II Applications of Differential equations.

Introduction
orthogonal trajectories
Newton's law of cooling
simple electrical circuits
Rectilinear motion
simple harmonic motion
Heat flow.

1. Trajectory: A curve which cuts every member of a given family of curves according to some definite law is called a trajectory of the family

2. Orthogonal Trajectory: A curve which cuts every member of a given family of curves at right angles is called orthogonal trajectory of the family.

3. Orthogonal trajectories: Two families of curves are said to be orthogonal if every member of the either family cuts each member of the other family at right angles.

Thus if the given family consists of straight lines $y = mx$ (m - constant) representing family of straight lines all passing through the origin then the family of circles $x^2 + y^2 = a^2$ (a is a parameter), with centres at $(0,0)$ represents a family of orthogonal trajectories to the family $y = mx$.

working rule, to find the eq^n of orthogonal trajectories.

Step 1) → Given $f(x, y, a) = 0$, where a is variable parameter

Step 2) → Differentiate $f(x, y, a) = 0$ w.r.t. x and eliminate 'a', thus form a diff. eq^n of the family of the form $\phi\left(x, y, \frac{dy}{dx}\right) = 0$

replace, $\frac{dy}{dx}$ by $\frac{-dx}{dy}$. then the diff. eq^n of the family of orthogonal trajectories will be

$$\phi\left(x, y, \frac{-dx}{dy}\right) = 0$$

Step 4 : The sol^n of step 3 is the family of orthogonal trajectories.

Q1) Find orthogonal trajectories of the family of staight lines $y = mx$.

→ Given, $y = mx$ —— (I)

diff. w.r.t. x

$$\therefore \frac{dy}{dx} = m$$

put in (I)

$$\therefore y = \frac{dy}{dx} \times x$$

$$\therefore \frac{y}{x} = \frac{dy}{dx}$$

is diff^n eq^n of given family

replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$

$$\therefore \frac{y}{x} = \frac{-dx}{dy}$$

$$\therefore y\,dy = -x\,dx \quad \text{—— (II)}$$

$$\therefore x\,dx + y\,dy = 0$$

this is the diff. eq^n of the orthogonal trajectories.

now: ~~diff.~~ (II) integrating

(II)

$$\therefore \int y\,dy = \int -x\,dx$$

$$\therefore \frac{y^2}{2} = \frac{-x^2}{2} + c$$

$$\therefore \frac{y^2}{2} + \frac{x^2}{2} = c$$

$$\therefore \frac{y^2 + x^2}{2} = c$$

$$\therefore x^2 + y^2 = 2c = c_1$$

$$\therefore x^2 + y^2 = c_1$$

req^d orthogonal trajectories of (I)

Q7) Find the orthogonal trajectories of the curves given by $x^2 + cy^2 = 1$ — (1) [Dec14, M-4]

→ diff. (1) w.r.t. x

$\therefore 2x + c \cdot 2y\frac{dy}{dx} = 0$ — (1)

$\Rightarrow c = \frac{-\cancel{2}x}{\cancel{2}y\frac{dy}{dx}} = \left(\frac{-x}{y}\right)\left(\frac{1}{\frac{dy}{dx}}\right)$

put value of c in (1)

$\therefore \cancel{2x + \left(\frac{-x}{y} \times \frac{1}{\frac{dy}{dx}}\right) 2y\frac{dy}{dx} = 0}$

$\therefore x^2 + \left(\frac{-x}{y}\right)\left(\frac{1}{\frac{dy}{dx}}\right) y^2 = 1$

$\Rightarrow x^2 - \frac{x}{\cancel{y}} \times \frac{1}{\frac{dy}{dx}} \times y^{\cancel{2}} = 1$

$\Rightarrow x^2 - \frac{xy}{\frac{dy}{dx}} = 1$

$\Rightarrow \frac{x^2\frac{dy}{dx} - xy}{\frac{dy}{dx}} = 1$

$\Rightarrow x^2\frac{dy}{dx} - xy = \frac{dy}{dx}$

$\Rightarrow x^2\frac{dy}{dx} - \frac{dy}{dx} = xy$

$\Rightarrow (x^2-1)\frac{dy}{dx} = xy$

$\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2-1}$

this is diff. eqn of given family

replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$

$\Rightarrow \frac{-dx}{dy} = \frac{xy}{x^2-1}$

$\Rightarrow \frac{dx}{dy} = \frac{-xy}{x^2-1}$

this is diff. eqn of orthogonal trajectories

now solving this eqn.

$\Rightarrow (x^2-1)dx = (-xy)dy$

$\Rightarrow \frac{x^2-1}{x}dx = -y\,dy$

$\Rightarrow \int\frac{x^2-1}{x}dx = \int -y\,dy + c$

$\Rightarrow \int\left(\frac{x^2}{x} - \frac{1}{x}\right)dx = -\int y\,dy + c$

$\Rightarrow \int\left(x - \frac{1}{x}\right)dx = -\int y\,dy + c$

$\Rightarrow \frac{x^2}{2} - \log x = -\frac{y^2}{2} + c$

$\Rightarrow \frac{x^2}{2} - \log x + \frac{y^2}{2} = c$

This is reqd family of orthogonal trajectories.

Find the orthogonal trajectories of the curves $x^2+2y^2=c^2$.

→ given, $x^2+2y^2=c^2$ —— (i)

step 1) form d.E. of given eqn (i)

$\therefore 2x+2\times 2y\frac{dy}{dx}=0$

$\Rightarrow 2x=-4y\frac{dy}{dx}$

$\Rightarrow x=-2y\frac{dy}{dx}$ —— (ii)

$\Rightarrow \frac{x}{-2y}=\frac{dy}{dx}$

is reqd d.e. of given family (i)

now d.e. of orthogonal trajectories is,

step 2) replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$ in (ii)

$\therefore x=-2y\left(\frac{-dx}{dy}\right)$

$\Rightarrow x=+2y\frac{dx}{dy}$

$\Rightarrow \frac{dx}{dy}=\frac{x}{2y}$

d.e. of orthogonal trajectories.

step 3) now we solve it,

we have $\frac{dx}{dy}=\frac{x}{2y}$

$\Rightarrow \frac{dx}{x}=\frac{dy}{2y}$

$\Rightarrow \int\frac{dx}{x}=\int\frac{dy}{2y}+\log c$

$\Rightarrow \log x=\frac{1}{2}\log y+\log c$

~~$\Rightarrow 2\log x=\log(yc)$~~

~~$\Rightarrow \boxed{x=cy}$~~

$\Rightarrow 2\log x=\log y+2\log c$

$\Rightarrow \log x^2=\log y+\log c^2$

$\Rightarrow \log x^2=\log(yc^2)$

$\Rightarrow x^2=c^2y$

or

$x^2=c_1y$.

This is reqd eqn of orthogonal trajectories.

Find orthogonal trajectories of the following families.

a) $y = cx^2$ b) $xy = c^2$ c) $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ where λ is a parameter

a) given, $y = cx^2$ —— (1)

step (1) → to form diff. eqn of (1)

$\therefore \frac{dy}{dx} = c \cdot 2x$

$\Rightarrow \frac{dy}{dx} \times \frac{1}{2x} = c$

put value of c in (1)

$\Rightarrow y = cx^2$

$\Rightarrow y = \left(\frac{dy}{dx} \times \frac{1}{2x}\right) \times x^2$

$\Rightarrow y = \frac{x}{2} \frac{dy}{dx}$

$\Rightarrow \frac{2y}{x} = \frac{dy}{dx}$ —— (11)

this is d.e. of (1)

step (11) → diff. eqn of orthogonal trajectories

just replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$

put in (11)

$\Rightarrow \cancel{2y} \cdot \frac{2y}{x} = \frac{-dx}{dy}$

this is de. of orthogonal trajectories.

$\Rightarrow 2y\,dy = -x\,dx$

integrating b.s.

$\Rightarrow \int 2y\,dy = \int -x\,dx$

$\Rightarrow \cancel{2} \times \frac{y^2}{\cancel{2}} = \frac{-x^2}{2} + c \Rightarrow y^2 + \frac{x^2}{2} = c$ —— (111) or $2y^2 + x^2 = 2c$

i.e. $2y^2 + x^2 = c_1$ orthogonal traj.

or from (iii) we can write,

$$\frac{x^2}{2c} + \frac{y^2}{c} = 1$$

$$\frac{x^2}{(\sqrt{2c})^2} + \frac{y^2}{(\sqrt{c})^2} = 1$$

This is family of ellipses.

b) $xy = c^2$ — (i) (hyperbola)

step (i) diff. w.r.t. x.

$$\therefore x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x} \quad \text{— (ii)}$$

this is diff. eq^n of given family (i)

step (ii), diff. eq^n of orthogonal trajectories is obtained by, replacing $\frac{dy}{dx}$ by $\frac{-dx}{dy}$

put in (ii)

$$\therefore \frac{-dx}{dy} = \frac{-y}{x}$$

$$\therefore \frac{dx}{dy} = \frac{y}{x}$$

this is diff. eq^n of orthogonal trajectories.

step (iii) solving diff. eq^n.

$$\frac{dx}{dy} = \frac{y}{x}$$

$$\therefore x\,dx = y\,dy$$

int. both sides

$$\therefore \int x\,dx = \int y\,dy + c$$

$$\Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + c$$

$$\Rightarrow \frac{x^2}{2} - \frac{y^2}{2} = c$$

$$\Rightarrow \frac{x^2 - y^2}{2} = c$$

$$\Rightarrow x^2 - y^2 = 2c = c_1$$

$$\Rightarrow x^2 - y^2 = c_1$$

this is req^d orthogonal traj.

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c) $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ where λ is a parameter. —— (1)

step (1) → diff. (1) w.r.t. x.

$$\therefore \frac{1}{a^2} \times 2x + \frac{1}{b^2+\lambda} \cdot 2y \cdot \frac{dy}{dx} = 0$$

$$\therefore 2\left(\frac{x}{a^2} + \frac{y}{b^2+\lambda}\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{y}{b^2+\lambda}\frac{dy}{dx} = \frac{-x}{a^2}$$

$$\Rightarrow \frac{1}{b^2+\lambda} = \frac{\frac{-x}{a^2 \times y}}{\frac{dy}{dx}}$$

put in (1)

$$\therefore \frac{x^2}{a^2} + y^{\cancel{2}}\left(\frac{\frac{-x}{a^2 \cancel{y}}}{dy/dx}\right) = 1$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{xy}{a^2(dy/dx)} = 1$$

replacing $\frac{dy}{dx}$ by $\frac{-dx}{dy}$, we get,

$$\frac{x^2}{a^2} - \frac{xy}{a^2\left(\frac{-dx}{dy}\right)} = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{xy}{a^2} \times \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{xy}{a^2} \times \frac{dy}{dx} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow y\frac{dy}{dx} = \frac{\left(1 - \frac{x^2}{a^2}\right)}{x/a^2} = \frac{\left(\frac{a^2-x^2}{a^2}\right)}{x/a^2} = \frac{\cancel{a^2}}{x} \times \left(\frac{a^2-x^2}{\cancel{a^2}}\right) = \frac{a^2-x^2}{x}$$

$\therefore y\frac{dy}{dx} = \frac{a^2-x^2}{x}$. $\Rightarrow y\,dy = \frac{a^2-x^2}{x}dx$

integrating we get,

$\int y\,dy = \int\left(\frac{a^2-x^2}{x}\right)dx$

$= \int \frac{a^2}{x}dx - \int \frac{x^2}{x}dx$

$= a^2\int \frac{1}{x}dx - \int x\,dx$

$\Rightarrow \frac{y^2}{2} = a^2\log x - \frac{x^2}{2} + c$

$\Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = a^2\log x + c$

$\Rightarrow y^2 + x^2 = 2a^2\log x + 2c$

$\Rightarrow x^2 + y^2 = 2a^2\log x + c_1$

d) $y^2 = 4ax$ — (i)

Step (i) → diff. w.r.t. x.

