Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) The kernel of an $\mathbb{R}$-linear map $f\colon \mathbb{R}^5 \to \mathbb{R}^5$ with $f^2 = f \circ f = 0$ has dimension at least three.

b) If $\mathbf{P} \in \mathbb{R}^{n\times n}$ affords an orthogonal projection onto a subspace of $\mathbb{R}^n$ then $\mathbf{P}^2 = \mathbf{P}^\mathsf{T} = \mathbf{P}$.

c) If $\mathbf{A} \in \mathbb{R}^{3\times 3}$ satisfies $\mathbf{A}^4 = 2\,\mathbf{A}^2 - \mathbf{I}_3$ then $\mathbf{A}$ is diagonalizable. ($\mathbf{I}_3$ denotes the 3×3 identity matrix.)

d) Every matrix $\mathbf{B} \in \mathbb{R}^{3\times 3}$ has an eigenvector $\mathbf{v} \in \mathbb{R}^3$.

e) If U, V, W are subspaces of $\mathbb{R}^{2022}$ with $U \cap V \cap W = \{0\}$ then one of U, V, W has dimension at most 1011.

f) If $\mathbf{C} \in \mathbb{R}^{3\times 3}$ is skew-symmetric ($\mathbf{C}^\mathsf{T} = -\mathbf{C}$) then $\lambda = 0$ is an eigenvalue of $\mathbf{C}$.

Question 2 (ca. 8 marks)

Depending on $a \in \mathbb{R}$, determine all solutions of the linear system of equations

$$\begin{array}{rcrcrcrcr}
-4\,x_1 & - & x_2 & + & (a+11)\,x_3 & & & = & -7,\\
 & - & x_2 & + & (2a-1)\,x_3 & + & (2-2a)\,x_4 & = & -1,\\
2\,x_1 & + & x_2 & - & (a+5)\,x_3 & + & (a-1)\,x_4 & = & 4,\\
x_1 & - & x_2 & + & (a-4)\,x_3 & + & (3-3a)\,x_4 & = & 0.
\end{array}$$

Question 3 (ca. 8 marks)

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix} \in \mathbb{R}^{4\times 4}.$$

a) Determine the characteristic polynomial and the minimum polynomial of $\mathbf{A}$.

b) Determine all eigenvectors of $\mathbf{A}$.

c) Determine the Jordan canonical form $\mathbf{J}$ of $\mathbf{A}$ and an invertible matrix $\mathbf{S} \in \mathbb{R}^{4\times 4}$ satisfying $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{J}$.

Question 4 (ca. 9 marks)

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

a) Determine the entries of

$$\mathbf{A}^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \qquad n = 0, 1, 2, 3, \dots,$$

i.e. the sequences (a_n), (b_n), (c_n), (d_n), in closed (explicit) form.

b) With $\mathbf{I} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\mathbf{E} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, show that $B = \{\mathbf{I}, \mathbf{A}, \mathbf{E}, \mathbf{AE}\}$ is a basis of $\mathbb{R}^{2\times 2}/\mathbb{R}$.

c) For the (linear) map $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$, $\mathbf{X} \mapsto \mathbf{A}^{-1}\mathbf{X}\mathbf{A}$ compute the representing matrix ${}_B(f)_B$.

Question 5 (ca. 11 marks)

For $a \in \mathbb{R}$ consider the quadric surface Q_a in $\mathbb{R}^3$ with equation

$$x^2 + y^2 + z^2 + (ay - 1)(x + z) = 0.$$

a) Show that all quadrics Q_a are nondegenerate.

b) For which $a \in \mathbb{R}$ is Q_a central? Determine the (affine) type of Q_a in these cases.

c) Determine the type of Q_a in the remaining cases.

Solutions

1 a) True. The rank-nullity formula gives $\dim(\ker f) + \dim(\operatorname{range} f) = 5$. Since $f^2 = 0$, we have $\operatorname{range}(f) \subseteq \ker(f)$ and hence $\dim(\operatorname{range} f) \leq \dim(\ker f)$. Hence $2\dim(\ker f) \geq 5$, which implies the assertion. $\boxed{2}$

b) True. The range U of the corresponding linear map $f_{\mathbf{P}} \colon \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{x} \mapsto \mathbf{Px}$ is the column space of $\mathbf{P}$. For $\mathbf{x} \in \mathbb{R}^n$ the vector $\mathbf{x} - \mathbf{Px}$ must be orthogonal to U by the definition of "orthogonal projection", and hence orthogonal to the columns of $\mathbf{P}$. This means $\mathbf{P}^\mathsf{T}(\mathbf{x} - \mathbf{Px}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$, which in turn is equivalent to $\mathbf{P}^\mathsf{T} = \mathbf{P}^\mathsf{T}\mathbf{P}$. Since $\mathbf{P}^\mathsf{T}\mathbf{P}$ is symmetric, we obtain from this $\mathbf{P}^\mathsf{T} = \mathbf{P}$ and $\mathbf{P} = \mathbf{P}^2$. $\boxed{2}$

c) False. Since $(X-1)^2$ divides $(X-1)^2(X+1)^2 = (X^2-1)^2 = X^4 - 2X^2 + 1$, any matrix $\mathbf{A} \in \mathbb{R}^{3\times 3}$ satisfying $(\mathbf{A} - \mathbf{I})^2 = \mathbf{0}$ has this property. But there are non-diagonalizable matrices among these, e.g., $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $\boxed{2}$

d) True. The characteristic polynomial $\chi_{\mathbf{B}}(X) \in \mathbb{R}[X]$ has degree 3 and hence at least one real root. Thus $\mathbf{B}$ has a real eigenvalue and corresponding real eigenvector. $\boxed{2}$

e) False. We can divide a basis of $\mathbb{R}^{2022}$ into three subsets B_1, B_2, B_3 of size $2022/3 = 674$. Then, setting $U = \langle B_1 \cup B_2 \rangle$, $V = \langle B_1 \cup B_3 \rangle$, $W = \langle B_2 \cup B_3 \rangle$, we have $\dim(U) = \dim(V) = \dim(W) = 2 \cdot 674 = 1348$ and $U \cap V \cap W = \{0\}$, as is easily seen: $U \cap V = \langle B_1 \rangle$ is complementary to W, since $B_1 \cup B_2 \cup B_3$ is a basis of $\mathbb{R}^{2022}$. (Hence the statement is false even if we replace 1011 by 1347. It is not necessary to identify the extreme case, of course. If you don't want to compute 2022/3, you can apply the same reasoning to a subspace of $\mathbb{R}^{2022}$ of dimension 1800, say.) $\boxed{2}$

f) True. That $\lambda = 0$ is an eigenvalue of $\mathbf{C}$ means $\mathbf{C}$ is not invertible. We have $\det(\mathbf{C}) = \det(\mathbf{C}^\mathsf{T}) = \det(-\mathbf{C}) = (-1)^3 \det \mathbf{C} = -\det \mathbf{C}$, and hence $\det(\mathbf{C}) = 0$. Thus $\mathbf{C}$ cannot be invertible. (Knowing the fact that eigenvalues of skew-symmetric matrices must be purely imaginary, also leads to an easy proof: Use d) and the obvious fact that the only complex number which is both real and purely imaginary is zero.) $\boxed{2}$

