# 1D Systems. Flows in the line

# Session 2.

23/04/2020.

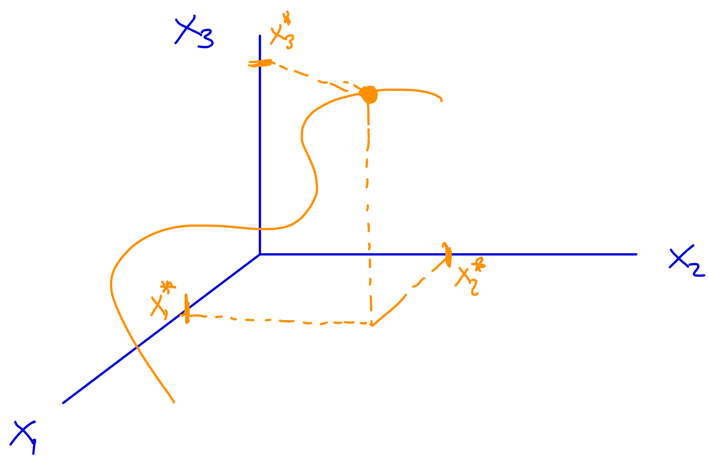
## 1D SYSTEMS. FLOWS ON THE LINE.

In the first session, we saw that a dynamical system can be described, in general, by a set of ODE:

$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2 \dots x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2 \dots x_n \end{aligned}\right\}$$

$n \equiv$ dimension of the system.

We also saw that the solutions of these systems can be represented in a phase space defined by the coordinates $(x_1 \dots x_n)$. For instance, for $n=3$:



This "geometric" way of looking at dynamical systems will allow us to gain a lot of information about the system.

Here $\rightarrow$ $\boxed{n=1}$ ONE-DIMENSIONAL SYSTEMS or FIRST-ORDER SYSTEMS.

The object of our study will be:

$$\boxed{\dot{x} = f(x) \quad , \quad x(0) = x_0}$$

Existence and uniqueness theorem.

If $f(x)$ and $f'(x)$ are continuous on an open interval of the x-axis and $x_0$ is a point in $\mathbb{R}$, then the initial value problem has a solution $x(t)$ on some time interval $(-\tau, \tau)$ about $t=0$ and that solution is unique.

E.g.

$\dot{x} = \sin x$ ✓

$\dot{x} = x^2 + \alpha x^3 - \cos x$ ✓

$\dot{x} = x^{-1/3}$ ✗

$\rightarrow$ Notice that we do not allow a explicit dependence of $f$ on time, $t$. That leads to non-autonomous systems that are much harder to analyze.

# FORMULAS VS PICTURES

## FORMULAS

Let's consider one example:

$$\dot{x} = \sin x$$

which can be solved analytically:

$$\frac{dx}{\sin x} = dt \quad ; \quad \csc(x)\,dx = dt \Rightarrow t = \int \csc(x)\,dx$$

$$t = -\ln\left|\csc(x) + \cot(x)\right| + C$$

and finally

$$\boxed{x(t) = 2\,\mathrm{ArcCot}\left[e^{-t}\cot\left(\frac{x_0}{2}\right)\right]}$$

where $x_0 = x(t=0)$.

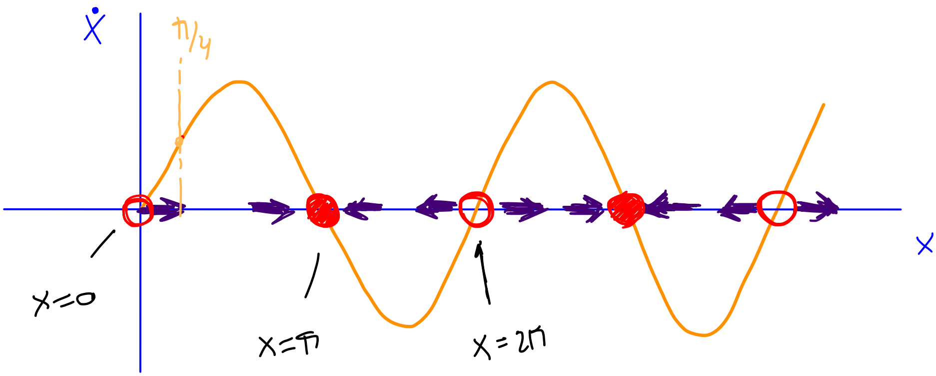
We got a closed expression for $x(t)$ $\forall t$, but

but... HOW USEFUL IT IS ?

a) If $x_0 = \pi/4$ what is the behavior of $x(t)$ $\forall t > 0$. Q1

b) What happens for an arbitrary initial condition in the limit $t \to \infty$?. Q2

# PICTURES




In graphical analysis, $t$ is time, $x$ is the position of an imaginary particle moving along the real line (phase space) and $\dot{x}$ is the velocity of that particle.

