1 Complex Numbers

Definition 1.1 (Complex Numbers)**.** Complex numbers are numbers of the form $z = x + iy$ with $x, y \in \mathbb{R}$.
We denote the set of complex numbers by $\mathbb{C} := \{x + iy \mid x, y \in \mathbb{R}\}$.
For $z = x + iy$, wecan x the **real part** of z and y the **imaginary part**. We usually write

$$x = \Re z, \quad y = \Im z$$

If $y = 0$, then z is real. If $x = 0$, then z is purely imaginary. It is obvious that $\mathbb{R} \subset \mathbb{C}$.
Two complex numbers are equal if and only if their real parts and imaginary parts are both equal.

Definition 1.2 (Addition, Subtraction, Multiplication)**.** For $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we define

- Addition/Subtraction: $(x_1+iy_1)+(x_2+iy_2) = (x_1+x_2) + i(y_1 + y_2)$
- Multiplication: $(x_1+iy_1)\cdot(x_2+iy_2) = (x_1x_2-y_1y_2)+i(x_1y_2 + y_1x_2)$.

Theorem 1.1. For any $z_1, z_2, z_3 \in \mathbb{C}$, we have

- Commutative law: $z_1 + z_2 = z_2 + z_1$, $z_1 \cdot z_2 = z_2 \cdot z_1$.
- Associative law: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$.

Definition 1.3 (Division)**.** For $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \neq 0$, we have

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$$

Definition 1.4 (Modulus)**.** The modulus $|z|$ of a complex number $z = x + iy$ is the distance of z from the origin 0 in the complex plane:

$$|z| = \sqrt{x^2 + y^2}$$

Remark:

- $|z| \geq 0$; $|z| = 0 \Leftrightarrow z = 0$.
- $|\Re z| \leq |z|$, $|\Im z| \leq |z|$.
- Distance between z_1 and z_2 is $|z_1 - z_2|$.

Definition 1.5 (Complex Conjugate)**.** The **complex conjugate** $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\bar{z} = x - iy$$

Theorem 1.2 (Consequences involving Complex Conjugate)**.**

- $\Re z = \frac{z+\bar{z}}{2}$.
- $\Im z = \frac{z-\bar{z}}{2}$
- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.
- $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$.
- $\overline{\frac{z_1}{z_2}} = \frac{\bar{z}_1}{\bar{z}_2}$.
- $z\bar{z} = |z|^2$.
- $\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{|z_2|^2}$

Theorem 1.3. $|z_1z_2| = |z_1| \cdot |z_2|$.

Theorem 1.4 (Triangle Inequality)**.**

$$||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$$

Definition 1.6 (Polar Form, Exponential Form)**.** Let $z = x + iy$, Then, with (r, θ) satisfying $x = r\cos\theta$ and $y = r\sin\theta$, we can write

$$z = r(\cos\theta + i\sin\theta)$$

We call the above expression **polar form** of z.
We define $e^{i\theta} := \cos\theta + i\sin\theta$. So we have

$$z = re^{i\theta}$$

We call the above expression the **exponential form** of z. Here, $r = \sqrt{x^2 + y^2}$, $\theta = \arctan\frac{y}{x} + (2n+1)\pi$.

Definition 1.7 (Argument)**.** Any real value that makes the polar form holds is called the **argument** of z. We have

$$\arg z = \theta + 2n\pi, n \in \mathbb{Z}$$

Due to this result, we define the general polar form of z to be

$$z = r[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)], n = 0, \pm 1, \ldots$$

and general exponential form as

$$z = r\exp[i(\theta + 2n\pi)], n = 0, \pm 1, \ldots$$

Definition 1.8 (Principal Argument)**.** $\operatorname{Arg} z = \{x \mid x \in \arg z, -\pi < x \leq \pi\}$.

Theorem 1.5. $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

Theorem 1.6 (De Moivre's Formula)**.** $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Definition 1.9 (Principal nth root)**.** Principle nth root is the nth root of the principle argument.

2 Analytic Functions

Definition 2.1 (Polynomial)**.** A polynomial in z is a function of the form

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

with each $a_i \in \mathbb{C}$.

Definition 2.2 (Rational Function)**.** A rational function in z is a function of the form $f(z) = \frac{p(z)}{q(z)}$, where p, q are polynomial in z.

Definition 2.3 (Limits)**.** We write $\lim_{z\to z_0} f(z) = w_0$ if the following condition is satisfied: For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta$$

Theorem 2.1 (Limit of Real/Imaginary-Part Separable Function)**.** If $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$, then

$$\lim_{z\to z_0} f(z) = w_0 \Leftrightarrow \begin{cases} \lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0, \\ \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0 \end{cases}$$

Theorem 2.2. If there are two curves C_1 and C_2 passing through z_0 such that

$$\lim_{z\to z_0 \text{ along } C_1} f(z) \neq \lim_{z\to z_0 \text{ along } C_2} f(z)$$

then $lim_{z\to z_0} f(z)$ does not exist.