Remarks: For answers without justification no marks for this question were assigned.

a) Some students argued with a representing matrix $\mathbf{N}$ of f in JCF, which must satisfy $\mathbf{N}^2 = \mathbf{0}$. Thus Jordan boxes can have size at most 2, and hence there must be at least 3 of them, resulting in a (right) kernel of dimension at least 3. Many students asserted without justification that $f^2 = 0$ implies $\operatorname{rank}(f) \leq 2$. This was penalized by 1 mark.

b) For 2 marks a detailed argument like in the sample solution was required, including a proof that $\mathbf{P}^2 = \mathbf{P}$. One can also use the formula $\mathbf{P} = \mathbf{A}\left(\mathbf{A}^\mathsf{T}\mathbf{A}\right)^{-1}\mathbf{A}^\mathsf{T}$ from one of our exercises for the proof.

c) For answers without an explicit counterexample at most 1 mark was assigned. Several students asserted falsely that $\chi_{\mathbf{A}}(X)$ must contain both $X - 1$ and $X + 1$ as factors, or an equivalent statement about the algebraic multiplicities of the eigenvalues. Some students concluded that $\mathbf{A}$ can't be diagonalizable, which is wrong (consider the 3×3 identity matrix). The negation of the statement is "there exists at least one matrix satisfying the equation that is not diagonalizable".

d) For real matrices in general only complex eigenvalues/eigenvectors are guaranteed to exist. In the case under consideration (and, more generally, if the dimension n of the matrix is odd) the answer is True, and the proof ultimately relies on the Intermediate Value Theorem of Calculus.

e) This admittedly difficult question was answered correctly only by a handful of students, mostly arguing with subspaces of dimension 1012 intersecting in 2-dimensional subspaces. Many students didn't understand the logic of the statement. This is an assertion about all triples U, V, W of subspaces of $\mathbb{R}^{2022}$ satisfying $U \cap V \cap W = \{0\}$. To prove it false, you need to exhibit at least one triple satisfying $U \cap V \cap W = \{0\}$ and such that each of U, V, W has dimension ≥ 1012.

f) This subquestion was answered correctly by most students. One can also use the explicit form of a 3×3 skew-symmetric matrix (there are 3 undetermined entries) and compute the determinant directly. However, stating that the determinant is zero without computing it was not accepted (1 mark penalty). A few students used Gaussian elimination to show that $\mathbf{C}$ has rank at most 2, and a few exhibited directly a nonzero vector in the (right) kernel of $\mathbf{C}$.

$$\sum_1 = 12$$

2

$$\left(\begin{array}{cccc|c} 1 & -1 & a-4 & 3-3a & 0 \\ -4 & -1 & a+11 & 0 & -7 \\ 0 & -1 & 2a-1 & 2-2a & -1 \\ 2 & 1 & -a-5 & a-1 & 4 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & a-4 & 3-3a & 0 \\ 0 & -5 & 5a-5 & 12-12a & -7 \\ 0 & -1 & 2a-1 & 2-2a & -1 \\ 0 & 3 & 3-3a & 7a-7 & 4 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & -1 & a-4 & 3-3a & 0 \\ 0 & -1 & 2a-1 & 2-2a & -1 \\ 0 & -5 & 5a-5 & 12-12a & -7 \\ 0 & 3 & 3-3a & 7a-7 & 4 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & a-4 & 3-3a & 0 \\ 0 & -1 & 2a-1 & 2-2a & -1 \\ 0 & 0 & -5a & 2-2a & -2 \\ 0 & 0 & 3a & a-1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & -1 & a-4 & 3-3a & 0 \\ 0 & -1 & 2a-1 & 2-2a & -1 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 3a & a-1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & a-4 & 3-3a & 0 \\ 0 & 1 & 1-2a & 2a-2 & 1 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a-1 & 1 \end{array}\right) \qquad \boxed{3}$$

For $a = 1$ there is no solution, since the last equation then reads $0 = 1$. $\boxed{1}$

For $a = 0$ the variable x_3 is free, and we obtain $x_4 = -1$, $x_2 = 1 - x_3 + 2x_4 = -1 - x_3$, $x_1 = x_2 + 4x_3 - 3x_4 = -1 - x_3 + 4x_3 + 3 = 2 + 3x_3$, and the solution in this case is

$$\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \end{pmatrix} ; x_3 \in \mathbb{R} \right\}. \qquad \boxed{2}$$

For $a \notin \{0, 1\}$ there is a unique solution, viz., $x_4 = \frac{1}{a-1}$, $x_3 = 0$, $x_2 = 1 - (2a-2)x_4 = -1$, $x_1 = x_2 + (3a - 3)x_4 = 2$, or

$$\mathbf{x} = \left(2, -1, 0, \frac{1}{a-1}\right). \qquad \boxed{2}$$

$$\sum_2 = 8$$

3 a) Since $\mathbf{A}$ is a block diagonal matrix, $\chi_{\mathbf{A}}(X)$ is the product of the characteristic polynomials of the two blocks, which are $X^2 + 2X + 1 = (X+1)^2$ (since both blocks have trace -2 and determinant 1.
$\Longrightarrow \chi_{\mathbf{A}}(X) = (X+1)^4$. $\boxed{2}$

$\mu_{\mathbf{A}}(X)$ must be a divisor of $\chi_{\mathbf{A}}(X)$ and hence of the form $(X+1)^k$ with $k \in \{1,2,3,4\}$. Here k is the smallest positive integer satisfying $(\mathbf{A}+\mathbf{I})^k = \mathbf{0}$.

$$\mathbf{A}+\mathbf{I} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix} \neq \mathbf{0},$$

$$(\mathbf{A}+\mathbf{I})^2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix} = \mathbf{0}.$$

$\Longrightarrow \mu_{\mathbf{A}}(X) = (X+1)^2$. $\boxed{2}$

b) By a), the only eigenvalue of $\mathbf{A}$ is $\lambda = -1$. It is obvious that $\mathbf{A}+\mathbf{I}$ has rank 2 and right kernel spanned by $(1,-1,0,0)^\mathsf{T}$, $(0,0,0,1)$. Hence the set of eigenvectors of $\mathbf{A}$ is

$$\bigl\{(a,-a,0,b); a,b \in \mathbb{R},\ \text{not both zero}\bigr\}. \quad \boxed{1}$$

c) By b) the size of the largest Jordan block of $\mathbf{A}$ is 2, and by c) the number of the Jordan blocks, which is equal to the dimension of the single eigenspace of $\mathbf{A}$, is 2 as well. This leaves 2,2 as the only possibility for the sizes of the Jordan blocks, i.e.,

$$\mathbf{J} = \left(\begin{array}{rr|rr} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right). \quad \boxed{1}$$