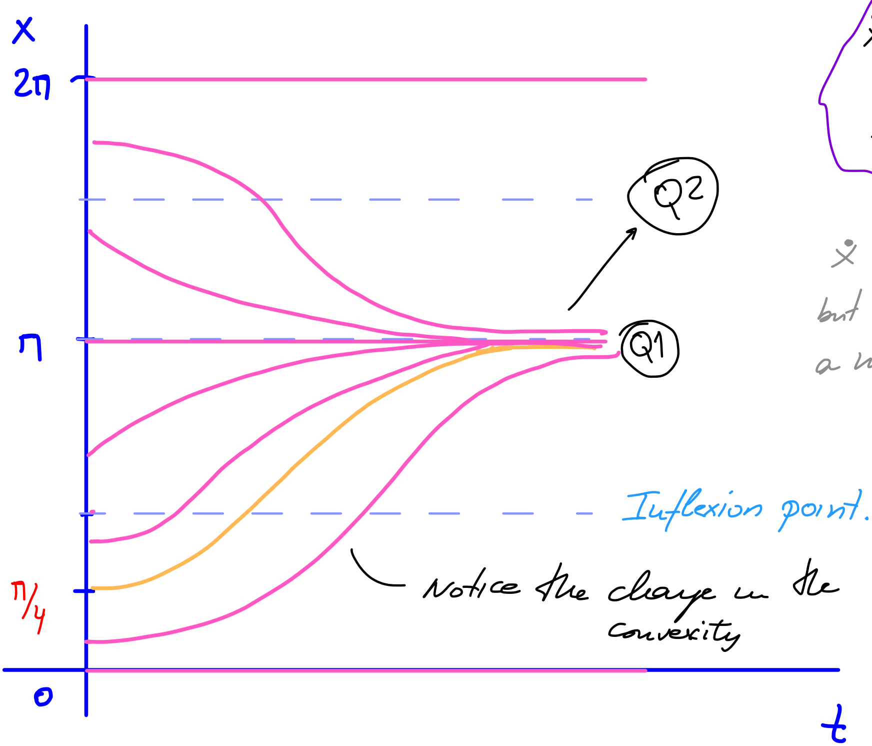
In this representation, the dynamical system $\dot{x} = \sin x$ represents a vector field on the line $x$. Let's sketch and analyse this vector field.

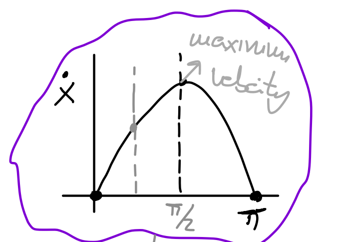
1. Plot $\dot{x}$ vs $x$. (now draw the orange line in the plot).
2. What is the sign of $\dot{x}$? Draw arrows on the $x$ axis. (purple arrows)
3. The points in which there is a change in the velocity $\rightarrow$ $\dot{x} = 0$ and thus FIXED POINTS

④ We identify two different types of fixed points

- In one of them, the flow of the velocity field attracts the particle to the fixed point → → STABLE FIXED POINT / SINK / ATTRACTOR.
- In the other, the flow takes the particle away from the point → UNSTABLE / SOURCE / REPELLER.

So what if we go back to the previous question with this new information?




$\dot{x}$ is always positive but $\ddot{x}$ is not ($\dot{x}$ reaches a maximum at $x = \pi/2$)

But of course the figures provide us with limited information and there are certain things we cannot answer with the diagram. For instance, at which time do we get the maximum $\dot{x}$?

