CSE 2813
Discrete Structures

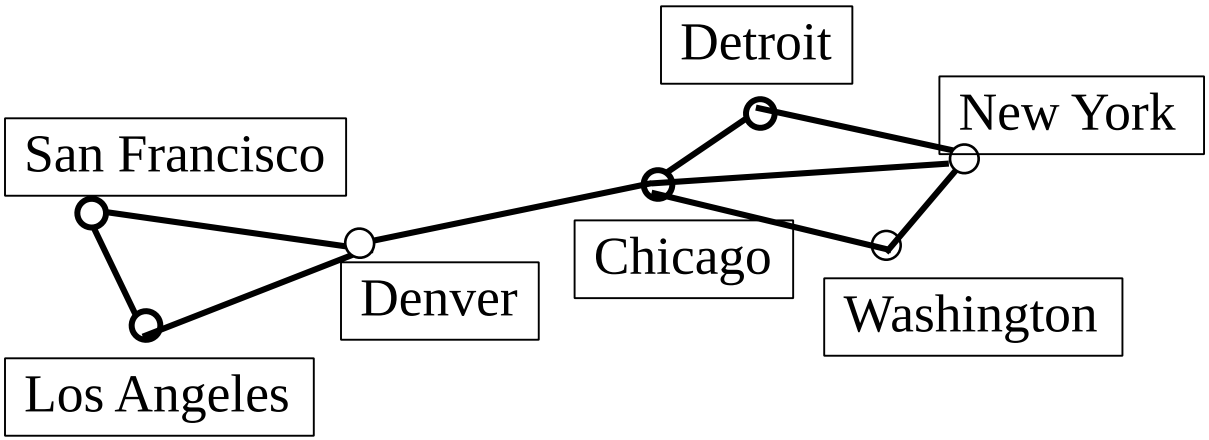
Chapter 4, Section 9.1
Introduction to Graphs

These class notes are based on material from our textbook, **Discrete Mathematics and Its Applications**, 6^{th} ed., by Kenneth H. Rosen, published by McGraw Hill, Boston, MA, 2006. They are intended for classroom use only and are **not** a substitute for reading the textbook.

Simple Graph

- A *simple graph* consists of
  - a nonempty set of *vertices* called V
  - a set of edges (unordered pairs of distinct elements of V) called E
- Notation: $G = (V,E)$

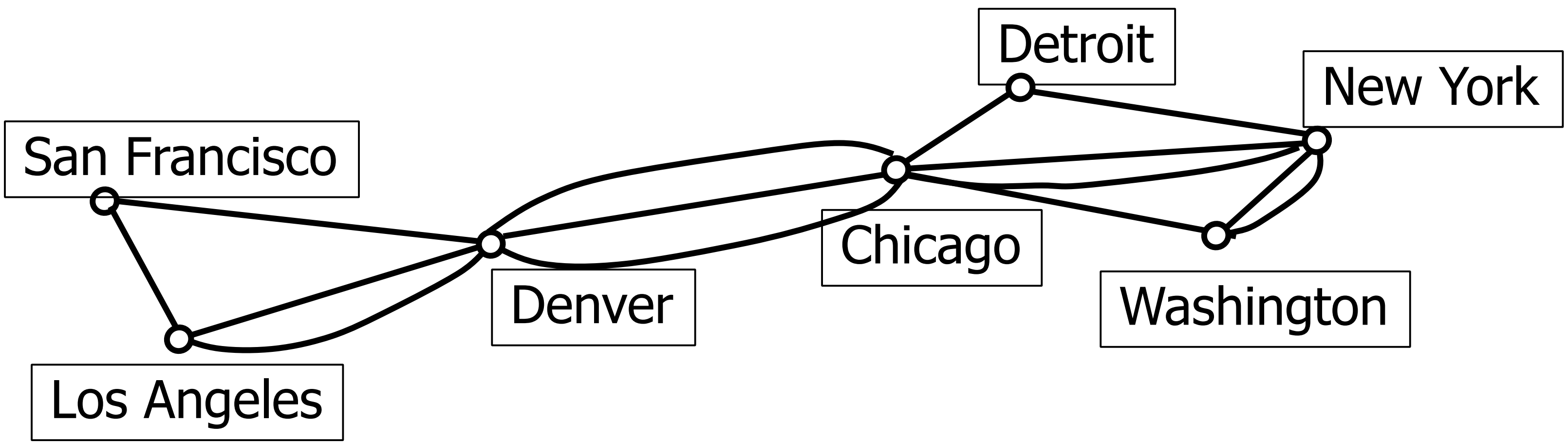
Simple Graph Example



- This simple graph represents a network.
- The network is made up of computers and telephone links between computers

Multigraph

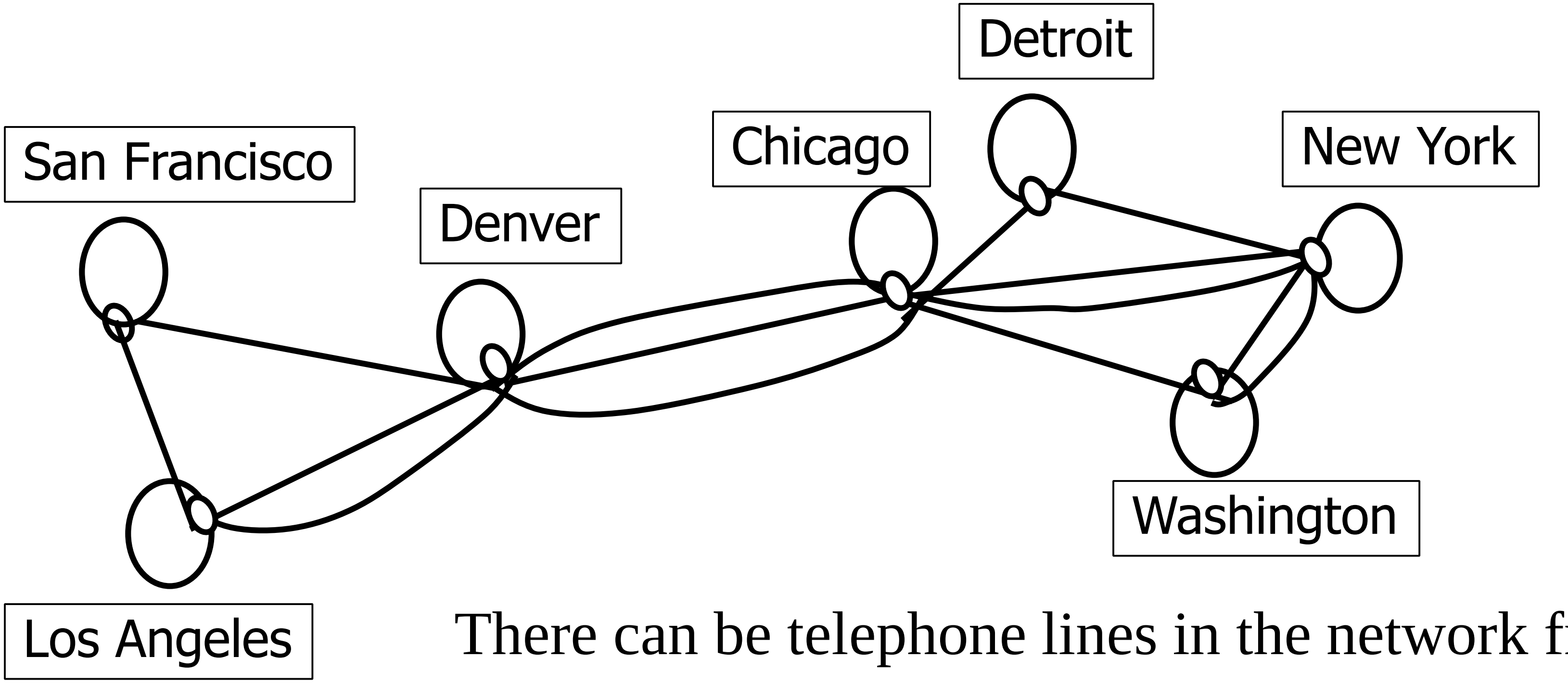
- A *multigraph* can have *multiple edges* (two or more edges connecting the same pair of vertices).



There can be multiple telephone lines between two computers in the network.

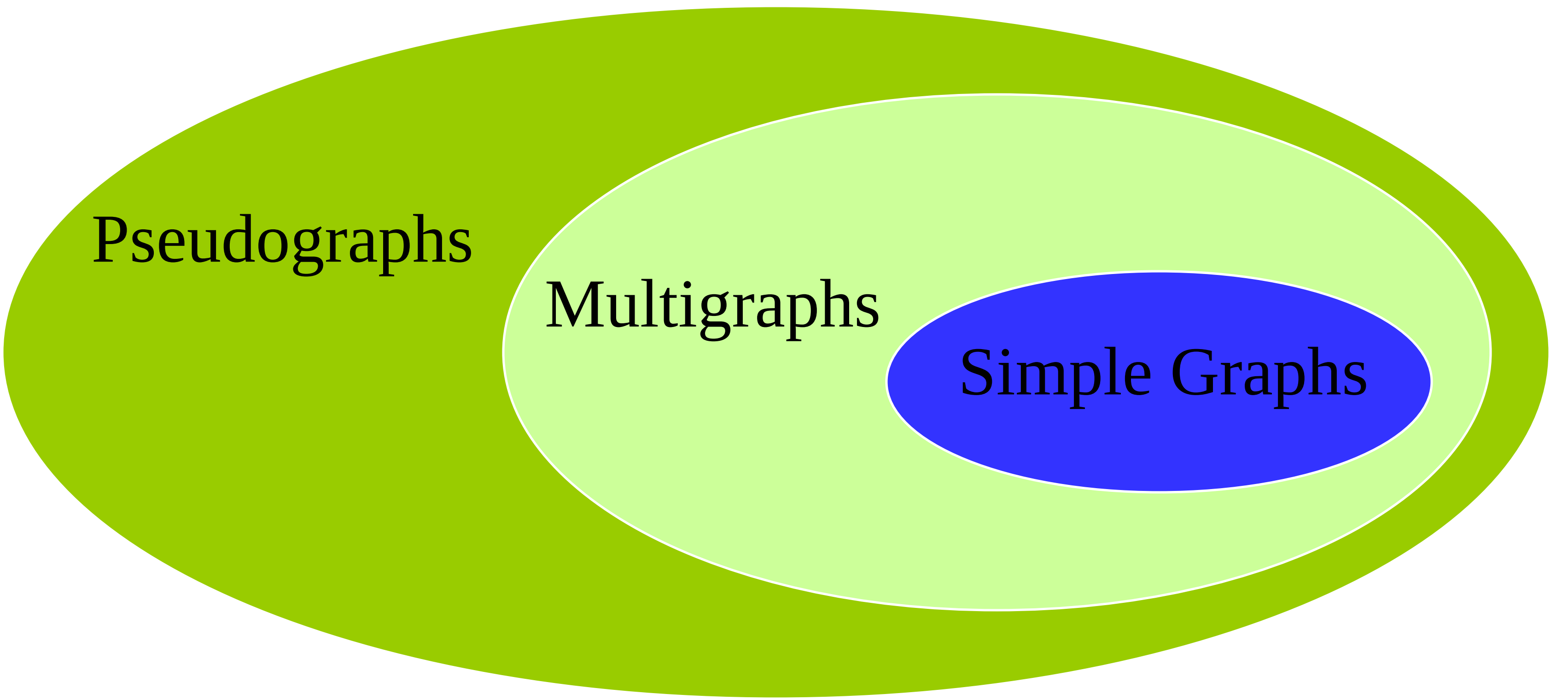
Pseudograph

- A *Pseudograph* can have multiple edges and *loops* (an edge connecting a vertex to itself).



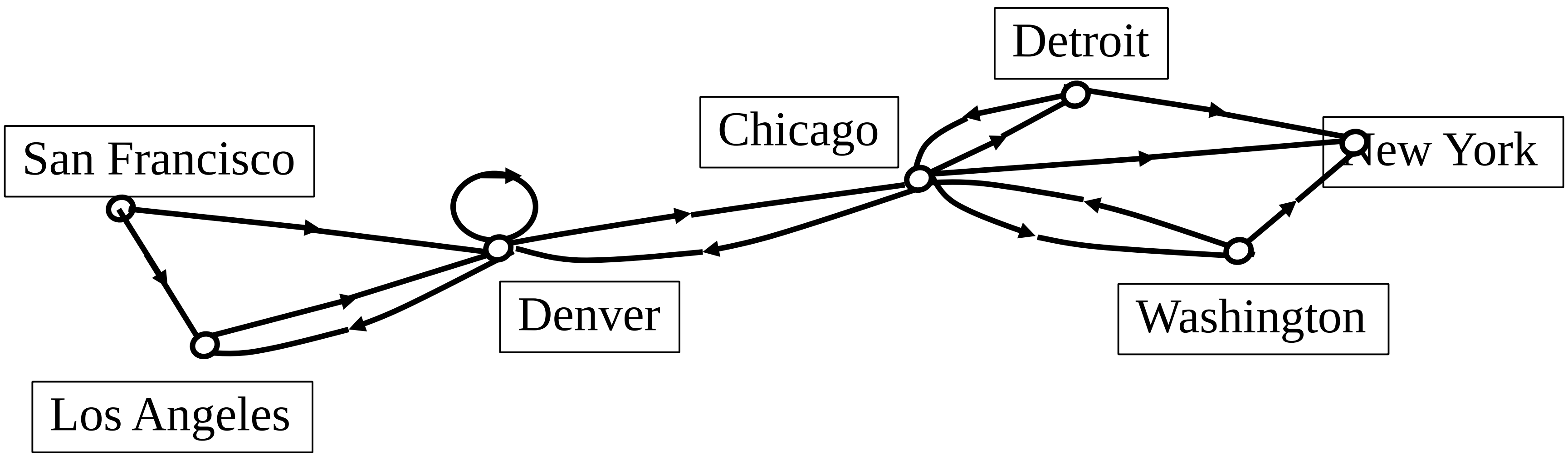
There can be telephone lines in the network from a computer to itself.

Types of Undirected Graphs



Directed Graph

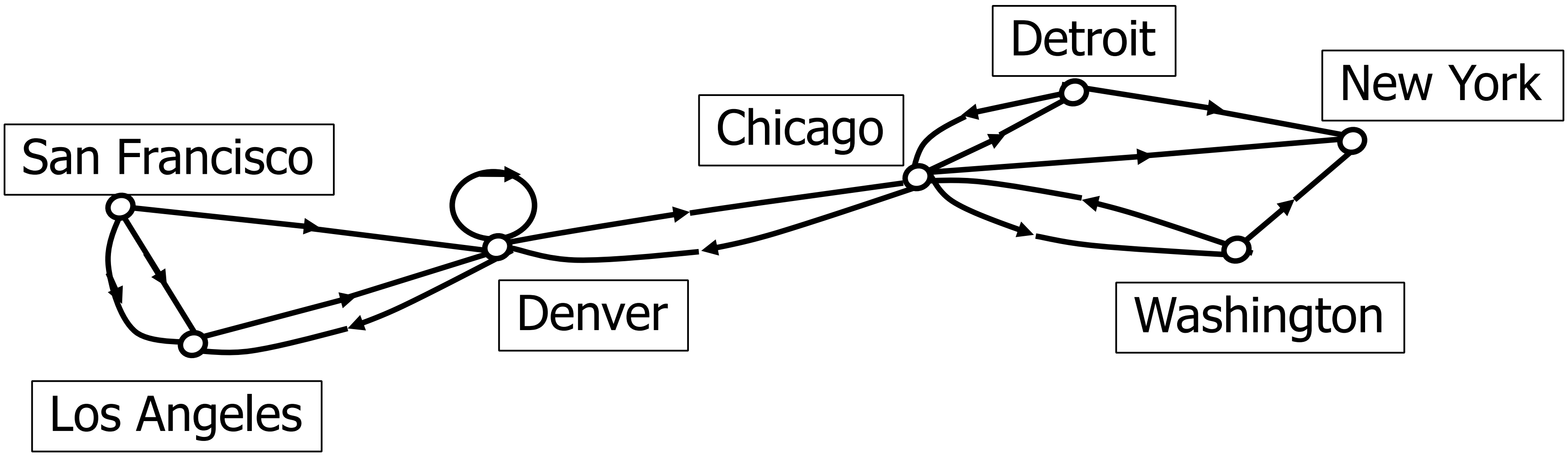
The edges are ordered pairs of (not necessarily distinct) vertices.



Some telephone lines in the network may operate in only one direction. Those that operate in two directions are represented by pairs of edges in opposite directions.

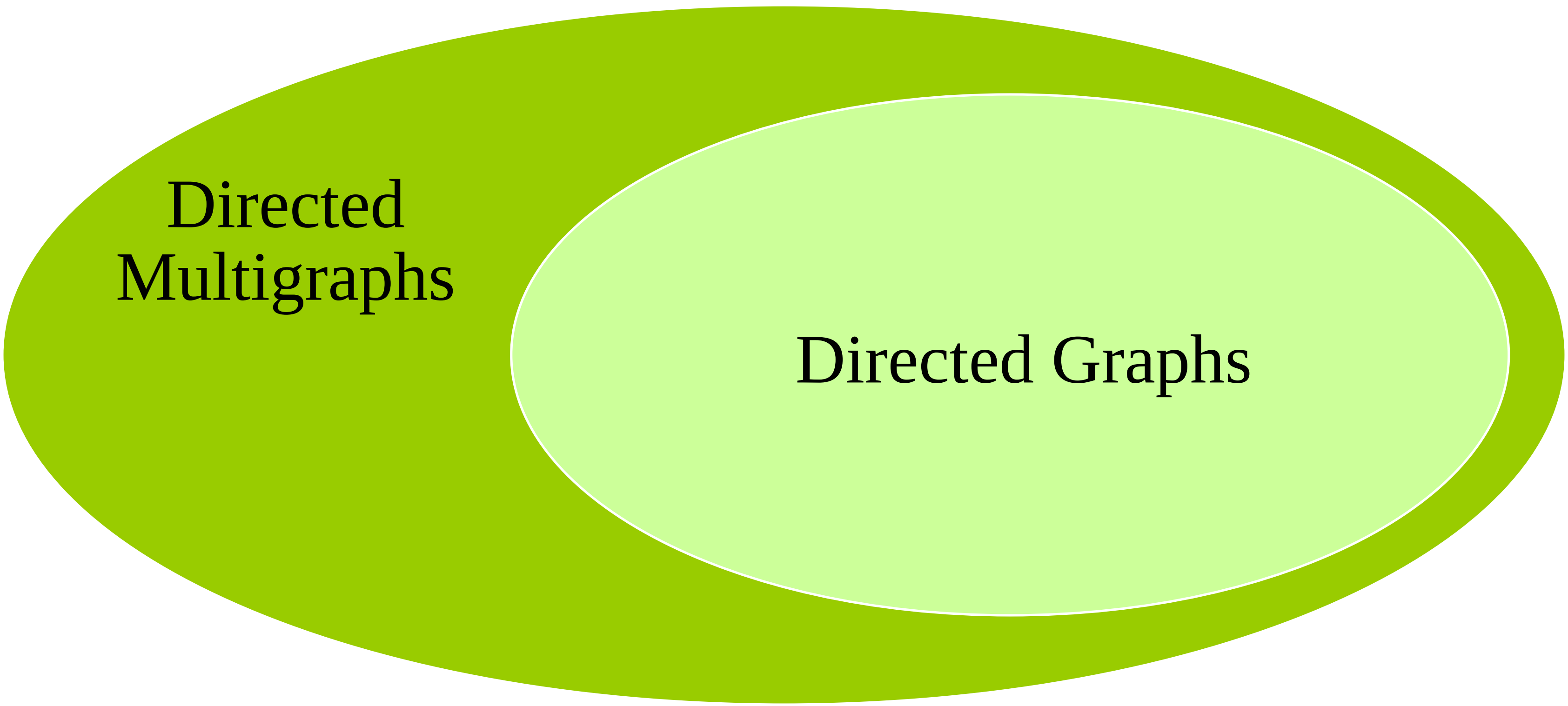
Directed Multigraph

A directed multigraph is a directed graph with multiple edges between the same two distinct vertices.



There may be several one-way lines in the same direction from one computer to another in the network.

