Bifurcations, imperfect bifurcations and catastrophes in 1D systems

Nonlinear dynamical systems and applications

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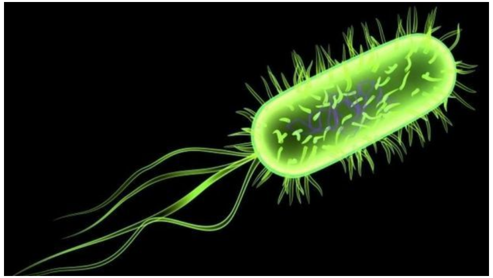
Outline

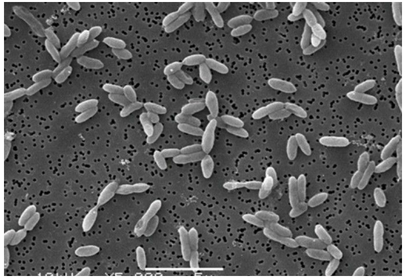
- Review on fixed points and local stability
- What is a bifurcation?
- Kinds of bifurcation:
    - Saddle-node bifurcation
    - Transcritical bifurcation
    - Pitchfork bifurcation
        - Histereses
- Normal modes
- Imperfect bifurcations
- Catastrophes
- Example with insects outbreak
- Take-away messages

Parameters

E. coli population doubles in 20 min



S. fumaroxidans population doubles in 140 h



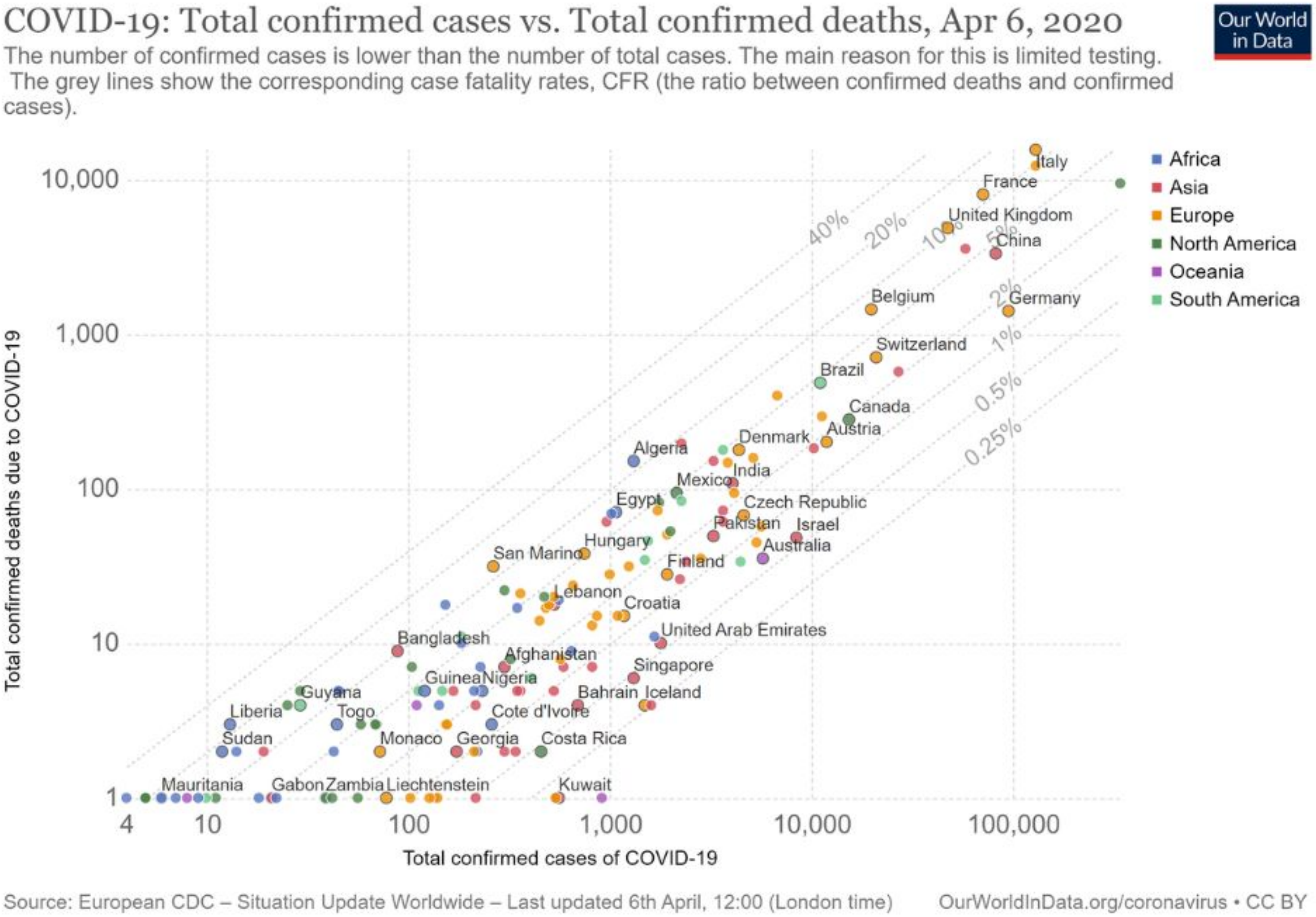
One week after having 1 bacteria: 10^(150) *E.coli* and, 2 or 3 *S. fumaroxidans*