$2y\frac{dy}{dx} = 4a$

put in (i),

$\therefore y^2 = \left(2y\frac{dy}{dx}\right)\times x = 2xy\frac{dy}{dx}$

$\therefore y^{\cancel{2}} = 2x\cancel{y}\frac{dy}{dx} \Rightarrow y = 2x\frac{dy}{dx}$ — (ii)

this dae. of (i)

Step (ii) → replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$

$\Rightarrow y = 2x\left(\frac{-dx}{dy}\right)$

$\Rightarrow y = -2x\frac{dx}{dy}$ this is d.e. of orthogonal trajectory

$\Rightarrow y\,dy = -2x\,dx$

int. both sides,

$\int y\,dy = \int -2x\,dx + c$

$\Rightarrow \frac{y^2}{2} = -\cancel{2}\times\frac{x^2}{\cancel{2}} + c$

$\Rightarrow \frac{y^2}{2} + x^2 = c$

$\Rightarrow y^2 + 2x^2 = 2c = c_1$

$\Rightarrow 2x^2 + y^2 = c_1$.

Working Rules to find orthogonal trajectories of polar curves $f(r, \theta, c) = 0$

stepI: Differentiate $f(r,\theta,c) = 0$ w.r.t. θ and eliminate c.

Thus we form the DE. of the family of curves of the form say $\phi\left(r, \theta, \frac{dr}{d\theta}\right) = 0$ —— (i)

stepII: Replacement.

replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in eqn (i).

$\therefore$ DE of orthogonal trajectories will be,

$\phi\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0$ —— (ii)

step III: Family of orthogonal trajectories:

solve DE (ii), we will get the reqd family of orth. traj.

Q1) Find orthogonal trajectories of $r = a(1+\cos\theta)$

→ given, $r = a(1+\cos\theta)$ —— (i)

stepI: diff. w.r.t. θ.

$\therefore \frac{dr}{d\theta} = a(0 + (-\sin\theta))$

$\therefore \frac{dr}{d\theta} = -a\sin\theta$

$\therefore a = \frac{-1}{\sin\theta}\frac{dr}{d\theta}$

put in (i)

$\therefore r = \left(\frac{-1}{\sin\theta}\frac{dr}{d\theta}\right)(1+\cos\theta)$

$$\Rightarrow -r\sin\theta = \left(\frac{dr}{d\theta}\right)(1+\cos\theta)$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{-r\sin\theta}{1+\cos\theta} \quad \text{—— (ii)}$$

this is DE of family (i)

step II) to obtain DE of orthogonal trajectories replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (ii)

$$\therefore -r^2\frac{d\theta}{dr} = \frac{-r\sin\theta}{1+\cos\theta}$$

$$\therefore r\frac{d\theta}{dr} = \frac{\sin\theta}{1+\cos\theta} \quad \text{—— (iii)}$$

this is DE of orthogonal traj.

step III) now solving (iii)

$$\therefore \frac{1+\cos\theta}{\sin\theta}d\theta = \frac{dr}{r}$$

int. b.s.

$$\therefore \int \frac{1+\cos\theta}{\sin\theta}d\theta = \int\frac{dr}{r}$$

$$\therefore \int \frac{\cancel{2}\cos^{\cancel{2}}\theta/2 \, d\theta}{\cancel{2}\sin\theta/2 \times \cancel{\cos\theta/2}} = \int\frac{dr}{r}$$

$$\therefore \int \cot\theta/2 \, d\theta = \int\frac{dr}{r}$$

$$\therefore \frac{\log\sin\frac{\theta}{2}}{1/2} = \log r + \log c$$

$$\therefore 2\log\sin\theta/2 = \log(rc)$$

$$\therefore \log(\sin\theta/2)^2 = \log(rc)$$

$$\therefore (\sin\theta/2)^2 = rc$$

$$\therefore \sin^2\theta/2 = rc$$

$$\therefore rc = \left(\frac{1-\cos\cancel{2}(\theta/\cancel{2})}{2}\right) \quad \left[\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$\therefore rc = \dfrac{1-\cos\theta}{2}$

or

$$r = \frac{1-\cos\theta}{2c} = \frac{1}{2c} \times (1-\cos\theta) = c_1 (1-\cos\theta)$$

$\therefore r = c_1(1-\cos\theta)$

is orthogonal trajectories of given eqn (1)

Find O.T. of $r = a\cos\theta$ which all pass through the origin & have their centres on the initial line, a being the variable parameter. [May 05]

→ given $r = a\cos\theta$ —— (1)

step I. $\dfrac{dr}{d\theta} = a.-\sin\theta$

$$\therefore a = \frac{-1}{\sin\theta}\frac{dr}{d\theta}$$

put in (1)

$$\therefore r = \left(\frac{-1}{\sin\theta}\frac{dr}{d\theta}\right)\cos\theta$$

$$\therefore r = -\cot\theta \frac{dr}{d\theta}$$

$$\therefore \frac{dr}{d\theta} = \frac{-r}{\cot\theta} = -r\tan\theta \quad \text{—— (11)}$$

this is DE of (1)

step II : replace $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$ in (11)

$$\therefore \cancel{-}r^{\cancel{2}}\frac{d\theta}{dr} = \cancel{-}\cancel{r}\tan\theta$$

$$\therefore \cancel{r}\frac{d\theta}{dr} = \tan\theta \quad \text{—— (111)}$$

this is DE of orth. traj.

step III : solve (111)

$$\therefore \frac{d\theta}{\tan\theta} = \frac{dr}{r}$$

$$\therefore \cot\theta\, d\theta = \frac{dr}{r}$$

int. b.s.

$\therefore \int \cot\theta \, d\theta = \int \frac{dr}{r} + \log c$

$\therefore \log(\sin\theta) = \log r + \log c$

$\therefore \log(\sin\theta) = \log(rc)$

$\therefore \sin\theta = rc$

Or

$\frac{1}{c} \sin\theta = r$

$\Rightarrow c_1 \sin\theta = r$

This is reqd orth. Traj.

Q3) Find the OT of $r^2 = a\sin 2\theta$. —(1)

given $r^2 = a\sin 2\theta$

step I: $\cancel{2} r \frac{dr}{d\theta} = a.(\cos 2\theta)(\cancel{2})$

$\therefore r\frac{dr}{d\theta} = a\cos 2\theta$

$\therefore a = \frac{r}{\cos 2\theta} \frac{dr}{d\theta}$

put in (1)

$\therefore \cancel{r^2} = \left(\frac{\cancel{r}}{\cos 2\theta}\right) \frac{dr}{d\theta} \times \sin 2\theta$

$\therefore r = \tan 2\theta \frac{dr}{d\theta}$

$\therefore \frac{r}{\tan 2\theta} = \frac{dr}{d\theta}$ —(II)

de of family (1)

step II: replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$

$\therefore \frac{\cancel{r}}{\tan 2\theta} = -r^2 \frac{d\theta}{dr}$

$\therefore \frac{1}{\tan 2\theta} = -r \frac{d\theta}{dr}$ this is de of OT. —(III)

Step III : solving (iii)

$\therefore \frac{1}{r} = -\tan 2\theta \frac{d\theta}{dr}$

$\Rightarrow \frac{1}{r} dr = -\tan 2\theta \, d\theta$

int. b.s.

$\Rightarrow \int \frac{dr}{r} = -\int \tan 2\theta \, d\theta$

$\Rightarrow \log r + \log c = \frac{-\log(\sec 2\theta)}{2}$

$\Rightarrow \log(rc) = \cancel{\frac{-1}{2}} \frac{\log(\sec 2\theta)^{-1}}{2}$

$\Rightarrow 2\log(rc) = \log \frac{1}{\sec 2\theta}$

$\Rightarrow \log(rc)^2 = \log(\cos 2\theta)$

$\Rightarrow (rc)^2 = \cos 2\theta$

$\Rightarrow r^2 c^2 = \cos 2\theta$

$\Rightarrow r^2 = \frac{1}{c^2} \cos 2\theta$

$\Rightarrow r^2 = c_1 \cos 2\theta$

this is reqd O.T.

Newton's Law of cooling:-

Statement → The rate of change of temperature of body is proportional to the diff. betn temp of the body and the temp of surrounding.

Let,

$\theta \rightarrow$ temp of body at any time 't'.

$\theta_0 \rightarrow$ temp of surrounding medium

then

$$\frac{d\theta}{dt} \propto [\theta(t) - \theta_0]$$

$$\frac{d\theta}{dt} = -k(\theta - \theta_0) \; ; \; k > 0$$

k is proportionality constant.

The cooling rate depends on the parameter $k = \frac{\alpha A}{C}$

A → surface area of body through which heat is transfered.

α → heat transfer coe. depending on the geometry of the body, state of the surface, heat transfer mode and other factors.

c → heat capacity

Here negative sign is attached to k because as t increases, θ decreases, with increase of the parameter k, the cooling occures faster.

Limitations of Newton's Law of cooling:-

~~i) The major limitation of Newton's law of cooling is that the temp of surr medium must contremain constant durin~~

i) The major limitation of Newton's law of cooling is that the temperature of surrounding medium must remain constant during the cooling of the body.

ii) The loss of heat from the body should by by radiation only.

iii) The difference betn the temp. of the body and surrounding medium must be small.