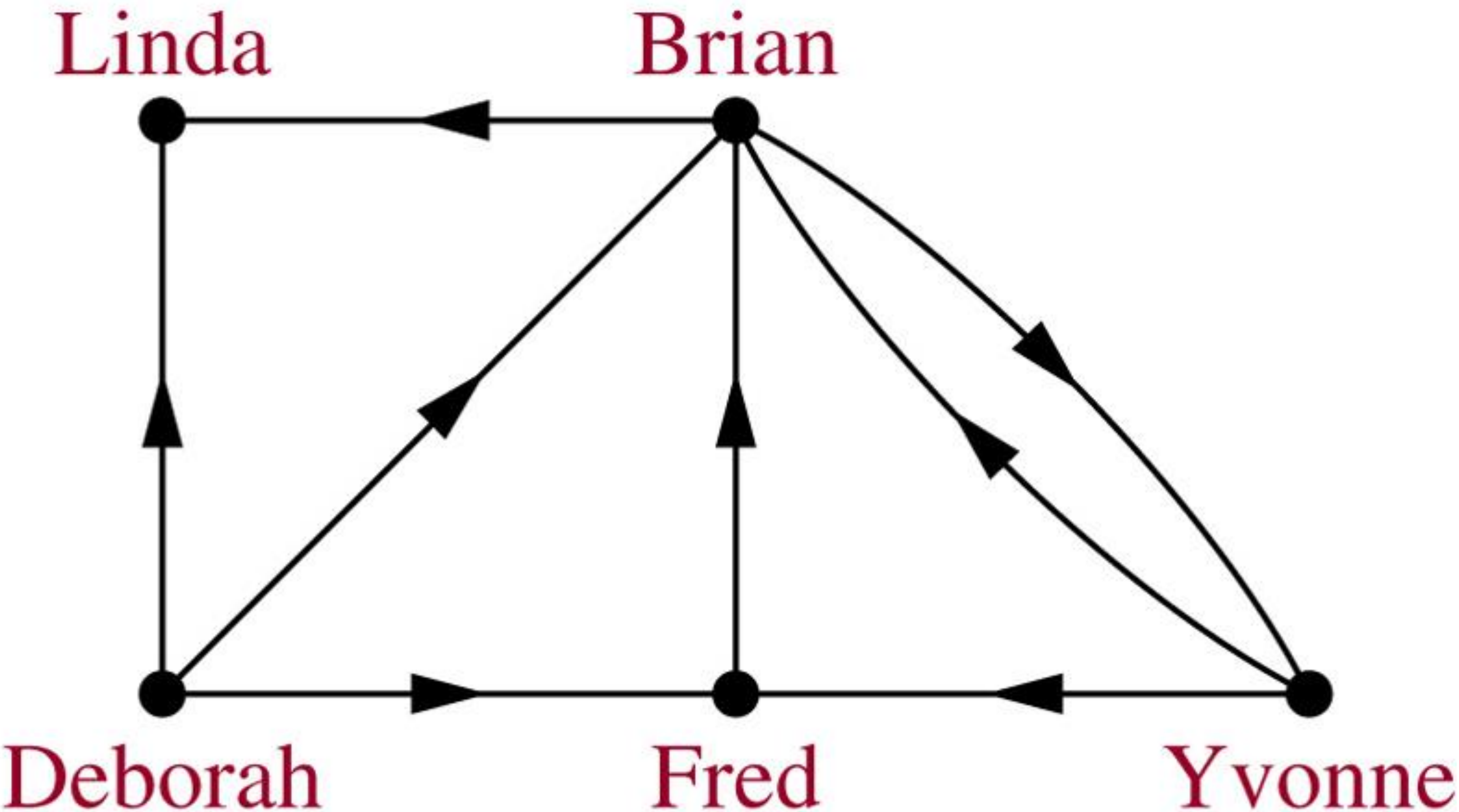
Types of Directed Graphs



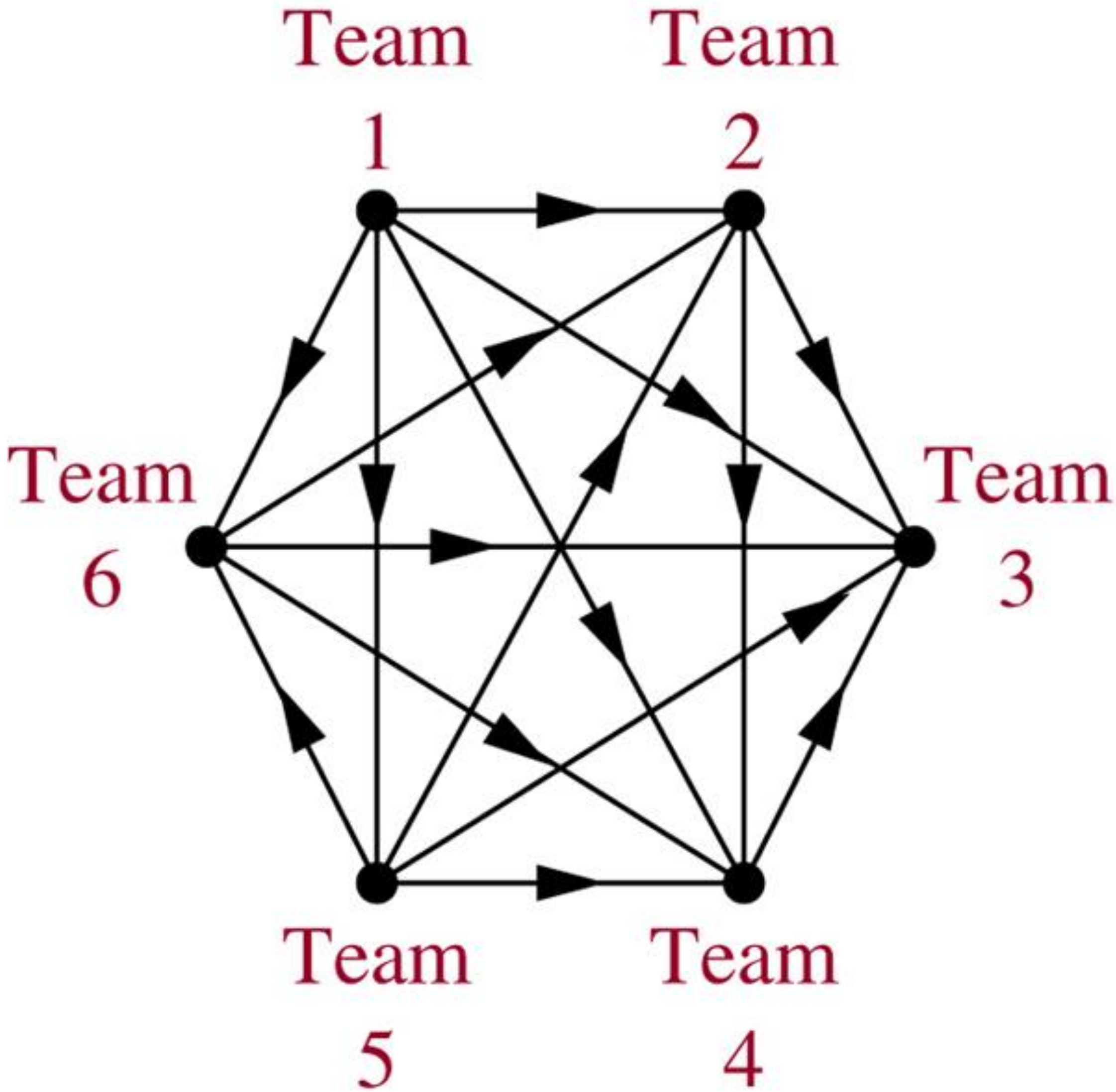
Graph Models

- Graphs can be used to model structures, sequences, and other relationships.
- Example: ecological niche overlay graph
  - Species are represented by vertices
  - If two species compete for food, they are connected by a vertex

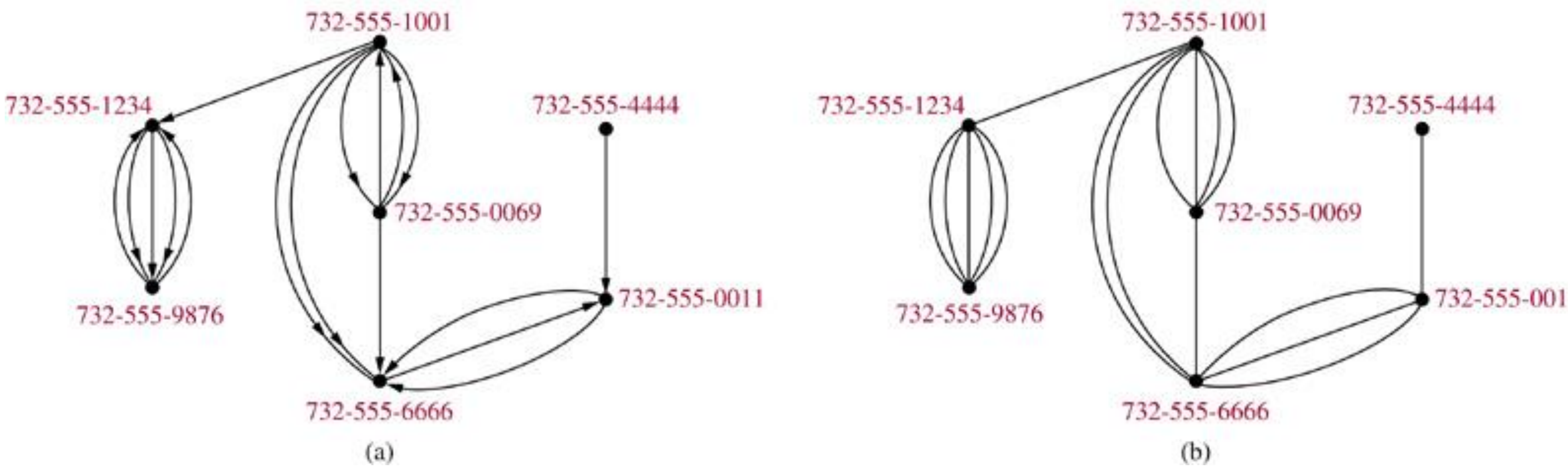
Influence Graph



Round-Robin Tournament Graph



Call Graphs



Directed graph (a) represents calls *from* a telephone number *to* another.

Undirected graph (b) represents called *between* two numbers.

Precedence Graphs

S_1 $a := 0$

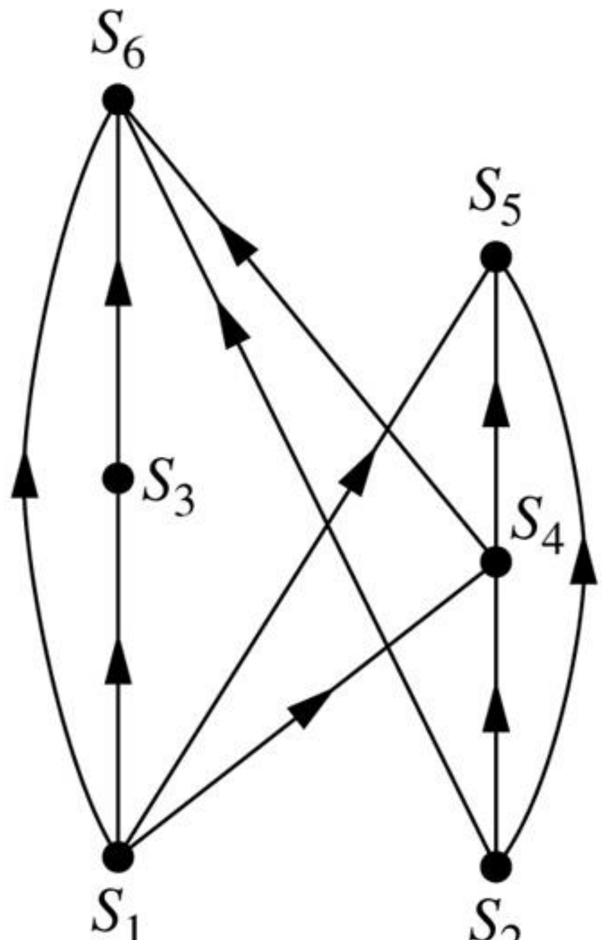
S_2 $b := 1$

S_3 $c := a + 1$

S_4 $d := b + a$

S_5 $e := d + 1$

S_6 $e := c + d$



In concurrent processing, some statements must be executed before other statements. A precedence graph represents these relationships.

Hollywood Graph

- In the Hollywood graph:
  - Vertices represent actors
  - Edges represent the fact that the two actors have worked together on some movie
- As of October 2007, this graph had 893,283 vertices, and over 20 million edges.

Shortest-path Algorithms

- A decade or so ago a game called "Six Degrees of Kevin Bacon" was popular on college campuses.
- The idea, based on the idea that “it’s a small world”, was to try to find the fewest number of connections to link any other actor with Kevin Bacon.
- It was discovered that you could connect Kevin Bacon with just about any other actor in 6 links or so.

Bacon Numbers

- In the Hollywood Graph, the Bacon Number of an actor *x* is defined as the <u>length of the shortest path</u> connecting *x* and the actor Kevin Bacon.
- The average Kevin Bacon number is 2.957
- For more information, see the Oracle of Bacon website at the University of Virginia Computer Science Department.

Bacon Numbers

Kevin Bacon Number	**Number of People**
0	1 (Kevin Bacon himself)
1	2030
2	190213
3	557245
4	133450
5	9232
6	958
7	137
8	17

Shortest-path Algorithms

- Writing an efficient program for finding the shortest path in a graph is an important optimizing task in Computer Science.
- For futher information about shortest-path algorithms, take CSE 4833, or read the course textbook, *Introduction to Algorithms* by Cormen, Leiserson, Rivest, and Stein.

Summary

Type	**Edges**	**Loops**	**Multiple Edges**
Simple Graph	Undirected	NO	NO
Multigraph	Undirected	NO	YES
Pseudograph	Undirected	YES	YES
Directed Graph	Directed	YES	NO

CSE 2813
Discrete Structures

Chapter 9.2
Graph Terminology

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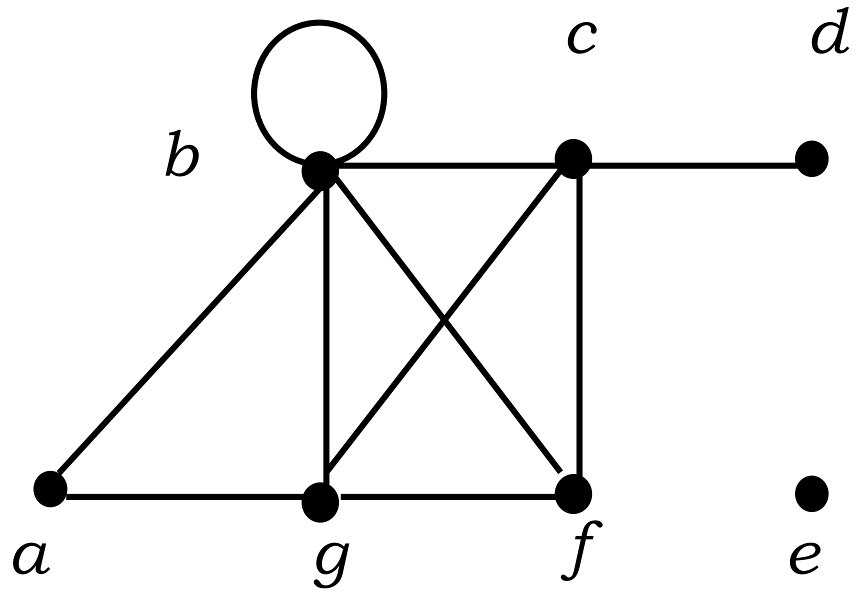
Adjacent Vertices in Undirected Graphs

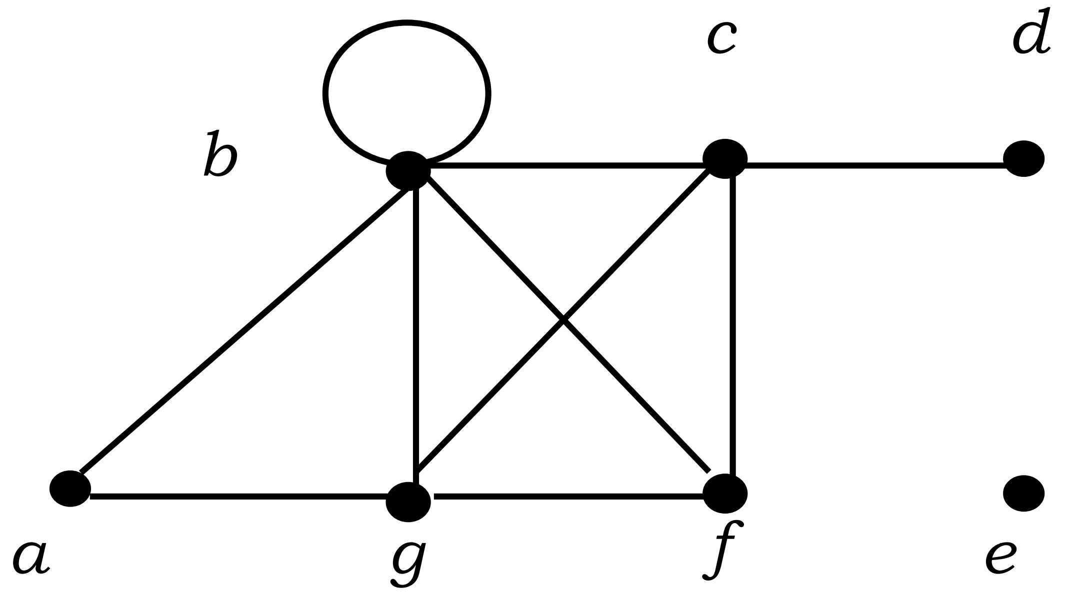
- Two vertices, *u* and *v* in an undirected graph *G* are called *adjacent* (or neighbors) in *G*, if {*u*,*v*} is an edge of *G*.
- An edge *e* connecting *u* and *v* is called *incident* with vertices *u* and *v*, or is said to connect *u* and *v*.
- The vertices *u* and *v* are called *endpoints* of edge {*u*,*v*}.

Degree of a Vertex

- The *degree of a vertex* in an undirected graph is the number of edges incident with it
  - except that a loop at a vertex contributes twice to the degree of that vertex
- The degree of a vertex v is denoted by $\deg(v)$.

Example





- Find the degrees of all the vertices:

deg(a) = 2, deg(b) = 6, deg(c) = 4, deg(d) = 1,

deg(e) = 0, deg(f) = 3, deg(g) = 4

Adjacent Vertices in Directed Graphs

When (u,v) is an edge of a directed graph G, u is said to be *adjacent to* v and v is said to be *adjacent from* u.