Same process, why different results?

Different parameters!

Universal behaviour → Time scales and Variable scales

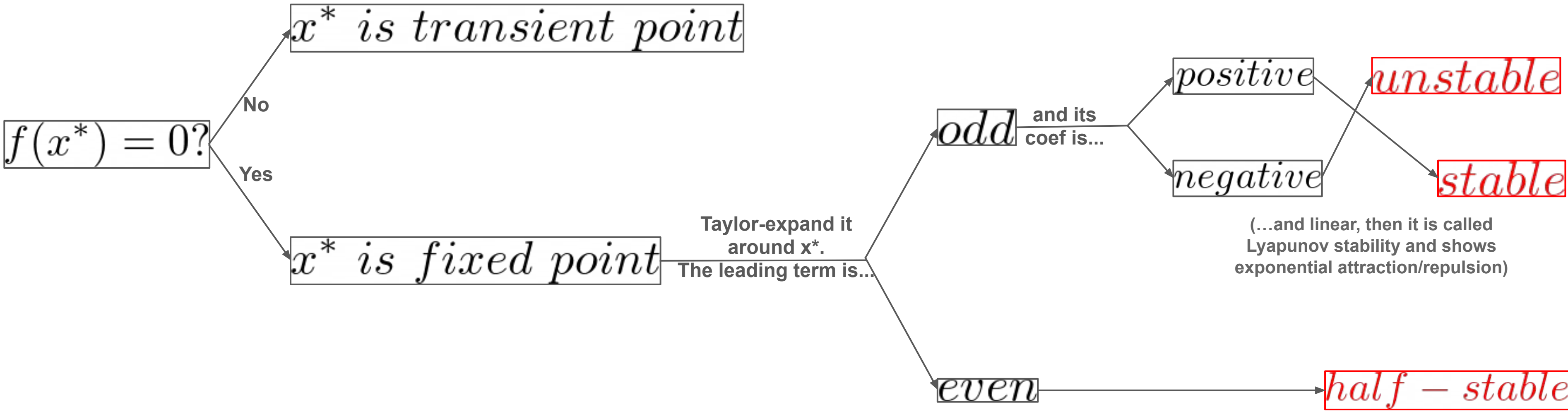
Parameters


COVID-19: Total confirmed cases vs. Total confirmed deaths, Apr 6, 2020
The number of confirmed cases is lower than the number of total cases. The main reason for this is limited testing.
The grey lines show the corresponding case fatality rates, CFR (the ratio between confirmed deaths and confirmed cases).
Our World in Data
Africa
Asia
Europe
North America
Oceania
South America
Total confirmed deaths due to COVID-19
Total confirmed cases of COVID-19
10,000
1,000
100
10
1
4
10
100
1,000
10,000
100,000
40%
20%
10%
5%
2%
1%
0.5%
0.25%
Italy
France
United Kingdom
China
Belgium
Germany
Switzerland
Brazil
Canada
Austria
Denmark
Algeria
India
Mexico
Egypt
Czech Republic
Pakistan
Israel
Australia
Hungary
San Marino
Finland
Lebanon
Croatia
United Arab Emirates
Bangladesh
Afghanistan
Singapore
Guinea
Nigeria
Guyana
Bahrain
Iceland
Liberia
Togo
Cote d'Ivoire
Sudan
Monaco
Georgia
Costa Rica
Mauritania
Gabon
Zambia
Liechtenstein
Kuwait
Source: European CDC – Situation Update Worldwide – Last updated 6th April, 12:00 (London time)
OurWorldInData.org/coronavirus • CC BY

Review on fixed points

$$\dot{x} = f(x; \vec{r})$$

1D autonomous dynamical system with parameters r given.



