Johns Hopkins Engineering

Applied Machine Learning for Mechanical Engineers

Optimization, Part 1, B

JOHNS HOPKINS
WHITING SCHOOL
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Introduction to Optimization Problems

- By the end of this lecture you will be able to:
  - Define an optimization problem.
  - Describe theories and formal mathematical formulations of optimization problems.
  - Describe different types of optimization problems.

Introduction to Optimization Problems

- An optimization **problem** has a problem statement asking to discover the best (optimum) **solution** to achieve an **objective** (purpose) considering every existing limitations.

  - What is the optimum proportion of ingredients to make the most delicious soup considering available ingredients at home?

  - What is the optimum time I need to put every week to achieve all the learning objectives of this course considering my busy schedule as a student?

Introduction to Optimization Problems

- Formal (standard) mathematical definition of an optimization problem:

$$\begin{aligned} &minimize\ f_i(\boldsymbol{x}) \quad i \in \{1, 2, \dots, I\} \\ &subject\ to \begin{cases} g_j(\boldsymbol{x}) \leq 0 & j \in \{1, 2, \dots, J\} \\ h_k(\boldsymbol{x}) = 0 & k \in \{1, 2, \dots, K\} \end{cases} \end{aligned} \qquad (1-1)$$

where $f_i(\boldsymbol{x})\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ is the i^{th} **objective** (fitness or cost) function, $\boldsymbol{x} = [x_1, x_2, \dots, x_N]$ include the optimization variables (**solution**), $g_j(\boldsymbol{x})\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ is the j^{th} inequality constraint, and $h_k(\boldsymbol{x})\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ is the k^{th} equality constraint.

- This is how you describe it: Find optimum $\boldsymbol{x}$ such all $f_i(\boldsymbol{x})$ become minimized while all $g_j(\boldsymbol{x})$ and $h_k(\boldsymbol{x})$ are satisfied.

Introduction to Optimization Problems

- Types of optimization problems
  - **Linear** when all f_i, g_j, and h_k are linear functions otherwise **nonlinear**.
  - **Discrete** when all x_i are integers otherwise **continues**.
  - **Minimization** when all f_i need to be minimized and **maximization** when all f_i needs to be maximized.
  - **Single-objective** when $I = 1$ in Eq. 1-1 otherwise **multi-objective**.
  - **Constrained** when $K \neq 0$ and/or $J \neq 0$ in Eq. 1-1, and **unconstrained** when $K = J = 0$.

Introduction to Optimization Problems

- An optimization problem might have no feasible solution or more than one solution.
- Feasible region is referred to as the hyperspace ($\mathbb{R}^N$) that satisfies all the constrains.

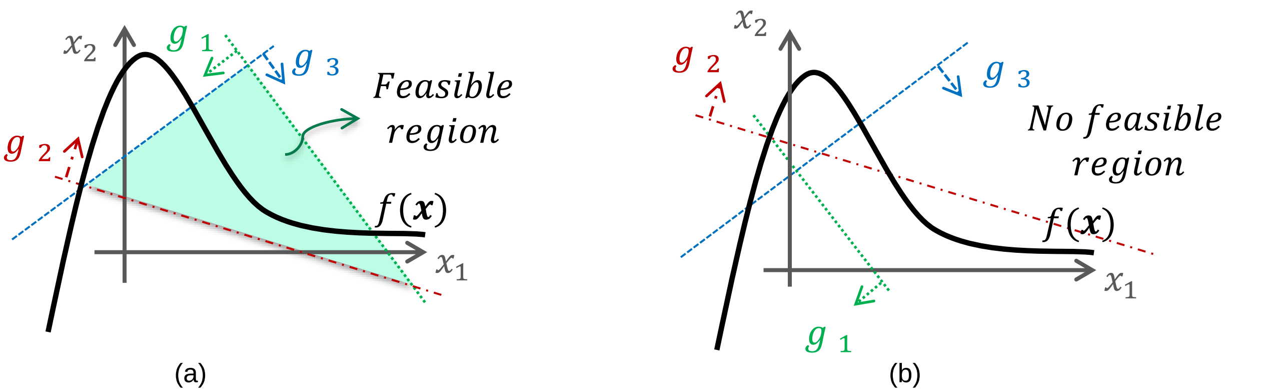

Figure 1-1 An optimization problem (a) with a feasible region of possible solutions, and (b) and without a feasible region

Introduction to Optimization Problems

- In linear optimization problems (also referred to as linear programming), objective functions, f_i, and all constraints, g_j and h_j, are linear functions.

  - Example

$$minimize\ f(\boldsymbol{x}) = x_1 + \frac{x_2}{3} \qquad \boldsymbol{x} = [x_1, x_2]$$

$$subject\ to \begin{cases} g_1(\boldsymbol{x}) = x_1 + x_2 - 2 \leq 0 \\ g_2(\boldsymbol{x}) = x_1 + \frac{x_2}{4} - 1 \leq 0 \\ g_3(\boldsymbol{x}) = x_1 - x_2 - 2 \leq 0 \\ g_4(\boldsymbol{x}) = -\frac{x_1}{4} - x_2 - 1 \leq 0 \\ g_5(\boldsymbol{x}) = -x_1 - x_2 + 1 \leq 0 \end{cases} \quad \begin{cases} g_6(\boldsymbol{x}) = -x_1 + x_2 - 2 \leq 0 \\ g_7(\boldsymbol{x}) = -x_1 - 1 \leq 0 \\ g_8(\boldsymbol{x}) = x_1 - 1.5 \leq 0 \\ g_9(\boldsymbol{x}) = -0.5 - x_2 \leq 0 \\ g_{10}(\boldsymbol{x}) = x_2 - 1.25 \leq 0 \\ h_1(\boldsymbol{x}) = x_1 + \frac{x_2}{4} - 0.5 = 0 \end{cases} \qquad (1-2)$$

https://www.mathworks.com/help/optim/ug/linprog.html

Introduction to Optimization Problems

- In nonlinear optimization problems, at least one of the objective functions, f_i, and/or constraints, g_j and h_j, are nonlinear functions.

  - Example

$$minimize\ f(\boldsymbol{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \qquad \boldsymbol{x} = [x_1, x_2]$$

$$subject\ to \begin{cases} g_1(\boldsymbol{x}) = -x_1 \leq 0 \\ g_2(\boldsymbol{x}) = -x_2 \leq 0 \\ g_3(\boldsymbol{x}) = x_1^2 - 1 \leq 0 \\ g_4(\boldsymbol{x}) = x_2 - 2 \leq 0 \end{cases} \qquad (1-3)$$

https://www.mathworks.com/help/optim/ug/fmincon.html#busowg2-1

Introduction to Optimization Problems

- In this lecture, you learned about:

  - Optimization problems in general and different types of optimization problems.

  - Theories and formal mathematical formulations of optimization problems.

  - How to make the least unhealthy and least distasteful soup using ingredients available at home!

- In the next lecture, we will talk about optimization algorithms.

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