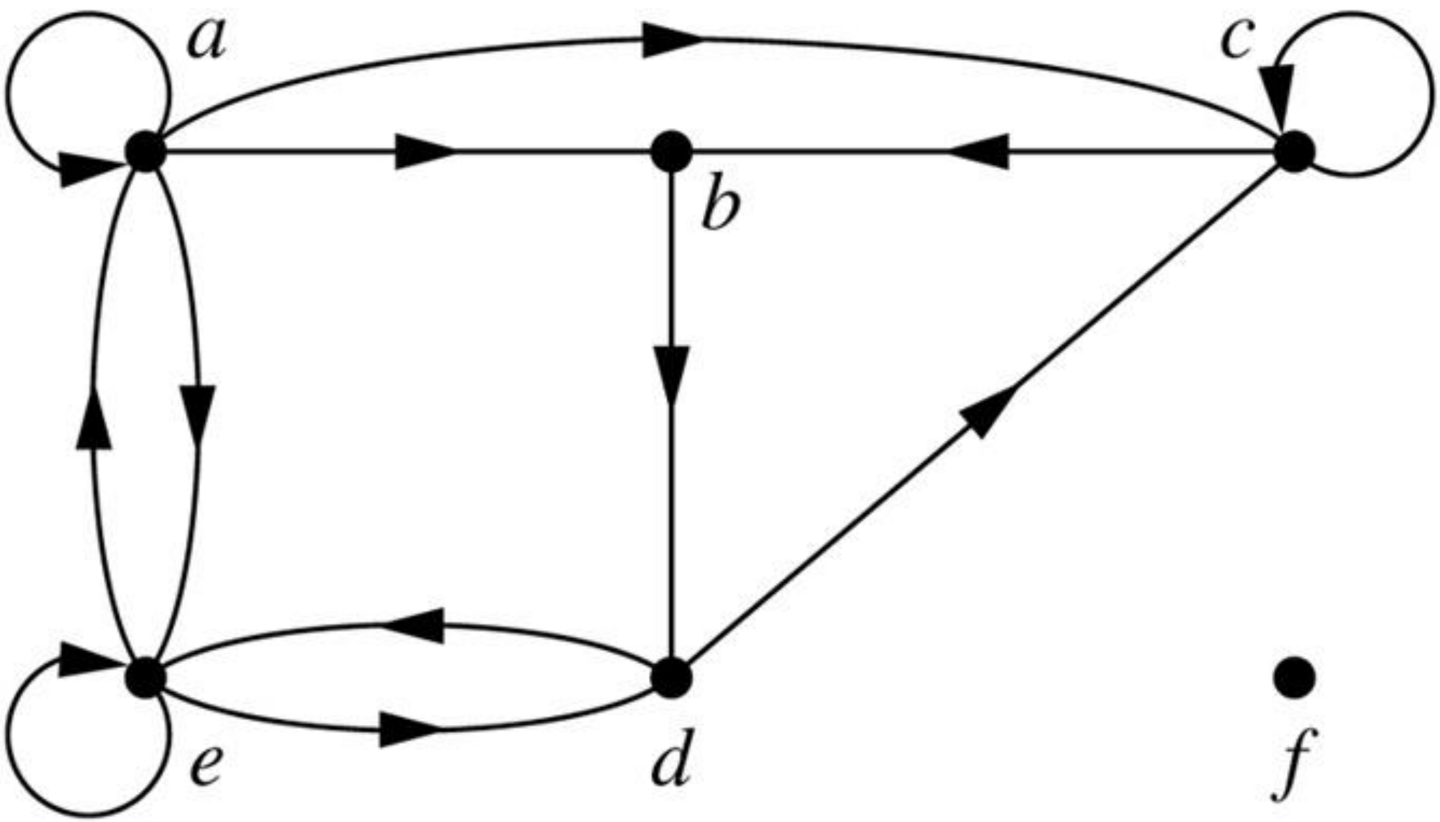
(u, v)

initial vertex   terminal vertex

Degree of a Vertex

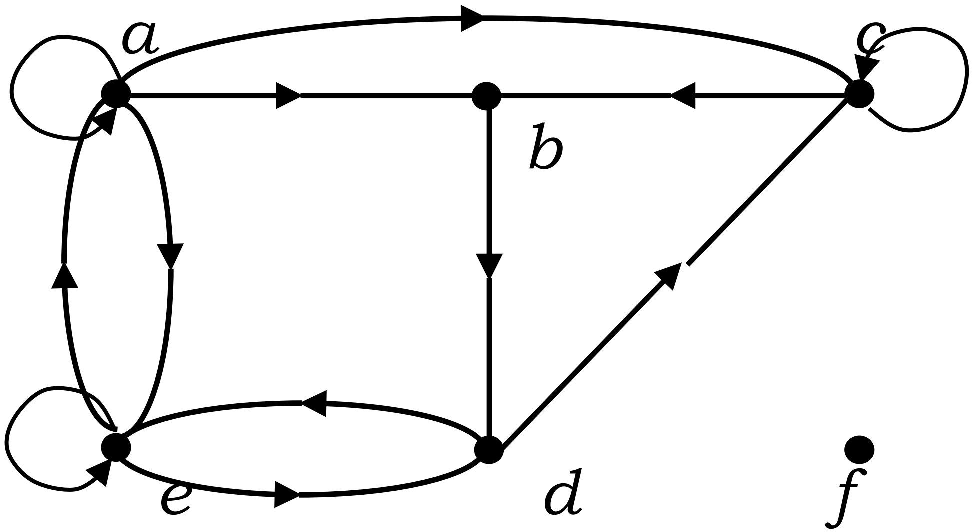
- *In-degree* of a vertex *v*
  - The number of vertices *adjacent to v* (the number of edges with *v* as their terminal vertex
  - Denoted by $\deg^{-}(v)$
- *Out-degree* of a vertex *v*
  - The number of vertices *adjacent from v* (the number of edges with *v* as their initial vertex)
  - Denoted by $\deg^{+}(v)$
- A loop at a vertex contributes 1 to both the in-degree and out-degree.

Example



G

Example



Find the in-degrees and out-degrees of this digraph.

In-degrees: $deg^-(a) = 2$, $deg^-(b) = 2$, $deg^-(c) = 3$, $deg^-(d) = 2$, $deg^-(e) = 3$, $deg^-(f) = 0$

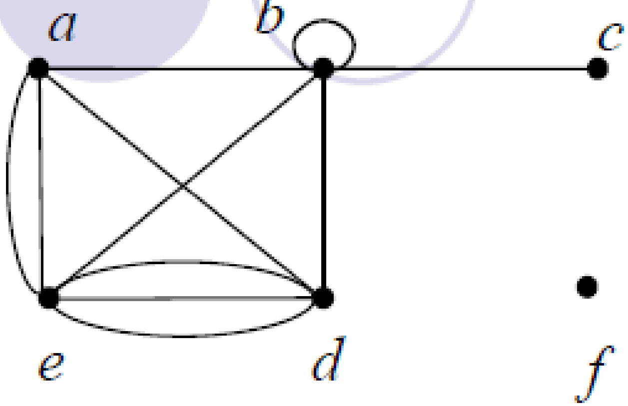
Out-degrees: $deg^+(a) = 4$, $deg^+(b) = 1$, $deg^+(c) = 2$, $deg^+(d) = 2$, $deg^+(e) = 3$, $deg^+(f) = 0$

Theorem 1. (The Handshaking Theorem)

- Let ***G* = *(V, E)* be an undirected graph with** *e edges (i.e.,* **|*E*| = *e*). Then**

$$\sum_{v \in V} \deg(v) = 2e$$

eg.



H

The graph H has 11 edges, and

$$\sum_{v \in V} \deg(v) = 22$$

Example 3. How many edges are there in a graph with 10 vertices each of degree six?

- **Sol :**

$$10 \cdot 6 = 2e \Rightarrow e = 30$$

Thm 2. An undirected graph *G = (V, E) has an even* number of vertices of odd degree.

Pf : Let $V_1 = \{v \in V \mid \deg(v) \text{ is even}\}$,

$V_2 = \{v \in V \mid \deg(v) \text{ is odd}\}$.

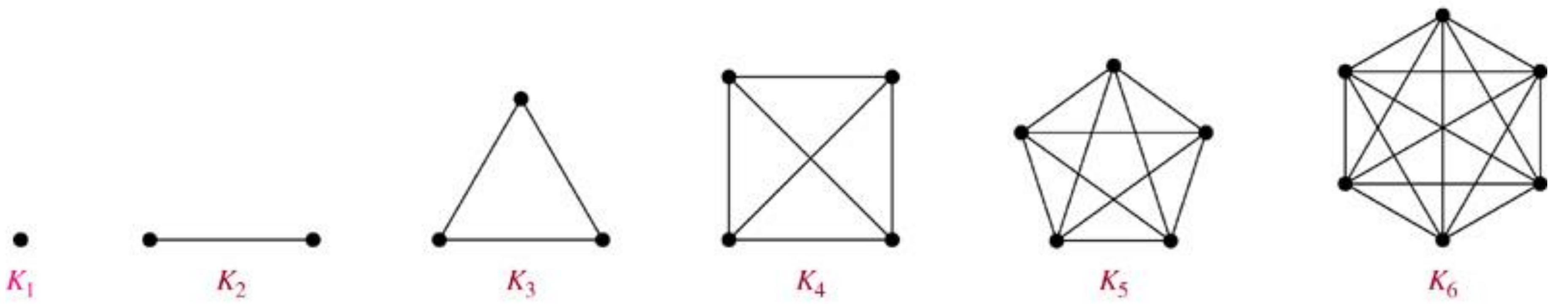
$$\sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v) = 2e \quad \Rightarrow \quad \sum_{v \in V_2} \deg(v) \text{ is even}$$

Theorem 3

- The sum of the in-degrees of all vertices in a digraph = the sum of the out-degrees = the number of edges.
- Let $G = (V, E)$ be a graph with directed edges. Then:

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|$$

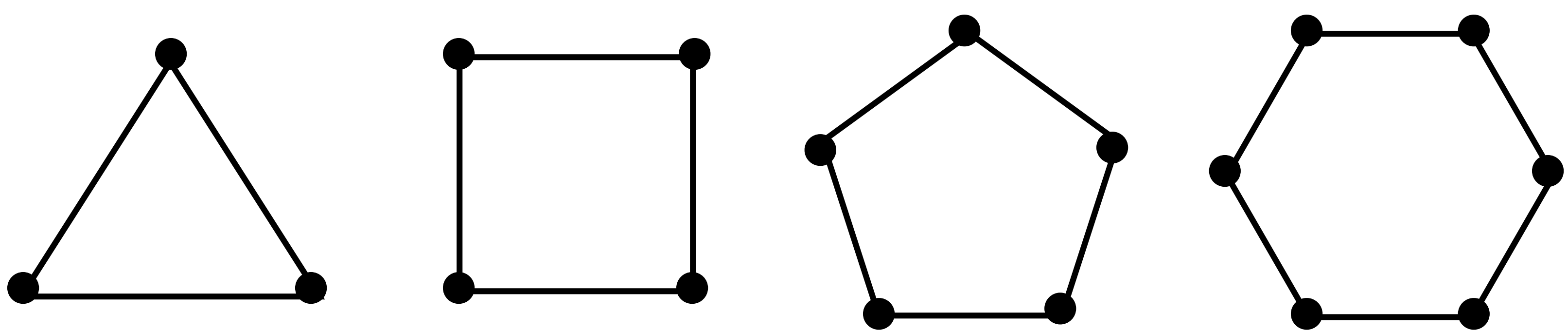
Complete Graph



- The *complete graph* on *n* vertices (K_n) is the simple graph that contains exactly one edge between each pair of distinct vertices.
- The figures above represent the complete graphs, K_n, for n = 1, 2, 3, 4, 5, and 6.

Cycle

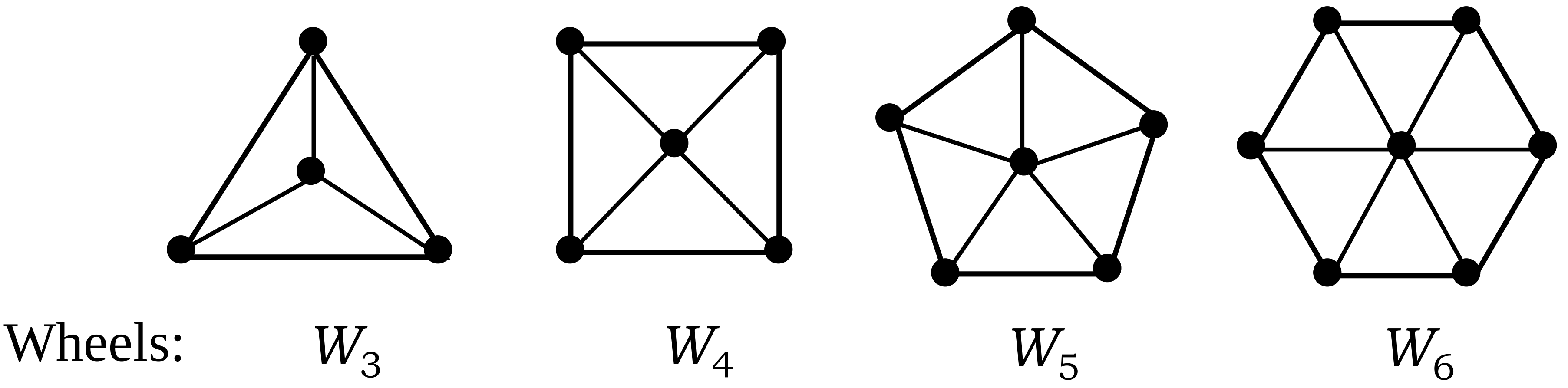
- The *cycle* C_n ($n \geq 3$), consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1,v_2\}$, $\{v_2,v_3\}$, …, $\{v_{n-1},v_n\}$, and $\{v_n,v_1\}$.



Cycles: C_3 C_4 C_5 C_6

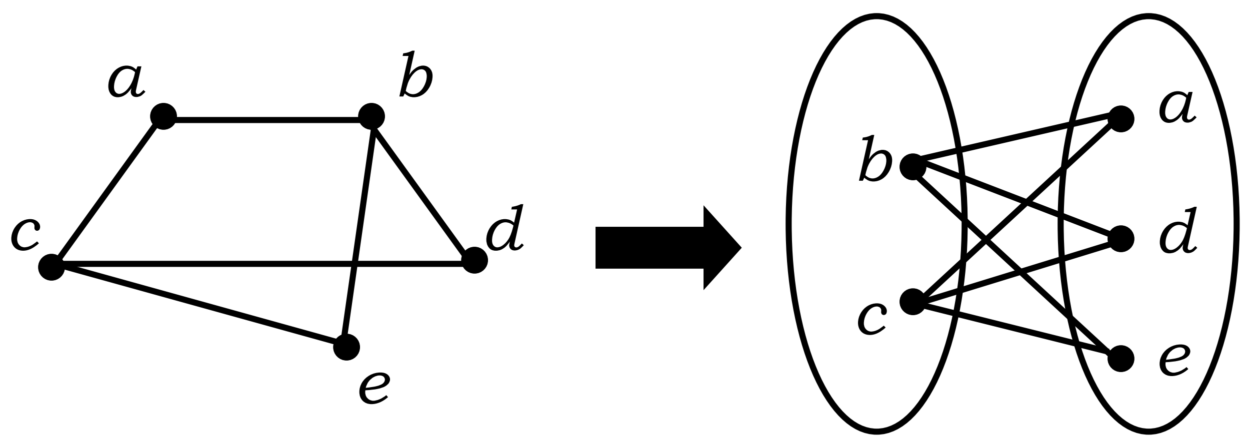
Wheel

When a new vertex is added to a cycle C_n and this new vertex is connected to each of the n vertices in C_n, we obtain a *wheel* W_n.

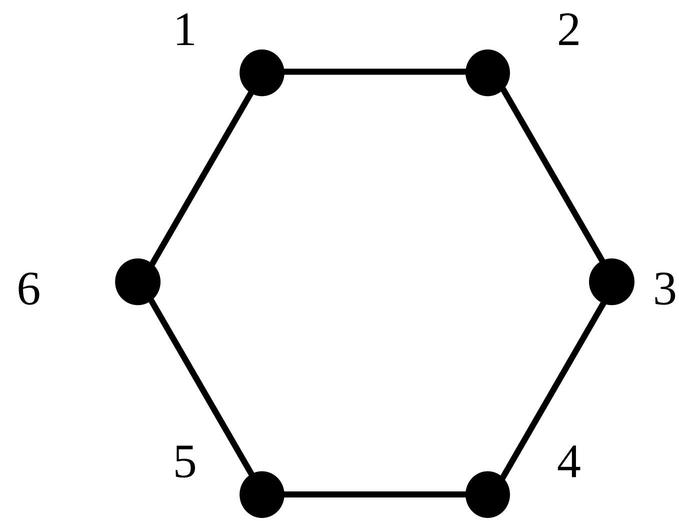


Bipartite Graph

A simple graph is called *bipartite* if its vertex set V can be partitioned into two disjoint nonempty sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).



Bipartite Graph (Example)



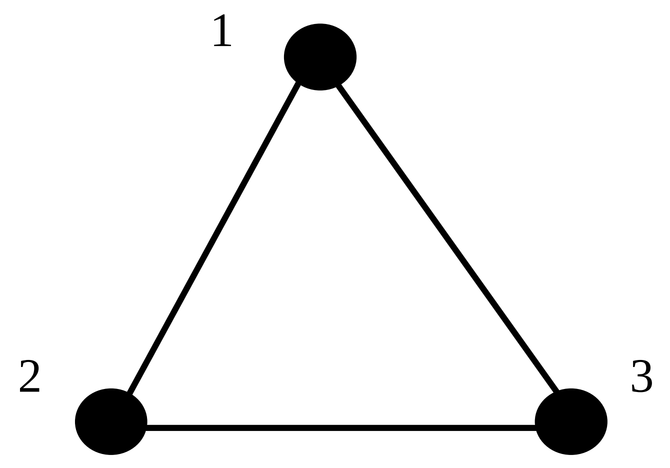
Is C_6 Bipartite?

Yes. Why?

Because:

- its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$
- every edge of C_6 connects a vertex in V_1 with a vertex in V_2

Bipartite Graph (Example)



Is K_3 Bipartite?

No. Why not?

Because:

• Each vertex in K_3 is connected to every other vertex by an edge

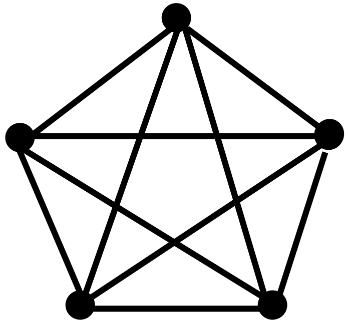
• If we divide the vertex set of K_3 into two disjoint sets, one set must contain two vertices

• These two vertices are connected by an edge

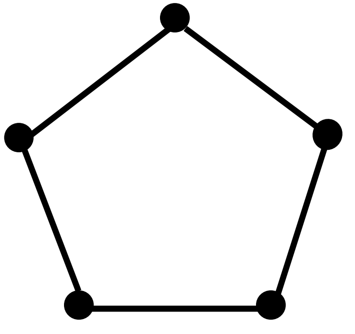
• But this can't be the case if the graph is bipartite

Subgraph

- A *subgraph* of a graph $G = (V,E)$ is a graph $H = (W,F)$ where $W \subseteq V$ and $F \subseteq E$.



K_5

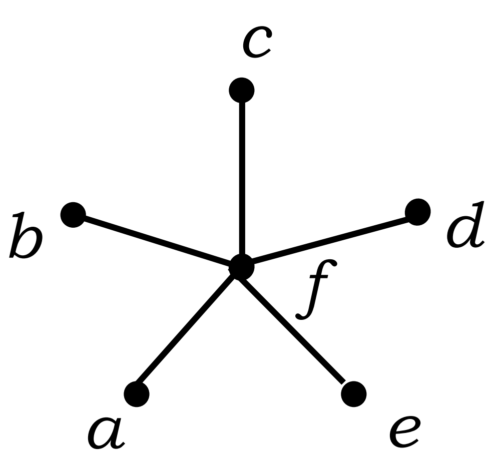


C_5

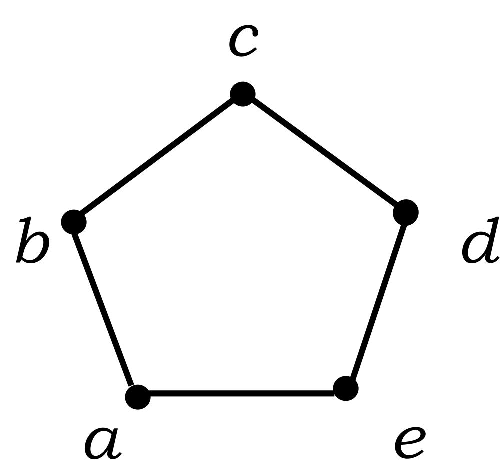
Is C_5 a subgraph of K_5?

Union

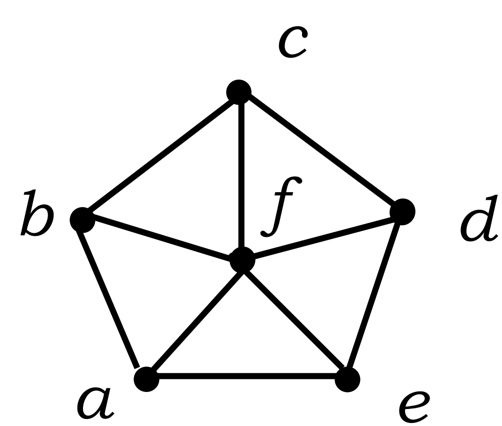
- The *union* of 2 simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The union is denoted by $G_1 \cup G_2$.