Bifurcations

**“Given the triviality of the dynamics,
what's interesting about one-dimensional systems?
Answer: Dependence on parameters.”**

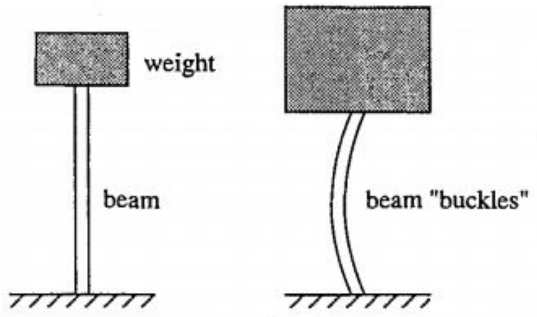


Figure 3.0.1

$$\dot{x} = f(x; \vec{r})$$

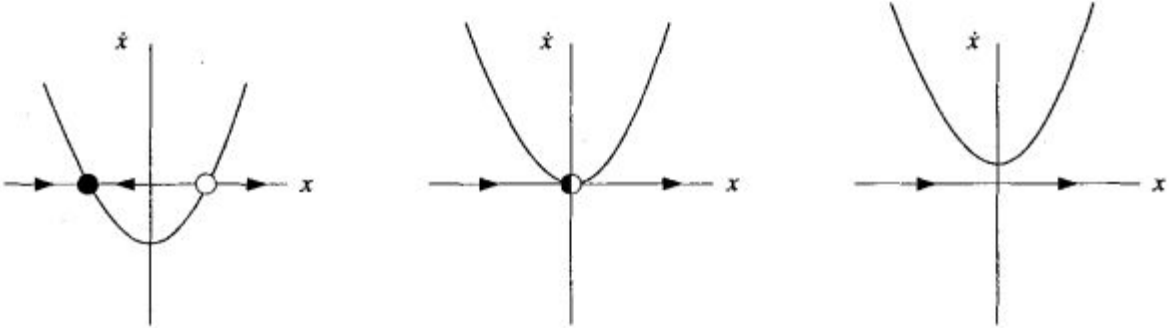
**How does the sets of stable, unstable and half-stable
fixed points (structure) behave wrt changes in the parameter r?**

Saddle-node bifurcations (aka. *fold* or *turning-point* bifurcations)

The saddle-node bifurcation is the basic mechanism by which fixed points are <u>created and destroyed</u>.

$$\dot{x} = f(x; \vec{r})$$

$$\dot{x} = r + x^2$$



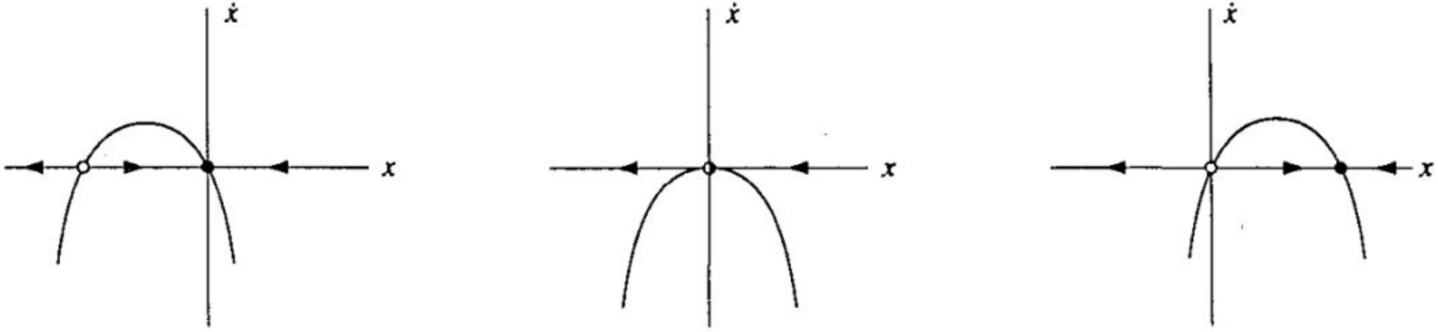
(a) $r<0$ (b) $r=0$ (c) $r>0$

Transcritical bifurcations

The transcritical bifurcation is the basic mechanism by which fixed points are switch stability.