Theorem 2.3. If $\lim_{z\to z_0} f(z) = w_1$ and $\lim_{z\to z_0} g(z) = w_2$, then

$$\lim_{z\to z_0} [f(z) + g(z)] = w_1 + w_2, \quad \lim_{z\to z_0} f(z) \cdot g(z) = w_1 \cdot w_2$$

and

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{w_1}{w_2} \text{ if } g(z), w_2 \neq 0$$

2.1 Infinity

Definition 2.4 (Extended Complex Plane)**.** $\mathbb{C} \cup \{\infty\}$ is called the extended complex plane.

Definitions of limits involving infinity is similar to that of real limits, and is omitted.

2.2 Continuity

Definition 2.5 (Continuous)**.** A complex-valued function $f(z)$ is continuous at z_0 if and only if $\lim_{z\to z_0} f(z) = f(z_0)$.

Theorem 2.4 (Some consequence)**.**

- If $f(z) = u(x, y) + iv(x, y)$, then $f(z)$ is continuous if and only if u, v are continuous.
- If $f(z), g(z)$ are continuous, then $f(z) \pm g(z)$, $f(z) \cdot g(z)$ are continuous at z_0.
- Also, $\frac{f}{g}(z)$ is continuous at any z_0 where $g(z_0) \neq 0$.

2.3 Differentiation

Definition 2.6 (Derivative)**.** The derivative of f at z_0 is defined as

$$\frac{\mathrm{d}}{\mathrm{d}z} f(z)|_{z=z_0} = f'(z_0) := \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If $f'(z_0)$ exists, then we say f is differentiable at z_0.

Theorem 2.5. If $f(z)$ is differentiable at z_0, then $f(z)$ is continuous at z_0.

The differentiation rule of complex functions are similar to that of real functions.

2.4 Cauchy Riemann Equations

Theorem 2.6 (A necessary condition for differentiability)**.** If $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$, then u and v satisfy Cauchy Riemann equations at (x_0, y_0) i,e,

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Theorem 2.7 (A Sufficient Condition for Differentiability)**.** Let $f(z) = u(x, y) + iv(x, y)$ be defined near the point $z_0 = x_0 + iy_0$. Suppose

1. u, v satisfies CR equations at (x_0, y_0):

$$u_x = v_y, u_y = -v_x$$

2. u_x, u_y, v_x, v_y continuous at (x_0, y_0).

Then f is differentiable at z_0.

2.5 Analytic Functions

Definition 2.7 (Analytic Function)**.** We say f is analytic at point z_0 if $f(z)$ differentiable everywhere in some open set U containing z_0.
We say f is analytic in an open set U if f is differentiable everywhere in U.

Theorem 2.8. If f, g are analytic in open set U, so are $f + g, f - g, f \times g$, and $\frac{f}{g}$, with $U \setminus \{z \in U \mid g(z) = 0\}$.

Definition 2.8 (Entire function)**.** An **entire** function is a function which is analytic in $\mathbb{C}$.

Theorem 2.9 (Constant Analytic Function)**.** Let $f(z)$ be analytic in domain D. If $f'(z) = 0$ on D, then $f(z)$ is constant in D.

2.6 Harmonic Function

Definition 2.9 (Laplace Equation)**.** Let $f(z) = u(x, y) + iv(x, y)$ be **analytic** in domain D. Then u, v satisfies Laplace equation:

$$u_{xx} + u_{yy} = 0$$
$$v_{xx} + v_{yy} = 0$$

on D.

Definition 2.10 (Real Harmonic Function)**.** Let D be a domain in $\mathbb{R}^2$. A function $u : D \to \mathbb{R}$ is said to be **harmonic** in $D \subset \mathbb{R}^2$ if

1. u has continuous first and second partial derivative
2. u satisfies Laplace equation in D.

Definition 2.11 (Harmonic Conjugate)**.** Let u, v be two harmonic functions in a domain D. We say v is a **harmonic conjugate** of u if

$$u_x = v_y \text{ and } u_y = -v_x \text{ on } D$$

Theorem 2.10 (Harmonic Conjugate composing Analytic Function)**.** Let u be a harmonic function in domain D. Let v be a harmonic conjugate of u. Then $f(z) = u + iv$ is an analytic function in D.

3 Elementary Functions

3.1 Exponential Function

Definition 3.1 (Exponential Function)**.** Let $z = x + iy \in \mathbb{C}$,

$$e^z = e^x(\cos y + i \sin y) = e^x e^{iy}$$

It is worth noting $e^{2\pi i} = 1$, so the expoential function is periodic with period $2\pi i$.

Theorem 3.1.

$$e^{z_1+z_2} = e^{z_1} e^{z_2}$$

Theorem 3.2 (Modulus and Argument of Exponential Function)**.** $|e^z| = e^x$, and $\arg(e^z) = y + 2n\pi, n \in \mathbb{Z}$.

3.2 Logarithmic Function

Definition 3.2 (Logarithmic Function)**.** We define, for $z \in \mathbb{C} \setminus \{0\}$,

$$\log z = \ln |z| + i \arg z$$

Definition 3.3 (Principal Logarithmic Function)**.**

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z$$

$\operatorname{Log} z$ has image $\{x + yi \mid x \in \mathbb{R}, y \in (-\pi, \pi]\}$.

