Mathematic Preliminaries

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Matrix

- A matrix is a rectangular array of numbers. An example of 3×4 (3 rows and 4 columns) is shown below:

$$\begin{bmatrix} 1 & 5 & 2 & -3 \\ 12 & -4 & 9 & 6 \\ -8 & 11 & -3 & 9 \end{bmatrix}$$

- Generally, we use a_{ij} to refer to the element at row i column j. An example of an $m \times n$ matrix $\mathbf{A}$ is shown below:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- From the above example, $a_{11} = 1$, $a_{32} = 11$, and $a_{24} = 6$.

Matrix

- Given a matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The first column of $\mathbf{A}$ is $\begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{m1} \end{bmatrix}$.
- The second row of $\mathbf{A}$ is $\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$.

Scalar Multiplication

- Given a matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- $\alpha\mathbf{A}$ where α is a scalar, is a new matrix which can be obtained by multiplying every entry of $\mathbf{A}$ by α.

$$\alpha\mathbf{A} = \alpha \times \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{bmatrix}$$

- For simplicity, we write $-\mathbf{A}$ instead of $(-1)\mathbf{A}$

Matrix Additions

- Two matrices which are the **same size** can be added.
- For example, given two matrices $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

- $\mathbf{A} + \mathbf{B}$ is defined as follows:

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix} \end{aligned}$$

Properties

- Two matrices are equal exactly when they are the same size and the corresponding entries are identical.
- Properties
  - Commutative law of addition:
  $$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$
  - Associative law of addition:
  $$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$
  - Existence of an additive identity:
  $$\mathbf{A} + \mathbf{0} = \mathbf{A}$$
  - Existence of an additive inverse:
  $$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$$
  - For any scalar α and β:
  $$\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B} \qquad \alpha(\beta\mathbf{A}) = \alpha\beta(\mathbf{A})$$
  $$(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A} \qquad \mathbf{1A} = \mathbf{A}$$

Matrices as Vectors

- $n \times 1$ or $1 \times n$ matrices are called **vectors**

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad u = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

- We generally call v as a **column vector** and u as a **row vector**.

Multiplication with Column Vector

- Multiplying an $m \times n$ matrix by an $n \times 1$ column vector as defined as follows:

$$
\begin{aligned}
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}
&= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \\
&= v_1 \mathbf{a}_1 + v_2 \mathbf{a}_2 + \cdots + v_n \mathbf{a}_n \\
&= \sum_{j=1}^{n} v_j \mathbf{a}_j
\end{aligned}
$$

  where $\mathbf{A}_j$ is the j^{th} column of $\mathbf{A}$.

- This is called **linear combination** of the columns.

Matrix Multiplications

- Given two matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{AB}$ is possible if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$.
- To compute,

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{bmatrix}$$

we can use the same concept as column vector which gives

$$\left[\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right]$$

which is

$$\left[1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]$$

- Note that **order** matters in matrix multiplication $(\mathbf{AB} \neq \mathbf{BA})$.

Properties of Matrix Multiplication

- The following properties hold for matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and scalar α and β:
  - $\mathbf{A}(\alpha\mathbf{B} + \beta\mathbf{C}) = \alpha(\mathbf{AB}) + \beta(\mathbf{AC})$
  - $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$
  - $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

Transpose

- The transpose of a matrix $\mathbf{A}$ is denoted by $\mathbf{A}^T$
- For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

- The first column because the first row and the second column because the second row
- Suppose $\mathbf{A}$ and $\mathbf{B}$ are matrices and α and β are scalar, the following properties hold:
  - $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$
  - $(\alpha\mathbf{A} + \beta\mathbf{B})^T = \alpha\mathbf{A}^T + \beta\mathbf{B}^T$

Transpose

- An $n \times n$ matrix $\mathbf{A}$ is
  - Symmetric if $\mathbf{A} = \mathbf{A}^T$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & -4 \\ 3 & -4 & 9 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & -4 \\ 3 & -4 & 9 \end{bmatrix}$$

  - Skew Symmetric if $\mathbf{A}^T = -\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix}$$

- For simplicity, sometimes we use transpose to represent a **column vector** since

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}^T$$

Identity Matrix

- An identity matrix $\mathbf{I}_n$ is an $n \times n$ as shown below:

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

- Formally, if δ_{ij} is an element of $\mathbf{I}_n$,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- For an $m \times n$ matrix $\mathbf{A}$, the following property hold:

$$\mathbf{A}\mathbf{I}_n = \mathbf{A}$$

- An $n \times n$ matrix $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$ if and only if there exists a matrix, denoted as $\mathbf{A}^{-1}$ such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$.
- For any two matrices $\mathbf{A}$ and $\mathbf{B}$, if $\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A} = \mathbf{I}$, then $\mathbf{B} = \mathbf{A}^{-1}$.
- If an inverse of a matrix exists, the matrix is called **invertible**
- The inverse of a matrix is **unique**

  Proof: Suppose $\mathbf{B}$ and $\mathbf{B}'$ are inverses of $\mathbf{A}$

$$\mathbf{B}' = \mathbf{B}'\mathbf{I} = \mathbf{B}'(\mathbf{A}\mathbf{B}) = (\mathbf{B}'\mathbf{A})\mathbf{B} = \mathbf{I}\mathbf{B} = \mathbf{B}$$

Finding the Inverse of a Matrix

- Consider a matrix $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$, its inverse $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exists if

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} a & b \\ a-c & b-d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  The above equality gives us four equations with four variables a, b, c, and d that we need to solve ($a = 1$, $b = 0$, $c = 1$, and $d = -1$)

- Check:

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} (1)(1)+(0)(1) & (1)(0)+(0)(-1) \\ (1)(1)+(-1)(1) & (0)(1)+(-1)(-1) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse of a 4 $\times$ 4 Matrix

- We are going to use 4×4 matrices a lot in this class
- Some procedures require inverse of matrices
- There are various way to find the inverse of a 4×4 matrix
  - Some methods are easy to calculate but hard to program
  - Some require a lot of calculations but easy to program
- For us, we just want to implement a function that returns the inverse of a 4×4 matrix

Inverse of a Matrix

- To calculate the inverse of a matrix $\mathbf{A}$, perform the following steps:
  1. Calculate the matrix of minor of $\mathbf{A}$
  2. Turn the matrix of minor of $\mathbf{A}$ into the matrix of cofactor
  3. Transpose the result of previous step
  4. Multiply the result of previous step by $1/(\text{determinant of } \mathbf{A})$
- This method can be use with any square matrices.
- Easy to program if we focus specifically for 4×4 matrix

```
mat4 a = ...;
mat4 minor = m4_minor(a);
mat4 cofactor = m4_cofactor(minor);
mat4 transpose = m4_transpose(cofactor);
GLfloat determinant = m4_determinant(a);
mat4 a_inv = sm4_multiplication(1/determinant, transpose);
```

Matrix

- A 4×4 matrix $\mathbf{A}$ is a follows: $\mathbf{A}$ is shown below:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

- Generally, we use a_{ij} to refer to the element at row i column j.

Matrix of Minor

- The matrix of minor of a 4×4 matrix $\mathbf{A}$ is a 4×4 matrix

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

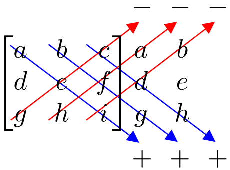
  where m_{ij} is the determinant of the matrix $\mathbf{A}$ where row i and column j are removed

- For example,

$$m_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} \text{ and } m_{43} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}$$

Determinant of a 3×3 matrix

- For a 3×3 matrix, we can find its determinant using the **rule of Sarrus**



- For example,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb$$

- **Note** if there is a column or a row in a matrix that contains all 0s, the determinant of that matrix will be 0.