The single generalized eigenspace of $\mathbf{A}$ is $\mathbb{R}^4$ itself. The vectors $\mathbf{e}_1 = (1,0,0,0)^\mathsf{T}$, $\mathbf{e}_3 = (0,0,0,2)^\mathsf{T}$ are such that $(\mathbf{A}+\mathbf{I})\mathbf{e}_1 = (1,-1,0,2)^\mathsf{T}$, $(\mathbf{A}+\mathbf{I})\mathbf{e}_3 = (0,0,0,2)^\mathsf{T}$ are linearly independent eigenvectors of $\mathbf{A}$. According to the general theory we can then take

$$\mathbf{S} = (\mathbf{e}_1 | (\mathbf{A}+\mathbf{I})\mathbf{e}_1 | \mathbf{e}_3 | (\mathbf{A}+\mathbf{I})\mathbf{e}_3) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}. \quad \boxed{2}$$

$$\sum_3 = 8$$

4 a) This task can be accomplished by diagonalizing $\mathbf{A}$; cf. a previous homework exercise. $\chi_{\mathbf{A}}(X) = X^2 - 2X - 3 = (X-3)(X+1) \implies$ The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 3$, $\lambda_2 = -1$. Corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ are computed as usual:

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}, \quad \implies \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad \boxed{1}$$

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}, \quad \implies \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \boxed{1}$$

are possible choices.

$$\begin{aligned}
\implies \quad & \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}^{-1} \mathbf{A} \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \\
\implies \mathbf{A} &= \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}^{-1} \\
\implies \mathbf{A}^n &= \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}^{-1} \\
&= -\frac{1}{4} \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -1 & 2 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 2 \cdot 3^n & 2(-1)^n \\ 3^n & (-1)^{n+1} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{3^n + (-1)^n}{2} & 3^n + (-1)^{n+1} \\ \frac{3^n + (-1)^{n+1}}{4} & \frac{3^n + (-1)^n}{2} \end{pmatrix}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
a_n = d_n &= \frac{3^n + (-1)^n}{2}, \\
b_n &= 3^n + (-1)^{n+1}, \\
c_n &= \frac{3^n + (-1)^{n+1}}{4}.
\end{aligned} \qquad \boxed{3}$$

(All powers of $\mathbf{A}$ have $a_n = d_n$, $c_n = b_n/4$.)

b) If

$$\begin{aligned}
a\mathbf{I} + b\mathbf{A} + c\mathbf{E} + d\mathbf{A}\mathbf{E} &= a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} a+b+c+d & b \\ b+d & a+b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

then, in order, $b = 0$, $a = d = 0$, $c = 0$. Thus the four matrices are linearly independent and, since $\dim(\mathbb{R}^{2\times 2}) = 4$, form a basis of $\mathbb{R}^{2\times 2}$. $\boxed{1}$

c) We have

$$\begin{aligned}
f(\mathbf{I}) &= \mathbf{I},\\
f(\mathbf{A}) &= \mathbf{A},\\
f(\mathbf{E}) &= -\frac{1}{3}\begin{pmatrix} 1 & -4\\ -1 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} 1 & 4\\ 1 & 1\end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 & -4\\ -1 & 1\end{pmatrix}\begin{pmatrix} 1 & 4\\ 0 & 0\end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 & 4\\ -1 & -4\end{pmatrix}\\
&= -\frac{1}{3}\left(\mathbf{A}+\begin{pmatrix} 0 & 0\\ -2 & -5\end{pmatrix}\right) = -\frac{1}{3}\left(-5\,\mathbf{I}+\mathbf{A}+\begin{pmatrix} 5 & 0\\ -2 & 0\end{pmatrix}\right)\\
&= -\frac{1}{3}\left(-5\,\mathbf{I}+\mathbf{A}+7\,\mathbf{E}-2\,\mathbf{AE}\right),\\
f(\mathbf{AE}) &= \mathbf{EA} = \begin{pmatrix} 1 & 4\\ 0 & 0\end{pmatrix} = \mathbf{A}+\begin{pmatrix} 0 & 0\\ -1 & -1\end{pmatrix} = -\mathbf{I}+\mathbf{A}+\begin{pmatrix} 1 & 0\\ -1 & 0\end{pmatrix}\\
&= -\mathbf{I}+\mathbf{A}+2\,\mathbf{E}-\mathbf{AE}.
\end{aligned}$$

$$\implies {}_B(f)_B = \begin{pmatrix} 1 & 0 & 5/3 & -1\\ 0 & 1 & -1/3 & 1\\ 0 & 0 & -7/3 & 2\\ 0 & 0 & 2/3 & -1\end{pmatrix} \qquad \boxed{3}$$

Remarks: Many students did not remember the corresponding exercise (H41 of Homework 8) and tried to determine the entries of $\mathbf{A}^n$ from recurrence relations. The correct formulae, together with a proof by induction, were honored by full marks (5).

The linear independence of the 4 matrices in b) is not obvious and needs to be justified.

$$\sum_4 = 9$$

5 a) Multiplying the equation by 2 and expanding, we obtain $2x^2+2y^2+2z^2+2axy+2ayz-2x-2z=0$, i.e.,

$$\begin{pmatrix} x\\ y\\ z\end{pmatrix}^{\mathsf{T}} \underbrace{\begin{pmatrix} 2 & a & 0\\ a & 2 & a\\ 0 & a & 2\end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x\\ y\\ z\end{pmatrix} + 2\underbrace{\begin{pmatrix} -1\\ 0\\ -1\end{pmatrix}}_{\mathbf{b}}^{\mathsf{T}} \begin{pmatrix} x\\ y\\ z\end{pmatrix} = 0. \qquad \boxed{1}$$

For testing nondegeneracy we use the test from the lecture (note that $c=0$ in all cases): Since

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{b}\\ \hline \mathbf{b}^{\mathsf{T}} & c\end{array}\right) = \begin{pmatrix} 2 & a & 0 & -1\\ a & 2 & a & 0\\ 0 & a & 2 & -1\\ -1 & 0 & -1 & 0\end{pmatrix} \to \begin{pmatrix} 0 & a & -2 & -1\\ 0 & 2 & 0 & 0\\ 0 & a & 2 & -1\\ -1 & 0 & -1 & 0\end{pmatrix} \to \begin{pmatrix} 0 & 0 & -2 & -1\\ 0 & 2 & 0 & 0\\ 0 & 0 & 2 & -1\\ -1 & 0 & -1 & 0\end{pmatrix}$$

(using elementary row and column operations), the rank of $\begin{pmatrix} \mathbf{A} & \mathbf{b}\\ \mathbf{b}^{\mathsf{T}} & c\end{pmatrix}$ is 4, independently of a. This proves the assertion. $\boxed{1}$

b) Since $\det(\mathbf{A}) = 2^3+0+0-0-2a^2-2a^2 = 4(2-a^2) = 0 \iff a=\pm\sqrt{2}$, the quadric Q_a is central iff $a\notin\{\sqrt{2},-\sqrt{2}\}$. $\boxed{1}$

