $S_1, \ldots, S_n$ are *finite* sets, if IESDS eliminates all but the strategy $(s_1^*, \ldots, s_n^*)$, then these strategies are unique Nash equilibrium of the game.

In general, to compute Nash Equilibrium, find out expression $\pi_i(s_i, s_j)$ (usually in the form of piecewise functions), and take maximum to get a equation $s_i(s_j)$. Similarly, compute π_j and get a equation $s_j(s_i)$. Find all intersections.

1.2 Mixed Strategies

Definition 1.5 (Mixed Strategy).
In the normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$. Suppose $S_i = \{s_{i1}, \ldots, s_{iK}\}$. Then

- Each strategy s_{ik} in $S_i$is called a pure strategy for player i.
- A mixed strategy for player i is a probability distribution $p_i(p_{i1}, \ldots, p_{iK})$, where $\sum_{k=1}^K p_{ik} = 1$ and $p_{ik} \geq 0$.

We define the expected payoff for player 1 to play mixed strategy $p_1 := (p_{11}, \ldots, p_{1J}$ is

$$v_1(p_1, p_2) = \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j} p_{2k} u_1(s_{1j}, s_{2k})$$

Definition 1.6 (Nash Equilibrium).
In the two-player normal-form game $G = \{S_1, S_2; u_1, u_2\}$, the mixed strategies (p_1^*, p_2^*) are a **Nash equilibrium** if

each player's mixed strategy is a best response to the other player's mixed strategy, i.e.,

$$v_1(p_1^*, p_2^*) \geq v_1(p_1, p_2^*)$$

and

$$v_2(p_1^*, p_2^*) \geq v_2(p_1^*, p_2)$$

for all probability distribution p_1, p_2 on S_1, S_2.

Since it is completely known to us the value of $u_{1,2}(s_{1j}, s_{2k})$, the mixed strategy Nash Equilibrium only concerns solving the probability distribution. In a simplified setting where each player has only 2 strategies, let $p_1 := (r, 1-r)$ and $p_2 = (q, 1-q)$, then

$$v_1(p_1, p_2) = rv_1(s_{11}, p_2) + (1-r)v_1(s_{12}, p_2)$$

As you can see, r here is dependent on q. So we can solve $r^*(q)$ by maximising the above equation. Specifically, we can have

$$r^*(q) = \begin{cases} 1, & \text{if } v_1(s_{11}, p_2) > v_1(s_{12}, p_2) \\ 0, & \text{if } v_1(s_{11}, p_2) < v_1(s_{12}, p_2) \\ [0,1], & \text{if } v_1(s_{11}, p_2) = v_1(s_{12}, p_2) \end{cases}$$

And this is also true for $q^*(r)$. Then find intersections.

Theorem 1.3 (Strategies eliminated by IESDS)**.**
If a pure strategy $s_{kj} \in S_{kj}$ is eliminated by IESDS, then this strategy will be played with zero probability $p_{kj} = 0$, in any mixed strategy Nash Equilibrium. If there are only 2 strategies left for each player, then we can use the approach discussed before.

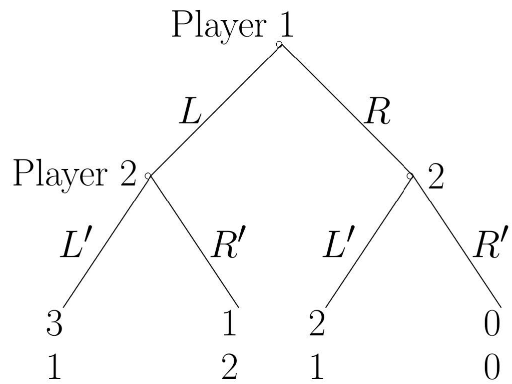
Theorem 1.4 (Existence Theorem on Nash Equilibrium)**.**

In the n-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, if n is *finite* and S_i is *finite* for every i, then there eixsts at least one Nash equilibrium, possibly involving mixed strategies.

2 Dynamic Games on Complete Information

Definition 2.1 (Dynamic Game of Complete and Perfect Information)**.**
A dynamic game of complete and perfect information is a game where

- Players move in sequence,
- (Perfect Information) All previous moves are observed before next move is chosen,
- (Complete Information): Payoffs are common knowledge



Such games can be represented by a game tree.
In this chapter, we will see games with perfect, and imperfect information in sequence.

Definition 2.2 (Backward Induction)**.**
The steps are as follow:

1. At the second stage, player 2 observes the action chosen by player 1 at the first stage, say a_1, and then chooses an action by solving

$$\arg\max_{a_2 \in A_2} u_2(a_1, a_2)$$

   Assume this optimization problem has a unique solution, denoted by $R_2(a_1)$. This will be the best response.

2. Player 1 will then solve $\max_{a_1 \in A_1} u_1(a_1, R_2(a_1))$. Assume it has a unique solution a_1^*, we call $(a_1^*, R_2(a_1^*))$ the backwards-induction outcome of the game.

2.1 Two Stage Games of Complete, Imperfect Information

Definition 2.3 (Subgame Perfect Outcome)**.**
Let players 1 and 2 simultaneously choose actions a_1 and a_2 from the feasible set A_1, A_2.
Let players 3 and 4 observe the outcome of the first stage (a_1, a_2) and then simultaneously choose action a_3, a_4 from the feasible sets A_3, A_4 respectively.
Payoffs are $u_i(a_1, a_2, a_3, a_4)$ for $i = 1, 2, 3, 4$.
For each given (a_1, a_2), player 3 and 4 try to find Nash equilibrium in stage 2. Assume the second-stage game has a unique Nash Equilibrium $(a_3(a_1, a_2), a_4(a_1, a_2))$, then playe 1 and player 2 play a simultaneous-move game with payoffs $u_i(a_1, a_2, a_3(a_1, a_2), a_4(a_1, a_2))$. Suppose (a_1^*, a_2^*) is the Nash equilibrium of the simultaneous-move game, then

$$(a_1^*, a_2^*, a_3(a_1^*, a_2^*), a_4(a_1^*, a_2^*))$$

is the **subgame-perfect** outcome of the 2-stage game.