Note, that $\operatorname{Log} z$ is continuous on the **cut complex plane** $\mathbb{C} \setminus (-\infty, 0]$.

Theorem 3.3 (Principal Log is analytic on cut complex plane)**.** The function $\operatorname{Log} z$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$, and

$$\frac{\mathrm{d}}{\mathrm{d}z} \operatorname{Log} z = \frac{1}{z}$$

for all z in cut complex plane.

Theorem 3.4.

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

Definition 3.4 (Complex Exponents)**.** For $z, c \in \mathbb{C}$ with $z \neq 0$,

$$z^c := e^{c \log z}$$

Note, z^c is a multivalued function. So we define the **principal value of** z^c to be $P.V.z^c = e^{c \operatorname{Log} z}$.

3.3 Trigonometric Function

Definition 3.5 (Sine, Cosine)**.** For $z \in \mathbb{C}$,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

We have, for $z = x + iy$, $|\sin z|^2 = \sinh^2 y + \sin^2 x$, and $|\cos z|^2 = \sinh^2 y + \cos^2 x$.
We have $\sin(z + \pi) = -\sin z$, $\cos(z + \pi) = -\cos z$, and $\tan(z + \pi) = \tan z$.

Definition 3.6 (Hyperbolic Sine, Hyperbolic Consine)**.** For $z \in \mathbb{C}$,

$$\cosh x = \frac{e^z + e^{-z}}{2} \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$

We have, $\sinh z = i \sin(-iz)$, $\cosh = \cos(-iz)$, $\cosh^2 z - \sinh^2 z = 1$.

Definition 3.7 (Inverse Trigonometirc Function)**.**

$$\cos^{-1} z = -i \log(z + (z^2 - 1)^{1/2})$$
$$\sin^{-1} z = -i \log(iz + (1 - z^2)^{1/2})$$
$$\tan^{-1} z = \frac{i}{2} \log \frac{i + z}{i - z}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}z} \cos^{-1} z = \frac{z}{(1 - z^2)^{1/2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}z} \sin^{-1} z = \frac{1}{(1 - z^2)^{1/2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}z} \tan^{-1} z = \frac{1}{1 + z^2}$$

4 Contour Integration

Definition 4.1 (Integration of Complex Function of a real variable)**.** Let $w : [a,b] \to \mathbb{C}$, and $t \in [a,b]$. Let $w(t) = u(t) + iv(t)$.

- If $u'(t)$ and $v'(t)$ both exist, then we define

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t) = w'(t) := u'(t) + iv'(t)$$

- If u and v are integrable on $[a,b]$, then we define

$$\int_a^b w(t)\mathrm{d}t = \int_a^b u(t)\mathrm{d}t + i\int_a^b v(t)\mathrm{d}t$$

Theorem 4.1. Let $w, w_1, w_2 : [a,b] \to \mathbb{C}$ and $z_0 \in \mathbb{C}$.

- $\frac{\mathrm{d}}{\mathrm{d}t}[w_1(t) + w_2(t)] = w_1'(t) + w_2'(t)$
- $\frac{\mathrm{d}}{\mathrm{d}t}[z_0 w(t)] = z_0 w'(t)$
- $\int_a^b [w_1(t) + w_2(t)]\mathrm{d}t = \int_a^b w_1(t)\mathrm{d}t + \int_a^b w_2(t)\mathrm{d}t$
- $\int_a^b z_0 w(t)\mathrm{d}t = z_0 \int_a^b w(t)\mathrm{d}t$

Theorem 4.2. If $w : [a,b] \to \mathbb{C}$, then

$$\left|\int_a^b w(t)\mathrm{d}t\right| \leq \int_a^b |w(t)|\mathrm{d}t$$

Theorem 4.3 (Fundamental Theorem of Calculus)**.** Suppose $F(t)$ and $f(t)$ continuous such that

$$F'(t) = f(t)(a \leq t \leq b)$$

Then

$$\int_a^b f(t)\mathrm{d}t = F(b) - F(a)$$

Definition 4.2 (Curve)**.** A **curve** in the complex plane is a continuous function

$$\gamma : [a,b] \to \mathbb{C}$$

That is if we write $\gamma(t) = x(t) + iy(t)$, then $x(t)$ and $y(t)$ are continuous on $[a,b]$.
The set of points on γ is called the **track** of γ.

Definition 4.3 (Simple, Closed, and Simple Closed curve)**.**
γ is simple if $t_1 \neq t_2 \implies \gamma(t_1) \neq \gamma(t_2)$.
γ is closed if $\gamma(a) = \gamma(b)$.
γ is a simple closed curve if it is closed and $a \leq t_1 < t_2 < b \implies \gamma(t_1) \neq \gamma(t_2)$

Definition 4.4 (Smooth curve)**.** A curve $\gamma : [a,b] \to \mathbb{C}$ is smooth if

- $\gamma'(t) = x'(t) + iy'(t)$ exists and is continuous on $[a,b]$, and
- $\gamma'(t) \neq 0$ for all $t \in [a,b]$.