Matrix of Cofactor

- Cofactor of a matrix can be easily obtained by simply apply a 4×4 checkerboard of **minuses**

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \rightsquigarrow \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} = \begin{bmatrix} m_{11} & -m_{12} & m_{13} & -m_{14} \\ -m_{21} & m_{22} & -m_{23} & m_{24} \\ m_{31} & -m_{32} & m_{33} & -m_{34} \\ -m_{41} & m_{42} & -m_{43} & m_{44} \end{bmatrix}$$

Transpose of a Cofactor

- Next simply transpose the cofactor:

$$\begin{bmatrix} m_{11} & -m_{12} & m_{13} & -m_{14} \\ -m_{21} & m_{22} & -m_{23} & m_{24} \\ m_{31} & -m_{32} & m_{33} & -m_{34} \\ -m_{41} & m_{42} & -m_{43} & m_{44} \end{bmatrix}^T = \begin{bmatrix} m_{11} & -m_{21} & m_{31} & -m_{41} \\ -m_{12} & m_{22} & -m_{32} & m_{42} \\ m_{13} & -m_{23} & m_{33} & -m_{43} \\ -m_{14} & m_{24} & -m_{34} & m_{44} \end{bmatrix}$$

Multiply by 1/Determinant

- The last step is to multiply by 1/(Determinant of $\mathbf{A}$)

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} m_{11} & -m_{21} & m_{31} & -m_{41} \\ -m_{12} & m_{22} & -m_{32} & m_{42} \\ m_{13} & -m_{23} & m_{33} & -m_{43} \\ -m_{14} & m_{24} & -m_{34} & m_{44} \end{bmatrix}$$

- Since we already of the matrix of minor of $\mathbf{A}$, the determinant of $\mathbf{A}$ can be easily calculated by

$$|\mathbf{A}| = a_{11}m_{11} - a_{12}m_{12} + a_{13}m_{13} - a_{14}m_{14}$$

- Note that if the determinant of the matrix $\mathbf{A}$ is 0, no inverse.

Example

- Let $\mathbf{A} = \begin{bmatrix} -4 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -4 & 4 & 4 & 0 \\ -1 & -9 & -1 & 1 \end{bmatrix}$. Find the inverse of $\mathbf{A}$.
- Recall that

$$\begin{aligned}
m_{11} &= \begin{vmatrix} 2 & 0 & 0 \\ 4 & 4 & 0 \\ -9 & -1 & 1 \end{vmatrix} \\
&= (2)(4)(1) + (0)(0)(-9) + (0)(4)(-1) - (-9)(4)(0) - (-1)(0)(2) - (1)(4)(0) \\
&= 8 \\
m_{12} &= \begin{vmatrix} -1 & 0 & 0 \\ -4 & 4 & 0 \\ -1 & -1 & 1 \end{vmatrix} \\
&= (-1)(4)(1) + (0)(0)(-1) + (0)(-4)(-1) - (-1)(4)(0) - (-1)(0)(-1) \\
&\quad - (1)(-4)(0) \\
&= -4 \\
&\vdots
\end{aligned}$$

Example

- Matrix of minor of $\mathbf{A}$ is as follows:

$$\begin{bmatrix} 8 & -4 & 4 & -48 \\ 0 & -16 & -16 & -128 \\ 0 & 0 & -8 & 8 \\ 0 & 0 & 0 & -32 \end{bmatrix}$$

- Recall that the determinant of $\mathbf{A}$ can be calculated by:

$$\begin{aligned} |\mathbf{A}| &= a_{11}m_{11} - a_{12}m_{12} + a_{13}m_{13} - a_{14}m_{14} \\ &= (-4)(8) - (0)(-4) + (0)(4) - (0)(-48) \\ &= -32 \end{aligned}$$

Example

- The cofactor is as follows:

$$\begin{bmatrix} 8 & -4 & 4 & -48 \\ 0 & -16 & -16 & -128 \\ 0 & 0 & -8 & 8 \\ 0 & 0 & 0 & -32 \end{bmatrix} \rightsquigarrow \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} = \begin{bmatrix} 8 & 4 & 4 & 48 \\ 0 & -16 & 16 & -128 \\ 0 & 0 & -8 & -8 \\ 0 & 0 & 0 & -32 \end{bmatrix}$$

- Transpose

$$\begin{bmatrix} 8 & 4 & 4 & 48 \\ 0 & -16 & 16 & -128 \\ 0 & 0 & -8 & -8 \\ 0 & 0 & 0 & -32 \end{bmatrix}^T = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 4 & -16 & 0 & 0 \\ 4 & 16 & -8 & 0 \\ 48 & -128 & -8 & -32 \end{bmatrix}$$