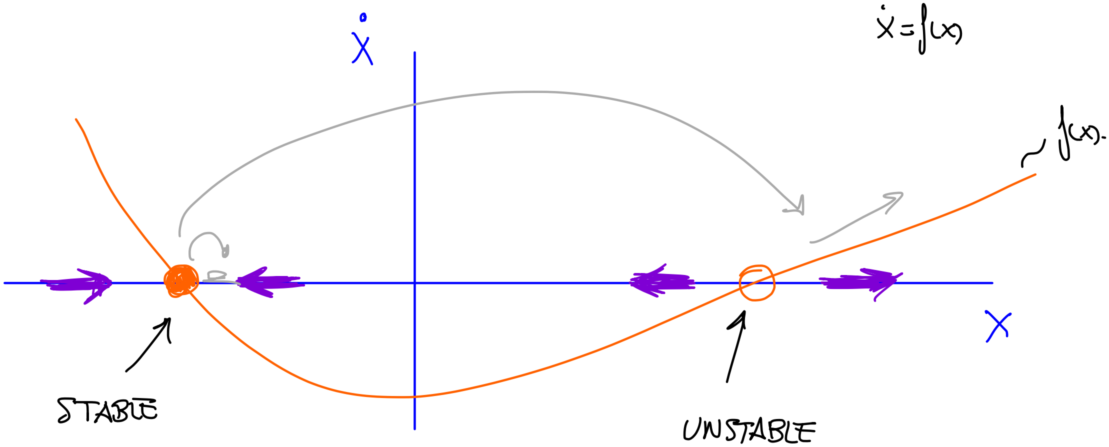
[ But so far, we are going to be happy with the qualitative picture provided by the geometrical analysis.

?

## GENERALIZATION TO AN "ARBITRARY" $f(x)$:

The previous analysis can be generalized to arbitrary systems $\dot{x} = f(x)$ [provided $f(x)$; $f'(x)$ are ✓]

Let's see this general case and introduce some definitions.




•) REAL LINE: phase space.

•) PHASE FLUID: we can think that fluid is flowing steadily along the x-axis with a x-dependent velocity given by $\dot{x} = f(x)$. This imaginary fluid is the phase fluid.

•) The flow goes to the right if $\underline{f(x) > 0}$ and to the left if $\underline{f(x) < 0}$. To sketch a solution we place a "particle" at a starting point $x_0$ and follow its path carried by the flow $\longrightarrow$ TRAJECTORY.

•) The whole diagram is a PHASE PORTRAIT.

•) key $\to$ fixed points $x^* / \underline{\underline{f(x^*) = 0}}$ EQUILIBRIUM POINTS.

If the flow goes towards the point $\Rightarrow$ STABLE
from the point $\Rightarrow$ UNSTABLE

(LOCALLY!! Big perturbations)

EXERCISE

Sketch the phase portrait of $\dot{x} = x - \cos x$ and determine the stability of all fixed points

## LINEAR STABILITY ANALYSIS

Lets be more quantitative or systematic in the determination of the stability of the fixed points:

Consider a dynamical system:

$\dot{x} = f(x)$ with a fixed point at $x^*$

Lets define a perturbation around $x^* \Rightarrow x = x^* + \eta$

Hence $\dot{\eta} = \dot{x}$ and $\dot{\eta} = f(x^* + \eta) = f(x^*) + f'(x^*)\eta + O(\eta^2)$

$$\boxed{\dot{\eta} \simeq f'(x^*)\,\eta}$$

LINEARIZATION OF THE DYNAMICAL SYSTEM AROUND $x^*$

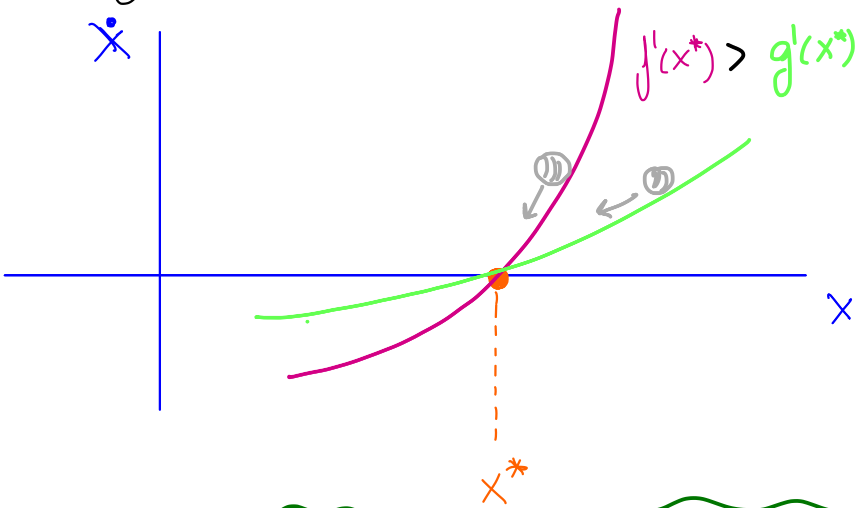
$\longrightarrow$ If $f'(x^*) < 0 \Rightarrow \eta$ DECAYS

If $f'(x^*) > 0 \Rightarrow \eta$ EXPLODES.

$$\eta(t) \propto e^{f'(x^*)t}$$

$|f'(x^*)|^{-1} \Rightarrow$ CHARACTERISTIC TIME SCALE FOR PERTURBATION DYNAMICS.