Definition 4.5 (Length of Curve)**.** Length of γ equals

$$\int_a^b |\gamma'(t)|\mathrm{d}t$$

Definition 4.6 (Integral of f along γ)**.** The integral of f along γ is defined by

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t$$

Theorem 4.4 (Reparametrization does not change integral value)**.** Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth curve, and let $\phi : [c,d] \to [a,b]$ such that

- $\phi'(t)$ exists and is continuous on $[c,d]$, and
- $\phi(c) = a, \phi(d) = b$

Define $\alpha(t) = \gamma(\phi(t)), c \leq t \leq d$, then for any function f continuous on $\alpha = \gamma$,

$$\int_\gamma f(z)\mathrm{d}z = \int_\alpha f(z)\mathrm{d}z$$

Definition 4.7 (Opposite curve)**.** *Let $\gamma : [a,b] \to \mathbb{C}$ be a curve. Define opposite curve $-\gamma$ by*

$$(-\gamma)(t) = \gamma(-t), -b \leq t \leq -a$$

We have, for any smooth curve γ, $\int_{-\gamma} f(z)\mathrm{d}z = -\int_\gamma f(z)\mathrm{d}z$

Definition 4.8 (Contour Integral)**.** If f is continuous on γ, then we define contour integral of f along γ to be

$$\int_\gamma f(z)\mathrm{d}z = \sum_{j=1}^n \int_{\gamma_j} f(z)\mathrm{d}z$$

Theorem 4.5 (ML Inequality)**.** Suppose that f is continuous on an open set containing the track of a contour γ, and

$$|f(z)| \leq M \text{ for all } z \in \gamma$$

Then

$$|\int_\gamma f(z)\mathrm{d}z| \leq ML$$

where L is the length of γ.

4.1 Antiderivatives

Definition 4.9 (Antiderivative)**.** Let f be a continuous function on a domain D. A function F such that

$$F'(z) = f(z) \text{ for all } z \in D$$

is called an **antiderivative** of f in D.

Theorem 4.6 (Fundamental Theorem of Calculus For Contour Integrals)**.** Suppose f has antiderivative F on a domain D.

1. If $z_1, z_2 \in D$ and γ a contour in D joining z_1 to z_2, then
$$\int_\gamma f(z)\mathrm{d}z = F(z_2) - f(z_1)$$
This indicate the contour integral of such f is path independent.

2. In particular, if γ is closed in D, then $\int_\gamma f(z)\mathrm{d}z = 0$.

Theorem 4.7. *Let f be continuous on a domain D. The following are equivalent:*

1. *f has an antiderivative in D*

2. *For any closed contour γ in D, $\int_\gamma f(z)\mathrm{d}z = 0$.*

3. *The contour integral of f are independent of paths in D, that is if $z_1, z_2 \in D$, and γ_1, γ_2 are contours in D joining z_1 to z_2 then*
$$\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z$$

4.2 Cauchy-Goursat Theorem

Theorem 4.8 (Jordan Curve Theorem)**.** Any simple closed contour γ separates the plane into two domains, each having γ as its boundary.

Definition 4.10. A simple closed contour γ is positively oriented if the interior domain lies ot the left of an observer tracing out the points in order.

Theorem 4.9 (Cauchy Goursat)**.** If a function f is analytic at all points interior to and on a simple closed contour γ, then

$$\int_\gamma f(z)\mathrm{d}z = 0$$

Theorem 4.10. Let γ_1 and γ_2 be positively oriented simple closed contours with γ_2 interior to γ_1. If f is analytic on the closed region containing γ_1 and γ_2 and the points between them, then

$$\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z$$

Definition 4.11 (Simple Connected Domain)**.** A domain D is **simply connected** if every simple closed contour in D encloses only points in D.

Theorem 4.11 (Cauchy Roursat Theorem For Simply Connected Domain)**.** If f is analytic is a simply connected domain D, then

$$\int_\gamma f(z)\mathrm{d}z = 0$$

for every closed contour γ in D.

4.3 Cauchy Integral Formula

Definition 4.12 (Cauchy Integral Formula)**.** Let γ be a positively oriented simple closed contour and let f be analytic everywhere within and on γ. Then for any z_0 interior to γ,

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\mathrm{d}z$$

Theorem 4.12 (Cauchy Integral Formula for Derivatives)**.** Let $f(z)$ be analytic everywhere inside and on a positively oriented simple closed contour γ. Then for any z_0 inside γ, and any integer $n \geq 1$, $f^{(n)}(z_0)$ exists, and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\mathrm{d}z$$

Theorem 4.13. If f is analytic in domain D, then all its derivative exists and are analytic in D.

Theorem 4.14 (Morera)**.** If f continuous on a domain D, and

$$\int_\gamma f(z)\mathrm{d}z = 0$$

for every closed contour γ in D, then f is analytic in D.

4.4 Cauchy's Inequality

Theorem 4.15 (Cauchy's Inequality)**.** Let $f(z)$ be analytic within and on the circle γ_R centred at z_0 and of radius $R (R > 0)$. Write $M_R = \max_{z \in \gamma_R} |f(z)|$. Then for any integer $n \geq 1$, we have

$$|f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n}$$

Definition 4.13 (Bounded)**.** Let $S \subset \mathbb{C}$. A function $f : S \to \mathbb{C}$ is **bounded** if there exists some $K > 0$ such that

$$|f(z)| \leq K \text{ for all } z \in S$$

Definition 4.14 (Liouville)**.** If an entire function f is bounded, then it must be a constant function.