Solution of diff. eqn of Newton's law of cooling

we have, $\frac{d\theta}{dt} \propto (\theta - \theta_0)$

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\therefore \frac{d\theta}{\theta-\theta_0} = -k\,dt$$

$$\int \frac{d\theta}{\theta-\theta_0} = \int -k\,dt + c_1$$

$$\Rightarrow \log(\theta-\theta_0) = -kt + c_1$$

$$\Rightarrow \theta-\theta_0 = e^{-kt+c_1} = e^{c_1} \times e^{-kt}$$

$$\Rightarrow \theta-\theta_0 = ce^{-kt}$$ proportionality constant.

$$\Rightarrow \boxed{\theta = \theta_0 + ce^{-kt}}$$ c and k are constants,

temp. of body, temp. of surrounding

procedure to solve NLC problems :-

Step I $\longrightarrow$ sol^n of NLC

Get sol^n $\rightarrow \theta = \theta_0 + ce^{-kt}$ —— (1)

Step II $\longrightarrow$ Find c : put $\theta = \theta_1$ at $t=0$ in eq^n (1)

Step III $\longrightarrow$ Find k : put $\theta = \theta_2$ at $t = T_1$ in eq^n (1)

Step IV $\longrightarrow$ Find θ or t

Find the req^d values of θ or t.

Q Water at temp. 100°c cools in 10 minutes to 88°c in a room temperature 25°c. Find the temperature of the water after 20 minutes. —— [May 17, M4]

$\rightarrow$ $\theta \rightarrow 100°c \longrightarrow$ temp of body

$\theta_0 \rightarrow 25°c \longrightarrow$ surrounding temp.

by Newton's law of cooling (NLC)

$$\frac{d\theta}{dt} \propto (\theta-\theta_0)$$

and sol^n is, $\theta = \theta_0 + ce^{-kt}$ —— (I)

Step 2) Given that, $\theta_0 = 25°c$

Find c

$\theta = 100°c$

$t = 0$ min

put in (I)

$$\Rightarrow 100 = 25 + ce^{-k(0)}$$
$$= 25 + ce^{0}$$
$$100 = 25 + c$$
$$\therefore c = 100 - 25 = 75$$
$$\therefore \boxed{c = 75}$$

$$\Rightarrow \theta = \theta_0 + 75e^{-kt} \text{ —— (II)}$$

Step 3) Find k, use, $\theta = 88°c$ at $t = 10$ min. and $c = 75$, $\theta_0 = 25$ in (II)

$$\therefore \cancel{88 = 25 + 75e^{-k(20)}}$$
$$\therefore \cancel{88 = 25 + 75e^{-20k}}$$
$$\therefore \cancel{88 - 25 = 75e^{-20k}}$$
$$\therefore \cancel{63 = 75e^{-20k}}$$

θ_0 = Temp. of surrounding is fixed only 'θ' = Temp of body varies

$$\therefore 88 = 25 + 75e^{-k(10)}$$
$$\therefore 88 - 25 = 75e^{-10k}$$
$$\therefore 63 = 75e^{-10k}$$
$$\therefore e^{-10k} = \frac{63}{75}$$
$$\therefore \log e^{-10k} = \log\left(\frac{63}{75}\right)$$
$$\therefore -10k = \log\left(\frac{63}{75}\right)$$
$$\therefore k = \frac{-1}{10} \times \log\left(\frac{63}{75}\right)$$
$$= -0.01744.$$

Step 4): Find θ at $t = 20$ min.

put $t = 20$ and values of c and k in eq^n (I) we get,

$$\theta = \theta_0 + ce^{-kt} \quad -\left[\frac{-1}{10}\log\left(\frac{63}{75}\right)\right](20)$$
$$= 25 + 75e$$
$$= 25 + 75e^{\frac{20}{10}\log\left(\frac{63}{75}\right)}$$

$$= 25 + 75e^{2\log(63/75)}$$

$$= 25 + 75e^{\log(63/75)^2}$$

$$= 25 + 75 \times \left(\frac{63}{75}\right)^2$$

$$= 25 + \cancel{75} \times \left(\frac{63}{\cancel{75}}\right) \times \left(\frac{63}{75}\right)$$

$\therefore \theta = 77.9151^\circ c$

$\therefore$ the temp. of water after 20 minutes will be $77.9151^\circ c$.

Ex. 2) A body at temp $100^\circ c$ is placed at a room. whose temp. is $25^\circ c$ and cools to $80^\circ c$ in 10 minutes. Find the time. when the temp. will be $60^\circ c$. —[May 18]

$\theta \rightarrow$ body temp. at any time t.

$\theta_0 \rightarrow$ temp. of the surrounding medium.

step I.). By NLC.

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

$$\therefore \theta = \theta_0 + ce^{-kt} \quad \text{—— (1)}$$

step 2) Find c ; Given, $\theta_0 = 25^\circ c$

put, $\theta = 100^\circ c$ at $t = 0$ min in eqn (1)

$$100 = 25 + ce^{-K(0)}$$

$$\therefore 100 - 25 = c$$

$$\therefore \boxed{c = 75}$$

Step 3) find k.

we have, $\theta = 80^\circ c$ at $t = 10$ min. and $\theta_0 = 25^\circ c$

$\therefore$ (1) becomes,

$$80 = 25 + 75e^{-10k}$$

$$\therefore 80 - 25 = 75e^{-10k}$$

$\therefore 55 = 75 e^{-10k}$

$\therefore \frac{55}{75} = e^{-10k}$

$\therefore \log_e\left(\frac{55}{75}\right) = \log_e e^{-10k}$

$\log\left(\frac{55}{75}\right) = -10k$

$\therefore k = \frac{-1}{10}\log\left(\frac{55}{75}\right)$

$= \frac{-1}{10}\log\left(\frac{11}{15}\right)$

$\therefore \boxed{k = 0.03101}$

Step 4) Find t at $\theta = 60°c$, $\theta_0 = 25$, $c = 75$, $k = 0.03101$

$\therefore$ eqn ① becomes,

$60 = 25 + 75 e^{-\left[\frac{-1}{10}\log\left(\frac{11}{15}\right)\right]t}$

$\therefore 60 - 25 = 75 e^{\left[\frac{1}{10}\log\left(\frac{11}{15}\right)\right]t}$

$\therefore \frac{35}{75} = e^{\left[\frac{1}{10}\log\left(\frac{11}{15}\right)\right]t}$

$\therefore \log\left(\frac{35}{75}\right) = \log_e e^{\left[\frac{1}{10}\log\left(\frac{11}{15}\right)\right]t}$

$\therefore \log\left(\frac{7}{15}\right) = \left[\frac{1}{10}\log\left(\frac{11}{15}\right)\right]t$

$\therefore 10\log\left(\frac{7}{15}\right) = \left[\log\left(\frac{11}{15}\right)\right]t.$

$\Rightarrow t = \frac{(10)\log\left(\frac{7}{15}\right)}{\log\left(\frac{11}{15}\right)} = 24.5772 \text{ min}$

Q A body originally at 80°c cools to 60°c in 20 minutes and the temp of air being 40°c. what will be the temp of the body after 40 minutes? —— May 19, M-4.

→ step ① by NLC.

$$\boxed{\theta = \theta_0 + ce^{-kt}} \quad \text{①}$$

step ② Find c, $\theta = 80°c$, $\theta_0 = 40°c$, $t = 0$ ∴ put in ①

$$80 = 40 + ce^{-k(0)}$$

$$\therefore 80 - 40 = c$$

$$\therefore \boxed{c = 40}$$

step ③ Find k, $\theta = 60°c$, $\theta_0 = 40°c$, $t = 20 min$

∴ put in ①

$$\therefore 60 = 40 + 40e^{-k(20)}$$

$$\therefore \frac{60-40}{40} = e^{-k(20)}$$

$$\therefore \frac{20}{40} = e^{-20k}$$

$$\therefore \frac{1}{2} = e^{-20k}$$

$$\therefore \log\left(\frac{1}{2}\right) = \log e^{-20k}$$

$$\therefore \log\left(\frac{1}{2}\right) = -20k$$

$$\therefore k = \frac{-1}{20}\log\left(\frac{1}{2}\right)$$

$$\therefore k = 0.034657.$$

step ④ Find θ, if $t = 40 min$, $\theta_0 = 40°c$, $c = 40$, $k = 0.034657$

put in ①

$$\therefore \theta = 40 + 40e^{-k(40)}$$

$$= 40 + 40e^{-(0.034657)(40)}$$

$$= 40 + 40(0.25) = 50.0001°c$$

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Q A body of temp. 100°c is placed in a room whose temp. is 20°c and cool to 60°c in 5 minutes. what will be its temp. after a further interval of time minutes. [May 05, 09, Dec-06]

→ step 1) by NLC., $\theta = \theta_0 + ce^{-kt}$ —— ①

step 2) find c, $\theta = 100°c$, $t = 0\,min$, $\theta_0 = 20°c$ put in ①

$\therefore 100 = 20 + ce^{-k(0)}$

$\therefore \boxed{80 = c}$

step 3) put, $\theta = 60°c$, $\theta_0 = 20°c$, $c = 80$ and $t = 5\,min$ in ①

$\therefore 60 = 20 + (80)e^{-k(5)}$

$\therefore 40 = 80e^{-5k}$

$\therefore \frac{40}{80} = e^{-5k}$

$\therefore \frac{1}{2} = e^{-5k}$

$\therefore \log\left(\frac{1}{2}\right) = \log e^{-5k}$

$\therefore \log\left(\frac{1}{2}\right) = -5k$

$\therefore k = -\frac{1}{5}\log\left(\frac{1}{2}\right)$

$\boxed{k = 0.138629}$

step 4) put $\theta_0 = 20°c$, $c = 80$, $k = 0.138629$ and $t = 10\,min$ in ① and find θ.

$\therefore \theta = 20 + (80)e^{-(0.1386)(10)}$

$\therefore \theta = 20 + (80)e^{-1.386}$

$= 20 + (80)(0.2500736) = 40.00588°c \approx 40°c$

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~~Simple electrical circuits.~~

Q If the temp. of a body drops from $100^\circ c$ to $60^\circ c$ in one minute when the temp. of the surrounding is $20^\circ c$, what will be the temp of the body at the end of the second minute?

→ step I, we have, $\theta = \theta_0 + c e^{-kt}$ —— (1)

Step II, put, $\theta = 100^\circ c$, $t = 0$, $\theta_0 = 20^\circ c$ in (1)

$\therefore 100 = 20 + c e^{-k(0)}$

$\therefore 100 - 20 = c$

$\therefore \boxed{c = 80}$

Step III: put, $\theta = 60^\circ c$, $t = 1$ min., $c = 80$, $\theta_0 = 20^\circ c$ in (1)

$\therefore 60 = 20 + 80 e^{-k(1)}$

$\therefore 40 = 80 e^{-k}$

$\therefore \frac{1}{2} = e^{-k}$

$\therefore \log\left(\frac{1}{2}\right) = \log\left(e^{-k}\right)$

$\therefore \log\left(\frac{1}{2}\right) = -k$

$\therefore k = -\log\left(\frac{1}{2}\right)$

Step IV : $\theta = ?$ if $\theta_0 = 20^\circ c$, $t = 2$ min, $c = 80$, $k =$ in (1)

$\therefore \theta = 20 + 80 e^{-(\quad)(2)}$

Simple electrical circuits.

we shall consider circuits made up of

i) three passive elements - resistance, inductance, capacitance

ii) An active element - voltage source which may be a battery or generator.