S_5



C_5



W_5

$S_5 \cup C_5 = W_5$

CSE 2813
Discrete Structures

Chapter 9.3
Representing Graphs and Graph Isomorphism

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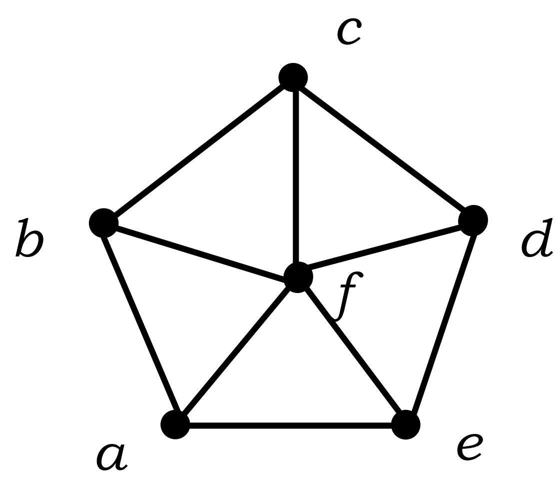
Adjacency Matrix

- A simple graph $G = (V,E)$ with n vertices can be represented by its *adjacency matrix*, A, where the entry a_{ij} in row i and column j is:

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge in } G \\ 0 & \text{otherwise} \end{cases}$$

Adjacency Matrix Example



W_5

$$
\begin{array}{c|c}
 & \text{To} \\
\text{From} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$

$\{v_1, v_2\}$

row column

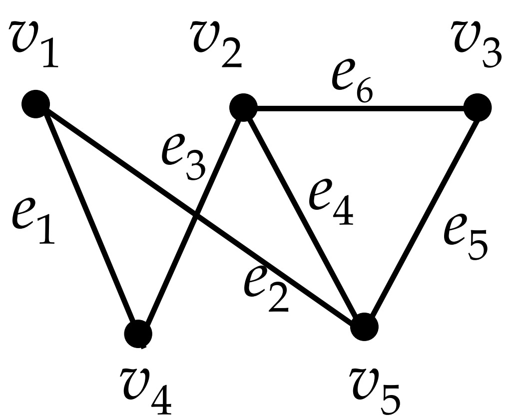
Incidence Matrix

- Let $G = (V,E)$ be an undirected graph. Suppose $v_1,v_2,v_3,\ldots,v_n$ are the vertices and $e_1,e_2,e_3,\ldots,e_m$ are the edges of G. The *incidence matrix* w.r.t. this ordering of V and E is the $n\times m$ matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

Incidence Matrix Example

- Represent the graph shown with an incidence matrix.



$$
\begin{array}{c|cccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\hline
v_1 & 1 & 1 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 1 & 1 & 0 & 1 \\
v_3 & 0 & 0 & 0 & 0 & 1 & 1 \\
v_4 & 1 & 0 & 1 & 0 & 0 & 0 \\
v_5 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}
$$

edges

vertices

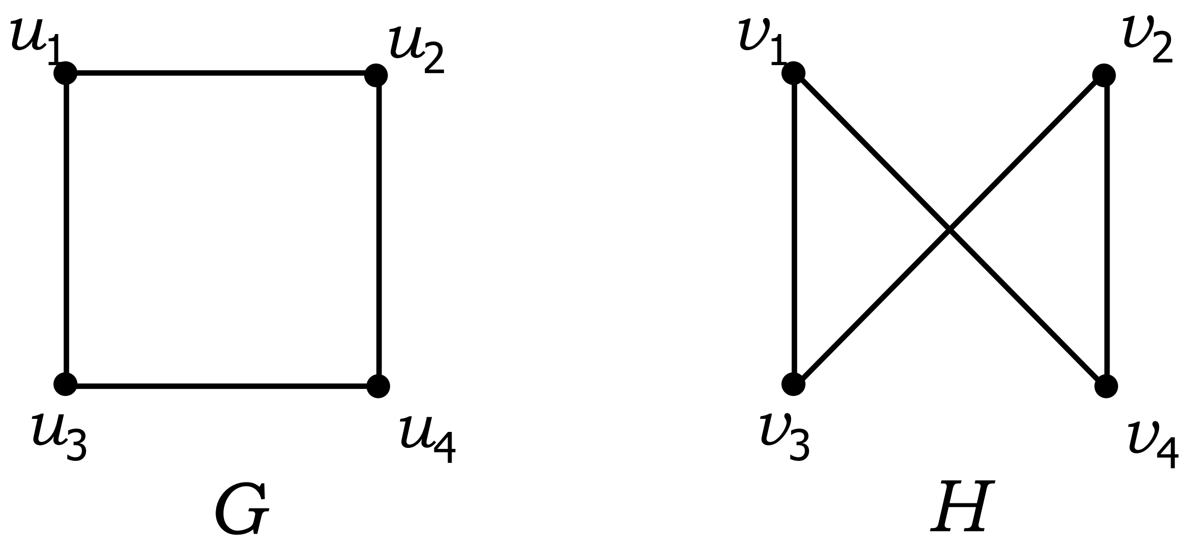
Isomorphism

- Two simple graphs are isomorphic if:
  - there is a one-to one correspondence between the vertices of the two graphs
  - the adjacency relationship is preserved

Isomorphism (Cont.)

- The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are *isomorphic* if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1.

Example



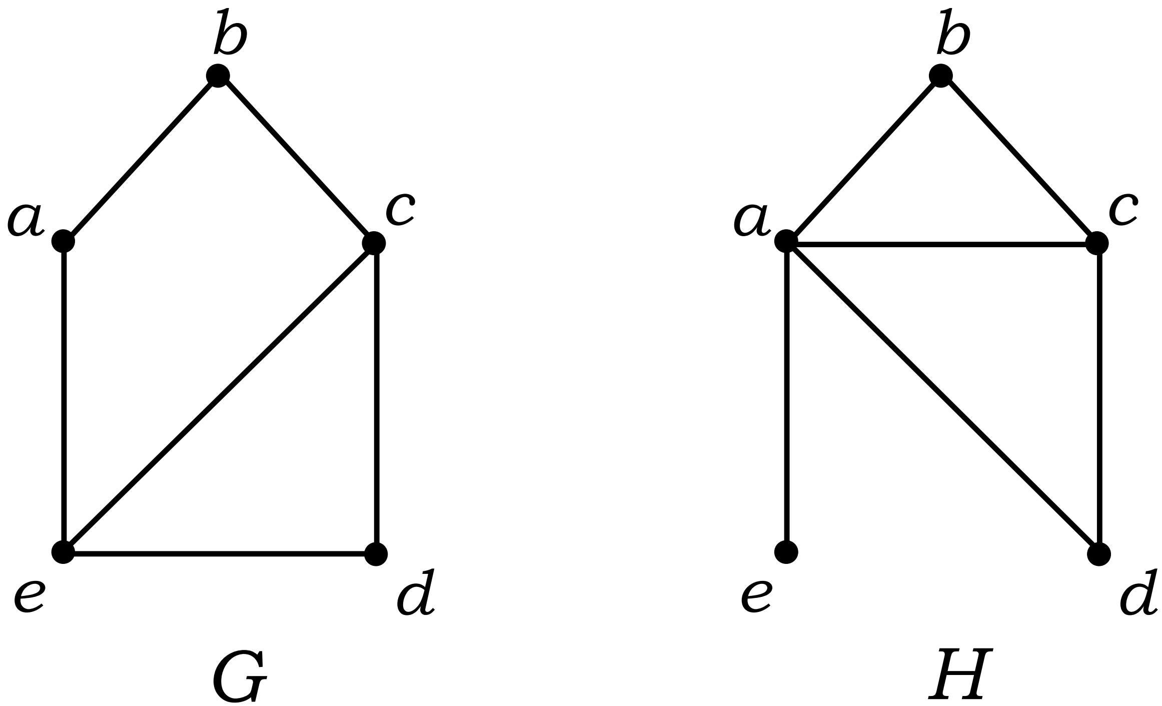
Are G and H isomorphic?

$f(u_1) = v_1,\ f(u_2) = v_4,\ f(u_3) = v_3,\ f(u_4) = v_2$

Invariants

- Invariants – properties that two simple graphs must have in common to be isomorphic
  - Same number of vertices
  - Same number of edges
  - Degrees of corresponding vertices are the same
  - If one is bipartite, the other must be; if one is complete, the other must be; and others …

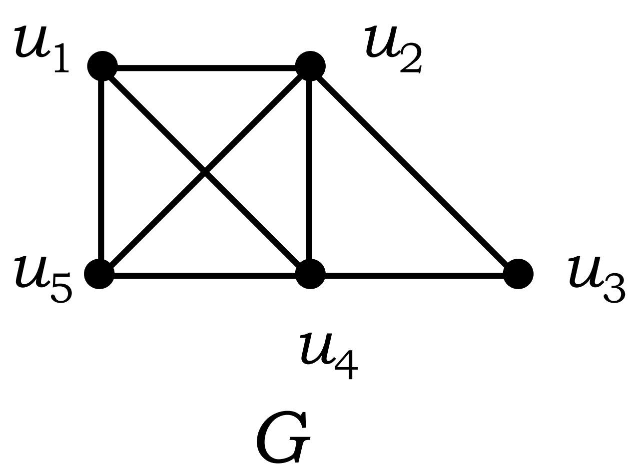
Example

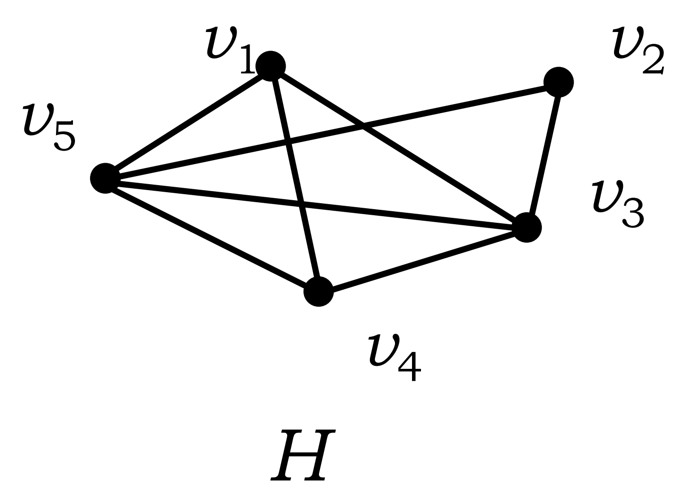


Are G and H isomorphic?

Example

- Are these two graphs isomorphic?





  - They both have 5 vertices
  - They both have 8 edges
  - They have the same number of vertices with the same degrees: 2, 3, 3, 4, 4.

Example (Cont.)

G	H	H → G?
$\begin{array}{c|ccccc} & u_1 & u_2 & u_3 & u_4 & u_5 \\ \hline u_1 & 0 & 1 & 0 & 1 & 1 \\ u_2 & 1 & 0 & 1 & 1 & 1 \\ u_3 & 0 & 1 & 0 & 1 & 0 \\ u_4 & 1 & 1 & 1 & 0 & 1 \\ u_5 & 1 & 1 & 0 & 1 & 0 \end{array}$	$\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 0 & 1 & 1 & 1 \\ v_2 & 0 & 0 & 1 & 0 & 1 \\ v_3 & 1 & 1 & 0 & 1 & 1 \\ v_4 & 1 & 0 & 1 & 0 & 1 \\ v_5 & 1 & 1 & 1 & 1 & 0 \end{array}$	$\begin{array}{c|ccccc} & v_1 & v_3 & v_2 & v_5 & v_4 \\ \hline v_1 & & & & & \\ v_3 & & & & & \\ v_2 & & & & & \\ v_5 & & & & & \\ v_4 & & & & & \end{array}$

– G and H don't appear to be isomorphic.

– However, we haven't tried mapping vertices from G onto H yet.

Example (Cont.)

- Start with the vertices of degree 2 since each graph only has one:

  $\deg(u_3) = \deg(v_2) = 2$ therefore $f(u_3) = v_2$

Example (Cont.)

- Now consider vertices of degree 3

  $$\deg(u_1) = \deg(u_5) = \deg(v_1) = \deg(v_4) = 3$$

  therefore we must have either one of

  $f(u_1) = v_1$ and $f(u_5) = v_4$

  $f(u_1) = v_4$ and $f(u_5) = v_1$

Example (Cont.)

- Now try vertices of degree 4:

  $\deg(u_2) = \deg(u_4) = \deg(v_3) = \deg(v_5) = 4$

  therefore we must have one of:

  $f(u_2) = v_3$ and $f(u_4) = v_5$ or

  $f(u_2) = v_5$ and $f(u_4) = v_3$

Example (Cont.)

- There are four possibilities (this can get messy!)

  $f(u_1) = v_1, f(u_2) = v_3, f(u_3) = v_2, f(u_4) = v_5, f(u_5) = v_4$

  $f(u_1) = v_4, f(u_2) = v_3, f(u_3) = v_2, f(u_4) = v_5, f(u_5) = v_1$

  $f(u_1) = v_1, f(u_2) = v_5, f(u_3) = v_2, f(u_4) = v_3, f(u_5) = v_4$

  $f(u_1) = v_4, f(u_2) = v_5, f(u_3) = v_2, f(u_4) = v_3, f(u_5) = v_1$

Example (Cont.)

G H H'

$$\begin{array}{c|ccccc} & u_1 & u_2 & u_3 & u_4 & u_5 \\ \hline u_1 & 0 & 1 & 0 & 1 & 1 \\ u_2 & 1 & 0 & 1 & 1 & 1 \\ u_3 & 0 & 1 & 0 & 1 & 0 \\ u_4 & 1 & 1 & 1 & 0 & 1 \\ u_5 & 1 & 1 & 0 & 1 & 0 \end{array} \qquad \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 0 & 1 & 1 & 1 \\ v_2 & 0 & 0 & 1 & 0 & 1 \\ v_3 & 1 & 1 & 0 & 1 & 1 \\ v_4 & 1 & 0 & 1 & 0 & 1 \\ v_5 & 1 & 1 & 1 & 1 & 0 \end{array} \qquad \begin{array}{c|ccccc} & v_1 & v_3 & v_2 & v_5 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 1 & 1 \\ v_3 & 1 & 0 & 1 & 1 & 1 \\ v_2 & 0 & 1 & 0 & 1 & 0 \\ v_5 & 1 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 1 & 0 & 1 & 0 \end{array}$$

We permute the adjacency matrix of *H* (per function choices above) to see if we get the adjacency of *G*. Let's try:

$$f(u_1) = v_1, f(u_2) = v_3, f(u_3) = v_2, f(u_4) = v_5, f(u_5) = v_4$$

Does G = H'? Yes!

Example (Cont.)

G

$$\begin{array}{c|ccccc} & u_1 & u_2 & u_3 & u_4 & u_5 \\ \hline u_1 & 0 & 1 & 0 & 1 & 1 \\ u_2 & 1 & 0 & 1 & 1 & 1 \\ u_3 & 0 & 1 & 0 & 1 & 0 \\ u_4 & 1 & 1 & 1 & 0 & 1 \\ u_5 & 1 & 1 & 0 & 1 & 0 \end{array}$$

H

$$\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 0 & 1 & 1 & 1 \\ v_2 & 0 & 0 & 1 & 0 & 1 \\ v_3 & 1 & 1 & 0 & 1 & 1 \\ v_4 & 1 & 0 & 1 & 0 & 1 \\ v_5 & 1 & 1 & 1 & 1 & 0 \end{array}$$

H’

$$\begin{array}{c|ccccc} & v_4 & v_3 & v_2 & v_5 & v_1 \\ \hline v_4 & 0 & 1 & 0 & 1 & 1 \\ v_3 & 1 & 0 & 1 & 1 & 1 \\ v_2 & 0 & 1 & 0 & 1 & 0 \\ v_5 & 1 & 1 & 1 & 0 & 1 \\ v_1 & 1 & 1 & 0 & 1 & 0 \end{array}$$

It turns out that

$$f(u_1) = v_4,\ f(u_2) = v_3,\ f(u_3) = v_2,\ f(u_4) = v_5,\ f(u_5) = v_1$$

also works.

CSE 2813
Discrete Structures

Chapter 9.4
Connectivity

These class notes are based on material from our textbook, **Discrete Mathematics and Its Applications**, 6th ed., by Kenneth H. Rosen, published by McGraw Hill, Boston, MA, 2006. They are intended for classroom use only and are **not** a substitute for reading the textbook.

Paths in Undirected Graphs

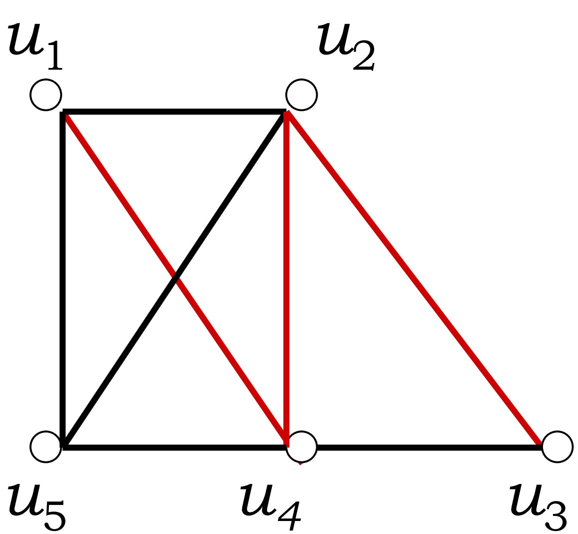
- There is a *path* from vertex v_0 to vertex v_n if there is a sequence of edges from v_0 to v_n
  - This path is labeled as $v_0,v_1,v_2,\ldots,v_n$ and has a length of n.
- The path is a *circuit* if the path begins and ends with the same vertex.
- A path is *simple* if it does not contain the same edge more than once.

Paths in Undirected Graphs

- A path or circuit is said to *pass through* the vertices v_0, v_1, v_2, …, v_n or *traverse* the edges e_1, e_2, …, e_n.

Example

- u_1, u_4, u_2, u_3

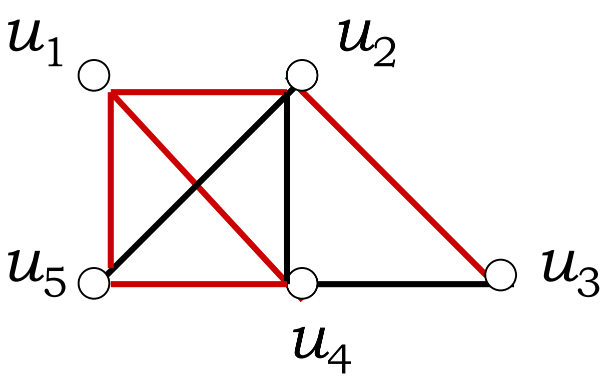


- Is it simple?
- *yes*

- What is the length?
- *3*

- Does it have any circuits?
- *no*

Example



- $u_1, u_5, u_4, u_1, u_2, u_3$

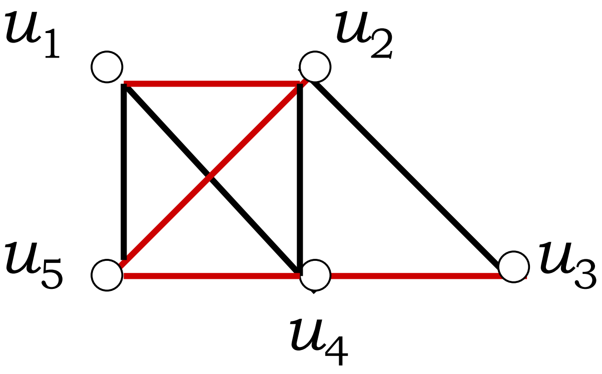
  – Is it simple?
  – *yes*

  – What is the length?
  – *5*

  – Does it have any circuits?
  – Yes; u_1, u_5, u_4, u_1

Example

- u_1, u_2, u_5, u_4, u_3
  - Is it simple?
  - *yes*
  - What is the length?
  - *4*
  - Does it have any circuits?
  - *no*

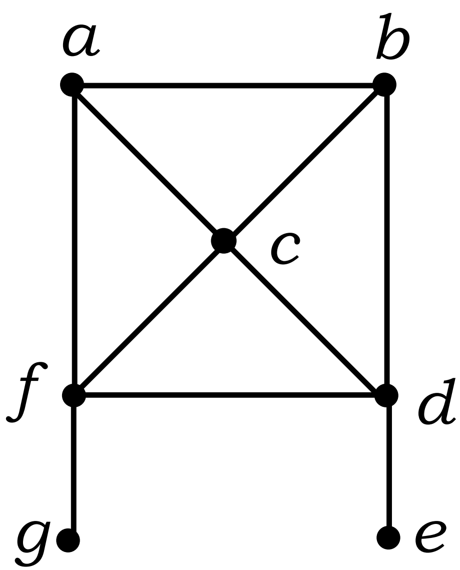


Connectedness

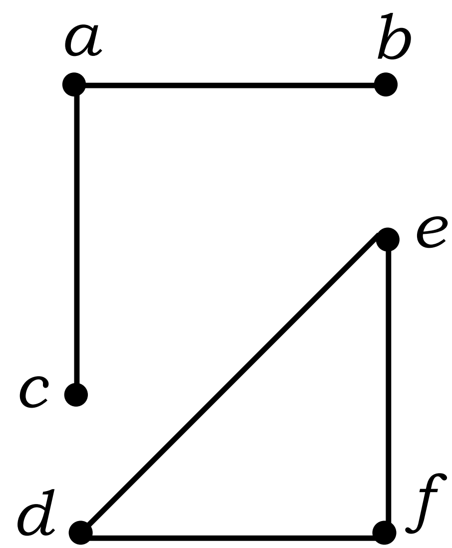
- An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph.
- There is a simple path between every pair of distinct vertices of a connected undirected graph.

Example

Are the following graphs connected?



Yes



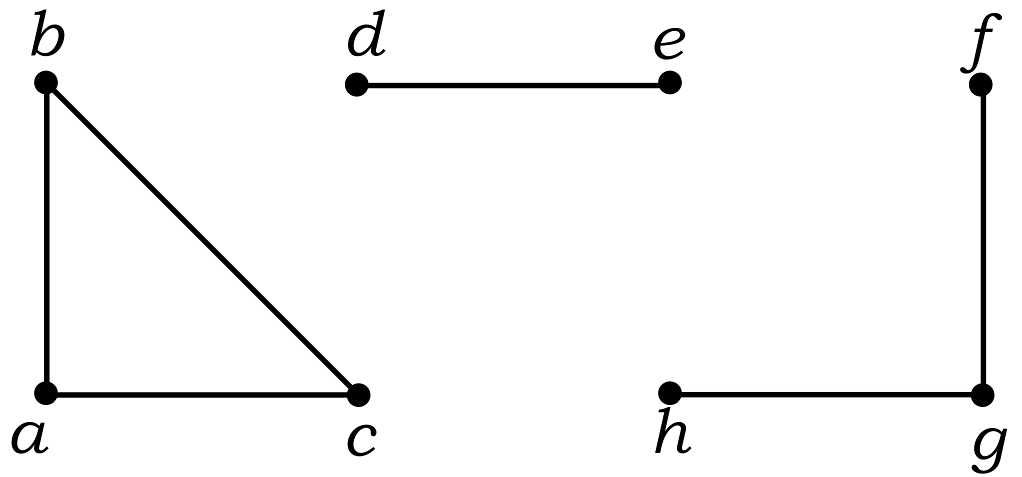
No

Connectedness (Cont.)