Theorem 4.16 (Fundamental Theorem of Algebra)**.** If $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, then $p(z) = 0$ has a solution in $\mathbb{C}$.

Definition 4.15.
- *A subset S of $\mathbb{C}$ is said to be closed if $\mathbb{C} \setminus S$ is open.*
- *A subset S of $\mathbb{C}$ is said to be bounded if there exists a number $K > 0$ such that*

$$|z| \leq K \textit{ for all } z \in S$$

Theorem 4.17 (Extreme Value Theorem)**.** Let S be a non-empty closed bounded subset of $\mathbb{C}$, and let $g : S \to \mathbb{R}$ be a continuous function. Then there exists $z_1, z_2 \in S$ such that $g(z_1) \leq g(z) \leq g(z_2)$ for all $z \in S$.

5 Series Representation of Analytic Functions

5.1 Sequences and Series of Complex Numbers

A sequence of complex number is an ordered list of complex numbers, denote dby $\{z_n\}_{n=1}^{\infty}$.

Definition 5.1 (Limit of Sequence)**.** A sequence $\{z_n\}$ has a limit z if for any $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that

$$|z_n - z| < \epsilon \text{ whenever } n \geq N$$

In this case, we say $\{z_n\}$ converges to z.

If $\{z_n\}$ has no limit, then $\{z_n\}$ diverges.

Theorem 5.1. If $z_n = x_n + iy_n$ for all n and $z = x + iy$, then

$$\lim_{n\to\infty} z_n = z \Leftrightarrow \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

Theorem 5.2. A convergent sequence $\{z_n\}$ has a unique limit.

Theorem 5.3. Let $\{z_n\}$ and $\{w_n\}$ be two convergent sequences of complex numbers, and let $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$, then

- $\lim_{n\to\infty}(z_n + w_n) = z + w$
- $\lim_{n\to\infty}(z_n - w_n) = z - w$
- $\lim_{n\to\infty}(z_n \cdot w_n) = zw$
- $\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{z}{w}$ if $w \neq 0$ and each $w_n \neq 0$.

Definition 5.2 (Cauchy Sequence)**.** A sequence $\{z_n\}$ is called a Cauchy sequence if for any $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that

$$|z_n - z_m| < \epsilon \text{ whenever } n, m \geq N$$

Theorem 5.4 (Cauchy Criterion)**.** A sequence $\{z_n\}$ is convergent if and only if $\{z_n\}$ is a Cauchy sequence.

Definition 5.3 (Series)**.** A series of complex numbers is of the form

$$\sum_{n=1}^{\infty} z_n$$

We define $S_n = \sum_{k=1}^{n} z_n$ to be the nth partial sum of the series, and we say $\sum_{n=1}^{\infty} z_n$ converges to S, if $\lim_{n\to\infty} S_n = S$.
If $\{S_n\}$ diverges, the the series diverges.

Theorem 5.5.
- $\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$ where $z_n = x_n + iy_n$ and $S = X + iY$.
- That $\sum_{n=1}^{\infty} z_n$ converges implies $\lim_{n\to\infty} z_n = 0$
- If $\lim_{n\to\infty} z_n \neq 0$, or does not exist, then $\sum_{n=1}^{\infty} z_n$ diverges.

Definition 5.4 (Absolute Convergence)**.** If $\sum_{n=1}^{\infty} |z_n|$ converges, we say that $\sum_{n=1}^{\infty} z_n$ converges absolutely.

Theorem 5.6. $\sum_{n=1}^{\infty} z_n$ converges absolutely implies $\sum_{n=1}^{\infty}$ converges.

5.2 Sequence and Series of Functions

A sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined a set $D \subset \mathbb{C}$ in an ordered list of functions, where each $f_i : D \to \mathbb{C}$.

Definition 5.5 (Pointwise Convergence)**.** Let $\{f_n\}$ be a sequence of functions. Suppose for each $z \in D$, the sequence of complex numbers $f_n(z)$ converges. Then we can define a function $f : D \to \mathbb{C}$ given by

$$f(z) = \lim_{n\to\infty} f_n(z) \text{ for all } z \in D$$

And we say that f_n **converges pointwise** to f on D.

Definition 5.6 (Uniform Convergence)**.** We say a sequence $\{f_n\}$ converges uniformly to f on D if for any $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$, such that

$$|f_n(z) - f(z)| < \epsilon \text{ for all } n > N \text{ and all } z \in D$$

Theorem 5.7. Suppose $\{f_n\}$ converges uniformly to f and g bounded on D. Then $\{f_n \cdot g_n\}$ converges uniformly to function $f \cdot g$.

Theorem 5.8 (Cauchy Criterion)**.** A sequence of functions $\{f_n\}$ converges uniformly on D if and only if for any $\epsilon > 0$, there exists $N = N(\epsilon)$ such that

$$|f_n(z) - f_m(z)| < \epsilon \text{ for all } z \in D \text{ and all } m, n \geq N$$

Note, here N does not depend on z.