Example

- Finally, multiply by $1/|\mathbf{A}|$:

$$\mathbf{A}^{-1} = \frac{1}{-32}\begin{bmatrix} 8 & 0 & 0 & 0 \\ 4 & -16 & 0 & 0 \\ 4 & 16 & -8 & 0 \\ 48 & -128 & -8 & -32 \end{bmatrix} = \begin{bmatrix} -1/4 & 0 & 0 & 0 \\ -1/8 & 1/2 & 0 & 0 \\ -1/8 & -1/2 & 1/4 & 0 \\ 3/2 & 4 & 1/4 & 1 \end{bmatrix}$$

- Let's check

$$\begin{bmatrix} -4 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -4 & 4 & 4 & 0 \\ -1 & -9 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} -1/4 & 0 & 0 & 0 \\ -1/8 & 1/2 & 0 & 0 \\ -1/8 & -1/2 & 1/4 & 0 \\ 3/2 & 4 & 1/4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Vector Spaces

- Given a set of all vectors V and vectors u, v, and w:
  - Vector-vector addition is closed
$$u + v \in V$$
  - Vector-vector addition is commutative
$$u + v = v + u$$
  - Vector-vector addition is associative
$$u + (v + w) = (u + v) + w$$
- A special vector $\mathbf{0} \in V$ such that for any vector $u \in V$,
  - $u + \mathbf{0} = u$, and
  - $u + (-u) = \mathbf{0}$.

Vector Spaces

- Given a set of all vectors V and a set of all scalar $\mathcal{R}$
  - Scalar-vector multiplication is distributive

$$\forall\, u\ v\ \alpha\ (u \in V) \wedge (v \in V) \wedge (\alpha \in \mathcal{R}) \rightarrow \alpha(u + v) = \alpha u + \alpha v$$

$$\forall\, u\ \alpha\ \beta\ (u \in V) \wedge (\alpha \in \mathcal{R}) \wedge (\beta \in \mathcal{R}) \rightarrow (\alpha + \beta)u = \alpha u + \beta u$$

- A vector can be represented by n-tuple of scalars:

$$u = (x_1, x_2, \ldots, x_n)$$

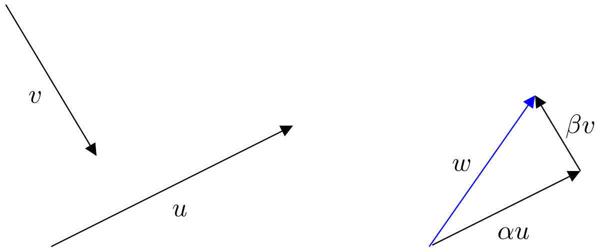
$$v = (y_1, y_2, \ldots, y_n)$$

- Operations are defined in a straightforward way:

$$\begin{aligned} u + v &= (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) \\ &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ \alpha u &= \alpha(x_1, x_2, \ldots, x_n) \\ &= (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \end{aligned}$$

Vector Spaces

- Given two vectors u and v (not parallel to each other), a vector w can be represented by these two vectors



$$w = \alpha u + \beta v$$

- Given n vectors $u_1, u_2, \ldots, u_n$ we can represent a vector u by a **linear combination** of these n vector as follows:

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

where each α_i is a scalars.

Linearly Independent

- Given n vectors $u_1, u_2, \ldots, u_n$ and n scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, if

  $$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = \mathbf{0}$$

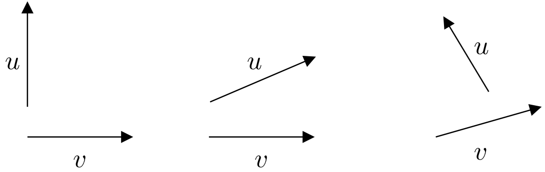
  only if

  $$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

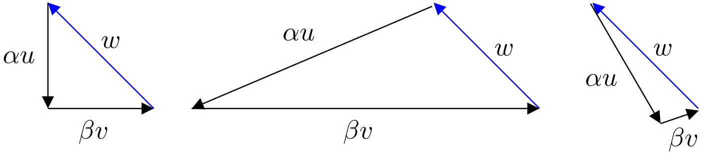
  $u_1, u_2, \ldots, u_n$ are said to be **linearly independent**.
- The **dimension** of the space is the maximum number of linearly independent vectors in the space.
- In n-dimension vector space, n linearly independent vectors are called **basis**.