Element	symbol	unit
Time	t	second
current (time rate of flow of change)	$i = \frac{dq}{dt}$	ampere (A)
quantity of electricity (electric charge)	q	coulumb
resistance (R)	R	ohm (Ω)
Inductance	L	Henry (H)
capacitance	C	farad (F)
Electromotive force or voltage (constant),	Battery, E = constant	Volt
variable voltage generator	generator, - E = variable voltage	volt.

Basic Relations:

(i) $i = \dfrac{dq}{dt}$ or $q = \int i\,dt$ [$\because$ current is the rate of flow of electricity]

(ii) voltage drop across resistance $R = Ri$; (ohm's law)

(iii) voltage drop across inductance $L = L\dfrac{di}{dt}$

(iv) Voltage drop across capacitance $C = \dfrac{q}{C}$

Kirchoff's Law:

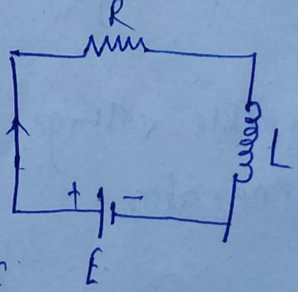
The formulⁿ of diff. eqⁿs for an electrical circuit depends on the following two kirchoff's laws which are of cardinal importance,

1) The algebraic sum of the voltage drops around any closed circuit is equal to the resultant EMF in the circuit.

2) The algebraic sum of the currents flowing into (or from) any node is zero.

Differential eqⁿ:-

i) Circuit involving L and R along with a voltage source (battery) E, all in series; consider a circuit containing resistance R and inductance L in series with a voltage source (battery) E.



Let i be the current flowing in the circuit at any time t. Then by Kirchoff's first law, we have sum of voltage drops across R and L = E.

i.e. $Ri + L\dfrac{di}{dt} = E$

or $\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$ which is Linear diff eqn.

$\therefore$ its I.F is $e^{\int \frac{R}{L} dt} = e^{Rt/L}$

$\therefore$ soln is,

$i \cdot I.F. = \int Q . I.F . dt + c$

$\Rightarrow i \times e^{\frac{Rt}{L}} = \int \frac{E}{L} \times e^{Rt/L} dt + c$

$= \frac{E}{L} \times \frac{e^{Rt/L}}{R/L} + c$

$= \frac{E}{\cancel{L}} \times \frac{\cancel{L}}{R} \times e^{Rt/L} + c$

$= \frac{E}{R} \times e^{Rt/L} + c$

$\Rightarrow i = \frac{\frac{E}{R} \times e^{Rt/L} + c}{e^{\frac{Rt}{L}}} = \frac{\frac{E}{R} \times \cancel{e^{Rt/L}}}{\cancel{e^{Rt/L}}} + \frac{c}{e^{Rt/L}}$

$\therefore i = \frac{E}{R} + c e^{-Rt/L}$ —— (1).

If initially there is no current in the circuit, i.e. $i=0$ when $t=0$, we have, from (1)

$0 = \frac{E}{R} + c e^{-\frac{R}{L}(0)}$

$\therefore 0 = \frac{E}{R} + c$

$\therefore c = \frac{-E}{R}$

put in (1)

$\therefore i = \frac{E}{R} + \left(\frac{-E}{R}\right) e^{-Rt/L} = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$ —— (ii) as $t \to \infty$, ~~$i = \frac{E}{R}$~~ (11) becomes

$$i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}(\infty)}\right)$$

$$= \frac{E}{R}\left(1 - \frac{1}{e^{+\frac{R}{L}(\infty)}}\right)$$

$$= \frac{E}{R}\left(1 - \frac{1}{e^{\infty}}\right)$$

$$= \frac{E}{R}(1 - 0)$$

$\therefore \boxed{i = \frac{E}{R}}$, which shows that i increases with t and attains the maximum value $\frac{E}{R}$

Remember → ~~$i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$~~

1) $\boxed{RI + L\frac{dI}{dt} = E}$

2) $\boxed{I = \frac{E}{R}\left(1 - e^{-Rt/L}\right)}$

3) I increases with t.
and as $t \to \infty$ it attains maximum value i.e. $\frac{E}{R}$

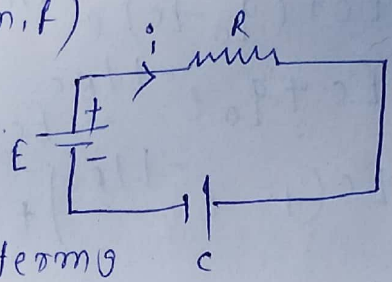
hence $\boxed{I = \frac{E}{R}, \text{ if } t \to \infty}$

ii) Circuits involving R and C along with a voltage source (battery) E, all in series:

Consider a circuit containing Resistance R and capacitance C in series with a voltage source (battery) E.

Let I be the current flowing in the circuit at any time t. Then by Kirchoff's first law, we have sum of voltage drops across R and C = E (e.m.f)

i.e. $RI + \frac{q}{C} = E.$



since, $I = \frac{dq}{dt}$, this eq^n in terms of q can be written as,

$$R\frac{dq}{dt} + \frac{q}{C} = E \text{ or } \frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R}$$

which is a linear Diff. eq^n.

$$\therefore I.F. = e^{\int \frac{1}{RC} dt} = e^{\frac{1}{RC} \times t} = e^{t/RC}.$$

and $\therefore$ G.S. is,

$$q.\, I.F. = \int Q \times I.F.\, dt + B$$

$$q \times e^{t/RC} = \int \frac{E}{R} \times e^{t/RC} dt + B$$

$$= \frac{\frac{E}{R} \times e^{t/RC}}{\frac{1}{RC}} + B$$

$$= \frac{E}{\cancel{R}} \times \cancel{R}C \times e^{t/RC} + B$$

$$\therefore q \times e^{t/RC} = EC\, e^{t/RC} + B$$

$$\therefore q = \frac{EC\, \cancel{e^{t/RC}}}{\cancel{e^{t/RC}}} + \frac{B}{e^{t/RC}} = EC + Be^{-t/RC} \quad \text{——— (1)}$$

assuming $q = q_0$ when $t=0$

$$q_0 = Ec + Be^{-\frac{0}{RC}}$$

$$\therefore q_0 = Ec + B$$

$$\Rightarrow B = q_0 - Ec \quad \text{put in (1)}$$

$$\therefore q = Ec + (q_0 - Ec)\, e^{-t/RC}$$

$$\therefore q = Ec + q_0 e^{-t/RC} - Ec e^{-t/RC}$$

$$\therefore q = Ec(1 - e^{-t/RC}) + q_0 e^{-t/RC}$$

$$\therefore I = \frac{dq}{dt} = \frac{d}{dt}\left\{ Ec(1-e^{-t/RC}) + q_0 e^{-t/RC} \right\}$$

$$= \frac{d}{dt}\left\{ Ec(1-e^{-t/RC})\right) + \frac{d}{dt} q_0 e^{-t/RC}$$

$$= (Ec)\frac{d}{dt}(1-e^{-t/RC}) + q_0 \frac{d}{dt} e^{-t/RC}$$

$$= (E\cancel{c})\left(0 - e^{-t/RC} \times \frac{-1}{R\cancel{C}}\right) + q_0 \times e^{-t/RC} \times \frac{-1}{RC}$$

$$= (E)\left(\frac{e^{-t/RC}}{R}\right) + \left(\frac{-q_0}{RC}\right) e^{-t/RC}$$

$$\therefore I = \left(\frac{E}{R} - \frac{q_0}{RC}\right) e^{-t/RC}$$

iii) Circuit involving L and c both in series, after removing source applied e.m.f.: consider a circuit containing inductance L and capacitance C in series without applied e.m.f.

Let I be the current flowing in the circuit at any time t, Then by kirchoff's first law, we have sum of voltage drops across L and c = 0.

i.e. $L\frac{dI}{dt} + \frac{q}{C} = 0$

$\therefore \frac{dI}{dt} = \frac{-q}{LC}$

$\therefore \frac{dI}{dq} \times \frac{dq}{dt} = \frac{-q}{LC}$

$\therefore I\frac{dI}{dq} = \frac{-q}{LC}$

$\therefore I\,dI = \frac{-q}{LC}\,dq$

Integrating both sides,

$\therefore \int I\,dI = \int \frac{-q}{LC}\,dq + A$

$\therefore \frac{I^2}{2} = \frac{-q^2}{2LC} + A$

$\therefore I^2 = \frac{-\cancel{2}q^2}{\cancel{2}LC} + 2A$

$\therefore I^2 = \frac{-q^2}{LC} + B$ —— (1)

assuming, $I = 0,\ q = q_0$ when $t = 0$ in (1)

$\therefore 0 = \frac{-q_0^2}{LC} + B$

$\therefore B = \frac{q_0^2}{LC}$

put in (1)

$\therefore I^2 = \frac{-q^2}{LC} + \frac{q_0^2}{LC}$

$\therefore I^2 = \frac{1}{LC}(q_0^2 - q^2)$

$\therefore I = \pm \frac{\sqrt{q_0^2 - q^2}}{\sqrt{LC}}$

Since q decreases as t increases,

$$\therefore I = \frac{dq}{dt} = -\frac{1}{\sqrt{LC}}\sqrt{q_0^2 - q^2}$$

$$\Rightarrow \frac{-dq}{\sqrt{q_0^2 - q^2}} = \frac{1}{\sqrt{LC}}\,dt$$

integrating b.s.

$$\therefore \int \frac{-dq}{\sqrt{q_0^2 - q^2}} = \int \frac{1}{\sqrt{LC}}\,dt + c$$

$$\therefore \cos^{-1}\left(\frac{q}{q_0}\right) = \frac{t}{\sqrt{LC}} + c \quad \text{--- (11)}$$

assuming $q = q_0$ when $t = 0$

$$\therefore \cos^{-1}\left(\frac{q_0}{q_0}\right) = \frac{0}{\sqrt{LC}} + c$$

$$\therefore \cos^{-1}(1) = 0 + c$$

$$\therefore 0 = 0 + c$$

$$\therefore \boxed{c = 0}$$

put $c = 0$ in (11)

$$\therefore \cos^{-1}\left(\frac{q}{q_0}\right) = \frac{t}{\sqrt{LC}} + 0$$

$$\therefore \cos^{-1}\left(\frac{q}{q_0}\right) = \frac{t}{\sqrt{LC}}$$

$$\therefore \frac{q}{q_0} = \cos\left(\frac{t}{\sqrt{LC}}\right)$$

$$\therefore \boxed{q = q_0 \cos\left(\frac{t}{\sqrt{LC}}\right)}$$

Useful formulae:-

1. $\int e^{at} \sin bt \, dt = \frac{e^{at}}{a^2+b^2}(a \sin bt - b \cos bt)$

2. $\int e^{at} \cos bt \, dt = \frac{e^{at}}{a^2+b^2}(a \cos bt + b \sin bt)$

3. $\int e^{at} \sin bt \, dt = \frac{e^{at}}{\sqrt{a^2+b^2}} \sin(bt - \phi)$, where $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

4. $\int e^{at} \cos bt \, dt = \frac{e^{at}}{\sqrt{a^2+b^2}} \cos(bt - \phi)$, where $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

Examples:-

i) A resistance of 100 ohms, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in a circuit as a fun of t.