Theorem 5.9. Let $D \subset \mathbb{C}$, and $\{f_n\}$ be a sequence of functions on D such that each f_n is continuous on D. If $\{f_n\}$ converges uniformly to f on D¡ then f is also continuous on D.

Theorem 5.10. Let γ be a contour and let $\{f_n\}$ be a sequence of continuous functions on the track of γ. If $\{f_n\}$ converges uniformly to f on γ, then

$$\lim_{n\to\infty} \int_\gamma f_n(z)\mathrm{d}z = \int_\gamma \lim_{n\to\infty} f_n(z)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}z$$

Theorem 5.11. Let $\{f_n\}$ be a sequence of analytic functions on a domain D. If $\{f_n\}$ converges uniformly to f on D, then f is also analytic on D. Moreover,

$$\lim_{n\to\infty} f_n^*(z) = f'(z) \text{ for all } z \in D$$

Definition 5.7 (Series of Functions)**.** Let $D \subset \mathbb{C}$. An expression of the form

$$\sum_{k=1}^{\infty} f_k(z) = f_1(z) + \cdots, z \in D$$

with each $f_k : D \to \mathbb{C}$, is said to be a **series of functions** on D.

Definition 5.8 (Partial Sum of Series of Functions)**.** The partial sum $\{S_n\}$ is given by

$$S_n(z) = \sum_{k=1}^{n} f_k(z)$$

We say that the series of fucntions **converges pointwise (resp. uniformly)** to a function S on D if its sequence of partial sums $\{S_n\}$ converges pointwise to S on D.

Theorem 5.12. Let γ be a contour and let $\sum_{k=1}^{\infty} f_k(z)$ be a series of continuous functions converging uniformly on γ, then

$$\sum_{n=1}^{\infty} \int_\gamma f_n(z)\mathrm{d}z = \int_\gamma \sum_{n=1}^{\infty} f_n(z)\mathrm{d}z$$

Theorem 5.13. Let $\sum_{k=1}^{\infty} f_k(z)$ be a series of analytic functions converging uniformly to some function $S(z)$ on a domain D. Then $S(z) = \sum_{k=1}^{\infty} f_k(z)$ is also analytic on D, and we have

$$\frac{\mathrm{d}}{\mathrm{d}z}(\sum_{k=1}^{\infty} f_k(z)) = S'(z) = \sum_{k=1}^{\infty} f_k'(z) \text{ for all } z \in D$$

Theorem 5.14 (Weierstrass M-test)**.** Consider a series of functions $\sum_{k=1}^{\infty} f_k$ on a set $D \subset \mathbb{C}$. Suppose that

- $|f_k(z)| \le M_k$ for all $z \in D$, $k = 1, 2, \ldots$ and
- $\sum_{k=1}^{\infty} M_k$ converges

Then $\sum_{k=1}^{\infty} f_k$ converges uniformly on S.

5.3 Power Series

An expression of the form $\sum_{k=0}^{\infty} a_k z^k$ is called a **power series** in z.
We can also form a power series in $z - z_0$.

Theorem 5.15 (Radius of Convergence)**.** Given any power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$, there is an associated number R, $0 \le R \le \infty$, called the radius of convergence, witht the following properties:

- The series of numbers $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges absolutely at each $z \in B(z_0, R)$.
- $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ diverges at each z satisfying $|z - z_0| > R$
- The series of of functions $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges uniformly on the closed ball $\overline{B(z_0, \rho)}$, for any ρ satisfying $0 < \rho < R$.

Moreover, R is given by

$$R = \frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}}$$

and is also given by

$$R = \frac{1}{\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}} \text{ if } \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} \text{ exists}$$

Theorem 5.16. Given a power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ with radius of convergence R, we have

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k \begin{cases} \text{converges pointwise} & \text{on } B(z_0, R) \\ \text{diverges} & \text{whenever } |z - z_0| > \end{cases}$$

Theorem 5.17. Let $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ be a power series with radius of convergence R. Then

1. $S(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ is an analytic function on $B(z_0, R)$.
2. Term by term differentiation:
$$S'(z) = \frac{\mathrm{d}}{\mathrm{d}z} \sum_{k=0}^{\infty} a_k(z-z_0)^k = \sum_{k=1}^{\infty} k a_k(z-z_0)^{k-1} \text{ on } B(z$$
3. Term by term integration: If γ a contour in $B(z_0, R)$ and $g(z)$ continuous on γ¡ then
$$\int_\gamma g(z)S(z)\mathrm{d}z = \int_\gamma g(z) \sum_{k=0}^{\infty} a_k(z-z_0)^k \mathrm{d}z = \sum_{k=0}^{\infty} a_k \int_\gamma g$$

5.4 Taylor Series

Theorem 5.18 (Taylor Theorem)**.** Suppose $f(z)$ is analytic in $B(z_0, R)$. Then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k, z \in B(z_0, R)$$

The power series is called the Taylor series of $f(z)$ at z_0, and coefficients Taylor coefficients.