- A graph that is not connected is the union of two or more disjoint connected subgraphs (called the *connected components* of the graph).

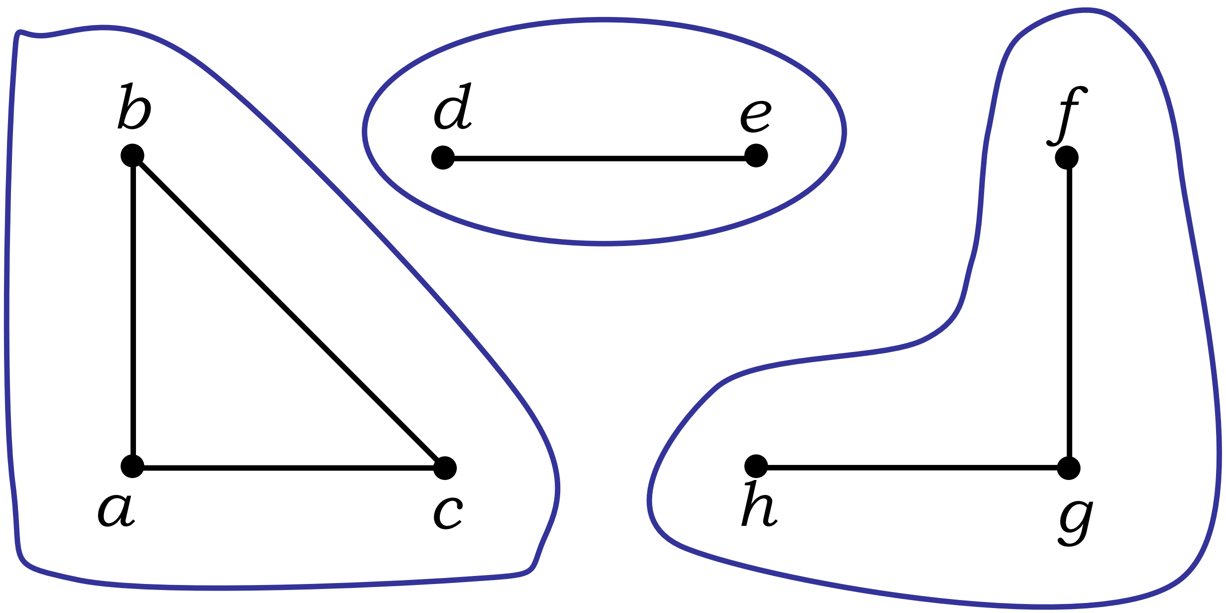
Example

- What are the connected components of the following graph?



Example

- What are the connected components of the following graph?



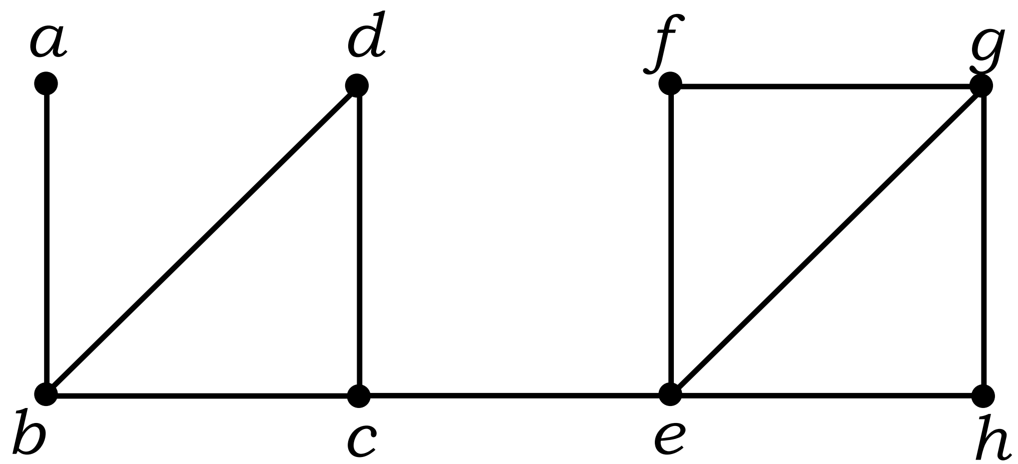
{a, b, c}, {d, e}, {f, g, h}

Cut edges and vertices

- If one can remove a vertex (and all incident edges) and produce a graph with more connected components, the vertex is called a *cut vertex*.
- If removal of an edge creates more connected components the edge is called a *cut edge* or *bridge*.

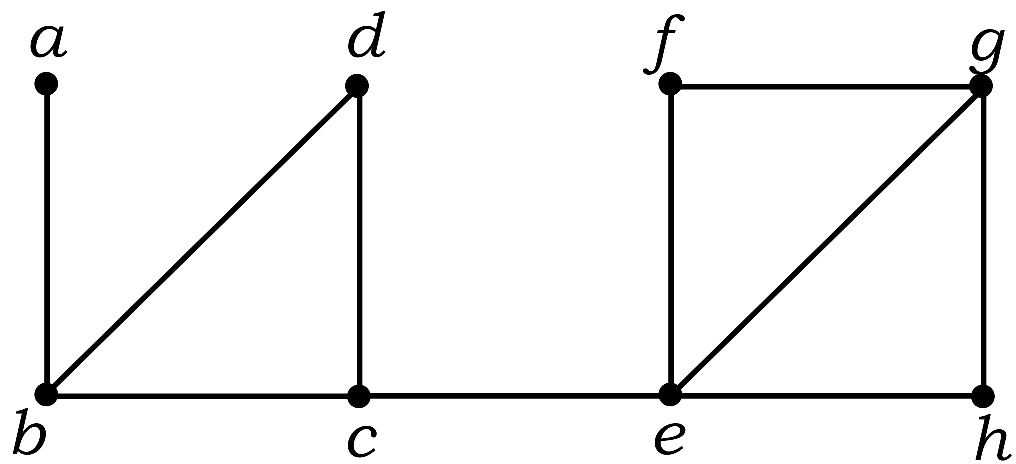
Example

- Find the cut vertices and cut edges in the following graph.



Example

- Find the cut vertices and cut edges in the following graph.



Cut vertices: c and e

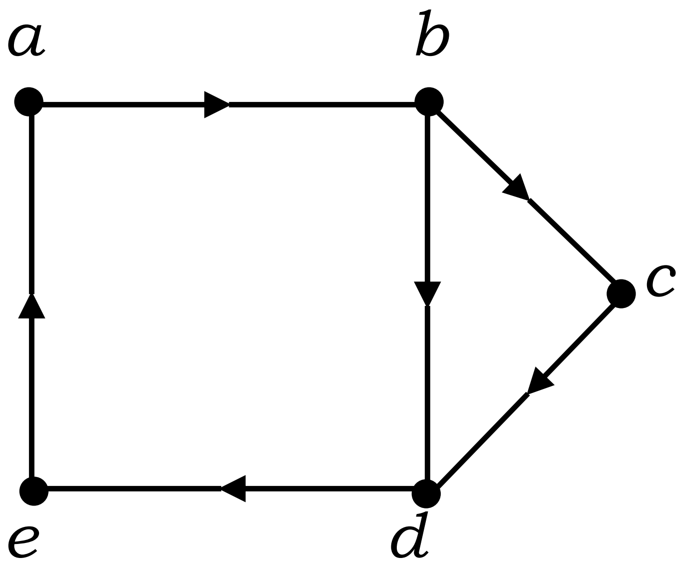
Cut edge: (c, e)

Connectedness in Directed Graphs

- A directed graph is *strongly connected* if there is a directed path between every pair of vertices.
- A directed graph is *weakly connected* if there is a path between every pair of vertices in the underlying undirected graph.

Example

- Is the following graph strongly connected? Is it weakly connected?

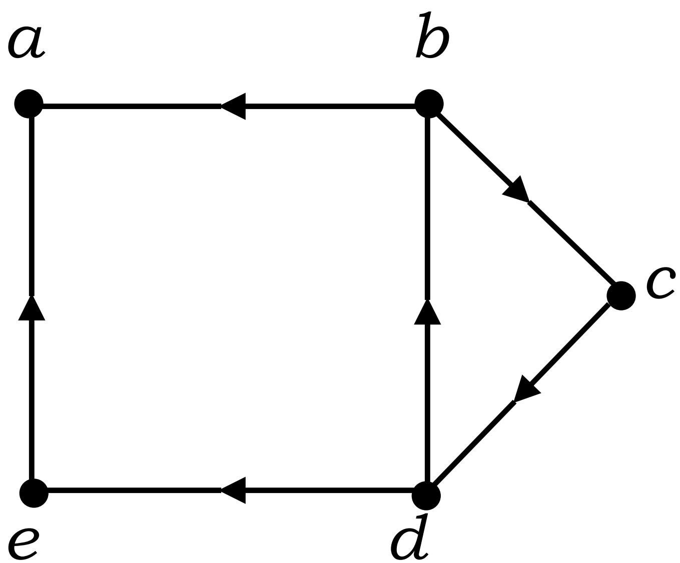


This graph is strongly connected. Why? Because there is a directed path between every pair of vertices.

If a directed graph is strongly connected, then it must also be weakly connected.

Example

- Is the following graph strongly connected? Is it weakly connected?



This graph is not strongly connected. Why not? Because there is no directed path between *a* and *b*, *a* and *e*, etc.

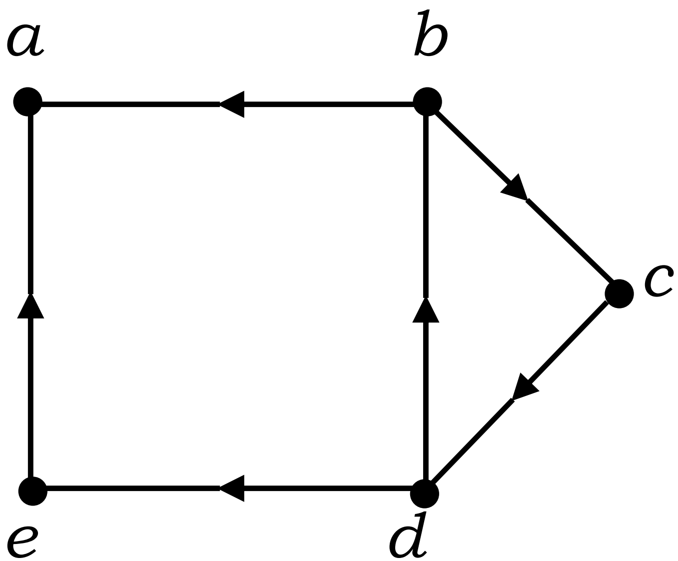
However, it *is* weakly connected. (Imagine this graph as an undirected graph.)

Connectedness in Directed Graphs

- The subgraphs of a directed graph G that are strongly connected but not contained in larger strongly connected subgraphs (the maximal strongly connected subgraphs) are called the *strongly connected components* or *strong components* of G.

Example

- What are the strongly connected components of the following graph?



This graph has three strongly connected components:
- The vertex *a*
- The vertex *e*
- The graph consisting of
  V = {*b*, *c*, *d*} and
  E = { (*b*, *c*), (*c*, *d*), (*d*, *b*)}

CSE 2813
Discrete Structures

Chapter 9.5
Euler and Hamilton Paths

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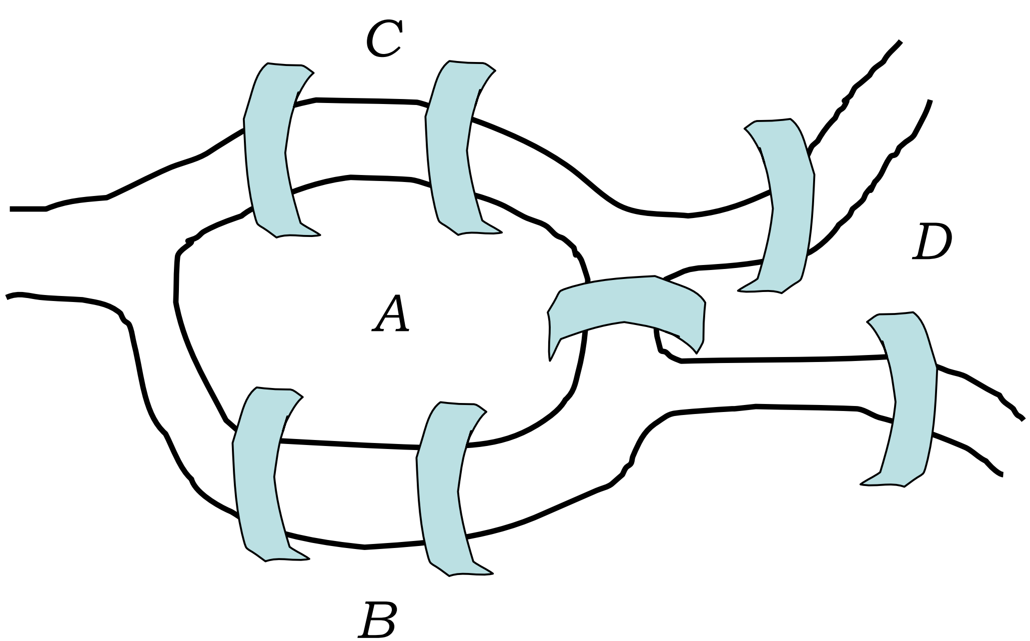
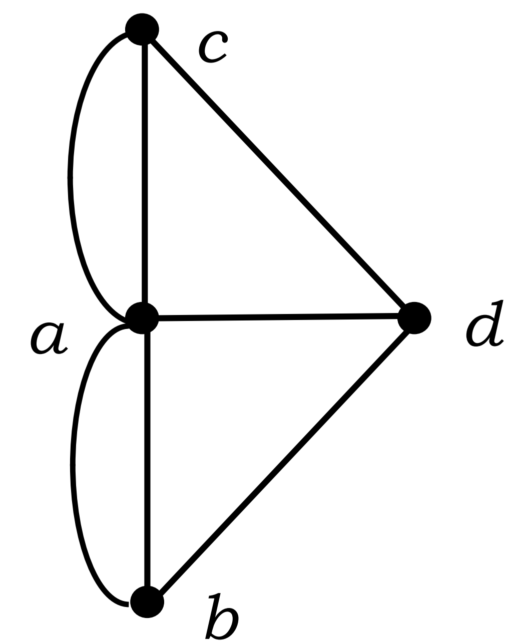
Euler Paths and Circuits

- The Seven bridges of Königsberg



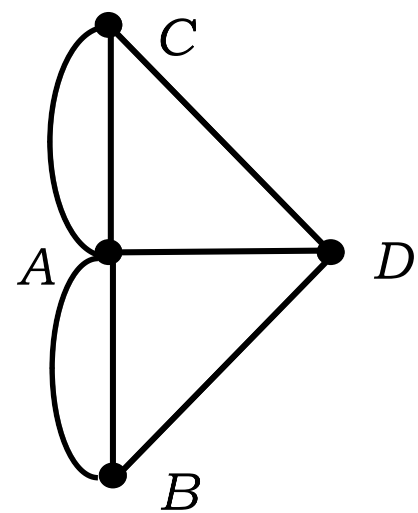


Euler Paths and Circuits

- An *Euler path* is a path using every edge of the graph *G* exactly once.
- An *Euler circuit* is an Euler path that returns to its start.

Does this graph have an Euler circuit?

No.

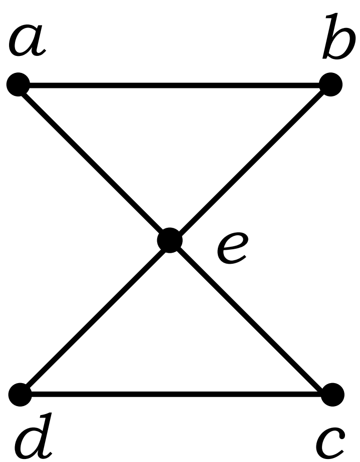


Necessary and Sufficient Conditions

- How about multigraphs?
- A connected multigraph has a Euler circuit iff *each of its vertices has an even degree*.
- A connected multigraph has a Euler path but not an Euler circuit iff *it has exactly two vertices of odd degree*.

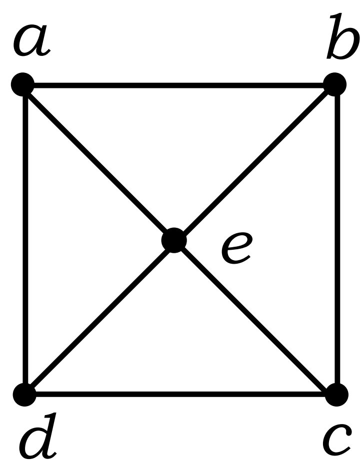
Example

- Which of the following graphs has an Euler *circuit*?

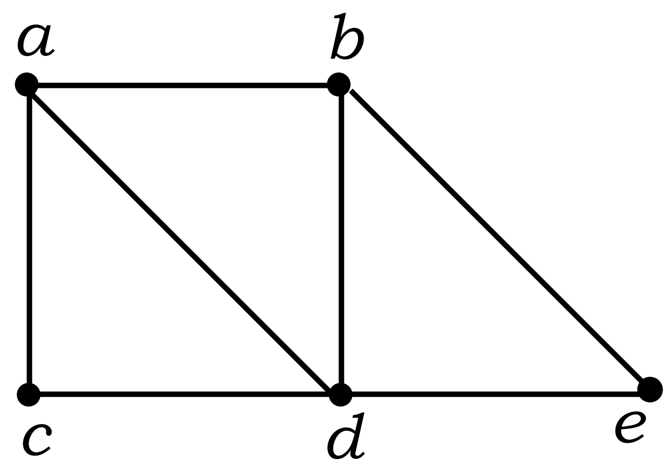


yes

(a, e, c, d, e, b, a)



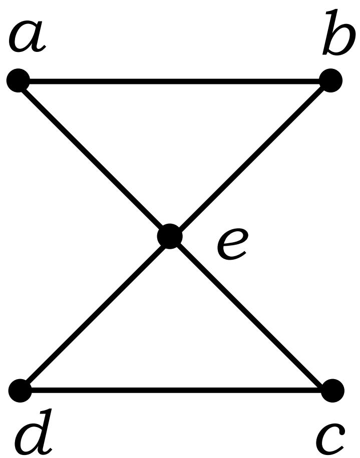
no

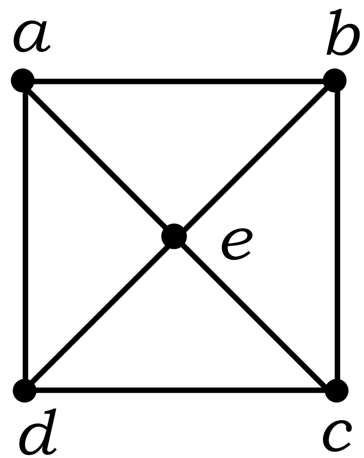


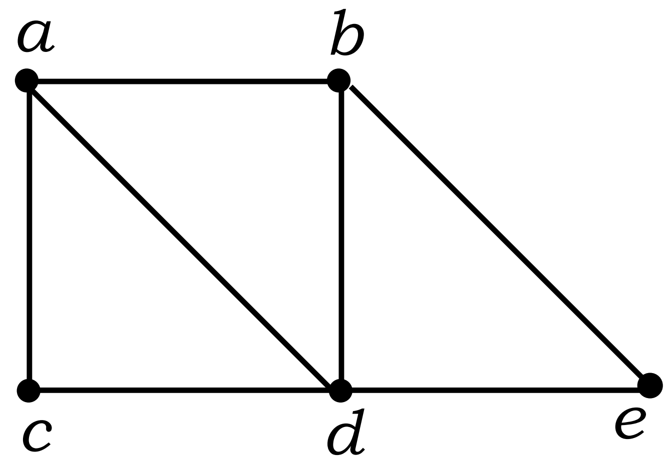
no

Example

- Which of the following graphs has an Euler *path*?