(Nov./Dec. 2019, May 2005, 2014, 2017, 2018)

given, $R = 100\ \Omega$, $L = 0.5 H$, $E = 20$ volts

→ By kirchoff's law, we have,

$$\cancel{L\frac{dI}{dt} + RI = E}.$$

$$\therefore \cancel{\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}} \text{ ~~which is~~}$$

$$I = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$$

$$\therefore I = \frac{20}{100}\left(1 - e^{\frac{-100 \times t}{0.5}}\right)$$

$$= \frac{1}{5}\left(1 - e^{\frac{-100t}{1/2}}\right)$$

$$\therefore \boxed{I = \frac{1}{5}\left(1 - e^{-200t}\right)}$$

ii) A voltage Ee^{-at} is applied at $t=0$ to a circuit containing inductance L and resistance R, show that the current at any time t is, $\frac{E}{R-aL}\left(e^{-at} - e^{\frac{-Rt}{L}}\right)$

---[May-10, 09, 07, Dec-06]

→ Let I → current at any time t.

voltage → Ee^{-at} applied at $t=0$

inductance → L

resistance → R.

∴ By kirchoff's Law,

$$L\frac{dI}{dt}+RI=Ee^{-at}$$

$$\therefore \frac{dI}{dt}+\frac{R}{L}I=\frac{E}{L}e^{-at}$$

$$\therefore I.F.=e^{\int\frac{R}{L}dt}=e^{Rt/L}$$

Sol^n,

$$I.\,I.F.=\int \varphi(I.F)\,dt+c$$

$$=\int\left(\frac{E}{L}e^{-at}\right)\left(e^{Rt/L}\right)dt+c$$

$$=\int\frac{E}{L}e^{-at+\frac{Rt}{L}}\,dt+c$$

$$=\frac{E}{L}\int\left(e^{\frac{Rt}{L}-at}\right)dt+c$$

$$=\frac{E}{L}\int e^{\left(\frac{R}{L}-a\right)t}\,dt+c$$

$$=\frac{E}{L}\times\frac{e^{\left(\frac{R}{L}-a\right)t}}{\left(\frac{R}{L}-a\right)}+c$$

$$=\frac{E}{L}\times\frac{e^{\left(\frac{R}{L}-a\right)t}}{\left(\frac{R-aL}{L}\right)}+c$$

$$=\frac{E}{\cancel{L}}\times\frac{\cancel{L}}{R-aL}\times e^{\left(\frac{R}{L}-a\right)t}+c$$

$$\therefore I\left(e^{\frac{Rt}{L}}\right)=\frac{E}{R-aL}e^{\left(\frac{R}{L}-a\right)t}+c \quad \text{—— } (1)$$

Given, $I=0$, $t=0$.

$$\therefore \quad 0 = \frac{E}{R-aL}e^{0} + B = \frac{E}{R-aL} + B$$

$$\therefore \quad B = \frac{-E}{R-aL} \quad \text{put in (1)}$$

$$\therefore \quad I e^{\frac{Rt}{L}} = \frac{E}{R-aL} e^{\left(\frac{R}{L}-a\right)t} + \left(\frac{-E}{R-aL}\right)$$

$$\therefore \quad I e^{\frac{Rt}{L}} = \frac{E}{R-aL}\left\{1 - e^{\left(\frac{R}{L}-a\right)t}\right\}$$

$$\therefore \quad I = \frac{E}{R-aL} \times e^{\frac{-Rt}{L}} \times \left\{1 - e^{\left(\frac{Rt}{L}-at\right)}\right\}$$

$$= \frac{E}{R-aL} \times e^{-Rt/L}\left\{1 - e^{Rt/L} \times e^{-at}\right\}$$

$$= \frac{E}{R-aL} \times \left(e^{-Rt/L} - e^{-Rt/L} \times e^{Rt/L} \times e^{-at}\right)$$

$$= \frac{E}{R-aL} \times \left(e^{-Rt/L} - e^{-at}\right)$$

$$\therefore \quad I e^{Rt/L} = \frac{E}{R-aL}\left\{e^{\left(\frac{R}{L}-a\right)t} - 1\right\}$$

$$\therefore \quad I = \frac{E}{R-aL} \times e^{-Rt/L} \times \left\{e^{\left(\frac{R}{L}-a\right)t} - 1\right\}$$

$$= \frac{E}{R-aL}\left\{e^{\left(\frac{R}{L}-a\right)t} \times e^{-Rt/L} - e^{-Rt/L}\right\}$$

$$= \frac{E}{R-aL}\left\{e^{\frac{Rt}{L}} \cdot e^{-at} \times e^{\frac{-Rt}{L}} - e^{-Rt/L}\right\}$$

$$\therefore I = \frac{E}{R-aL}\left\{ e^{-at} - e^{\frac{-Rt}{L}} \right\}$$

Q In a circuit containing inductance L, Resistance R and voltage E, the current I is given by; $E = RI + L\frac{dI}{dt}$. Given $L = 640$ H., $R = \cancel{250\,\Omega}$ $250\,\Omega$ and $E = 500$V. I being zero when $t=0$. Find the time that elapses, before it reaches 90% of its maximum value.

(May 2008, 2006, 2004, 2015)

→ $L = 640$H, $R = 250\,\Omega$, $E = 500$V

For given conditions,

$$I = \frac{E}{R}\left(1 - e^{-Rt/L}\right) \quad \text{—①}$$

maximum value of I be Imax which is obtained when $t \to \infty$.

$$\therefore I_{max} = \frac{E}{R}$$

$$\therefore I_{max} = \frac{90}{100}\frac{E}{R} \quad [90\% \text{ of maximum value}]$$

putting in ① if $t = t_1$ ~~for 90% Imax~~,

from ①

$$I_{max} = \frac{E}{R}\left(1 - e^{-Rt_1/L}\right)$$

$$\therefore \frac{9\cancel{0}}{10\cancel{0}} \times \frac{\cancel{E}}{\cancel{R}} = \frac{\cancel{E}}{\cancel{R}}\left(1 - e^{-Rt_1/L}\right)$$

$$\therefore \frac{9}{10} = \left(1 - e^{-Rt_1/L}\right)$$

$$\therefore e^{-Rt_1/L} = 1 - 9/10 = 1/10$$

$$\therefore \log_e e^{-Rt_1/L} = \log\left(\frac{1}{10}\right)$$

$\therefore \frac{-Rt_1}{L} = \log\left(\frac{1}{10}\right)$

put $L = 640H$, $R = 250\Omega$, $\underset{\text{not req}^d}{\underline{E = 500V}}$ — as given.

$\therefore \frac{-(25\cancel{0})t_1}{64\cancel{0}} = \log 1 - \log 10$

$\therefore \frac{-25}{64} t_1 = -\log 10$

$\therefore t_1 = \frac{64}{25} \times \log(10)$

$\therefore \boxed{t_1 = 5.89 \text{ sec.}}$ ← this is req^d time.

Q Show that the diff. eq^n for the current I in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force $E \sin \omega t$. Satisfies the eq^n $L\frac{dI}{dt} + RI = E\sin\omega t$, (Dec 2011, May 2019)

Find the value of the current at an time t, if initially there is no current in the circuit.

→ Given, $RI + L\frac{dI}{dt} = E\sin\omega t$.

$\Rightarrow L\frac{dI}{dt} + RI = E\sin\omega t$ ←

Convert into L.D.E

$\Rightarrow \frac{dI}{dt} + \frac{R}{L} I = \frac{E\sin\omega t}{L}$

which is LDE. Cont. value

$\int dt = t$

find I.F. → $I.F. = e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$

and its sol^m is,

$\boxed{I. I.F. = \int (I.F.) \times Q\, dt + C}$

$\therefore (I)\left(e^{\frac{Rt}{L}}\right) = \int \frac{E}{L} \sin\omega t \times e^{Rt/L} dt + C$

$$\therefore I \times e^{Rt/L} = \frac{E}{L}\int e^{Rt/L} \sin\omega t\,dt + c \quad , a = \frac{R}{L}, b = \omega$$

$$= \frac{E}{L} \times \frac{e^{Rt/L}}{\sqrt{\left(\frac{R}{L}\right)^2 + \omega^2}} \sin(\omega t - \phi) + c; \; \phi = \tan^{-1}\left(\frac{\omega}{R/L}\right)$$

$$= \tan^{-1}\left(\frac{L\omega}{R}\right)$$

using, $\int e^{at}\sin bt\,dt = \frac{e^{at}}{\sqrt{a^2+b^2}} \sin(bt - \phi)$,

where $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

$$= \frac{E}{L} \times \frac{e^{Rt/L}}{\sqrt{\frac{R^2}{L^2} + \omega^2}} \sin\left(\omega t - \tan^{-1}\left(\frac{L\omega}{R}\right)\right) + c$$

$$= \frac{E}{L} \times \frac{e^{Rt/L}}{\sqrt{\frac{R^2 + L^2\omega^2}{L^2}}} \sin\left(\omega t - \cancel{\phi}\tan^{-1}\left(\frac{L\omega}{R}\right)\right) + c$$

$$\therefore I e^{Rt/L} = \frac{E}{\cancel{L}} \times \cancel{L} \times \frac{e^{Rt/L}}{\sqrt{R^2 + L^2\omega^2}} \sin\left(\omega t - \tan^{-1}\left(\frac{L\omega}{R}\right)\right) + c$$

$$\therefore I \cancel{e} = \left\{ \frac{E \times e^{Rt/L}}{\sqrt{R^2 + L^2\omega^2}} \sin\left(\omega t - \tan^{-1}\left(\frac{L\omega}{R}\right)\right) + c \right\} e^{-\frac{Rt}{L}}$$

$$\therefore I = \frac{E}{\sqrt{R^2 + L^2\omega^2}} \sin\left(\omega t - \tan^{-1}\left(\frac{L\omega}{R}\right)\right) + c e^{-Rt/L} \quad \text{--- } (1)$$

from given cond^{n}s, $t = 0$, $I = 0$

$$\therefore 0 = \frac{E}{\sqrt{R^2 + L^2\omega^2}} \sin(0 - \phi) + c \cdot e^0$$

$$= \frac{E}{\sqrt{R^2 + L^2\omega^2}} \sin(-\phi) + c = \frac{-E\sin\phi}{\sqrt{R^2 + L^2\omega^2}} + c$$

$\therefore c = \dfrac{E \sin\phi}{\sqrt{R^2+\omega^2 L^2}}$

put in ①

$$\therefore I = \frac{E}{\sqrt{R^2+L^2\omega^2}} \sin(\omega t - \phi) + \frac{E \sin\phi}{\sqrt{R^2+\omega^2 L^2}} e^{-Rt/L}$$

$$\therefore \boxed{I = \frac{E}{\sqrt{R^2+L^2\omega^2}} \left[\sin(\omega t-\phi) + \sin\phi \cdot e^{-Rt/L}\right]}$$

which is given current at any time t.