Assuming this, we first compute the center $\mathbf{v}$ of Q_a, which is the solution of $\mathbf{A}\mathbf{v} = -\mathbf{b} = (1,0,1)^\mathsf{T}$.

$$\left(\begin{array}{ccc|c} 2 & a & 0 & 1 \\ a & 2 & a & 0 \\ 0 & a & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & a & 0 & 1 \\ 0 & 2-a^2/2 & a & -a/2 \\ 0 & a & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & a & 0 & 1 \\ 0 & 2 & 2a & 0 \\ 0 & a & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & a & 0 & 1 \\ 0 & 1 & a & 0 \\ 0 & 0 & 2-a^2 & 1 \end{array}\right)$$

$\Longrightarrow v_3 = \frac{1}{2-a^2}$, $v_2 = -a\,v_3 = -\frac{a}{2-a^2}$, $v_1 = (1-a\,v_2)/2 = \frac{1}{2}\left(1+\frac{a^2}{2-a^2}\right) = \frac{1}{2-a^2}$, so that

$$\mathbf{v} = \frac{1}{2-a^2}\begin{pmatrix} 1 \\ -a \\ 1 \end{pmatrix}. \qquad \boxed{2}$$

Translating the center into the origin gives the new equation $\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} + c' = 0$ with $c' = \mathbf{v}^\mathsf{T}\mathbf{A}\mathbf{v} + 2\mathbf{b}^\mathsf{T}\mathbf{v} = \mathbf{b}^\mathsf{T}\mathbf{v} = -\frac{2}{2-a^2}$. Hence Q_a is equivalent to the quadric with equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}^\mathsf{T} \begin{pmatrix} 2 & a & 0 \\ a & 2 & a \\ 0 & a & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{2}{2-a^2}. \qquad \boxed{1}$$

Next we compute the Sylvester canonical form of $\mathbf{A}$, using elementary row+column operations:

$$\begin{pmatrix} 2 & a & 0 \\ a & 2 & a \\ 0 & a & 2 \end{pmatrix} \to \begin{pmatrix} 2 & a & 0 \\ 0 & 2-a^2/2 & a \\ 0 & a & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2-a^2/2 & a \\ 0 & a & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4-a^2 & 2a \\ 0 & 2a & 4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4-a^2 & 2a \\ 0 & 0 & 4-\frac{(2a)^2}{4-a^2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4-a^2 & 0 \\ 0 & 0 & \frac{16-8a^2}{4-a^2} \end{pmatrix} \qquad (a \neq \pm 2)$$

$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4-a^2 & 0 \\ 0 & 0 & \frac{2-a^2}{4-a^2} \end{pmatrix} =: \mathbf{D} \qquad \boxed{1\tfrac{1}{2}}$$

For $|a| < \sqrt{2}$ all diagonal entries of $\mathbf{D}$ are positive and $\frac{2}{2-a^2} > 0$. Hence Q_a is equivalent to $x^2+y^2+z^2 = 1$, i.e., an ellipsoid. $\boxed{1}$

For $|a| > \sqrt{2}$ one diagonal entry of $\mathbf{D}$ is negative (since the product of the diagonal entries is $2-a^2 < 0$) and $\frac{2}{2-a^2} < 0$. Hence Q_a is equivalent to $x^2+y^2-z^2 = -1$ ($-x^2-y^2+z^2 = 1$), i.e., a hyperboloid of two sheets. $\boxed{1}$
This also holds in the case $a = \pm 2$, in which Q_a is equivalent to $x^2 + 4\,z^2 + 4a\,yz = x^2 + 4z(z \pm 2y) = x'^2 + y'z' = x''^2 + y''^2 - z''^2 = -1$. $\boxed{\tfrac{1}{2}}$

Alternative solution: The number of 1's on the diagonal of the Sylvester canonical form of $\mathbf{A}$ is equal to the number of positive eigenvalues of $\mathbf{A}$. Since $\mathbf{A} - 2\,\mathbf{I} = \left(\begin{smallmatrix} 0 & a & 0 \\ a & 0 & a \\ 0 & a & 0 \end{smallmatrix}\right)$ is singular, one eigenvalue of $\mathbf{A}$ is $\lambda_1 = 2$. The remaining eigenvalues λ_2, λ_3 must satisfy $\lambda_2 + \lambda_3 = \operatorname{tr}(\mathbf{A}) - 2 = 4$, $\lambda_2\lambda_3 = \det(\mathbf{A})/2 = (8 - 4a^2)/2 = 4 - 2a^2$. Thus λ_2, λ_3 are the roots of $X^2 - 4X + 4 - 2a^2$, which are $2 \pm a\sqrt{2}$. One sees that for $|a| < \sqrt{2}$ all eigenvalues are positive and for $|a| > \sqrt{2}$ there is one negative eigenvalue, viz. $2 - |a|\sqrt{2}$. The conclusion is thus the same as above.

c) The computation in b) shows that for $a = \pm\sqrt{2}$ the Sylvester canonical form of $\mathbf{A}$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\Longrightarrow Q_{\sqrt{2}}$ and $Q_{-\sqrt{2}}$ are elliptic paraboloids. $\boxed{1}$

Remarks: In a) many students confused the test for non-degeneracy of a quadric (invertibility of a certain 4×4 matrix constructed from $\mathbf{A}$, $\mathbf{b}$, c) with the test for centrality (invertibility of $\mathbf{A}$). Related with this, several students concluded in c) that Q_a is a cylinder, because $\mathbf{A}$ has the eigenvalue 0. But this cannote be seen from $\mathbf{A}$ alone. Rather it depends on the presence of a particular linear term in the equation, e.g., $x^2 + y^2 = 1$ and $x^2 + y^2 + x = 1$ are (elliptic) cylinders, but $x^2 + y^2 + z = 1$ is an elliptic paraboloid. This subtle difference in linear terms, which makes the first two quadrics degenerate and the third non-degenerate, is detected correctly by the non-degeneracy test.

In b) most students worked with the eigenvalues of $\mathbf{A}$ (like in the *alterative solution*, but again knowing the signs of the eigenvalues is generally not sufficient to determine the type of the quadric. A central quadric needs to be centered first in order to find out the sign of the constant term, e.g., it makes a difference whether the centered from is $x^2 + y^2 - z^2 = 1$ (hyperboloid of one sheet) or $x^2 + y^2 - z^2 = -1$ (equivalent to $-x^2 - y^2 + z^2 = 1$, and hence a hyperboloid of two sheets).