When $z_0 = 0$, Taylor series becomes the Maclaurin series of $f(z)$.
When $f(z)$ entire, $R = \infty$.
It follows that Taylor series has radius of convergence at least R, and it converges uniformly on the closed ball $\overline{B(z_0, \rho)}$ to f, whenever $0 < \rho < R$.

Theorem 5.19 (Uniqueness of Taylor Series)**.** Let $f(z)$ be an analytic function on some domain containing $B(z_0, R)$, where $R > 0$. If

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$$

for all $z \in B(z_0, R)$, then that power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ is the Taylor series of $f(z)$ at z_0, i.e.,

$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

for all $k = 0, 1, 2, \ldots$.

Theorem 5.20 (Multiplication/Division of Power Series)**.** Suppose

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n \text{ for } |z| < R$$

then

- $f(z) \cdot g(z) = (a_0 + a_1 z + \cdots)(b_0 + b_1 z + \cdots)$
- If $g(0) \neq 0$, we may use long divison to find maclaurin seris of $\frac{f}{g}$.

5.5 Laurent Series

We introduce $\text{Ann}(z_0, R_1, R_2) := \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$, and $\overline{\text{Ann}(z_0, R_1, R_2)} := \{z \in \mathbb{C} \mid R_1 \leq |z - z_0| \leq R_2\}$.

Theorem 5.21 (Laurent Theorem)**.** Suppose $f(z)$ analytic in the annulus $\text{Ann}(z_0, R_1, R_2)$, then

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, z \in \text{Ann}(z_0, R_1, R_2)$$

where

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s - z_0)^{n+1}}\mathrm{d}s$$

$$b_n = \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{(s - z_0)^{-n+1}}\mathrm{d}s$$

Theorem 5.22 (Uniqueness of Laurent Series Representation)**.** If an analytic function f in the annulus $Ann(z_0, R_1,$ has its Laurent series, then the expression in previous theorem is the unique Laurent series of $f(z)$ for the annulus $Ann(z_0, R_1, R_2)$.

6 Residues and Poles

6.1 Residue

Definition 6.1 (Residue)**.** A point z_0 is said to be a singlar point of a function f if

- f is not analytic at z_0, but
- f is analytic at some point in $B(z_0, \epsilon)$ for all $\epsilon > 0$

A singular point z_0 of f is isolated if there exists $R > 0$ such that f is analytic in $B(z_0, R) \setminus \{z_0\}$.
The residue of $f(z)$ at z_0 is

$$\lim_{z=z_0} f(z) = b_1$$

Theorem 6.1. If f analytic in $B(z_0, R) \setminus \{z_0\}$, then

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i b_1 = 2\pi i \lim_{z=z_0} f(z)$$

where γ is any positively oriented simple closed contour around z_0 in $B(z_0, R) \setminus \{z_0\}$.

6.2 Cauchy's Residue Theorem

Theorem 6.2 (Cauchy's Residue Theorem)**.** If γ is a positively oriented simple closed contour and $f(z)$ is analytic everywhere inside and on γ except for a finite number of singular points z_k inside γ, then

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \lim_{z=z_k} f(z)$$

6.3 Classification of Isolated Singular Poin

Definition 6.2 (Principal Part)**.** Suppose $f(z)$ has an isolated singular point at z_0. Consider Laurent series of $f(z)$ at z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, 0 < |z - z_0| < R$$

The second part $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ is called the **principal part** of $f(z)$ at z_0.

We calssify isolated singular points into 3 kinds:

- Removable singular points. If $b_n = 0$ for all $n = 1, 2, \ldots$, we say that z_0 is a **removable singular points** of $f(z)$. In this case, $\lim_{z=z_0} f(z) = 0$.

- Essential singular points. If $b_n \neq 0$ for infinitely many n, then we say z_0 is an essential singular point of $f(z)$.
- Pole. If there exists $m \in \mathbb{N}$ such that $b_m \neq 0$ but $b_n = 0$ for all $n > m$, then we say that z_0 is a **pole of order** m for $f(z)$.

Theorem 6.3 (Behaviour near Removable Singluar Point)**.** Suppose f has removable singular point at z_0. We will have

$$\lim_{z \to z_0} f(z) = a_0$$

Theorem 6.4 (Behaviour near Pole)**.** Suppose f has a pole of order m at z_0. Then

1. There exists $R > 0$ such that
$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \text{ for } 0 < |z - z_0| < R$$
where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.
2. In particular $\lim_{z \to z_0} f(z) = \infty$.

Theorem 6.5 (Behaviour near Essential Singular Point(Casorati-Weierstrass))**.** Suppose $f(z)$ has an essential singular point at $z = z_0$. Then for each small $r > 0$, the image $f(B(z_0, r) \setminus \{z_0\})$ is dense in $\mathbb{C}$.

6.4 Pole Orders for Quotients of Analytic Functions

Definition 6.3 (Zero of Order n)**.** We say analytic function $f(z)$ has zero of order n at z_0 if

$$f(z_0) = f^{(1)}(z_0) = \cdots f^{(n-1)}(z_0) = 0 \text{ and } f^{(n)}(z_0) \neq 0$$

Theorem 6.6. Suppose f_z has zero of order n_1 and f_2 has zero of order n_2 at some point z_0. THen the product $f_1 \cdot f_2$ has zero of order $n_1 + n_2$ at z_0.