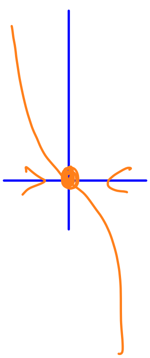
In the graphical representation, and using the example of an imaginary ball, this time scale emerges naturally:



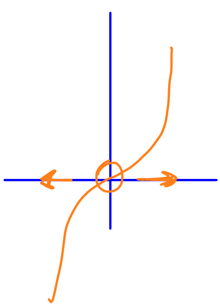

What happens if $f'(x^*) = 0$ ? $\longrightarrow$ NONLINEAR STABILITY ANALYSIS

Case by case

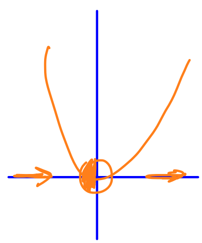
a) $\dot{x} = -x^3$



b) $\dot{x} = x^3$

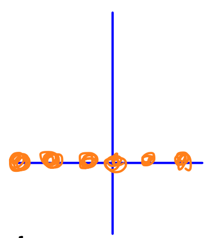


c) $\dot{x} = x^2$



HALF-STABLE

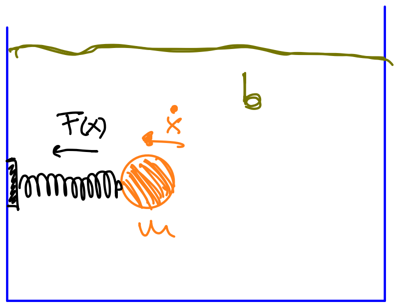
d) $\dot{x} = 0$



LINE OF FIXED POINTS

# MECHANICAL ANALOG.

Consider a particle of mass $m$ attached to a nonlinear spring with restoration force $F(x)$. and immersed in a fluid with a very high viscosity:




$\boxed{\begin{array}{l}\text{Newton's Law:}\\ m\ddot{x} = F(x) - b\dot{x}\end{array}}$

If the damping term is very strong, because $b \gg 1$, then

$b\dot{x} \gg m\ddot{x} \rightarrow$ overdamped system (inertia is negligible)

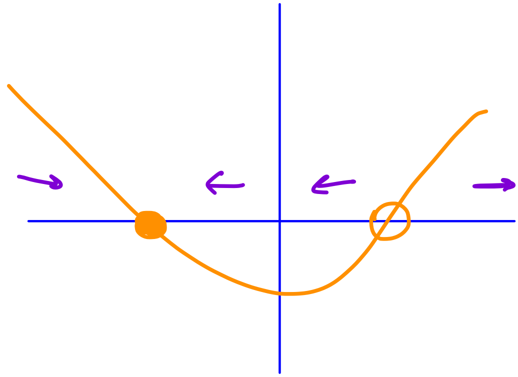
Then: $$b\dot{x} = F(x) \Rightarrow \dot{x} = F(x)/b \equiv f(x).$$

which is analogous to the dynamical systems we have been studying so far.

* From this mechanical analogy, we can obtain some conclusions about the expected solutions.

- No oscillations $\rightarrow$ If the mass is displaced a bit from equilibrium it slowly goes back

This can also be concluded from the phase portrait:



Trajectories in the phase space always end in a fixed point or diverge from it to $\pm\infty$. A phase point does not revert its movement.

* Continuing with this mechanical analog, we can consider that our force $F(x)$ or equivalently $f(x)$ come from the derivative of a potential

$$f(x) = -\frac{dV}{dx}$$

( standard convention → particle tends to minimum of potential or "move downhill")

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = -\left(\frac{dV}{dx}\right)^2 < 0$$

if $\frac{dx}{dt} = -\frac{dV}{dx}$

Hence we need the "−" sign to have "downhill movement".

Example

$$\dot{x} = x - x^3 = x(1-x^2) = -\frac{dV(x)}{dx} \Rightarrow V(x) = \frac{-x^2}{2} + \frac{x^4}{4} + C$$

as always irrelevant