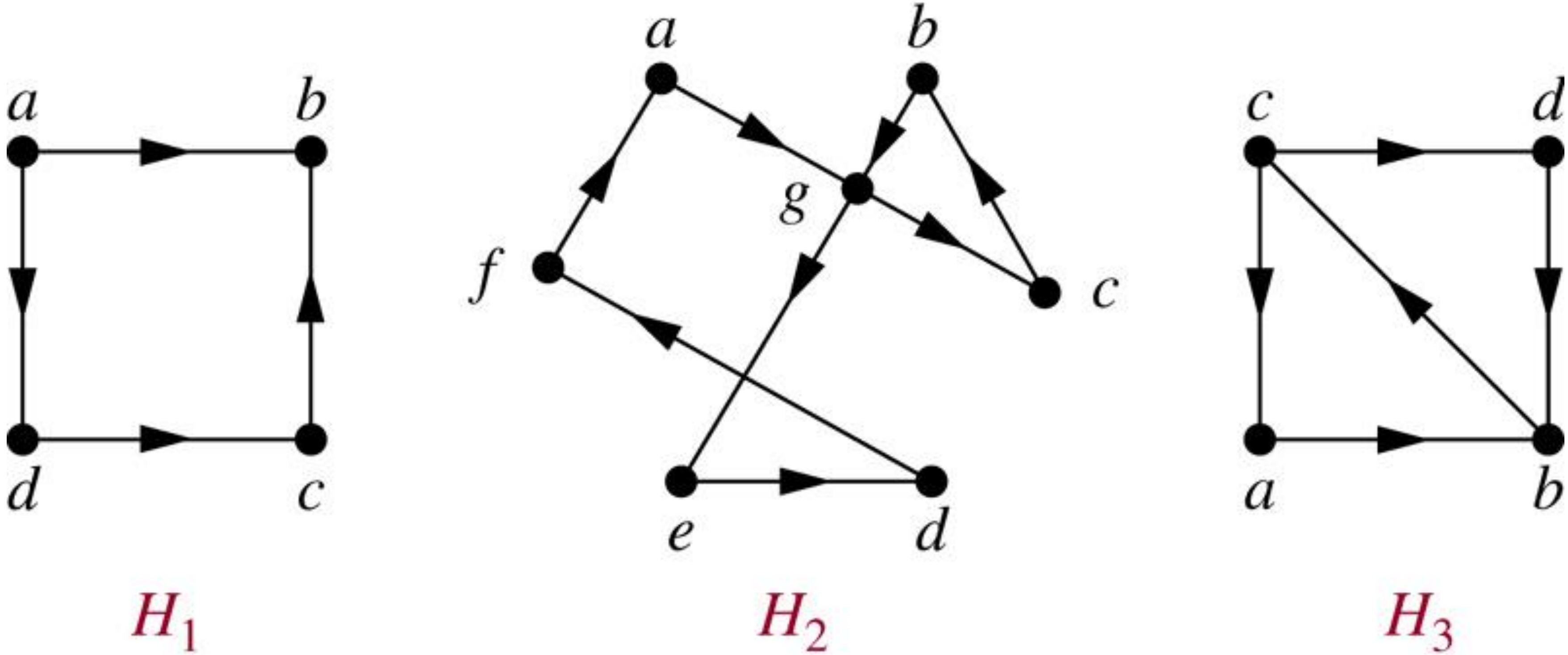


yes (a, e, c, d, e, b, a)

no

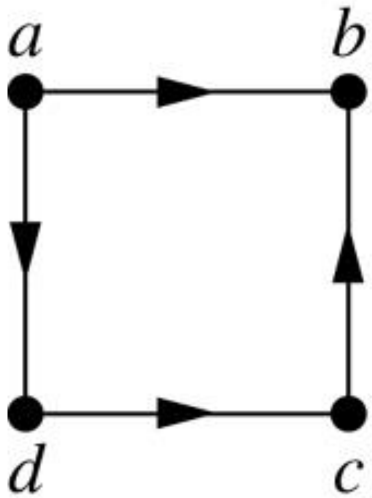
yes (a, c, d, e, b, d, a, b)

Euler Circuit in Directed Graphs



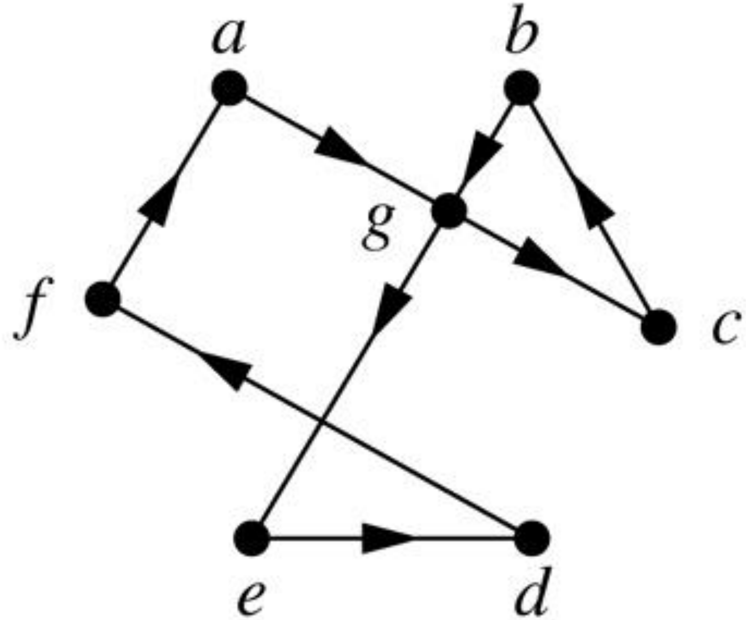
NO (a, g, c, b, g, e, d, f, a) NO

Euler Path in Directed Graphs



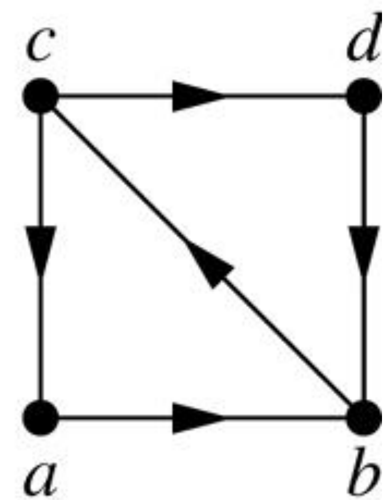
H_1

NO



H_2

(a, g, c, b, g, e, d, f, a)



H_3

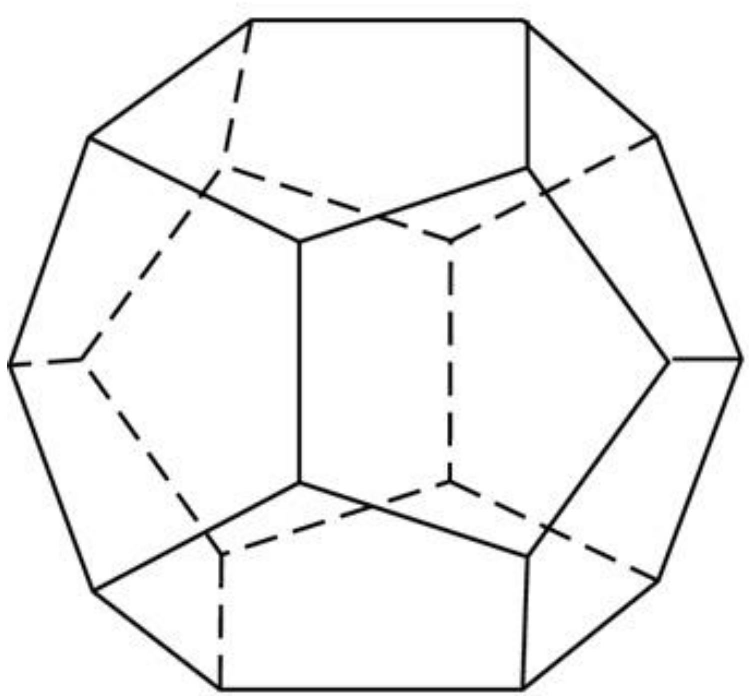
(c, a, b, c, d, b)

Hamilton Paths and Circuits

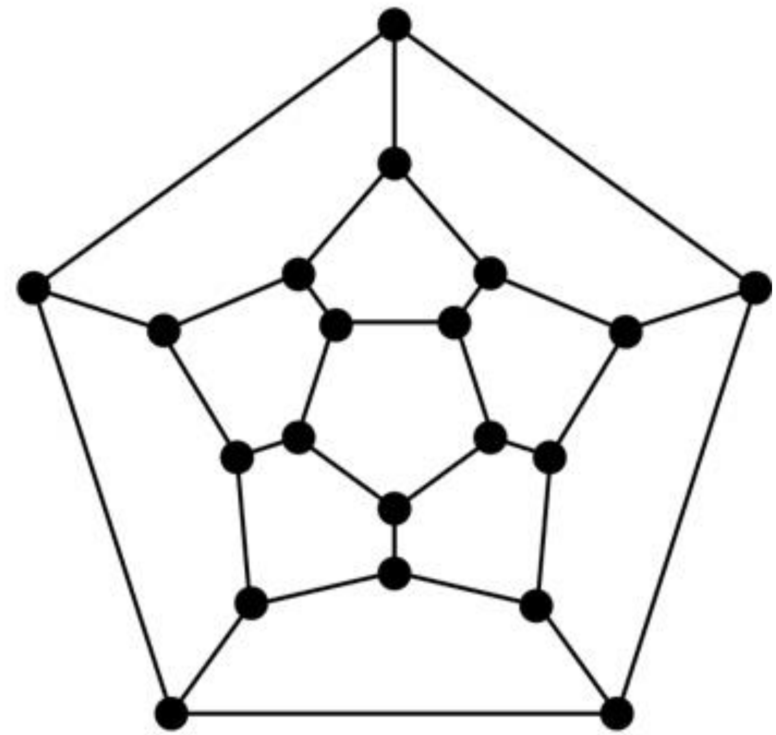
- A *Hamilton path* in a graph G is a path which visits every vertex in G exactly once.

- A *Hamilton circuit* is a Hamilton path that returns to its start.

Hamilton Circuits



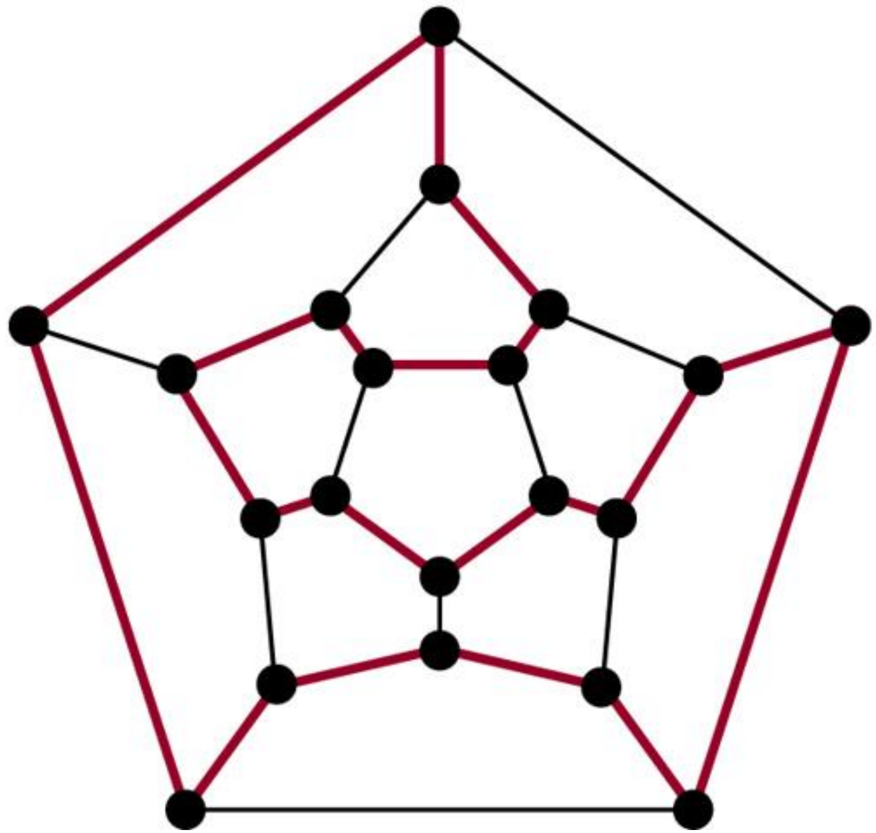
(a)



(b)

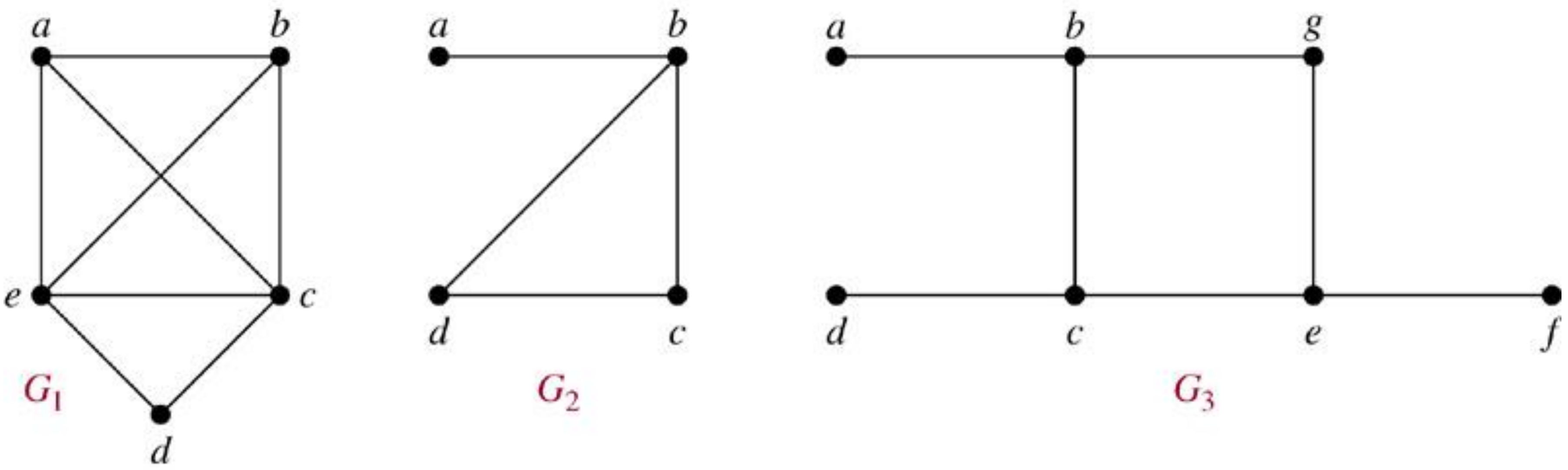
Is there a circuit in this graph that passes through each vertex exactly once?

Hamilton Circuits



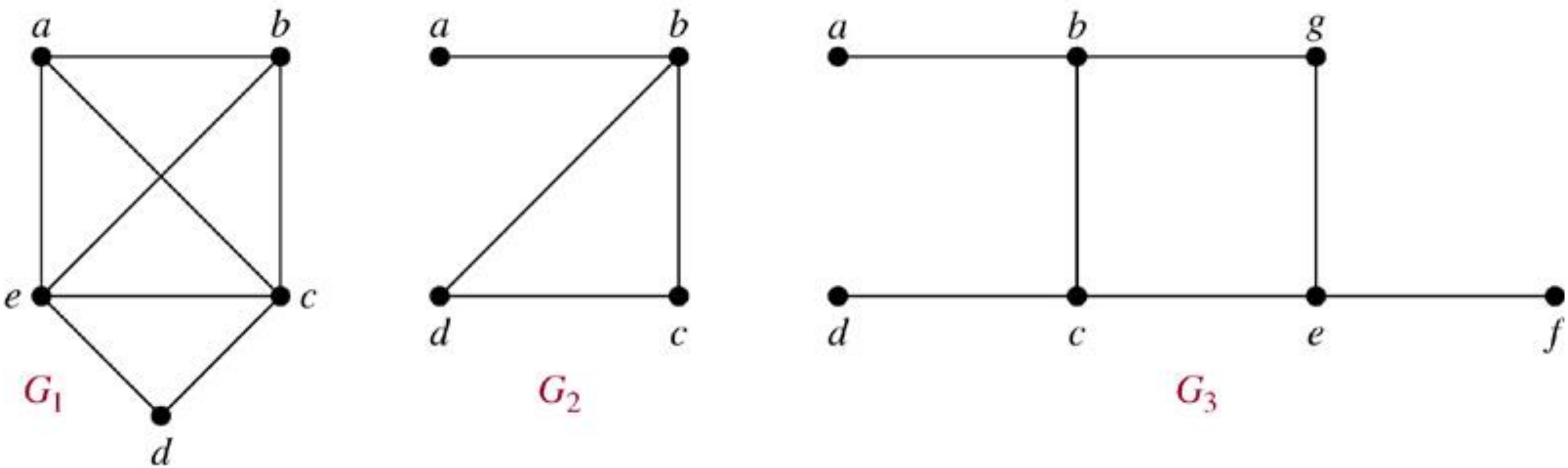
Yes; this is a circuit that passes through each vertex exactly once.

Finding Hamilton Circuits



Which of these three figures has a Hamilton circuit?
Of, if no Hamilton circuit, a Hamilton path?

Finding Hamilton Circuits



- G_1 has a Hamilton circuit: a, b, c, d, e, a
- G_2 does not have a Hamilton circuit, but does have a Hamilton path: a, b, c, d
- G_3 has neither.

Finding Hamilton Circuits

- Unlike the Euler circuit problem, finding Hamilton circuits is hard.
- There is no simple set of necessary and sufficient conditions, and no simple algorithm.

Properties to look for ...

- No vertex of degree 1
- If a node has degree 2, then both edges incident to it must be in any Hamilton circuit.
- No smaller circuits contained in any Hamilton circuit (the start/endpoint of any smaller circuit would have to be visited twice).

A Sufficient Condition

Let G be a connected simple graph with n vertices with $n \geq 3$.

G has a Hamilton circuit if the degree of each vertex is $\geq n/2$.

Travelling Salesman Problem

A Hamilton circuit or path may be used to solve practical problems that require visiting “vertices”, such as:

- road intersections
- pipeline crossings
- communication network nodes

A classic example is the Travelling Salesman Problem – finding a Hamilton circuit in a complete graph such that the total weight of its edges is minimal.