Q The eqⁿ of an L-R circuit is given by $L\dfrac{dI}{dt} + RI = 10\sin t$. If $I = 0$, at $t = 0$, express I as a funⁿ of t (Dec-2008, 05)

$\longrightarrow$ $L\dfrac{dI}{dt} + RI = 10\sin t$

$\therefore \dfrac{dI}{dt} + \dfrac{R}{L} I = \dfrac{10\sin t}{L}$

$I.F. = e^{\int \frac{R}{L} dt} = e^{Rt/L}$

and solⁿ is,

$$I \cdot I.F. = \int Q \times I.F.\, dt + c$$

$$= \int \frac{10\sin t}{L} \times e^{Rt/L} dt + c$$

$$= \frac{10}{L} \int e^{Rt/L} \sin t \, dt + c \quad , a = \frac{R}{L} \quad b = 1$$

using $\int e^{at} \sin bt \, dt = \dfrac{e^{at}}{\sqrt{a^2+b^2}} \sin(bt-\phi)$

$\phi = \tan^{-1}\left(\dfrac{b}{a}\right)$

$$\therefore I \times e^{Rt/L} = \frac{10}{L} \times \frac{e^{Rt/L}}{\sqrt{\left(\frac{R}{L}\right)^2 + 1^2}} \sin(t-\phi) + c, \quad \phi = \tan^{-1}\left(\frac{1}{R/L}\right) = \tan^{-1}\left(\frac{L}{R}\right)$$

$$\therefore I = e^{-Rt/L} \times \frac{10}{L} \times \frac{e^{Rt/L}}{\sqrt{\frac{R^2+L^2}{L^2}}} \sin(t-\phi), \; \phi = \tan^{-1}\left(\frac{L}{R}\right) + ce^{-Rt/L}$$

$$= e^{-Rt/L} \times \frac{10}{\cancel{L}} \times \cancel{L} \times \frac{e^{Rt/L}}{\sqrt{R^2+L^2}} \sin(t-\phi), \; \phi = \tan^{-1}\left(\frac{L}{R}\right) + ce^{-Rt/2}$$

$$\therefore I = \frac{10}{\sqrt{R^2+L^2}} \sin(t-\phi) + ce^{-Rt/L} \quad \text{--- (1)}$$

when $t=0$, $I=0$ put in (1)

$$\therefore 0 = \frac{10}{\sqrt{R^2+L^2}} \sin(0-\phi) + c.e^{0}$$

$$\therefore 0 = \frac{-\sin\phi \times 10}{\sqrt{R^2+L^2}} + c$$

$$\therefore c = \frac{10\sin\phi}{\sqrt{R^2+L^2}}$$

put in (1)

$$\therefore I = \frac{10}{\sqrt{R^2+L^2}} \sin(t-\phi) + \frac{10\sin\phi}{\sqrt{R^2+L^2}} e^{-Rt/L}$$

$$= \frac{10}{R^2+L^2}\left[\sin(t-\phi) + \sin\phi . e^{-Rt/L}\right]$$

Q A constant electromotive force E volts is applied to a circuit containing a constant resistance R ohms in series and a constant inductance L henries. If the initial current is zero, show that the current builds up to half its theoretical maximum in $\left(\frac{L \log 2}{R}\right)$ seconds.

(Dec. 2011, 09, 08, 06, 17, May-13)

→ Let I be the current in the ckt at any time t, by Kirchoff's law we have,

$$I = \frac{E}{R}\left(1 - e^{\frac{-Rt}{L}}\right) \quad \text{—— (1)}$$

This eqn gives the current in the circuit at any time t.

clearly I increases with t and attains the maximum value $\frac{E}{R}$.

given that, if $I = \frac{1}{2} \times \frac{E}{R}$ --- [half of its theoretical maximum.. from given]

put in (1)

$$\therefore \frac{1}{2}\cancel{\frac{E}{R}} = \cancel{\frac{E}{R}}\left(1 - e^{-Rt/L}\right)$$

$$\therefore \frac{1}{2} = \left(1 - e^{-Rt/L}\right)$$

$$\therefore 1 - \frac{1}{2} = e^{-Rt/L}$$

$$\therefore \frac{1}{2} = e^{-Rt/L}$$

$$\therefore \log\left(\frac{1}{2}\right) = \log e^{-Rt/L}$$

$$\therefore \log 1 - \log 2 = \frac{-Rt}{L}$$

$$\therefore -\log 2 = \frac{-Rt}{L}$$

$\Rightarrow \log 2 = \frac{Rt}{L}$

$\therefore \boxed{t = \frac{L\log 2}{R}}$ --- Hence proved.

Q An electrical circuit contains an inductance of 5 henries and a resistance of 12 ohms in series with an e.m.f. 120 sin(20t) volts.) Find the current at t=0.01. If it is zero when t=0.

$\longrightarrow$ $L\frac{dI}{dt} + RI = 120\sin(20t)$

$\Rightarrow \frac{dI}{dt} + \frac{R}{L}I = \frac{120\sin(20t)}{L}$

$I.F. = e^{\int \frac{R}{L}dt} = e^{Rt/L}$

G.S. is,

$I.\, I.F. = \int Q \times I.F.\, dt + c$

$\therefore I \times e^{\frac{Rt}{L}} = \int \frac{120\sin(20t)}{L} \times e^{Rt/L} dt + c$

$\therefore I \times e^{Rt/L} = \frac{120}{L}\int \sin(20t)\, e^{Rt/L} dt + c$

$= \frac{120}{L} \times \int e^{Rt/L} \sin(20t)\, dt + c$

$= \frac{120}{L} \times \frac{e^{Rt/L}}{\sqrt{\frac{R^2}{L^2} + 20^2}} \times \sin(20t - \phi) + A$

$\phi = \tan^{-1}\left(\frac{20}{R/L}\right)$

$= \tan^{-1}\frac{20L}{R}$

using $\int e^{at}\sin bt\, dt = \frac{e^{at}}{\sqrt{a^2+b^2}} \sin(bt-\phi)$, $\phi = \tan^{-1}\frac{b}{a}$

$$\therefore I e^{Rt/L} = \frac{120}{L} \frac{e^{Rt/L}}{\sqrt{\frac{R^2}{L^2} + 400}} \times \sin(20t - \phi) + c$$

$$= \frac{120}{L} \frac{e^{Rt/L}}{\sqrt{\frac{R^2 + 400L^2}{L^2}}} \sin(20t - \phi) + c$$

$$= \frac{120}{\cancel{L}} \times \frac{\cancel{L}}{\sqrt{R^2 + 400L^2}} \times e^{Rt/L} \sin(20t - \phi) + c$$

$$\therefore I = \frac{120}{\sqrt{R^2 + 400L^2}} \times e^{Rt/L} \times \sin(20 - \phi) \times e^{\frac{-Rt}{L}} + ce^{-Rt/L}$$

$$\therefore I = \frac{120 \sin(20 - \phi)}{\sqrt{R^2 + 400L^2}} + ce^{-Rt/L} \quad \text{—(1)}, \quad \phi = \tan^{-1}\left(\frac{20L}{R}\right)$$

~~To find I at t=0.01, given, L=5H, R=12Ω.~~

$$\cancel{\phi = \tan^{-1}\left(\frac{20 \times 5}{12}\right) = 1.451367401}$$

$$\cancel{\therefore I = \frac{120 \sin(20 - 1.451367401)}{\sqrt{12^2 + (400)(5^2)}}}$$

given that, $I = 0$ if $t = 0$

$$\therefore 0 = \frac{120 \sin(20 - \phi)}{\sqrt{R^2 + 400L^2}} + ce^0$$

$$\therefore c = \frac{-120 \sin(20 - \phi)}{\sqrt{R^2 + 400L^2}}$$

put in (1)

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$$\therefore I = \frac{120 \sin(20-\phi)}{\sqrt{R^2+400L^2}} - \frac{120 \sin(20-\phi)}{\sqrt{R^2+400L^2}} e^{-Rt/L}$$

$$\therefore I = \frac{120 \sin(20-\phi)}{\sqrt{R^2+400L^2}} \left[1 - e^{-Rt/L}\right] \quad \text{--- (11)}$$

To find I at $t = 0.01$ given, $L = 5H$, $R = 12\,\Omega$

$$\phi = \tan^{-1}\left(\frac{20L}{R}\right) = \tan^{-1}\left(\frac{20 \times 5}{12}\right) = 1.451367401$$

put in (11) we get $I = 0.023729630$ A.

Q The equation of electromotive force in terms of current I for an electrical circuit having resistance R and a condenser of capacity C in series, is $E = RI + \int \frac{I}{C}\, dt$. Find the current I at any time t, when $E = E_0 \sin \omega t$.