$$\dot{x} = f(x; \vec{r})$$

$$\dot{x} = rx - x^2$$



(a) $r<0$ (b) $r=0$ (c) $r>0$

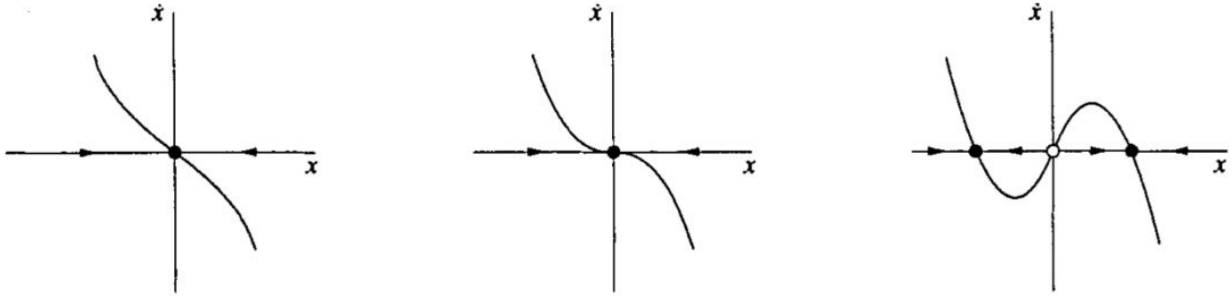
Pitchfork bifurcations (trifurcation)

The pitchfork bifurcation is the basic mechanism by which systems lose symmetry.

$$\dot{x} = f(x; \vec{r})$$

Supercritical

$$\dot{x} = rx - x^3$$



(a) $r < 0$ (b) $r = 0$ (c) $r > 0$

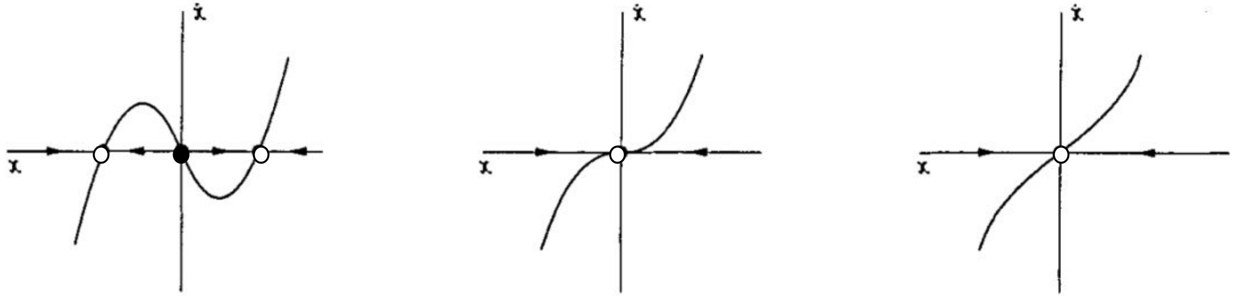
Pitchfork bifurcations

The pitchfork bifurcation is the basic mechanism by which systems lose symmetry.