Summary

Property	**Euler**	**Hamilton**
Repeated visits to a given node allowed?	Yes	No
Repeated traversals of a given edge allowed?	No	No
Omitted nodes allowed?	No	No
Omitted edges allowed?	No	Yes

CSE 2813
Discrete Structures

Chapter 9.7
Planar Graphs

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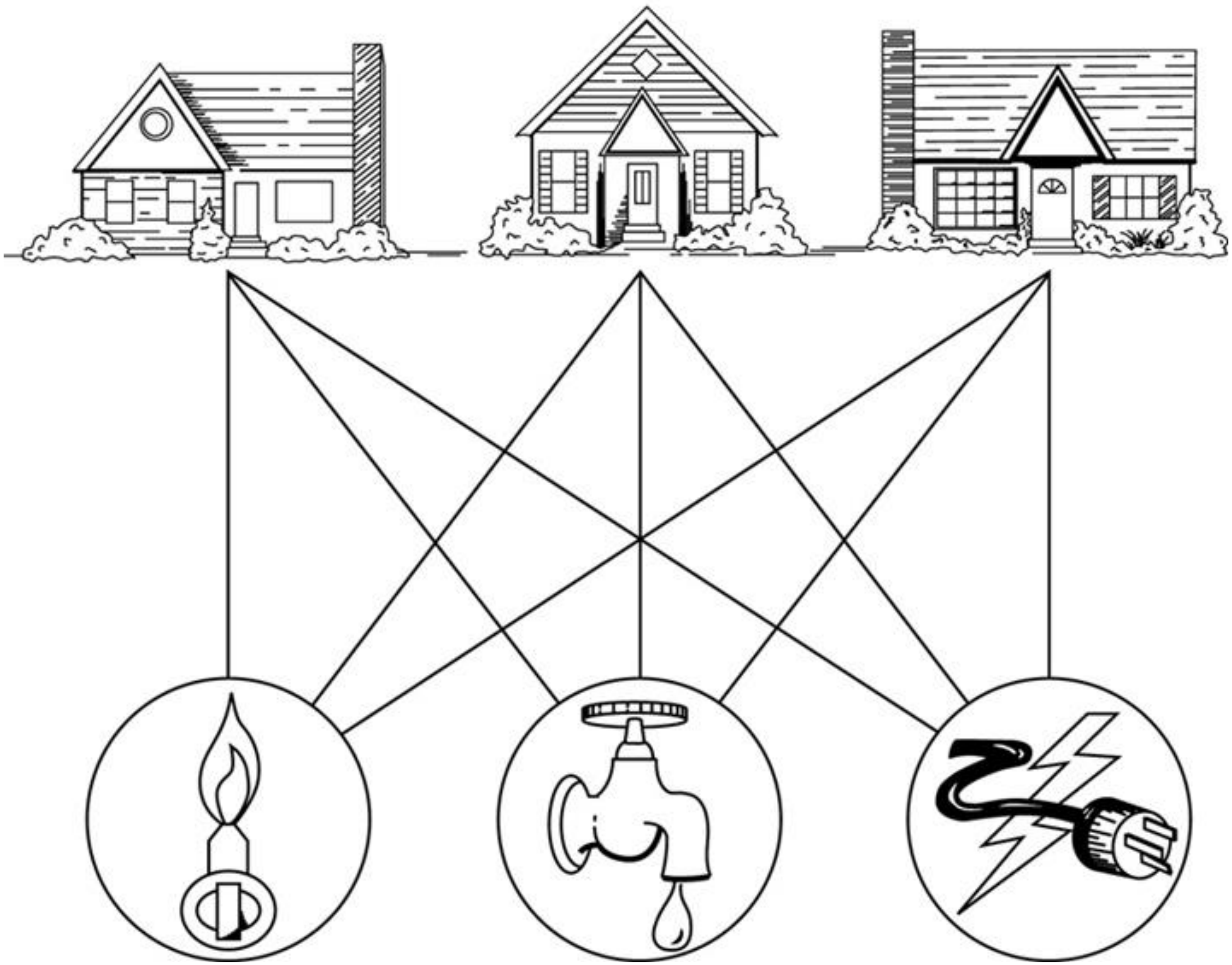
The House-and-Utilities Problem

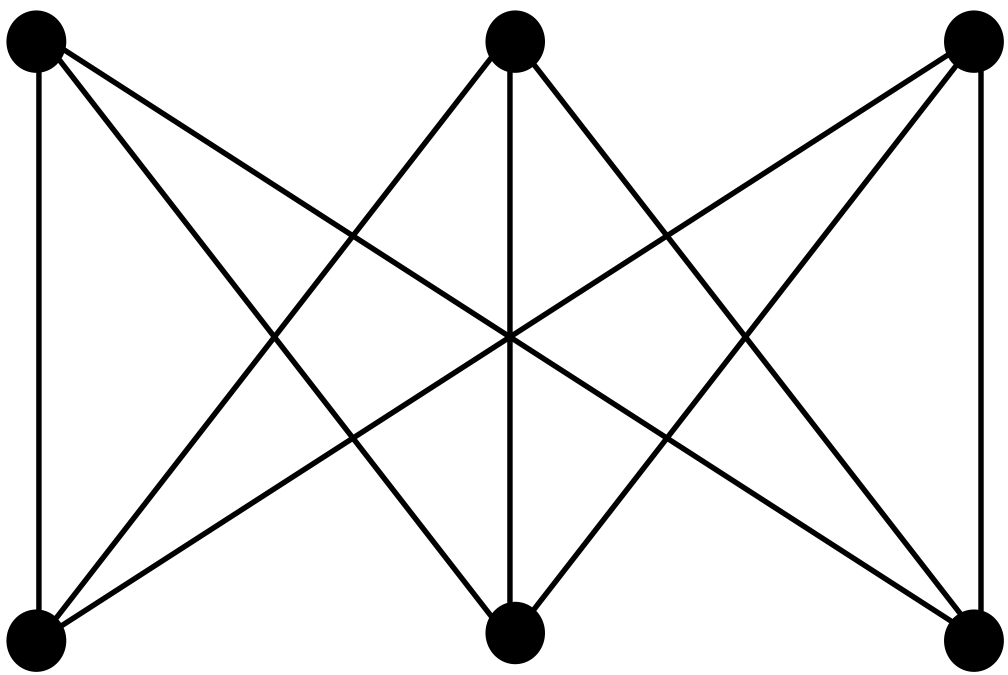


Planar Graphs

- Consider the previous slide. Is it possible to join the three houses to the three utilities in such a way that none of the connections cross?

Planar Graphs

- Phrased another way, this question is equivalent to: Given the complete bipartite graph $K_{3,3}$, can $K_{3,3}$ be drawn in the plane so that no two of its edges cross?



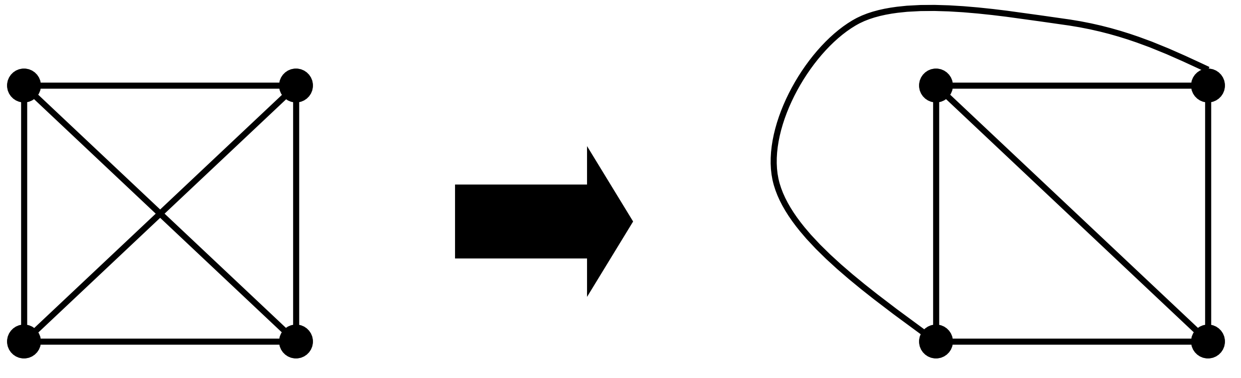
$K_{3,3}$

Planar Graphs

- A graph is called *planar* if it can be drawn in the plane without any edges crossing.
- A crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint.
- Such a drawing is called a *planar representation* of the graph.

Example

A graph may be planar even if it is usually drawn with crossings, since it may be possible to draw it in another way without crossings.



Example

A graph may be planar even if it represents a 3-dimensional object.

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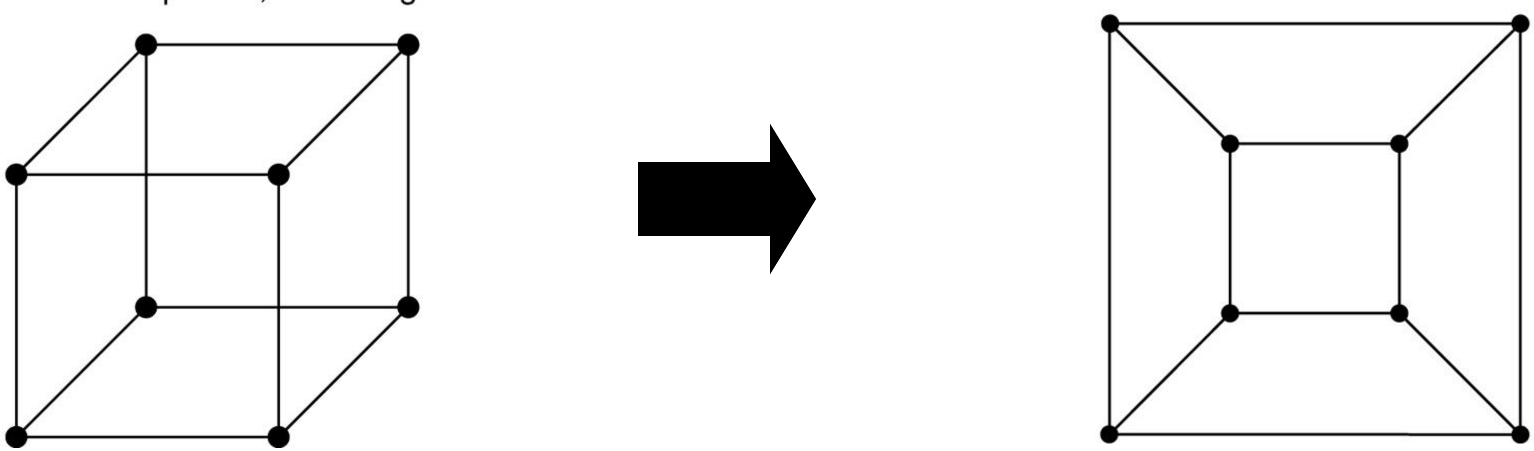
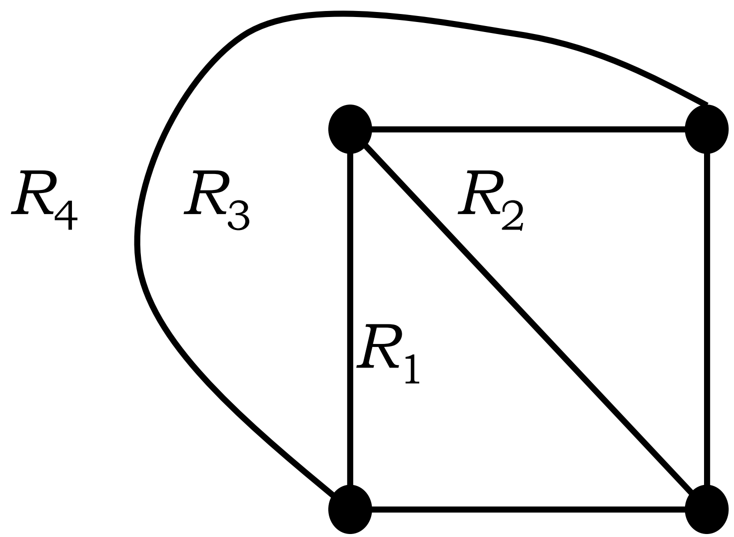


Planar Graphs

- We can prove that a particular graph is planar by showing how it can be drawn without any crossings.
- However, not all graphs are planar.
- It may be difficult to show that a graph is nonplanar. We would have to show that there is *no way* to draw the graph without any edges crossing.

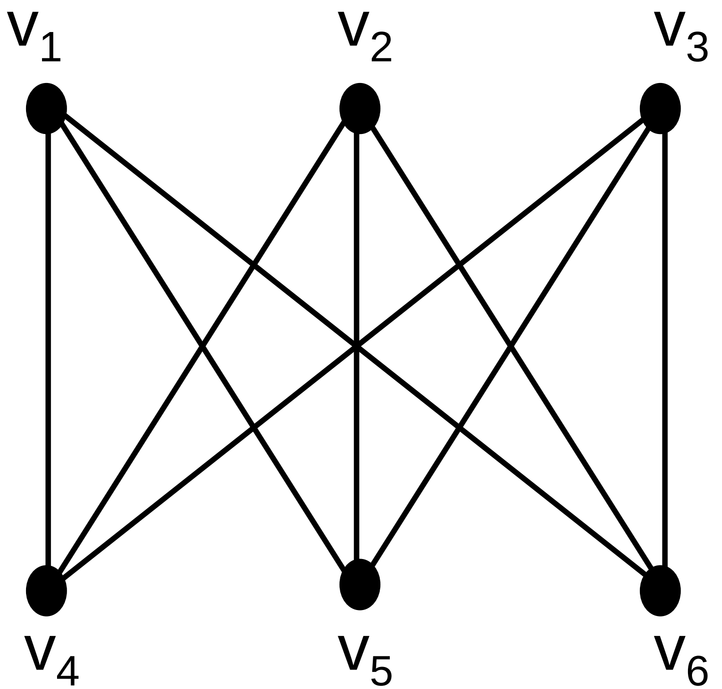
Regions

- Euler showed that all planar representations of a graph split the plane into the some number of *regions*, including an unbounded region.



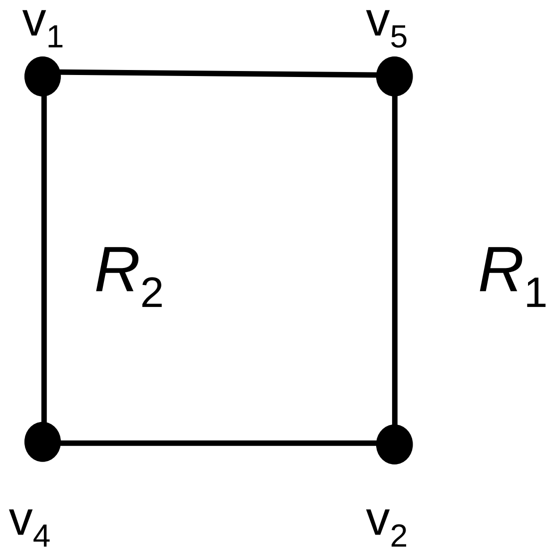
Regions

- In any planar representation of $K_{3,3}$, vertex v_1 must be connected to both v_4 and v_5, and v_2 also must be connected to both v_4 and v_5.



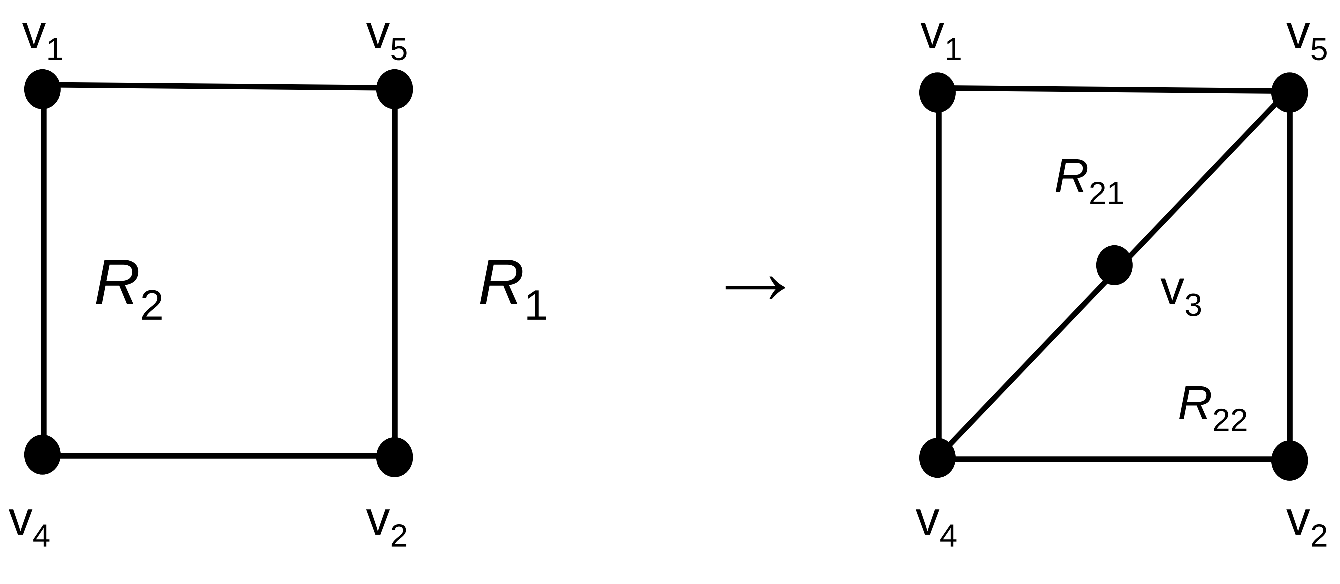
Regions

- The four edges $\{v_1, v_4\}$, $\{v_4, v_2\}$, $\{v_2, v_5\}$, $\{v_5, v_1\}$ form a closed curve that splits the plane into two regions, R_1 and R_2.



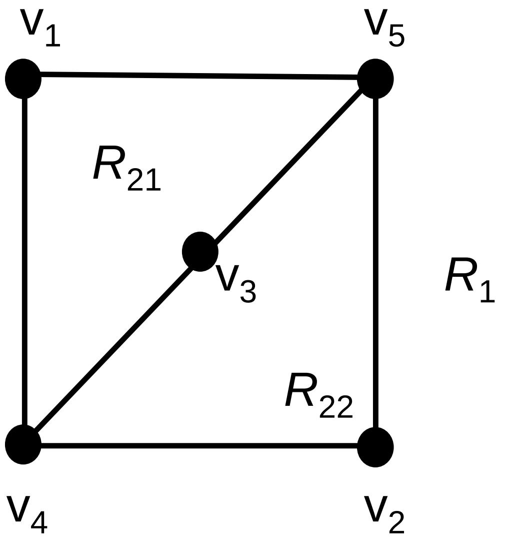
Regions

- Next, we note that v_3 must be in either R_1 or R_2.
- Assume v_3 is in R_2. Then the edges $\{v_3, v_4\}$ and $\{v_4, v_5\}$ separate R_2 into two subregions, R_{21} and R_{22}.



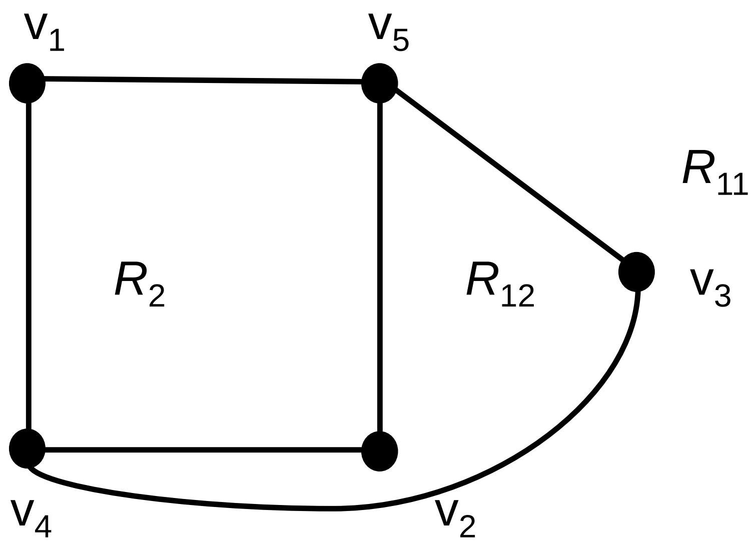
Regions

- Now there is no way to place vertex v_6 without forcing a crossing:
  - If v_6 is in R_1 then $\{v_6, v_3\}$ must cross an edge
  - If v_6 is in R_{21} then $\{v_6, v_2\}$ must cross an edge
  - If v_6 is in R_{22} then $\{v_6, v_1\}$ must cross an edge



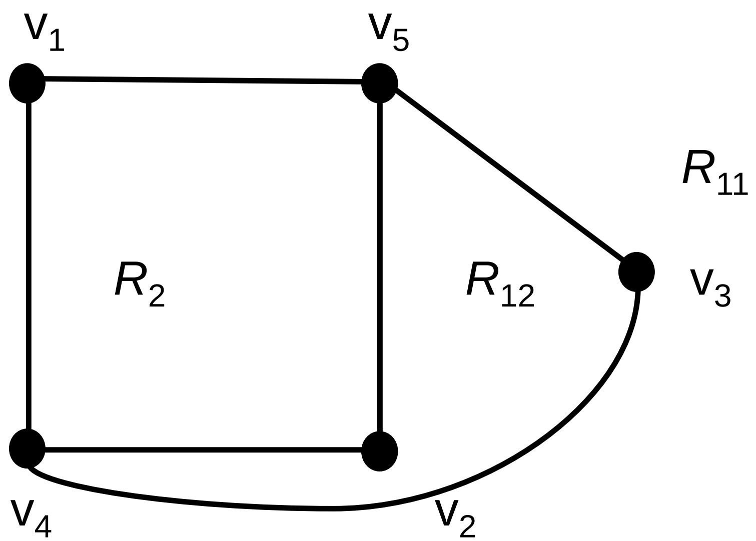
Regions

- Alternatively, assume v_3 is in R_1. Then the edges $\{v_3, v_4\}$ and $\{v_4, v_5\}$ separate R_1 into two subregions, R_{11} and R_{12}.



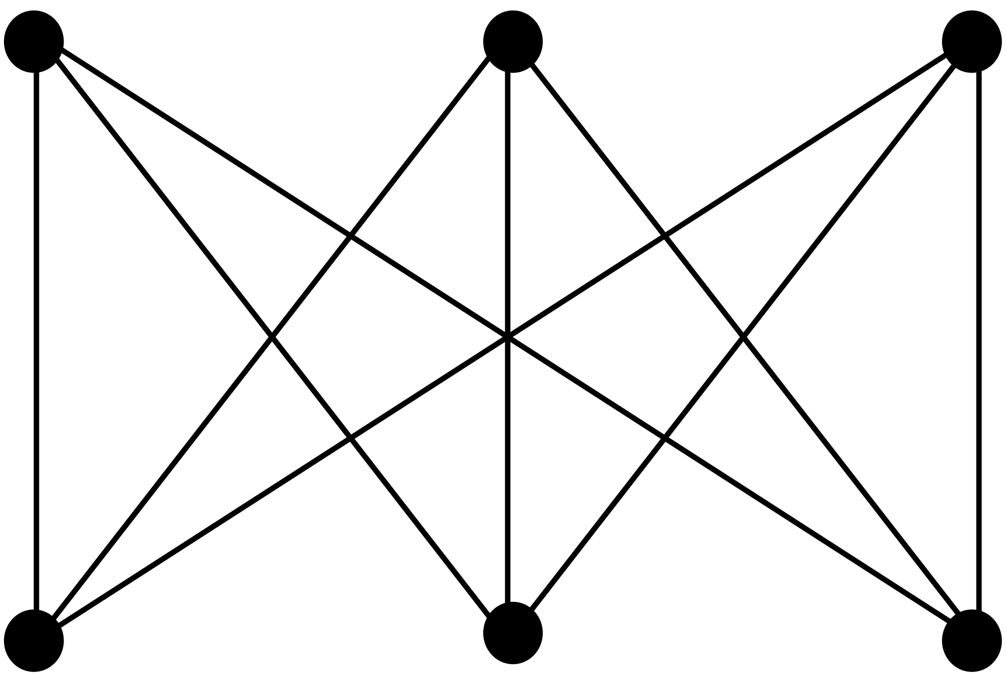
Regions

- Now there is no way to place vertex v_6 without forcing a crossing:
  - If v_6 is in R_2 then $\{v_6, v_3\}$ must cross an edge
  - If v_6 is in R_{11} then $\{v_6, v_2\}$ must cross an edge
  - If v_6 is in R_{12} then $\{v_6, v_1\}$ must cross an edge



Planar Graphs

- Consequently, the graph $K_{3,3}$ must be nonplanar.