Basis Vectors

- Examples of two linearly independent vectors in 2-dimension vector space



- If the third vector w is added



In all cases, $\alpha u + \beta v + w = \mathbf{0}$. Obviously α and β are not zero.

Basis Vectors

- Given a basis $u_1, u_2, \ldots, u_n$, a vector v can be represented by

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

- Given another basis $v_1, v_2, \ldots, v_n$, same vector v can be represented by

$$v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$$

- In other words

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$$

- If we consider $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $(\beta_1, \beta_2, \ldots, \beta_n)$ as representations of the vector v in two different bases, we can have $n \times n$ matrix **M** such that

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \mathbf{M} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

Basis Vectors

- Given two bases $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$
- Note that u_i is just a vector, therefore, it can be represented by

$$
\begin{aligned}
u_1 &= \gamma_{11} v_1 + \gamma_{12} v_2 + \cdots + \gamma_{1n} v_n \\
u_2 &= \gamma_{21} v_1 + \gamma_{22} v_2 + \cdots + \gamma_{2n} v_n \\
&\vdots \\
u_n &= \gamma_{n1} v_1 + \gamma_{n2} v_2 + \cdots + \gamma_{nn} v_n
\end{aligned}
$$

- If we look at the above equations in a form of matrices

$$
\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \ldots & \gamma 1n \\ \gamma_{21} & \gamma_{22} & \ldots & \gamma 2n \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \ldots & \gamma nn \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}
$$

Basis Vectors

- Given two bases $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$
- Let

$$\mathbf{A} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \ldots & \gamma 1n \\ \gamma_{21} & \gamma_{22} & \ldots & \gamma 2n \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \ldots & \gamma nn \end{bmatrix}$$

we have

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{A} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Basis Vectors

- Recall $v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$ be a vector in the basis $\{u_1, u_2, \ldots, u_n\}$
- The vector v can be represented by

$$v = \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

- Let $\mathbf{a}$ be a column matrix

$$\mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

  we have

$$v = \mathbf{a}^T \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Basis Vectors

- Similarly $v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$ be a vector in the basis $\{v_1, v_2, \ldots, v_n\}$
- The vector v can be represented by

$$v = \begin{bmatrix} \beta_1 & \beta_2 & \ldots & \beta_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- Let $\mathbf{b}$ be a column matrix

$$\mathbf{b} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

we have

$$v = \mathbf{b}^T \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Basis Vectors

- From previous two slides we have

$$\mathbf{b}^T \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \mathbf{a}^T \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

- Since

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{A} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

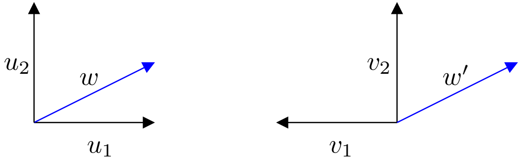
we have

$$\mathbf{b}^T \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \mathbf{a}^T \mathbf{A} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

in other words

$$\mathbf{b}^T = \mathbf{a}^T \mathbf{A}$$

- The matrix **A** can be used to change representation from one basis to another.
- For example, consider two bases in 2 dimension and a vector w



Let $u_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $v_1 = -u_1 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, and $v_2 = u_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

- If $w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, what would be w'?
    - Looks like it should be $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Basis Vectors

- First, we need to find the matrix **A** such that

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

- Let $A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$, from matrix multiplication, we have

$$\begin{aligned} u_1 &= \alpha_{11} v_1 + \alpha_{12} v_2 \\ u_2 &= \alpha_{21} v_1 + \alpha_{22} v_2 \end{aligned}$$

Replace all u_i and v_i, we obtain

$$\begin{aligned} \begin{bmatrix} 2 \\ 0 \end{bmatrix} &= \alpha_{11} \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \alpha_{12} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 2 \end{bmatrix} &= \alpha_{21} \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \alpha_{22} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{aligned}$$