$$\dot{x} = f(x; \vec{r})$$

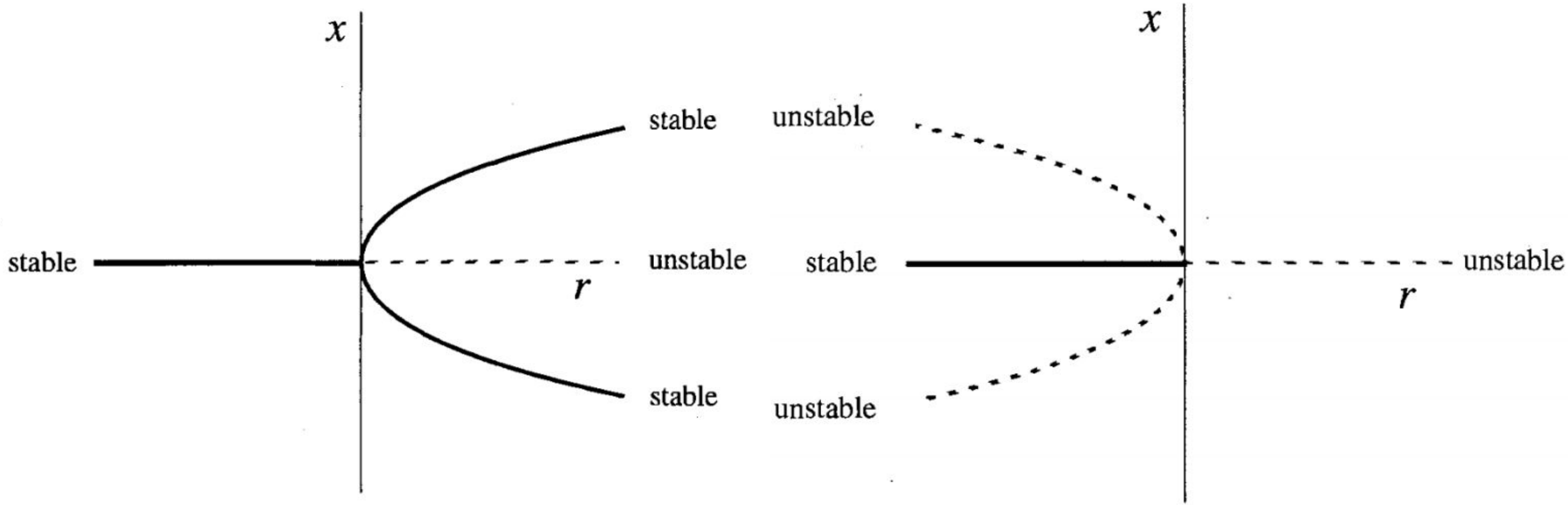
Subcritical

$$\dot{x} = rx + x^3$$



(a) $r < 0$ (b) $r = 0$ (c) $r > 0$

Supercritical vs Subcritical



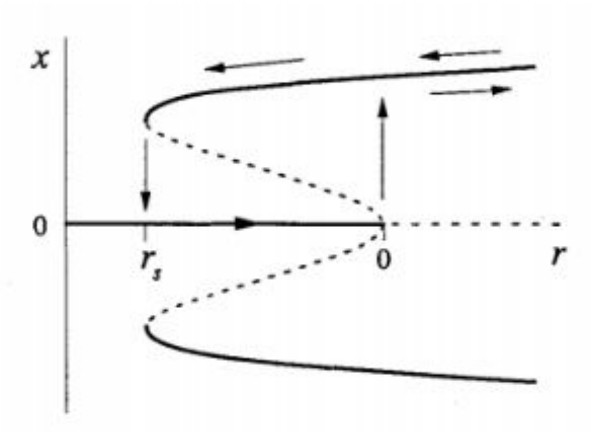
Parity symmetry → Broken symmetry

Bounded symmetry restoration → Broken symmetry

Hysteresis

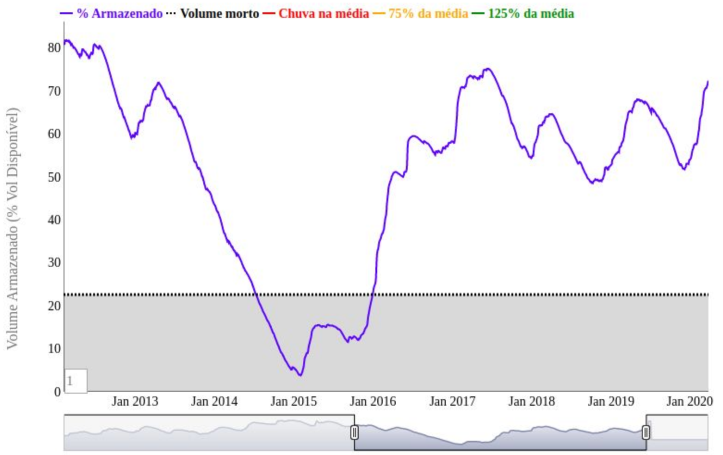
Represents a system with “memory” when parameters are changed.