Theorem 6.7. Suppose an analytic function $f(z)$ has a zero of order m at z_0. Ten there exists an analytic function $\phi(z)$ near z_0 such that

$$f(z) = (z - z_0)^m \phi(z) \text{ near } z_0, \text{ and } \phi(z_0) \neq 0$$

Theorem 6.8. Suppose $p(z)$ and $q(z)$ are analytic at z_0. Suppose that $p(z)$ has a zero of order α at z_0, and $q(z)$ has a zero of order β at z_0. Consider the function

$$f(z) = \frac{p(z)}{q(z)}$$

1. If $\beta > \alpha$, then $\frac{p}{q}$ has a pole of order $\beta - \alpha$ at z_0.
2. If $\beta \leq \alpha$, then $\frac{p}{q}$ has a removable singular point at z_0.

In particular, one sees that quotients of analytic functions do not have essential singular points.

6.5 Methods for Computing Residues

Theorem 6.9 (Method I)**.** Suppose $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{z - z_0} \text{ near } z_0$$

for some function $\phi(z)$ analytic at z_0. Then

$$\lim_{z = z_0} f(z) = \phi(z_0)$$

Theorem 6.10 (Method II)**.** Suppose $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \text{ near } z_0$$

for some function $\phi(z)$ analytic at z_0 and $m \geq 1$, Then

$$\lim_{z = z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

Theorem 6.11 (Method III)**.** If $p(z)$ and $q(z)$ are analytic at z_0, and $q(z)$ has a simple zero at z_0, then

$$\lim_{z = z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

7 Application of Residues

7.1 Evaluation of Improper Real Integrals

Recall the following definition of improper real integrals:

$$\int_0^\infty f(x) = \lim_{R \to \infty} \int_o^R f(x) \mathrm{d}x$$

also,

$$\int_{-\infty}^\infty f(x) \mathrm{d}x = \lim_{R_1 \to \infty} \int_0^{R_1} f(x) \mathrm{d}x + \lim_{R_2 \to \infty} \int_{-R_2}^0 f(x) \mathrm{d}x$$

There is also Cauchy Principle Value:

$$\text{P.V.} \int_{-\infty}^\infty f(x) \mathrm{d}x = \lim_{R \to \infty} \int_{-R}^R f(x) \mathrm{d}x$$

Note, if the improper integral converges, cauchy principal values also converges, and they equal.

Theorem 7.1. Let $f : \mathbb{R} \to \mathbb{R}$ be an even function

$$f(-x) = f(x) \text{ for all } x \in \mathbb{R}$$

If P.V. $\int_{-\infty}^\infty f(x) \mathrm{d}x$ converges, so does $\int_{-\infty}^\infty f(x) \mathrm{d}x$ and we have

$$\int_0^\infty f(x) \mathrm{d}x = \frac{1}{2} \int_{-\infty}^\infty f(x) \mathrm{d}x$$

and

$$\int_{-\infty}^\infty f(x) \mathrm{d}x = \text{P.V.} \int_{-\infty}^\infty f(x) \mathrm{d}x$$

Theorem 7.2. We describe a procedure to evaluate P.V. $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \mathrm{d}x$ where p, q are polynomials in x such that $\deg q(x) \geq \deg p(x) + 2$:

1. Replace x by z to get $f(z) = \frac{p(z)}{q(z)}$
2. Write P.V. $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\mathrm{d}x$
3. Draw a semi-circle C_R in the upper half plane centered at 0 and with diameter $[-R, R]$, and find singular points.
4. By residue theorem, we have $\int_{-R}^{R} f(x)\mathrm{d}x + \int_{C_R} f(z)\mathrm{d}z = 2\pi i \sum_{z_k} f(z)$
5. Letting $R \to \infty$. Try to show $\lim_{R\to\infty} \int_{C_R} f(z)\mathrm{d}z = 0$ by using ML.
6. Obtain P.V. $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$ by letting $R \to \infty$.

7.2 Integrals of the form $\int_{-\infty}^{\infty} f(x) \cos ax \mathrm{d}x$

We are interested in computing integrals of the form

$$\text{P.V.} \int_{-\infty}^{\infty} f(x) \cos ax \mathrm{d}x \text{ or P.V.} \int_{-\infty}^{\infty} f(x) \sin ax \mathrm{d}x$$

Instead of computing these integrals directly, we often consider

$$\text{P.V.} \int_{-\infty}^{\infty} f(x) e^{iax} \mathrm{d}x$$

whose real and imaginary parts give values of the two improper integrals.

7.3 Definite Integrals Involving Sines and Cosines

We would like to compute integrals of the form

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\mathrm{d}\theta$$

We can make the substitution $z = e^{i\theta}$, so that

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\mathrm{d}\theta = \int_\gamma f(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}) \frac{\mathrm{d}z}{iz}$$

where γ is the positively oriented unit circle $|z| = 1$.