→ the given eq^n can be written as,

$$RI + \int \frac{I}{C}\, dt = E_0 \sin \omega t. \quad \left[\because \text{given, } RI + \int \frac{I}{C} dt = E \text{ and } E = E_0 \sin \omega t\right]$$

diff. both sides w.r.t. t.

$$R\frac{dI}{dt} + \frac{d}{dt}\int \frac{I}{C}\, dt = \frac{d}{dt}\left(E_0 \sin \omega t\right)$$

$$R\frac{dI}{dt} + \frac{I}{C} = E_0 \times \cos \omega t \times \omega$$

$$\therefore R\frac{dI}{dt} + \frac{I}{C} = \omega E_0 \cos \omega t.$$

$$\therefore \frac{dI}{dt} + \frac{I}{RC} = \frac{\omega E_0 \cos \omega t}{R}$$

$$\therefore \frac{dI}{dt} + \left(\frac{1}{RC}\right) I = \frac{\omega E_0 \cos \omega t}{R}$$

$$I.F. = e^{\int \frac{1}{RC} dt} = e^{t/RC}$$

G.S. is,

$$I \times I.F. = \int \phi \times I.F. \, dt + c$$

$$= \int \frac{\omega E_0 \cos \omega t}{R} \times e^{t/RC} dt + c_1$$

$$= \frac{\omega E_0}{R} \int \cos \omega t . e^{t/RC} dt + c_1$$

$$= \frac{\omega E_0}{R} \int e^{t/RC} . \cos \omega t \, dt + c_1 , \quad a = \frac{1}{RC} , \quad b = \omega$$

$$\int e^{at} \cos bt \, dt = \frac{e^{at}}{\sqrt{a^2+b^2}} \cos(bt - \phi),$$

$$\phi = \tan^{-1}\left(\frac{b}{a}\right)$$

$$= \frac{\omega E_0}{R} \times \frac{e^{t/RC}}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}} \cos(\omega t - \phi) + c_1$$

$$\phi = \tan^{-1}\left(\frac{\omega}{1/RC}\right) = \tan^{-1} RC\omega$$

$$= \frac{\omega E_0}{R} \times \frac{e^{t/RC}}{\sqrt{\frac{1}{R^2C^2} + \omega^2}} \cos(\omega t - \phi) + c_1$$

$$= \frac{\omega E_0}{R} \times \frac{e^{t/RC}}{\sqrt{\frac{1 + \omega^2 R^2 C^2}{R^2 C^2}}} \cos(\omega t - \phi) + c_1$$

$$I \times e^{t/RC} = \frac{\omega E_0}{\cancel{R}} \times \cancel{R} C \times \frac{e^{t/RC}}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\omega t - \phi) + c_1$$

$$\therefore I = \frac{\omega C E_0}{\sqrt{1 + \omega^2 R^2 C^2}} e^{t/RC} \times e^{-t/RC} \cos(\omega t - \phi) + c_1 e^{-t/RC}$$

$$\therefore I = \frac{\omega C E_0}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\omega t - \phi) + c_1 e^{-t/RC}$$ which gives current at any time t.

Q Find the current I in the circuit having resistance R & condenser of capacity C in series with emf $E\sin\omega t$.

(May 2006, Dec-2010, 2007)

→ we have, $RI + \frac{q}{c} = E\sin\omega t$.

$$R\frac{dq}{dt} + \frac{q}{c} = E\sin\omega t.$$

$$\therefore \frac{dq}{dt} + \frac{q}{RC} = \frac{E\sin\omega t}{R}$$

$$I.F. = e^{\int \frac{1}{RC} dt} = e^{t/RC}$$

$\therefore$ G.S. is,

$$q.\,I.F. = \int Q \times I.F.\, dt + c_1$$

$$= \int \frac{E\sin\omega t}{R} \times e^{t/RC} dt + c_1$$

$$= \frac{E}{R}\int e^{t/RC}\sin\omega t\, dt + c_1$$

$$= \frac{E}{R}\,\frac{e^{t/RC}}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}} \sin(\omega t - \phi) + c_1$$

$$\phi = \tan^{-1}\left(\frac{\omega}{1/RC}\right) = \tan^{-1} RC\omega.$$

$$= \frac{E}{R} \times \frac{e^{t/RC}}{\sqrt{\frac{1 + R^2c^2\omega^2}{R^2c^2}}} \sin(\omega t - \phi) + c_1$$

$$= \frac{E}{\cancel{R}} \times \cancel{R}C \times \frac{e^{t/RC}}{\sqrt{1 + R^2c^2\omega^2}} \sin(\omega t - \phi) + c_1$$

$$\therefore q.e^{t/RC} = \frac{Ec\, e^{t/RC}}{\sqrt{1 + R^2c^2\omega^2}} \sin(\omega t - \phi) + c_1$$

$$\therefore q = \frac{Ec\, e^{t/RC}}{\sqrt{1 + R^2c^2\omega^2}} \times e^{-t/RC} \sin(\omega t - \phi) + c_1 e^{-t/RC}$$

$\therefore q = \frac{Ec}{\sqrt{1+R^2c^2w^2}} \sin(wt-\phi) + c_1 e^{-t/RC}, \quad \phi = \tan^{-1}(Rcw)$

$I = \frac{dq}{dt} = \frac{Ec}{\sqrt{1+R^2c^2w^2}} \times \cos(wt-\phi) \cdot w + c_1 e^{-t/RC} \times \frac{-1}{RC},$

$= \frac{Ecw}{\sqrt{1+R^2c^2w^2}} \cos(wt-\phi) - \frac{c_1}{RC} e^{-t/RC}. \quad \phi = \tan^{-1}(Rcw)$

Q A circuit contains resistance 'R' ohms and a condenser of 'c' farads connected to a constant e.m.f. E if $\frac{q}{c}$ is the voltage of the condenser at time t after closing the circuit, show that the voltage at time t is, $E(1-e^{-t/cR})$ -- [May-07, 05, Dec-09, 13]

→ The diff. eqn for the circuit is $RI + \frac{q}{c} = E$.

$\Rightarrow I + \frac{q}{RC} = \frac{E}{R} \Rightarrow \frac{dq}{dt} + \left(\frac{1}{RC}\right) q = \frac{E}{R}.$

this is LDE.

$I.F. = e^{\int \frac{1}{RC} dt} = e^{t/RC}$

G.S. is,

$q \cdot I.F. = \int Q \times I.F. \, dt + c_1$

$\therefore q \cdot e^{t/RC} = \int \frac{E}{R} \times e^{t/RC} dt + c_1$

$= \frac{E}{R} \times \frac{e^{t/RC}}{1/RC} + c_1$

$= \frac{E}{\cancel{R}} \times \cancel{R}C \times e^{t/RC} + c_1$

$= Ec e^{t/RC} + c_1$

$\therefore \frac{q}{c} e^{t/RC} = E e^{t/RC} + c_1/c$

$\Rightarrow \frac{q}{c} e^{t/RC} = E e^{t/RC} + c_2$ ——— (11)

at $t=0$, $q=0$ ———

$\therefore 0 = E e^0 + c_2$

$\therefore 0 = E + c_2$

$\therefore c_2 = -E$

put in (11)

$\Rightarrow \frac{q}{c} e^{t/RC} = E e^{t/RC} - E.$

$\therefore \cancel{\frac{q}{c} = E(e^{t/RC} - 1)}$

$\Rightarrow \frac{q}{c} = e^{-t/RC}(E e^{t/RC} - E)$

$= e^{-t/RC} \times E \times e^{t/RC} - E e^{-t/RC}$

$= E - E e^{-t/RC}$

$\Rightarrow \boxed{\frac{q}{c} = E(1 - e^{-t/RC})}$

Q The charge 'q' on the plate of a condenser of capacity 'c' charged through a resistor 'R' by a steady voltage 'V' satisfies the diff. eqn $R\frac{dq}{dt} + \frac{q}{c} = V$. If $q=0$ at $t=0$, show that $q = CV[1 - e^{-t/RC}]$. Find the current flowing into the plate (Dec 05, May-2008, 04)

→ Given, $R\frac{dq}{dt} + \frac{q}{c} = V.$

i.e. $\frac{dq}{dt} + \frac{q}{RC} = \frac{V}{R}.$

$\therefore \frac{dq}{dt} + \left(\frac{1}{RC}\right) q = \frac{V}{R}$ which is linear diff. eqn.

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$\therefore$ I.F. is, $e^{\int \frac{1}{RC} dt} = e^{t/RC}$

G.S. is, $q \cdot I.F. = \int \frac{V}{R} \times e^{t/RC} dt + c_1$

$$= \frac{V}{R} \cdot \frac{e^{t/RC}}{1/RC} + c_1$$

$$= \frac{V}{\cancel{R}} \times \cancel{R}C \times e^{t/RC} + c_1$$

$$\therefore q \times e^{t/RC} = VC e^{t/RC} + c_1$$

$$\therefore q = e^{-t/RC} [VC e^{t/RC} + c_1]$$

$$= VC e^{t/RC} \times e^{-t/RC} + c_1 e^{-t/RC}$$

$$\therefore q = VC + c_1 e^{-t/RC} \quad \text{—(1)}$$

For $t = 0$, $q = 0$

$$\therefore 0 = VC + c_1$$

$$\therefore c_1 = -VC$$

put in (1)

$$\therefore q = VC - VC e^{-t/RC}$$

$$= VC(1 - e^{-t/RC})$$

now, $I = \frac{dq}{dt} = \frac{d}{dt}[VC(1 - e^{-t/RC})]$

$$= VC \times (0 - e^{-t/RC} \times -1/RC)$$

$$= V\cancel{C} \times \frac{1}{R\cancel{C}} e^{-t/RC}$$

$$= \frac{V}{R} e^{-t/RC}$$

Q An voltage $200e^{-5t}$ is applied to a circuit containing resistance $R = 20\,\Omega$ and condenser of capacity $C = 0.01$ farads in series. Find the change and current at any time, assuming $t = 0$, $q = 0$.

18) already taken.

HEAT FLOW.

The fundamental principles involved in the problems of heat conduction are:

i) Heat flows from a higher temp. to a lower temp.

ii) The quantity of heat in a body is proportional to its mass and temp.

Fourier's law of heat condn: The rate of heat flow across an area is proportional to the area and to the rate of change of temp. with respect to its distance normal to the area.

Q1) A pipe 20 cm in diameter contains steam at 150°c and is protected with a covering 5 cm thick for which $K = 0.0025$, if the temp. of the outer surface of the covering is 40°c, find the temperature half-way through the covering under steady-state condition.

(May-2011, 09, 05, 18, Dec-09, 2014)

If q (cal/sec) be the quantity of heat flows across a slab of area A (cm^2) and thickness δx in one second, where the difference of temp. at the faces is δT,

then, q = Thermal conductivity x area x temp. gradient

$$q = -kA \frac{dT}{dx}$$

where k is a constant depending upon the material of the body and is called the thermal conductivity. negative sign is attached because T decreases as x increases.

ans. of Q1)

we have, by Fourier's law,

$$q = -KA\frac{dT}{dx}$$

$$\therefore q = -K \times 2\pi x \times \frac{dT}{dx} \longrightarrow x \text{ is radius}$$

lateral surface area of pipe $= 2\pi x$

$$\therefore dT = \frac{-q}{2\pi K} \times \frac{dx}{x}$$

$$\therefore \int dT = \int \frac{-q}{2\pi K} \times \frac{dx}{x}$$

$$= \frac{-q}{2\pi K} \int \frac{dx}{x}$$

$$\therefore T = \frac{-q}{2\pi K} \times \log_e x + c \quad \text{— (i)}$$

since, $T = 150°c$, when $x = 10cm$ [$\because$ given that diameter $= 20cm$ $\therefore$ radius $= x = 10cm$]

put in (i)

$$\therefore 150 = \frac{-q}{2\pi K} \times \log_e 10 + c \quad \text{— (ii)}$$

again since, $T = 40$, when $x = 15$ (10+5)

$\therefore$ put in (i)

$$\therefore 40 = \frac{-q}{2\pi K} \times \log_e 15 + c \quad \text{— (iii)}$$

Subtracting (iii) from (ii)

$$\therefore 150 - 40 = \left(\frac{-q \times \log_e 10}{2\pi K} + c\right) - \left(\frac{-q \times \log_e 15}{2\pi K} + c\right)$$

$$= \frac{-q \log_e 10}{2\pi K} + \cancel{c} + \frac{q \times \log_e 15}{2\pi K} - \cancel{c}$$

$$\therefore 110 = \frac{q}{2\pi K}\left[\log_e 15 - \log_e 10\right]$$

$$\therefore 110 = \frac{q}{2\pi K} \log_e\left(\frac{15}{10}\right) \quad \text{— (*)}$$

$\therefore 110 = \frac{q}{2\pi k} \log_e 1.5$ —— (iv)

Let $T = t$, when $x = 12.5$.