$$\dot{x} = rx + x^3 - x^5$$



ÁGUAS FUTURAS

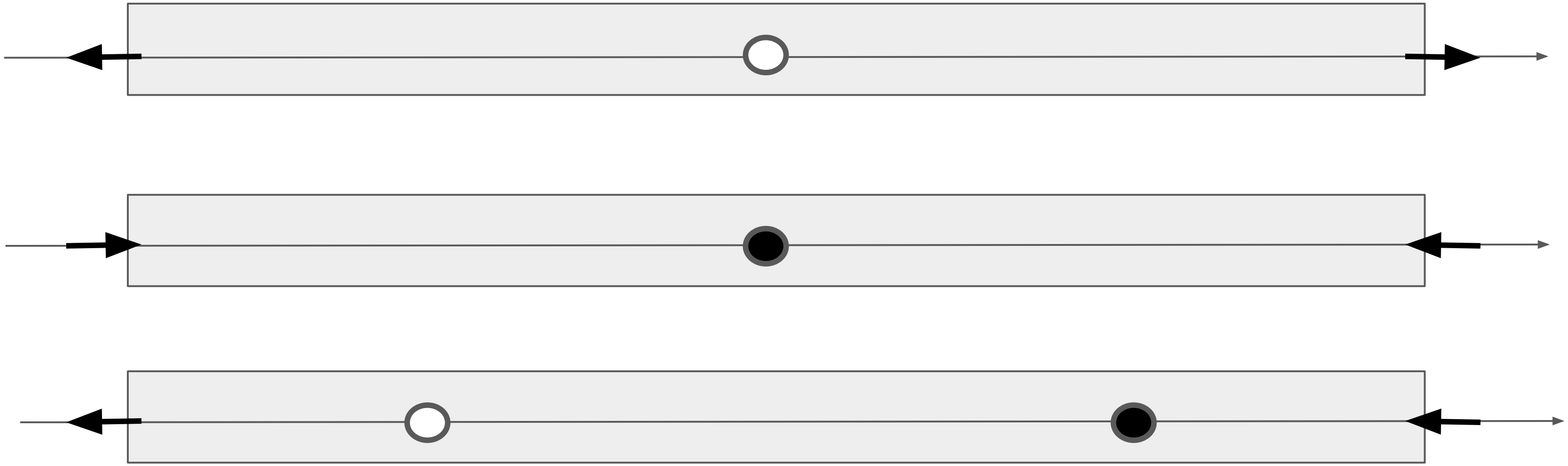
Usando 3 cenários de chuva até 19/05/2020



http://cantareira.github.io/

Why fixed points come in pairs?

Because of conservation of flux at the boundaries of the vector field



Normal forms

What if things are more complicated, such as non-polynomial functions?

$$\dot{x} = f(x, r)$$

$$= f(x^*, r_c) + (x - x^*) \left.\frac{\partial f}{\partial x}\right|_{(x^*, r_c)} + (r - r_c) \left.\frac{\partial f}{\partial r}\right|_{(x^*, r_c)} + \tfrac{1}{2}(x - x^*)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*, r_c)} + \cdots$$

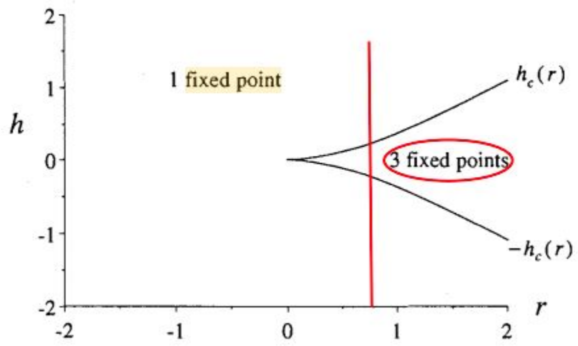
SN Bif. $\dot{x} = a(r - r_c) + b(x - x^*)^2$

TC Bif. $\dot{x} = a(r - r_c)(x - x^*) + b(x - x^*)^2$

PF Bif. $\dot{x} = a(r - r_c)(x - x^*) + b(x - x^*)^3$

Imperfect bifurcations

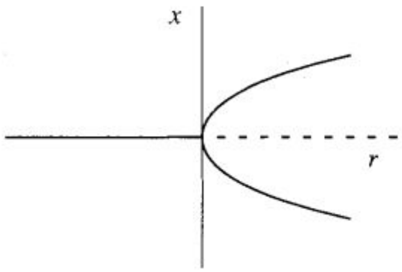
Parameter space of codimension 2



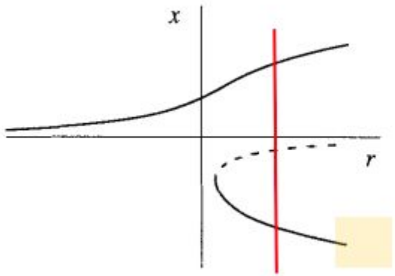
What happens when the **symmetry is only approximate**, i.e., an imperfection leads to a slight differences in parity?