The center of Q_a (in those cases where Q_a is central) can also be determined by first eliminating the linear terms x and z using "completing the square", e.g. $x^2 + (ay - 1)x = \left(x + \frac{ay-1}{2}\right)^2 - \frac{(ay-1)^2}{4}$, then eliminating in the resulting equation the linear term y in the same way, and finally setting the three linear forms whose squares appear in the last equation equal to zero.

$$\sum_{5} = 11$$

$$\sum_{\text{Final Exam}} = 48$$

Math 257 Linear Algebra
Prof. Honold

Midterm Exam 1

Nov 1, 2022
18:00–19:30

Name: ____________ Student ID: ____________ Major: ______

Instructions to candidates: This examination paper contains five (5) questions, of which you need to answer about four (4) for a full score. For your answers please use the space provided after each question. If this space is insufficient, continue on the back side. All answers must be JUSTIFIED. This is a CLOSED BOOK examination, except that you may bring 1 sheet of A4 paper (hand-written only) and a Chinese-English dictionary (paper copy only) to the examination.

Question 1 (ca. 6 marks)

For which $\alpha \in \mathbb{R}$ is the system

$$\begin{array}{rcrcrcr} \alpha\, x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & + & \alpha\, x_2 & + & x_3 & = & -1 \\ x_1 & + & x_2 & + & \alpha x_3 & = & -2 \end{array}$$

solvable, respectively, uniquely solvable? In the first case (i.e., if there is more than one solution) determine the solution in parametric form.

Question 2 (ca. 5 marks)

a) Find the inverse matrix of

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & 1 \\ -2 & 1 & -1 \end{pmatrix}.$$

b) Solve the linear system $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{b} = (2, 0, 1)^\mathsf{T}$.

Question 3 (ca. 5 marks)

a) Compute the determinant of the matrix

$$\begin{pmatrix} 2 & 1 & 1 & 2 \\ -11 & 1 & -6 & -16 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & -2 & 4 \end{pmatrix}$$

.

b) Suppose $\mathbf{A} \in \mathbb{R}^{n \times n}$ satisfies $\mathbf{A}^2 = -\mathrm{I}_n$ (where I_n denotes the $n \times n$ identity matrix). What can you conclude from this about n and $\det(\mathbf{A})$?

Question 4 (ca. 7 marks)

Consider the map $f \colon \mathrm{P}_3(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$f(p) = \begin{pmatrix} p(0) - 2\,p(1) \\ p(2) \end{pmatrix}.$$

a) Show that f is linear.

b) Determine the rank (dimension of the range) and the nullity (dimension of the kernel) of f.

c) Find a basis of the kernel of f.

d) Extend the basis you found in c) to a basis of $\mathrm{P}_3(\mathbb{R})$.

Question 5 (ca. 6 marks)

Suppose V is a vector space over $\mathbb{R}$, $f\colon V \to V$ a linear map, and $U_1, U_2 \subseteq V$ are defined by

$$U_1 = \bigl\{v \in V; f(v) = v\bigr\}, \quad U_2 = \bigl\{v \in V; f(v) = -v\bigr\}.$$

a) Show that U_1 and U_2 are subspaces of V with $U_1 \cap U_2 = \{0\}$.

b) Now suppose in addition that $f^2 = f \circ f = \mathrm{id}_V$ (the identity map on V). Show that in this case $V = U_1 \oplus U_2$.

Hint: Remember how it was proved in the lecture that $\mathbb{R}^{n\times n}$ is the direct sum of $\mathcal{S} = \bigl\{\mathbf{A} \in \mathbb{R}^{n\times n}; \mathbf{A}^\mathsf{T} = \mathbf{A}\bigr\}$ and $\mathcal{T} = \bigl\{\mathbf{A} \in \mathbb{R}^{n\times n}; \mathbf{A}^\mathsf{T} = -\mathbf{A}\bigr\}$.

Solutions

1 First we apply Gaussian elimination, avoiding "dependency on the parameter" for as long as possible:

$$\left[\begin{array}{ccc|c} \alpha & 1 & 1 & 1 \\ 1 & \alpha & 1 & -1 \\ 1 & 1 & \alpha & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \alpha & 1 & -1 \\ \alpha & 1 & 1 & 1 \\ 1 & 1 & \alpha & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \alpha & 1 & -1 \\ 0 & 1-\alpha^2 & 1-\alpha & 1+\alpha \\ 0 & 1-\alpha & \alpha-1 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & \alpha & 1 & -1 \\ 0 & 1-\alpha & \alpha-1 & -1 \\ 0 & 1-\alpha^2 & 1-\alpha & 1+\alpha \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \alpha & 1 & -1 \\ 0 & 1-\alpha & \alpha-1 & -1 \\ 0 & 0 & 2-\alpha-\alpha^2 & 2+2\alpha \end{array}\right] \qquad \boxed{2}$$

Since $2-\alpha-\alpha^2=(1-\alpha)(2+\alpha)$, for $\alpha \notin \{1,-2\}$ the last system is triangular with nonzero elements on the diagonal, and hence uniquely solvable (as is the original system). $\boxed{1}$

The solution in this case is

$$\begin{aligned} x_3 &= \frac{2+2\alpha}{2-\alpha-\alpha^2}, \\ x_2 &= x_3 - \frac{1}{1-\alpha} = \frac{2+2\alpha-(2+\alpha)}{2-\alpha-\alpha^2} = \frac{\alpha}{2-\alpha-\alpha^2}, \\ x_1 &= -1-\alpha x_2 - x_3 = \frac{-2+\alpha+\alpha^2-\alpha^2-2-2\alpha}{2-\alpha-\alpha^2} = \frac{-4-\alpha}{2-\alpha-\alpha^2} \end{aligned} \qquad \boxed{1}$$

For $\alpha=1$ and $\alpha=-2$ the system is unsolvable, since the last equation reads $0=4$, respectively, $0=-2$. $\boxed{1}$

Remarks:

$$\sum_1 = 5$$

---

2 a)

$$\left(\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ -2 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -1 & 1 & 1 & 0 & 1 & 0 \\ 3 & -1 & 1 & 1 & 0 & 0 \\ -2 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 3 & 0 \\ 0 & -1 & -3 & 0 & -2 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -3 & 0 & -2 & 1 \\ 0 & 2 & 4 & 1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -1 & 0 & -2 & 0 & -1 & 1 \\ 0 & -1 & -3 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 & -1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -\frac{3}{2} & -\frac{1}{2} & -2 \\ 0 & 0 & -2 & 1 & -1 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & \frac{3}{2} & \frac{1}{2} & 2 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -1 \end{array}\right) \qquad \boxed{3}$$

It follows that

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ \frac{3}{2} & \frac{1}{2} & 2 \\ -\frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}. \qquad \boxed{1}$$

b) The solution is

$$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}=\begin{pmatrix}1&0&1\\\frac{3}{2}&\frac{1}{2}&2\\-\frac{1}{2}&\frac{1}{2}&-1\end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}3\\5\\-2\end{pmatrix}. \qquad \boxed{1}$$

Remarks:

$$\sum_2 = 5$$

3 a) Probably the most economic way to evaluate this determinant is to eliminate the bottom right entry first, because this produces a row with three zeros and doesn't require a division.