Basis Vectors

- We obtain the following equations
  - $2 = \alpha_{11}(-2) + \alpha_{12}(0) \rightsquigarrow \alpha_{11} = -1$,
  - $0 = \alpha_{11}(0) + \alpha_{12}(2) \rightsquigarrow \alpha_{12} = 0$,
  - $0 = \alpha_{21}(-2) + \alpha_{22}(0) \rightsquigarrow \alpha_{21} = 0$, and
  - $2 = \alpha_{21}(0) + \alpha_{22}(2) \rightsquigarrow \alpha_{22} = 1$.
- Thus $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
- Recall $w'^T = w^T \mathbf{A}$

$$\begin{aligned} w'^T &= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (2 \times -1) + (1 \times 0) & (2 \times 0) + (1 \times 1) \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 \end{bmatrix} \\ w' &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} \end{aligned}$$

Dot Products (Algebraic Definition)

- Dot product of two vectors $u = [x_1, x_2, \ldots, x_n]$ and $v = [y_1, y_2, \ldots, y_n]$ is defined by

$$\begin{aligned} u \cdot v &= \sum_{i=1}^{n} x_i y_i \\ &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \end{aligned}$$

- Example: Let $u = [1, 3, 5]$ and $v = [-2, 4, -6]$

$$\begin{aligned} a \cdot b &= (1 \times -2) + (3 \times 4) + (5 \times -6) \\ &= -2 + 12 - 30 \\ &= -20 \end{aligned}$$

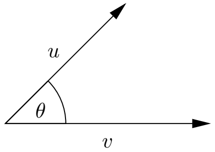
- Note that $[x_1, x_2, x_3] \cdot [y_1, y_2, y_3, y_4]$ is not defined.

Dot Products (Geometric Definition)

- Let $|u|$ be the magnitude (length) of the vector u.
- Dot product of two vectors u and v is defined by

$$u \cdot v = |u||v| \cos \theta$$

where θ is the angle between u and v



- If u and v are orthogonal $\theta = \frac{\pi}{2}$, since $\cos \frac{\pi}{2} = 0$,

$$u \cdot v = 0$$

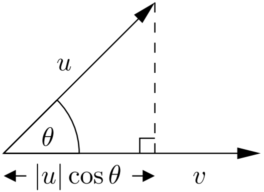
- If u and v are parallel to each other $\theta = 0$, since $\cos 0 = 1$,

$$u \cdot v = |u||v|$$

- Thus $u \cdot u = |u||u| = |u|^2$
- In other words, $|u| = \sqrt{u \cdot u}$

Dot Products (Geometric Definition)

- Consider the following vectors u and v



- According to trigonometry, the length of the projection of u onto v is $|u|\cos\theta$.
- Recall that $u \cdot v = |u||v|\cos\theta$
- The length of the projection of u onto v is

$$|u|\cos\theta = \frac{u \cdot v}{|v|}$$

- Properties:
  - $(\alpha u) \cdot v = \alpha(u \cdot v) = u \cdot (\alpha v)$
  - $u \cdot (v + w) = u \cdot v + u \cdot w$

Dot Products

- Given two vectors $u = [x_1, x_2, \ldots, x_n]$ and $v = [y_1, y_2, \ldots, y_n]$:

$$\sum_{i=1}^{n} x_i y_i \stackrel{?}{=} |u||v| \cos\theta$$

  where θ is the angle between u and v.

- Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be standard basis vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Dot Products

- Note that

$$u = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

$$= \sum_{i=1}^{n} x_i\mathbf{e}_i$$

- Similarly,

$$v = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} y_i\mathbf{e}_i$$

- Recall that $w \cdot w = |w|^2$ for any vector w, thus

$$\mathbf{e}_i \cdot \mathbf{e}_i = |\mathbf{e}_i|^2 = 1$$

  since the magnitude of $\mathbf{e}_i$ is 1.

- From the geometric definition,

$$u \cdot \mathbf{e}_i = |u||\mathbf{e}_i| \cos\theta = |u| \cos\theta = x_i$$

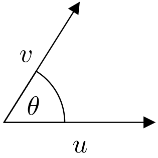
$$v \cdot \mathbf{e}_i = |v||\mathbf{e}_i| \cos\theta = |v| \cos\theta = y_i$$

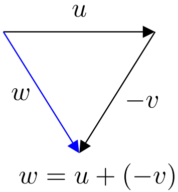
  the projection of u onto the basis vector $\mathbf{e}_i$