$$\dot{x} = h + rx - x^3$$

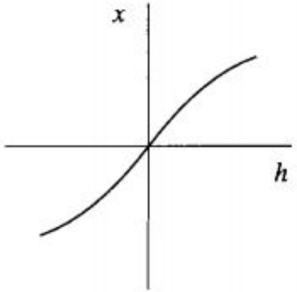
h is imperfection parameter



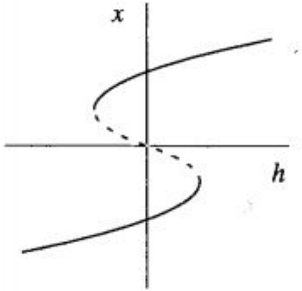
(a) $h = 0$



(b) $h \neq 0$



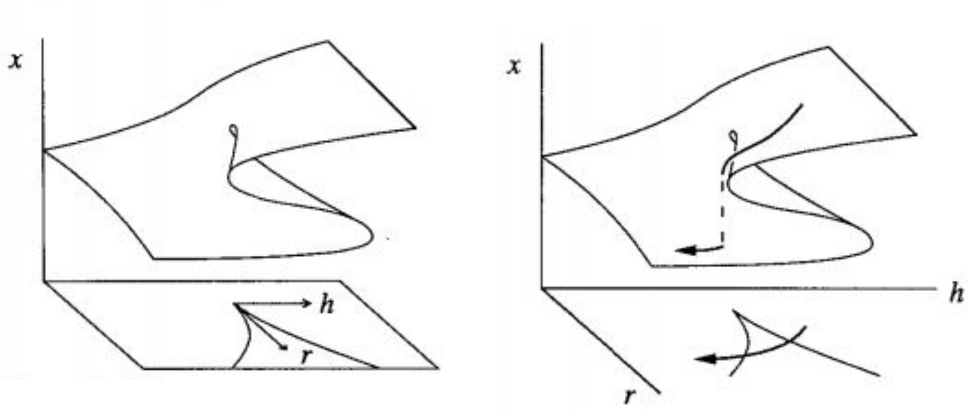
(a) $r \leq 0$



(b) $r > 0$

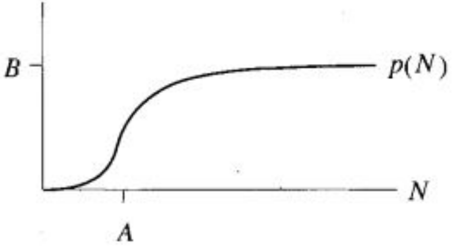
Catastrophes

Continuous changes in parameters result in discontinuous changes in x*.



Example: Outbreak of insects

1) In the absence of birds, budworms grow logistically.
2) Birds satiate.