$$\begin{vmatrix}2&1&1&2\\-11&1&-6&-16\\0&-1&-1&2\\0&0&-2&4\end{vmatrix}=\begin{vmatrix}2&1&1&4\\-11&1&-6&-28\\0&-1&-1&0\\0&0&-2&0\end{vmatrix}=-(-2)\begin{vmatrix}2&1&4\\-11&1&-28\\0&-1&0\end{vmatrix} \qquad \boxed{1}$$

$$=2(-1)(-1)\begin{vmatrix}2&4\\-11&-28\end{vmatrix} \qquad \boxed{1}$$

$$=2(-2\cdot 28+11\cdot 4)=-24 \qquad \boxed{1}$$

b) $\mathbf{A}^2=-\mathrm{I}_n \implies \det(\mathbf{A})^2=\det\left(\mathbf{A}^2\right)=\det\left(-\mathrm{I}_n\right)=(-1)^n\det(\mathrm{I}_n)=(-1)^n$. If follows that n must be even and $\det(\mathbf{A})=\pm 1$. $\boxed{2}$

Remarks:

$$\sum_3 = 5$$

4 a) For $p,q\in\mathrm{P}_3(\mathbb{R})$, $c\in\mathbb{R}$ we have

$$\begin{aligned}f(p+q)&=\begin{pmatrix}(p+q)(0)-2(p+q)(1)\\(p+q)(2)\end{pmatrix}=\begin{pmatrix}p(0)+q(0)-2\,p(1)-2\,q(1)\\p(2)+q(2)\end{pmatrix}\\&=f(p)+f(q), \qquad \boxed{1}\\f(cp)&=\begin{pmatrix}(cp)(0)-2(cp)(1)\\(cp)(2)\end{pmatrix}=\begin{pmatrix}c\,p(0)-2c\,p(1)\\c\,p(2)\end{pmatrix}=c\,f(p). \qquad \boxed{1}\end{aligned}$$

Thus f is linear.

b) Since $f(1)=(-1,1)^\mathsf{T}$ and $f(x^2)=(-2,4)^\mathsf{T}$ are linearly independent, the range of f must be $\mathbb{R}^2$, and hence $\mathrm{rk}(f)=2$. $\boxed{1}$

(*Caution:* $f(x)=(-2,2)$ cannot be used in the proof, since it is a multiple of $f(1)$.)

The rank-nullity formula then gives $\dim(\ker f)=\dim\mathrm{P}_3(\mathbb{R})-\mathrm{rk}(f)=4-2=2$. $\boxed{1}$

c) We need to find 2 linearly independent polynomials $p_1,p_2\in\mathrm{P}_3(\mathbb{R})$ with $f(p_1)=f(p_2)=0$. Obviously we can take $p_1(x)=x(x-1)(x-2)$. As 2nd basis vector we can take $p_2(x)=x-2$. (If you have computed $f(x)$ in a), you found that $f(x)=2\,f(1)$ and hence $f(x-2)=0$.) $\boxed{2}$

d) We can take $\{1, x-2, x^2, x(x-1)(x-2)\}$. This follows from the proof of the rank-nullity formula (in the notation used there, $L\langle 1, x^2\rangle$), but also from the observation that the basis polynomials have degrees $0, 1, 2, 3$ and hence must be linearly independent. $\boxed{1}$

Remarks:

$\sum_4 = 7$

5 1. Since $f(0) = 0 = -0$, the zero vector is in both U_1 and U_2. $\boxed{\frac{1}{2}}$

If $f(v_1) = v_1$, $f(v_2) = v_2$ then $f(v_1 + v_2) = f(v_1) + f(v_2) = v_1 + v_2$ and $f(c\,v_1) = c\,f(v_1) = c\,v_1$ for $c \in \mathbb{R}$. Thus U_1 is also closed with respect to vector addition and scalar multiplication, and hence a subspace of V. $\boxed{1}$

Similarly, $f(v_1) = -v_1$, $f(v_2) = -v_2$ implies $f(v_1+v_2) = f(v_1)+f(v_2) = -v_1-v_2 = -(v_1 + v_2)$ and $f(c\,v_1) = c\,f(v_1) = c(-v_1) = -(c\,v_1)$ for $c \in \mathbb{R}$, showing that U_2 is a subspace of V as well. $\boxed{\frac{1}{2}}$

For $v \in U_1 \cap U_2$ we have $v = f(v) = -v$ and hence $2v = 0$. Since $2 \neq 0$ in $\mathbb{R}$, this implies $v = 0$. $\boxed{1}$

2. Since $U_1 \cap U_2 = \{0\}$ was proved in a), we need only show $V = U_1 + U_2$. For $v \in V$ we have

$$v = \underbrace{\frac{1}{2}\big(v + f(v)\big)}_{u_1} + \underbrace{\frac{1}{2}\big(v - f(v)\big)}_{u_2} \qquad \boxed{1}$$

and

$$\begin{aligned} f(u_1) &= f\left(\frac{1}{2}\big(v + f(v)\big)\right) = \frac{1}{2}\big(f(v) + f^2(v)\big) \\ &= \frac{1}{2}\big(f(v) + v\big) && \text{(since } f^2 = \mathrm{id}_V\text{)} \\ &= u_1, && \boxed{1} \\ f(u_2) &= f\left(\frac{1}{2}\big(v - f(v)\big)\right) = \frac{1}{2}\big(f(v) - f^2(v)\big) \\ &= \frac{1}{2}\big(f(v) - v\big) && \text{(since } f^2 = \mathrm{id}_V\text{)} \\ &= -u_2. && \boxed{1} \end{aligned}$$

Thus $u_1 \in U_1$, $u_2 \in U_2$, and $v = u_1 + u_2 \in U_1 + U_2$.

Remarks:

$\sum_5 = 6$

$\sum_{\text{Midterm 1}} = 28 = 24 + 4$

Name: ____________________ Student ID : ____________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. The matrix $\begin{pmatrix} 2 & a & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 1-a \end{pmatrix}$ is diagonalizable for

   ☐ $a=-1$ ■ $a=0$ ☐ $a=1$ ☐ $a=2$ ☐ $a=3$

2. The largest possible rank of $\mathbf{N}\in\mathbb{R}^{5\times 5}$ with $\mathbf{N}^3=\mathbf{0}$ is

   ☐ 1 ☐ 2 ■ 3 ☐ 4 ☐ 5

3. The minimum rank of a real 4×4 matrix with seven 1's and nine -1's is

   ☐ 0 ☐ 1 ■ 2 ☐ 3 ☐ 4

4. The minimum polynomial of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is

   ☐ $(X-1)(X+1)^2$ ☐ $(X-1)^2$ ■ $(X-1)(X+1)$ ☐ $(X-1)^2(X+1)$ ☐ $X+1$

5. The characteristic polynomial of $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 4 & 1 \\ -1 & 4 & 2 \end{pmatrix}$ is