- Thus

$$u \cdot v = u \cdot \sum_{i=1}^{n} y_i \mathbf{e}_i = \sum_{i=1}^{n} y_i (u \cdot \mathbf{e}_i) = \sum_{i=1}^{n} y_i x_i = \sum_{i=1}^{n} x_i y_i$$

- Consider two vectors u and v

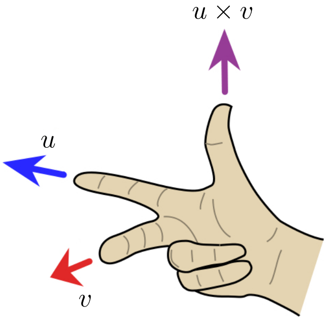


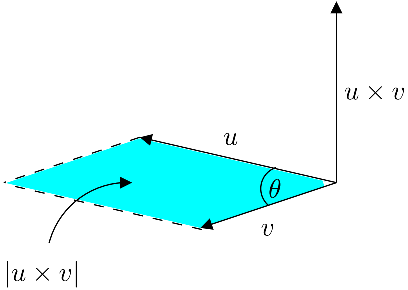


- Note that

$$
\begin{aligned}
w \cdot w &= (u - v) \cdot (u - v) \\
&= u \cdot u - u \cdot v - v \cdot u + v \cdot v \\
&= |u|^2 - 2(u \cdot v) + |v|^2 \\
|w|^2 &= |u|^2 + |v|^2 + 2(|u||v| \cos \theta)
\end{aligned}
$$

- $u \times v$ is a vector perpendicular to both u and v, with the direction given by the right-hand rule.
- Right-Hand Rule





- The magnitude of $u \times v$ is the area of the parallelogram constructed by u and v.

Cross Product

- The cross product of vectors u and v is defined by

$$u \times v = |u||v| \sin\theta \, \mathbf{n}$$

  where θ is the angle between u and v and **n** is a unit vector perpendicular to the plane containing u and v with the direction given by the right-hand rule.
- In three-dimension frame, consider three basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$:
  - $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ and $\mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3$
  - $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ and $\mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1$
  - $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$ and $\mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2$
  - $\mathbf{e}_i \times \mathbf{e}_i = \mathbf{0}$
- Let

$$u = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$$
$$v = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2 + \beta_3\mathbf{e}_3$$

Cross Product

- Consider $u \times v$

$$
\begin{aligned}
u \times v =& (\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3) \times (\beta_1 \mathbf{e}_1 + \beta_2 \mathbf{e}_2 + \beta_3 \mathbf{e}_3) \\
=& \alpha_1 \beta_1 (\mathbf{e}_1 \times \mathbf{e}_1) + \alpha_1 \beta_2 (\mathbf{e}_1 \times \mathbf{e}_2) + \alpha_1 \beta_3 (\mathbf{e}_1 \times \mathbf{e}_3) + \\
& \alpha_2 \beta_1 (\mathbf{e}_2 \times \mathbf{e}_1) + \alpha_2 \beta_2 (\mathbf{e}_2 \times \mathbf{e}_2) + \alpha_2 \beta_3 (\mathbf{e}_2 \times \mathbf{e}_3) + \\
& \alpha_3 \beta_1 (\mathbf{e}_3 \times \mathbf{e}_1) + \alpha_3 \beta_2 (\mathbf{e}_3 \times \mathbf{e}_2) + \alpha_3 \beta_3 (\mathbf{e}_3 \times \mathbf{e}_3) \\
=& \mathbf{0} + \alpha_1 \beta_2 \mathbf{e}_3 - \alpha_1 \beta_3 \mathbf{e}_2 + \\
& - \alpha_2 \beta_1 \mathbf{e}_3 + \mathbf{0} + \alpha_2 \beta_3 \mathbf{e}_1 + \\
& \alpha_3 \beta_1 \mathbf{e}_2 - \alpha_3 \beta_2 \mathbf{e}_1 + \mathbf{0} \\
=& (\alpha_2 \beta_3 - \alpha_3 \beta_2) \mathbf{e}_1 + (\alpha_3 \beta_1 - \alpha_1 \beta_3) \mathbf{e}_2 + (\alpha_1 \beta_2 - \alpha_2 \beta_1) \mathbf{e}_3
\end{aligned}
$$