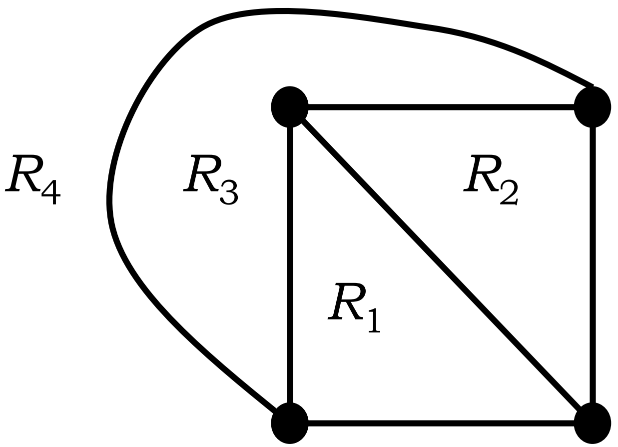
$K_{3,3}$

Regions

- Euler devised a formula for expressing the relationship between the number of vertices, edges, and regions of a planar graph.
- These *may* help us determine if a graph can be planar or not.

Euler's Formula

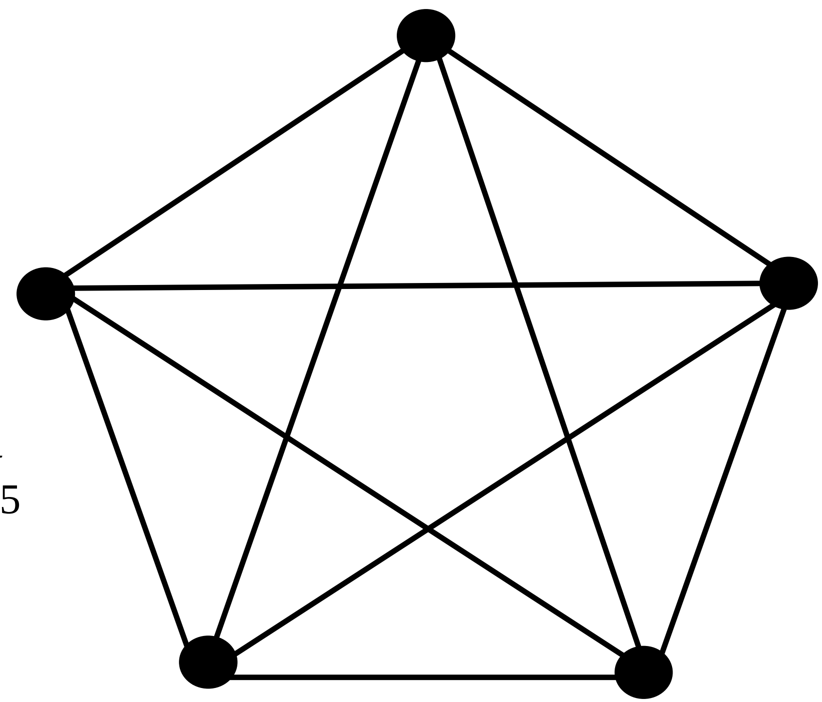
- Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.



# of edges, $e = 6$
# of vertices, $v = 4$
# of regions, $r = e - v + 2 = 4$

Euler's Formula (Cont.)

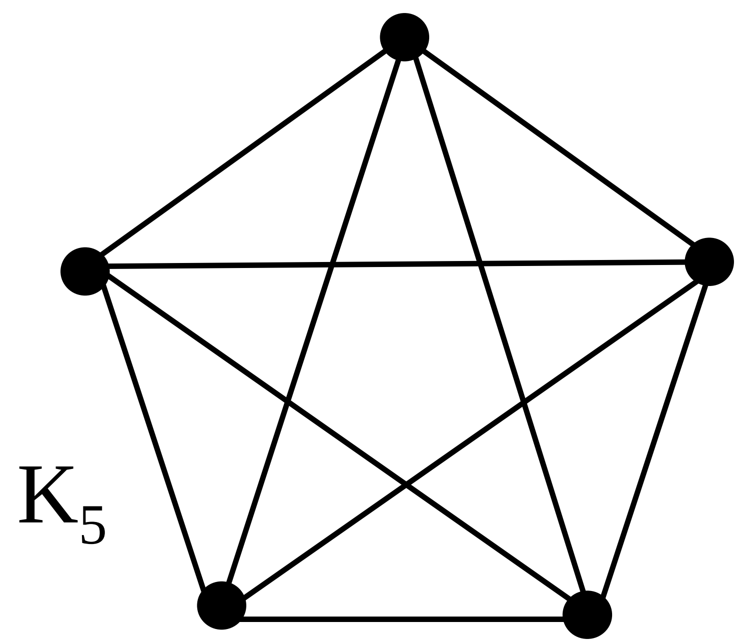
- Corollary 1: If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v - 6$.
- Is K_5 planar?



K_5

Euler's Formula (Cont.)

- K_5 has 5 vertices and 10 edges.
- We see that $v \geq 3$.
- So, if K_5 is planar, it must be true that $e \leq 3v - 6$.
- $3v - 6 = 3*5 - 6 = 15 - 6 = 9$.
- So e must be ≤ 9.
- But $e = 10$.
- So, K_5 is nonplanar.

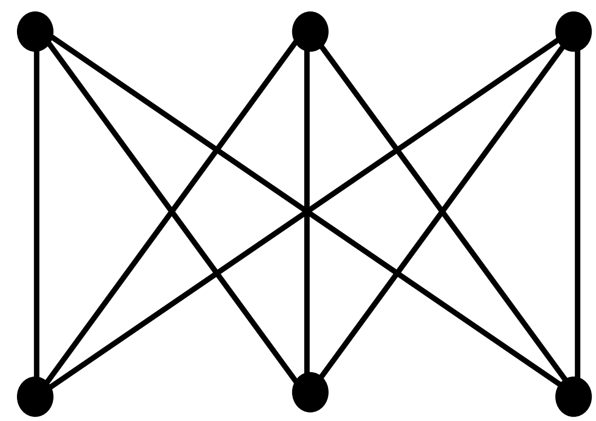


Euler's Formula (Cont.)

- Corollary 2: If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.

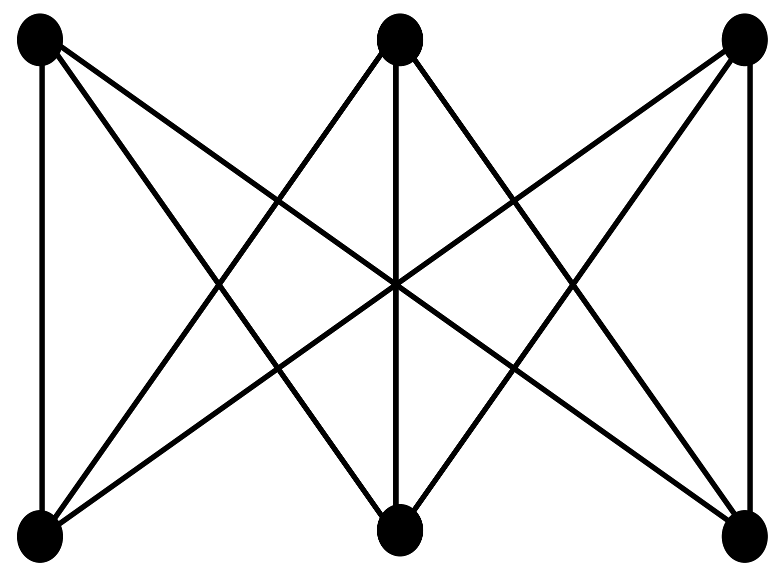
Euler's Formula (Cont.)

- Corollary 3: If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length 3, then $e \leq 2v - 4$.
- Is $K_{3,3}$ planar?



Euler’s Formula (Cont.)

- $K_{3,3}$ has 6 vertices and 9 edges.
- Obviously, v ≥ 3 and there are no circuits of length 3.
- If $K_{3,3}$ were planar, then $e \le 2v - 4$ would have to be true.
- $2v - 4 = 2*6 - 4 = 8$
- So e must be ≤ 8.
- But $e = 9$.
- So $K_{3,3}$ is nonplanar.



$K_{3,3}$

CSE 2813
Discrete Structures

Chapter 9.8
Graph Coloring

These class notes are based on material from our textbook, **Discrete Mathematics and Its Applications**, 6th ed., by Kenneth H. Rosen, published by McGraw Hill, Boston, MA, 2006. They are intended for classroom use only and are **not** a substitute for reading the textbook.

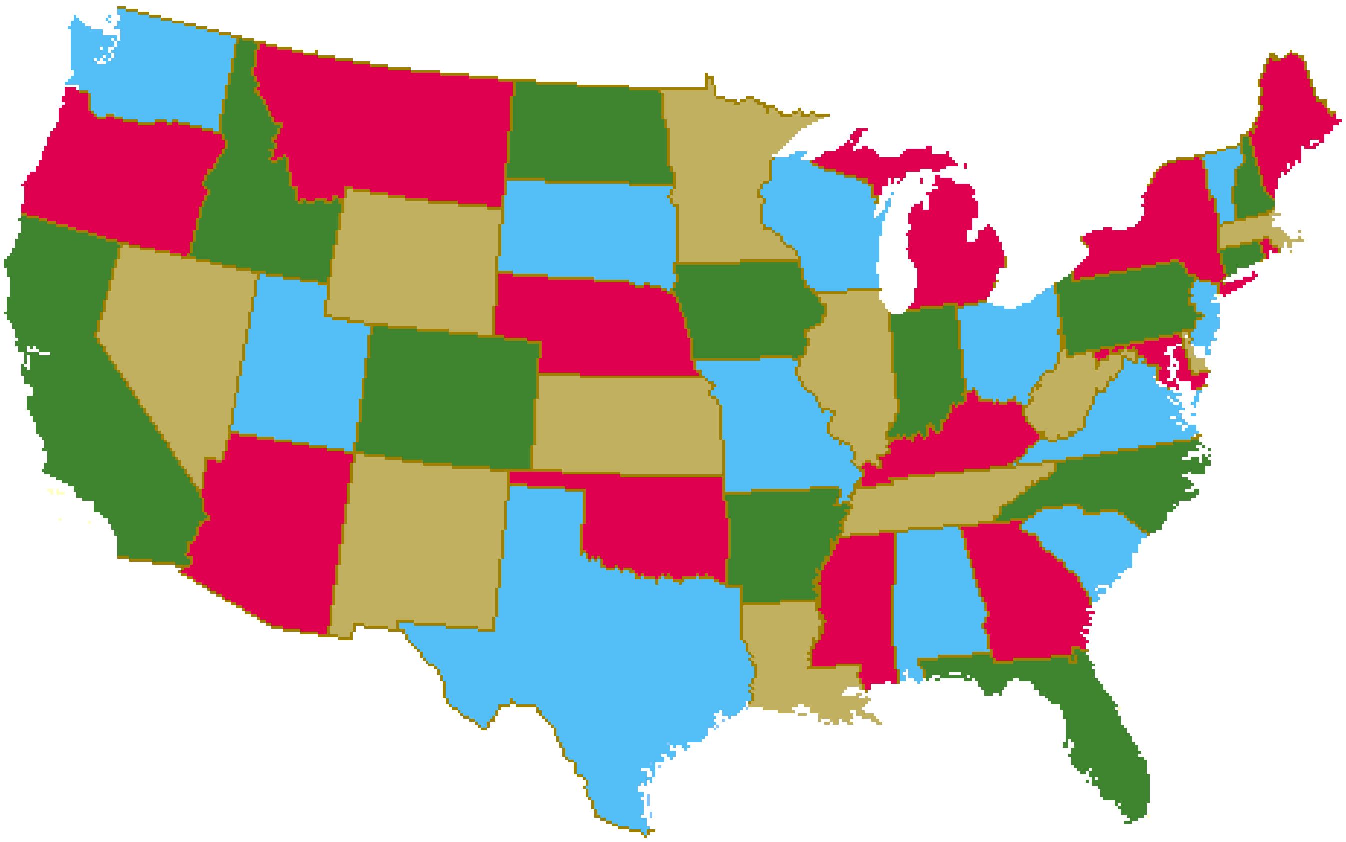
Introduction

- When a map is colored, two regions with a common border are customarily assigned different colors.
- We want to use the smallest number of colors instead of just assigning every region its own color.

4-Color Map Theorem

- It can be shown that any two-dimensional map can be painted using four colors in such a way that adjacent regions (meaning those which sharing a common boundary segment, and not just a point) are different colors.

Map Coloring

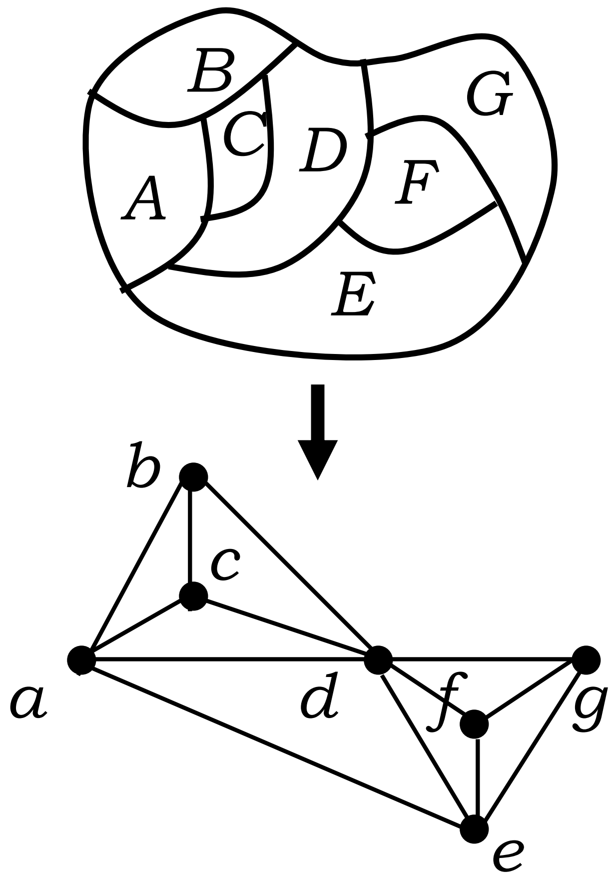


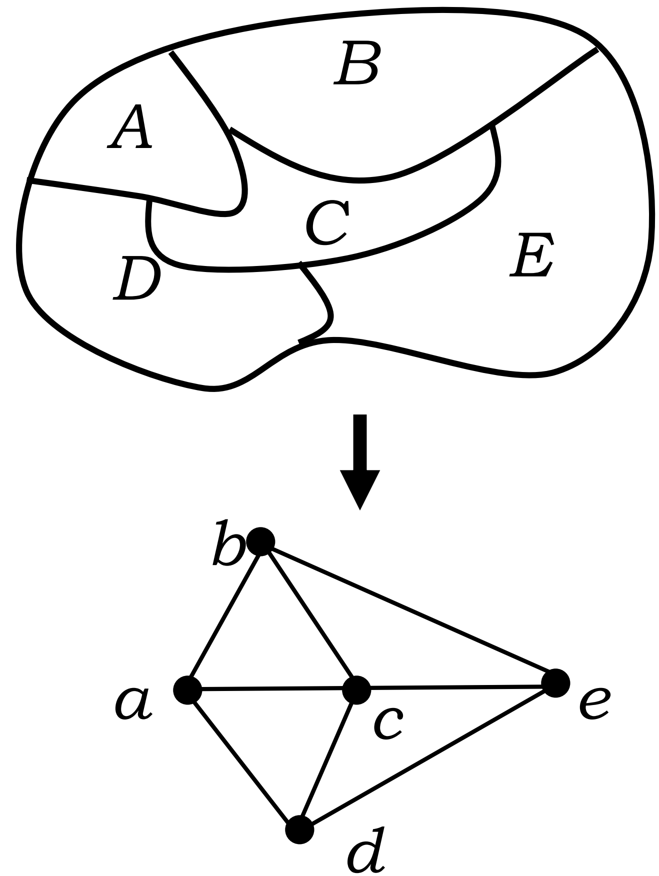
- Four colors are sufficient to color a map of the contiguous United States.
- *Source of map: http://www.math.gatech.edu/~thomas/FC/fourcolor.html*

Dual Graph

- Each map in a plane can be represented by a graph.
  - Each region is represented by a vertex.
  - Edges connect to vertices if the regions represented by these vertices have a common border.
  - Two regions that touch at only one point are not considered adjacent.
- The resulting graph is called the *dual graph* of the map.

Dual Graph Examples



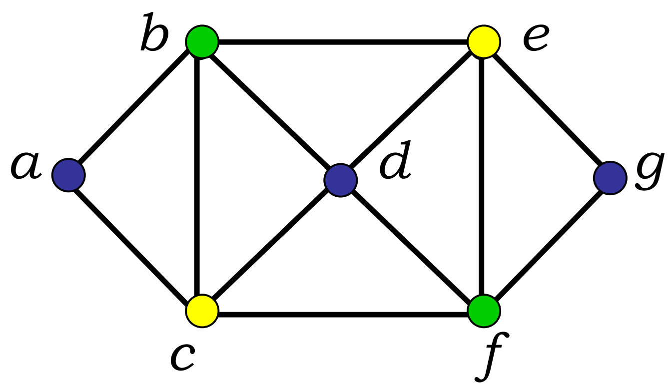


Graph Coloring

- A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.
- The *chromatic number* of a graph is the least number of colors needed for a coloring of the graph.
- The Four Color Theorem: *The chromatic number of a planar graph is no greater than four.*

Example

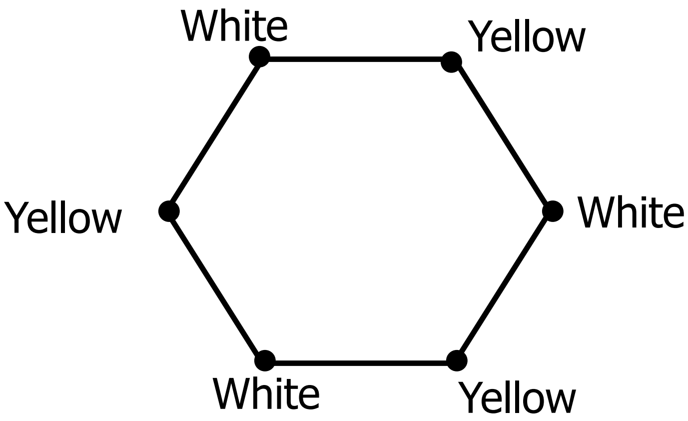
- What is the chromatic number of the graph shown below?

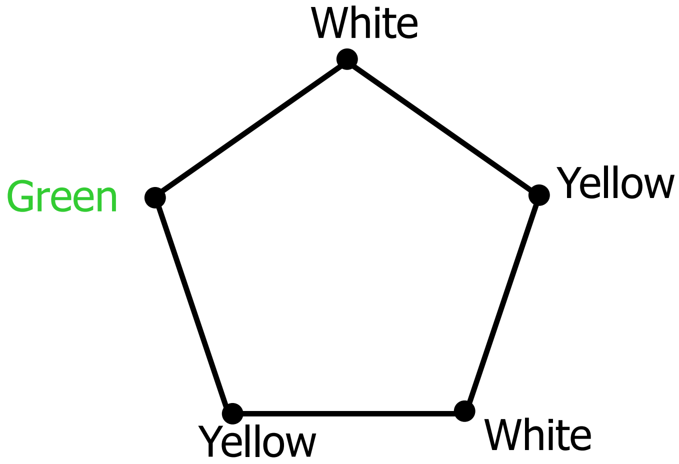


The chromatic number must be at least 3 since *a*, *b*, and *c* must be assigned different colors. So Let's try 3 colors first. 3 colors work, so the chromatic number of this graph is 3.

Example

- What is the chromatic number for each of the following graphs?





Chromatic number: 2

Chromatic number: 3

Conclusion

- In this chapter we have covered:
  - Introduction to Graphs
  - Graph Terminology
  - Representing Graphs and Graph Isomorphism
  - Graph Connectivity
  - Euler and Hamilton Paths
  - Planar Graphs
  - Graph Coloring