   ☐ X^3+7X^2+9X-1 ☐ X^3-7X^2+9X ■ X^3-7X^2+9X-1
   ☐ X^3+7X^2+9X+1 ☐ X^3-7X^2+9X+1

6. There exists a real matrix with characteristic polynomial X^5-X^3 and minimum polynomial

   ■ X^3-X ☐ X^2-X ☐ X^2-1 ☐ X^3 ☐ X^4-X

7. If $\mathbf{A}\in\mathbb{R}^{3\times 3}$ has the eigenvalue 2 then $\mathbf{A}^2-3\mathbf{A}+2\mathbf{I}$ has the eigenvalue

   ■ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4

8. The number of distinct real eigenvalues of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is equal to

   ☐ 0 ☐ 1 ☐ 2 ■ 3 ☐ 4

9. Suppose U,V are subspaces of $\mathbb{R}^{12}$ with $\dim(U)=5$, $\dim(V)=10$. The smallest possible dimension of $U\cap V$ is

   ☐ 0 ☐ 1 ☐ 2 ■ 3 ☐ 4

10. The matrix $\mathbf{S}^{-1}\begin{pmatrix} 18 & -15 \\ 20 & -17 \end{pmatrix}\mathbf{S}$ is diagonal for

    ☐ $\mathbf{S}=\begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix}$ ■ $\mathbf{S}=\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ ☐ $\mathbf{S}=\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$ ☐ $\mathbf{S}=\begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix}$ ☐ $\mathbf{S}=\begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}$

Time allowed: 45 min CLOSED BOOK ***Good luck!***

Notes

Notes are only provided for Midterm 2-A. Students of Group B should locate their questions in Midterm 2-A. Only Q1 was sligthly different, and the necessary modification in this case you can figure out yourself.

1 The algebraic multiplicity of the eigenvalue $\lambda = 2$ is $m = 2$ for $a \neq -1$ and $m = 3$ for $a = -1$. The matrix $\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} 0 & a & 3 \\ 0 & 0 & 3 \\ 0 & 0 & -1-a \end{pmatrix}$ has rank 1 for $a = 0$ and rank 2 for $a \neq 0$, and hence the eigenspace E_2 has dimenion m iff $a = 0$.

2 Since nilpotent matrices are not invertible, 5 is impossible. The matrix

$$\mathbf{N} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has rank 3 and satisfies $\mathbf{N}^3 = \mathbf{0}$. Adding a '1' in the first upper diagonal changes the nilpotency index to 4, hence rank 4 is impossible as well. (For this you need to know the classification of nilpotent matrices.)

3 The matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

has rank 2. Rank 1 is not possible, since in this case rows would differ only by sign, and hence contain i or $4 - i$ one's with i fixed in advance. But 7 is not a sum of four 4's and 0's or 3's and 1's, or 2's.

4 Reasoning as in Q1, the matrix is diagonalizable, and hence the minimum polynomial must be $\prod_{\lambda \in \Lambda}(X - \lambda)$, where Λ is the set of eigenvalues. Since $\Lambda = \{1, -1\}$, the minimum polynomial is as indicated. Another way to see this is

$$(\mathbf{A} - \mathbf{I})(\mathbf{A} + \mathbf{I}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathbf{A}$ is not equal to $\pm\mathbf{I}$, the minimum polynomial must be $(X - 1)(X + 1)$.

5 The coefficient of X^2 must be -7 (the negative trace of the matrix), and the constant term must be the negative determinant, which (add the 1st to the 3rd row) is $-\left|\begin{smallmatrix} 4 & 1 \\ 3 & 1 \end{smallmatrix}\right| = -1$. This leaves only (E).

6 The minimum polynomial must be a divisor of $X^5 - X^3$ and contain $X, X - 1, X + 1$ (the monic linear factors of $X^5 - X^3$) at least to the first power. This leaves only $X^3 - X$.

7 $\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \Longrightarrow (\mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{I})\mathbf{v} = \mathbf{A}^2\mathbf{v} - 3\mathbf{A}\mathbf{v} + 2\mathbf{v} = \lambda^2\mathbf{v} - 3\lambda\mathbf{v} + 2\mathbf{v} = \left(\lambda^2 - 3\lambda + 2\right)\mathbf{v}$.
For $\lambda = 2$ we have $\lambda^2 - 3\lambda + 2 = 0$.

8 The characteristic polynomial is

$$\begin{vmatrix} X-1 & -1 & -1 \\ -1 & X-1 & 0 \\ -1 & 0 & X \end{vmatrix} = (X-1)^2X - (X-1) - X = X^3 - 2X^2 - X + 1.$$

This polynomial $p(X)$ has 3 distinct real roots, since $p(0) = 1 > 0$, $p(1) = -1 < 0$. (Thus there must be roots x_1, x_2, x_3 with $x_1 < 0 < x_2 < 1 < x_3$.)

9 The dimension formula for subspaces gives $15 = \dim(U) + \dim(V) = \dim(U + V) + \dim(U \cap V)$. Since $\dim(U + V) \leq \dim \mathbb{R}^{12} = 12$, we must have $\dim(U \cap V) \geq 3$. Equality is possible, of course.

10 Since $\mathbf{A} = \left(\begin{smallmatrix} 18 & -15 \\ 20 & -17 \end{smallmatrix}\right)$ constant row sums 3, the vector $(1, 1)^\mathsf{T}$ is an eigenvector of $\mathbf{A}$ (for the eigenvalue 3). This leaves only the possibilities (A), (B). One finds that $(3, 4)^\mathsf{T}$ is an eigenvector of $\mathbf{A}$ (for the eigenvalue -2), so (B) is the correct answer.

Question 1 Depending on the parameter $t \in \mathbb{R}$, determine the solution of

$$\begin{array}{rcrcrcl} 2t \cdot x_1 & - & x_2 & - & x_3 & = & 2t-2 \\ 2x_1 & - & x_2 & - & t \cdot x_3 & = & 0 \\ -4x_1 & + & (3-t)\cdot x_2 & + & (1+t)\cdot x_3 & = & 1-t \end{array}$$

$$\begin{array}{cccc} 2t & -1 & -1 & 2t-2 \\ 2 & -1 & -t & 0 \\ -4 & 3-t & 1+t & 1-t \end{array}$$

$r_1, r_2, r_3 \Rightarrow r_2, r_3, r_1$

$$\Rightarrow \begin{array}{cccc} 2 & -1 & -t & 0 \\ -4 & 3-t & 1+t & 1-t \\ 2t & -1 & -1 & 2t-2 \end{array}$$

$$\Rightarrow \begin{array}{cccc} 2 & -1 & -t & 0 \\ 0 & 1-t & 1-t & 1-t \\ 0 & t-1 & t^2-1 & 2t-2 \end{array}$$

$$\Rightarrow \begin{array}{cccc} 2 & -1 & -t & 0 \\ 0 & 1-t & 1-t & 1-t \\ 0 & 0 & t^2-t & t-1 \end{array}$$