Definition 2.4 (Extensive Form Representation)**.**
The **extensive form** representation of a game specifies

- The players in teh game
- - When each player has the move
  - What each player can do at each move
  - What each player knows at each of his or her move
  - The payoff received by each player for each combinations of moves that could be chosen by the players

Definition 2.5 (Information Set)**.**
An **information set** for a player is a collection of decision nodes satisfying:

- The player needs to move at every node in the information set
- When the play of the game reached a node in the information set, the player with the move does not know which node in the set has been reached.

The second point impliees the player must have the **same set** of feasible actions at each decision node in an information set.

A game is said to have **imperfect information** if some of its information sets are *non-singletons*.
In an extensive-form game, a collection of decision nodes, which constitutes an information set, is connected by a dotted line.

Definition 2.6 (Strategy)**.**
A **strategy** for a player is a **complete plan of actions**. It specifies a feasible action for the player in every contingency in which the player might be called on to act.

Definition 2.7 (Payoffs)**.**
In the extensive-form representation, payoffs are given for **each sequence of actions**, namely

$$u_i(a_1, \dots, a_m), \quad i = 1, \dots, n$$

where $a_1, \dots, a_m$ are a sequence of actions.
Let $s = (s_1, \dots, s_n)$ be a combination of strategies of n players and $(a_1(s), \dots, a_m(s))$ be the sequence of actions specified by $s(s_1, \dots, s_n)$. Then the payoff received by playing $s = (s_1, \dots, s_n)$ is

$$\tilde{u}(s) = u(a_1(s), \dots, a_m(s))$$

where s on LHS is strategy while the parameters in the RHS are actions taken.

Definition 2.8 (Normal Form and Nash Equilibrium)**.**
The normal form of dynamic game specifies payoffs for each combination of ***strategies****. Nash Equilibrium is obtained from teh normal-form representation.*

Remark: The Nash Equilibrium for dynamic games concerns about players' respective best **strategies**.
In general, we are interested in finding the Nash Equilibrium (s_1, s_2) where $s_1 \in A_1$ and $s_2 = f : A_1 \to A_2$. This means

- Player 1 is interested in finding $\arg\max_{s_1} \tilde{u}_1(a_1 = s_1, s_2^*)$
- Player 2 is interested in finding $\arg\max_{s_2} \tilde{u}_2(a_1, s_2)$ for each $a_1 \in A_1$

Here, although R_2 gives an arg max of whatever player 1 plays, player 2 may *not* follow this strategy.

Theorem 2.1. (a_1^*, R_2) is a Nash equilibrium.

However, apart from a_1^*, there exists other Nash Equilibriums where player 1 not necessarily playing a_1^*.

Definition 2.9 (Subgame Perfect Nash Equilibrium)**.**
A **subgame** in an extensive-form game

- begins at a decision node n that is a singleton information set (but is not the game's first decision node)
- includes all the decision and terminal nodes following node n in teh game tree (but no nodes that do not follow n)
- does not cut any information sets (i.e., if a decision node n' follows n in the game tree, then all other nodes in the information set containing n' must also follow n, and so must be included in teh subgame)

A Nash Equilibrium is **subgame-perfect** if the players' strategies constitute a Nash Equilibrium in every subgame.

It can be shown that any finite dynamic game of complete information has a subgame-perfect Nash Equilibrium (which can be in mixed strategies).

2.2 Infinitely Repeated Games

Let π_t be the payoff in stage t. Given a discount factor $\delta \in (0, 1)$, the **present value** of sequence of payoff $\{\pi_1, \pi_2, \dots\}$ is

$$\pi_1 + \delta\pi_2 + \cdots = \sum_{t=1}^{\infty} \delta^{t-1} \pi_t$$

Here the period 1 is un-discounted.

Definition 2.10 (Infinitely Repeated Games)**.**
In the first stage, the player play the stage game G, and receive payoff $\pi_{1,1} and \pi_{2,1}$.
The game is repeated infinitely. In the tth stage, the players observe the actions chosen in the preceding $(t-1)$

stages, and then play G to receive $(\pi_{1,t}, \pi_{2,t})$
The payoff of infinitely repeated game is the **presetn value** of sequence of payoffs:

$$(\sum_{t=1}^{\infty} \delta^{t-1} \pi_{1,t}, \sum_{t=1}^{\infty} \delta^{t-1} \pi_{2,t})$$

Playing the stage game G does not mean having to play an equilibrium of G.
Denote by A_{it} the action space of player i in stage t. We have $A_t := A_{1t} \times A_{2t}$.
A strategy by player i is of the form $\{a_{i1}, a_{i2}, \ldots\}$ where $a_{it} : A_1 \times \cdots \times A_{t-1} \to A_{it}$.
The payoff received at stage t is $\pi_{it} = u_i(a_{it}, a_{jt})$.

Here, the non-cooperative strategy in Infinite Prisoner Dilemma is a Nash Equilibrium, whereas *trigger strategy* is a Nash Equilibrium if and only if $\delta \geq \frac{1}{4}$. A trigger strategy chooses cooperation until being betrayed.

Theorem 2.2. Trigger strategy Nash Equilibrium($\delta \geq 1/4$) is subgame perfect.