$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2}$$

Example: Outbreak of insects

Natural units

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2}$$

$$x = N/A$$

$$\tau = \frac{Bt}{A}, \qquad r = \frac{RA}{B}, \qquad k = \frac{K}{A}$$

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1 + x^2}$$

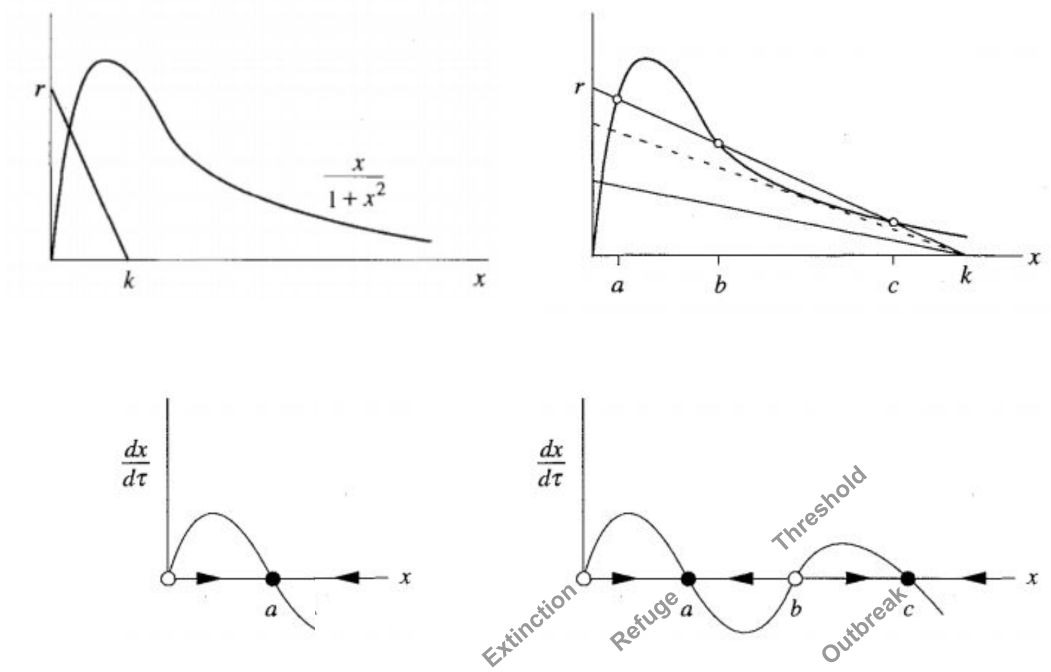
Example: Outbreak of insects

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$$

Analysing the fixed points

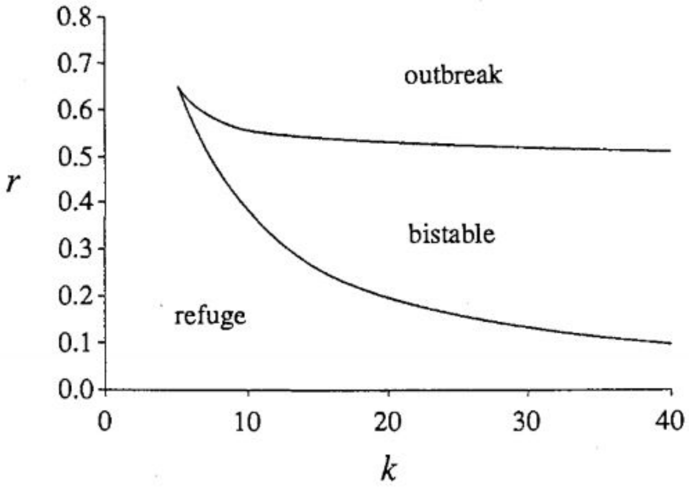
$$x^* = 0$$

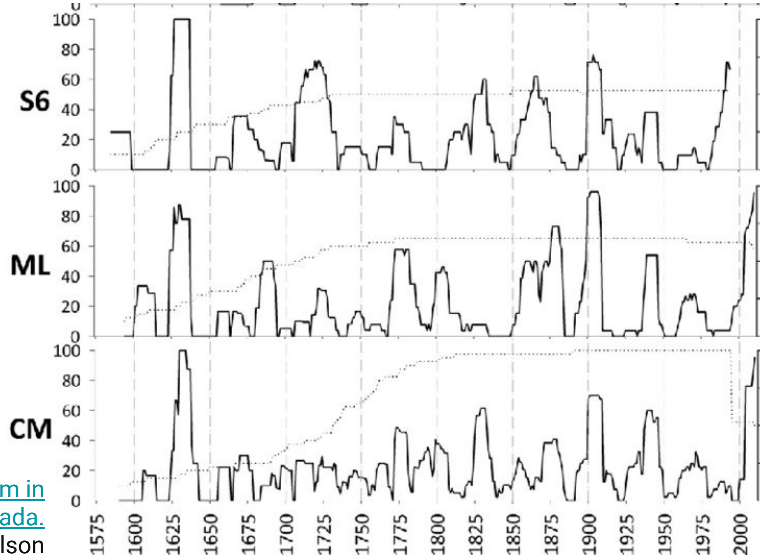
$$r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}$$



Example: Outbreak of insects

The catastrophe:





Periodicity of western spruce budworm in Southern British Columbia, Canada.
Rene I Alfaro, Jenny Berg, Jodi N. Axelson

Take-away messages

- **Fixed points are the long-term behaviour of systems.**
- **Bifurcations are changes in the configuration of FPs wrt parameters.**

- **Kinds of bifurcation:**
    - **Saddle-node bifurcation related to creation/destruction of FPs.**
    - **Transcritical bifurcation related to stability changes of FPs.**
    - **Pitchfork trifurcation related to symmetry breakings of FPs.**

- **Complicated functions can be Taylor expanded to normal modes.**
- **Imperfect bifurcations are parity breaking.**
- **Histereses/Catastrophes are discontinuous observations in continuous parameters displacements.**