① if $t=0$ $\quad r3 \rightarrow \quad 0=-1$

no solution

② if $t=1$ $\quad 2x_1 - x_2 - x_3 = 0$

$$\Rightarrow x = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \ \lambda_1, \lambda_2 \in R$$

③ if $t \neq 0$ and $t \neq 1$

then $Ax=b$ has only solution

$$x_3 = \frac{1}{t}, \ x_2 = \frac{t-1}{t}, \ x_1 = 1 - \frac{1}{2t}$$

Question 2 a) Compute the determinant of the matrix

$$\begin{pmatrix} 1 & 6 & 0 & -4 \\ -1 & 0 & 1 & 3 \\ 3 & 1 & -2 & 0 \\ -6 & -1 & 1 & 1 \end{pmatrix}$$

b) True or false? $\det(\lambda \mathbf{A}) = \lambda \det(\mathbf{A})$ holds for all matrices $\mathbf{A} \in \mathbb{R}^{4\times 4}$ and $\lambda \in \mathbb{R}$.

a) $\begin{vmatrix} 1 & 6 & 0 & -4 \\ -1 & 0 & 1 & 3 \\ 3 & 1 & -2 & 0 \\ -6 & -1 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 6 & 0 & -4 \\ 0 & 6 & 1 & -1 \\ 0 & -17 & -2 & 12 \\ 0 & 35 & 1 & -23 \end{vmatrix}$

$= - \begin{vmatrix} 1 & 0 & 6 & -4 \\ 0 & 1 & 6 & -1 \\ 0 & -2 & -17 & 12 \\ 0 & 1 & 35 & -23 \end{vmatrix}$

$= - \begin{vmatrix} 1 & 0 & 6 & -4 \\ 0 & 1 & 6 & -1 \\ 0 & 0 & -5 & 10 \\ 0 & 0 & 29 & -22 \end{vmatrix}$

$= - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} -5 & 10 \\ 29 & -22 \end{vmatrix}$

$= -1 \cdot (110 - 290) = 180$

b) false. $\det(\lambda A) = \lambda^n \det(A)$

Question 3 a) Find the inverse matrix of

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ -3 & -4 & -1 \\ 4 & 1 & 2 \end{pmatrix}.$$

b) True or false? Changing the middle entry $a_{22} = -4$ of $\mathbf{A}$ changes the top left entry b_{11} of the inverse $\mathbf{B} = \mathbf{A}^{-1}$.

a) $(A \mid I) \to (I \mid A^{-1})$

$$\left(\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ -3 & -4 & -1 & 0 & 1 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -3 & -4 & -1 & \frac{3}{2} & 1 & 0 \\ 4 & 1 & 2 & -2 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -1 & \frac{1}{2} & \frac{3}{2} & 1 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{2} & -1 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{2} & -1 & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{13}{2} & -3 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{2} & -1 & 0 \\ 0 & 0 & 1 & \frac{13}{3} & 2 & -\frac{2}{3} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & -\frac{7}{3} & -1 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{13}{3} & 2 & -\frac{2}{3} \end{pmatrix}$$

so $A^{-1} = \begin{pmatrix} -\frac{7}{3} & -1 & \frac{2}{3} \\ \frac{2}{3} & 0 & -\frac{1}{3} \\ \frac{13}{3} & 2 & -\frac{2}{3} \end{pmatrix}$

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b) $a_{21}\vec{b_1} + a_{22}\vec{b_2} + a_{23}\vec{b_3} = (0, 1, 0)$

a_{22} change $\to a_{22}\vec{b_2}$ change

$\to a_{21}\vec{b_1}$ change $\to \vec{b_1}$ change

so True

Question 4 Consider the map $f\colon \mathrm{P}_4(\mathbb{R}) \to \mathrm{P}_4(\mathbb{R})$ defined by

$$f(p) = p(1) + p'(1)(x-1).$$

a) Show that f is linear.

b) Determine a basis of the kernel of f.

c) Verify the rank-nullity formula for f and show that $\mathrm{P}_4(\mathbb{R}) = \ker(f) \oplus \operatorname{range}(f)$.

d) True or false? For every linear map $g\colon \mathrm{P}_4(\mathbb{R}) \to \mathrm{P}_4(\mathbb{R})$ we have $\mathrm{P}_4(\mathbb{R}) = \ker(g) \oplus \operatorname{range}(g)$.

Solution

4 a) For $p, q \in \mathrm{P}_4(\mathbb{R})$ and $c \in \mathbb{R}$ we have

$$\begin{aligned}
f(p+q) &= (p+q)(1) + (p+q)'(1)(x-1) = p(1) + q(1) + (p'+q')(1)(x-1) \\
&= p(1) + q(1) + \big(p'(1) + q'(1)\big)(x-1) \\
&= p(1) + p'(1)(x-1) + q(1) + q'(1)(x-1) = f(p) + f(q), \\
f(cp) &= (cp)(1) + (cp)'(1)(x-1) = c\,p(1) + (c\,p')(1)(x-1) = c\,p(1) + c\,p'(1)(x-1) \\
&= c\big(p(1) + p'(1)(x-1)\big) = c\,f(p).
\end{aligned}$$

Thus f is linear.

b) Since 1 and $x-1$ are linearly independent, $f(p) = 0$ is equivalent to $p(1) = p'(1) = 0$. Hence $\ker f$ consists of all polynomials in $\mathrm{P}_4(\mathbb{R})$ that have a double zero at 1, and a suitable basis is $\big\{(x-1)^2, x(x-1)^2, x^2(x-1)^2\big\}$.

c) Since a polynomial of degree ≤ 1 is its own 1st-order Taylor polynomial, the range of f is $\mathrm{P}_1(\mathbb{R})$ and has dimension 2. Thus $\dim(\ker f) + \dim(\operatorname{range} f) = 3 + 2 = 5 = \dim \mathrm{P}_4(\mathbb{R})$, in sync with the rank-nullity formula.

Since $1, x, (x-1)^2, x(x-1)^2, x^2(x-1)^2$ have degrees 0,1,2,3,4, they form a basis of $\mathrm{P}_4(\mathbb{R})$, which shows $\mathrm{P}_4(\mathbb{R}) = \langle 1, x\rangle \oplus \big\langle (x-1)^2, x(x-1)^2, x^2(x-1)^2\big\rangle = \operatorname{range}(f) \oplus \ker(f)$.

Remark: In fact we have $f^2 = f$ ("the Taylor polynomial of a Taylor polynomial is this Taylor polynomial"), so that the direct sum decomposition is an instance of H28 a) on Homework 6.

d) False. For example, we can easily construct a linear map $g\colon \mathrm{P}_4(\mathbb{R}) \to \mathrm{P}_4(\mathbb{R})$ whose range is contained in the kernel by setting $g(1) = g(x) = g(x^2) = 0$, $g(x^3) = 1$, $g(x^4) = x$, and extending linearly to $\mathrm{P}_4(\mathbb{R})$, i.e., $g\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4\right) = a_3 + a_4 x$.