$\therefore t = -\frac{q}{2\pi k} \log_e 12.5 + c$ —— (v)

Subtracting (ii) from (iv)

$\therefore t - 150 = \left(\frac{-q}{2\pi k} \log_e 12.5 + c\right) - \left(\frac{-q}{2\pi k} \log_e 10 + c\right)$

$= \frac{-q}{2\pi k} \log_e 12.5 \cancel{+c} + \frac{q}{2\pi k} \log_e 10 \cancel{-c}$

$= \frac{-q}{2\pi k} \left(\log_e 12.5 - \log_e 10\right)$

$= \frac{-q}{2\pi k} \left(\log_e \left(\frac{12.5}{10}\right)\right)$

$\therefore t - 150 = \frac{-q}{2\pi k} \log_e (1.25)$ —— (vi)

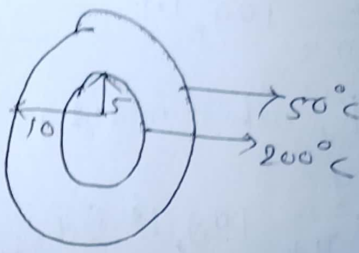
dividing (vi) by (iv)

$\therefore \frac{t-150}{110} = \frac{\frac{-\cancel{q}}{\cancel{2\pi k}} \log_e (1.25)}{\frac{\cancel{q}}{\cancel{2\pi k}} \log_e (1.5)} = \frac{-\log_e (1.25)}{\log_e (1.5)}$

$\therefore t = 89.5°c.$

Q A long hollow pipe has an inner diameter of 10cm and outer diameter of 20cm. The inner surface is kept at 200°c and the outer surface at 50°c. The thermal conductivity is 0.12. How much heat is lost per minute from a portion of the pipe 20 metres long? Find the temperature at a distance $x = 7.5$cm from the centre of the pipe. (Nov. Dec. 2019, Dec- 2011, 06, 05, May-07)

$\to$ $\varphi = -KA\frac{dT}{dx}$



$T = \frac{-\varphi}{2\pi K}\log x + c$ —★

$T = 200°c$, $x = 5cm$.

$200°c = \frac{-\varphi}{2\pi k}\log 5 + c$ — (I)

$T = 50°c$, $x = 10cm$

$50°c = \frac{-\varphi}{2\pi k}\log 10 + c$ — (II)

(I) — (II),

$150°c = \frac{-\varphi}{2\pi k}(\log 5 - \log 10)$

$= \frac{\varphi}{2\pi k}\log\left(\frac{10}{5}\right)$

$\therefore 150 = \frac{\varphi}{2\pi k}\log 2$ — (III)

$\varphi = \frac{150\times 2\pi k}{\log 2} = \frac{150\times 2\pi\times 0.12}{\log 2}$

$= 163$ cal/sec.

Hence the heat lost per minute through 20 metre length of the pipe $60\times 2000\,\varphi = 120000\times 163 = 19560000$ cal

now let, $T = t$, when $x = 7.5$ cm

$t = \frac{-\varphi}{2\pi k}\log 7.5 + c$ — from ★. — (IV)

(IV) — (I).

$t - 200 = \frac{-\varphi}{2\pi k}(\log 7.5 - \log 5)$

$t - 200 = \frac{-\varphi}{2\pi k}\log\left(\frac{7.5}{5}\right) = \frac{-\varphi}{2\pi k}\log(1.5)$ — (V)

now, (V) $\div$ (III)

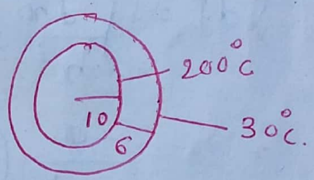
$$\frac{t-200}{150} = \frac{\frac{-\cancel{q}}{\cancel{2\pi k}}\log(1.5)}{\frac{\cancel{q}}{\cancel{2\pi k}}\log(2)}$$

$\therefore t = 113^\circ c$

$\therefore$ when, $x = 7.5$ cm then $t = 113^\circ c$

A steam pipe 20cm in diameter is protected with a covering 6 cm thick for which the coefficient of thermal conductivity is $k = 0.0003$ cal/cm deg. sec. in steady state. Find the heat lost per hour through a meter length of the pipe, if the surface of the pipe is at $200^\circ c$ and the outer surface of the covering is $30^\circ c$.

$\rightarrow$ $$Q = -kA\frac{dT}{dx}.$$



$$T = \frac{-q}{2\pi k}\log x + c$$

$T = 200^\circ c$, $x = 10$ cm

$$\cancel{T}\,200 = \frac{-q}{2\pi k}\log(10) + c \quad \text{--- (i)}$$

$T = \cancel{16}\ 30^\circ c$, $x = 16$ cm

$$\cancel{T}\,30 = \frac{-q}{2\pi k}\log(16) + c \quad \text{--- (ii)}$$

(i) $-$ (ii)

$$170 = \frac{-q}{2\pi k}(\log(10) - \log(16))$$

$$\therefore 170 = \frac{q}{2\pi k}\log(1.6)$$

$$\therefore q = \frac{170 \times 2\pi k}{\log(1.6)} \text{ cal/sec}$$

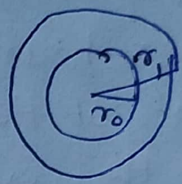
$$\therefore \text{req}^d \text{ heat loss} = \frac{170 \times 2 \times 3.14 \times 0.0003 \times 100 \times 3600}{\log(1.6)}$$

[$\because$ 1m = 100cm, 1hr = 3600sec]

$\therefore$ Heat loss $= 245443.3861$ cal

The inner and outer surfaces of a spherical shell are maintained at T_0 and T_1 temperatures resp. If the inner and outer radii of the shell are r_0 and r_1 resp. and thermal conductivity of the shell is k, find the amount of heat lost from the shell per unit time. Find also the temperature distribution through the shell. (Dec. 2007, Nov 15)

$\longrightarrow$ Let x be the thickness of the spherical shell. the heat flowing through the section is,



$$Q = -kA\frac{dT}{dx}$$

$$Q = -k(4\pi x^2)\frac{dT}{dx}$$

$$\therefore dT = \frac{dx}{x^2} \times \frac{-Q}{4\pi k}$$

$$\therefore \int dT = \int \frac{dx}{x^2} \times \frac{-Q}{4\pi k} + c$$

$$= \frac{-Q}{4\pi k}\int x^{-2}dx + c$$

$$\therefore T = \frac{-Q}{4\pi k}\frac{x^{-1}}{-1} + c$$

$$T = \frac{Q}{4\pi k x} + c \quad \text{--- (I)}$$

now, put $x = r_0$; $T = T_0$

$$\therefore T_0 = \frac{Q}{4\pi k r_0} + c \quad \text{--- (II)}$$

now, put $x = r_1$, $T = T_1$

$$\therefore T_1 = \frac{Q}{4\pi k r_1} + c \quad \text{--- (III)}$$

(II) $-$ (III)

$$\therefore T_0 - T_1 = \frac{Q}{4\pi k}\left[\frac{1}{r_0} - \frac{1}{r_1}\right]$$

$$\therefore T_0 - T_1 = \frac{Q}{4\pi k}\left(\frac{r_1 - r_0}{r_1 r_0}\right)$$

$$\therefore \varphi = \frac{(T_0 - T_1)(4\pi k)(r_1 r_0)}{r_1 - r_0} \quad \text{--- (iv)}$$

we have, from (i)

$$T = \frac{\varphi}{4\pi k x} + c$$

$$\therefore c = T - \frac{\varphi}{4\pi k} \cdot \frac{1}{x}$$

when, $T = T_0$, $x = r_0$

$$\therefore c = T_0 - \frac{\varphi}{4\pi k} \times \frac{1}{r_0}$$

$$\therefore T = \frac{\varphi}{4\pi k x} + \left(T_0 - \frac{\varphi}{4\pi k} \times \frac{1}{r_0}\right)$$

$$= \frac{\cancel{\dfrac{(T_0 - T_1)(4\pi k)(r_1 r_0)}{(r_1 - r_0)}}}{\cancel{4\pi k x}} + \cancel{\left(T_0 - \frac{\varphi}{4\pi k} \times \frac{1}{r_0}\right)} \text{ --- from (iv)}$$

put value of φ.

$$= \cancel{\frac{(T_0 - T_1)(r_1 r_0)}{(r_1 - r_0)x}} + T_0 -$$

$$= \frac{\varphi}{4\pi k}\left(\frac{1}{x} - \frac{1}{r_0}\right) + T_0$$

put, φ from (iv)

$$\therefore T = \frac{\dfrac{(T_0 - T_1)(4\cancel{\pi k})(r_1 r_0)}{r_1 - r_0}}{(4\cancel{\pi k})}\left(\frac{1}{x} - \frac{1}{r_0}\right) + T_0$$

$$\therefore T = \frac{(T_0 - T_1)(r_1 r_0)}{r_1 - r_0}\left(\frac{1}{x} - \frac{1}{r_0}\right) + T_0$$

3. solve the eq^n, $L\frac{dI}{dt} + RI = 200\cos(300t)$, where $R = 100\,\Omega$,

$L = 0.05\,H$ and Find I, given that.

* Rectilinear Motion :-

Velocity → $\frac{dx}{dt}$, x is distance.

accelaration → $\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = \frac{dv}{dx} \times v = v\frac{dv}{dx}$

D'Alembert's principle :

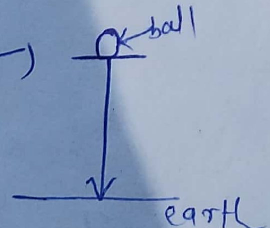
Net force = mass × accelaration

Statement → Algebraic sum of the forces acting on the body along a given direction is equal to the product of mass × accelairation in that direction.

Illustrations on rectillinear motion;

Q1) A body of mass m, falling from a rest is subjected to the force of gravity and an air resistance proportional to the square of the velocity 'kv^2'. If it falls through a distance x and possesses a velocity v at that instance, prove that $\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$ where $mg = ka^2$.

(May 06, 05, 04, Dec. 2011, 16, 17)

→ 

forces acting on ball

1) gravitational force → mg ✓ acting downward

2) the air resistance ~~acting~~ acting upward → $-kv^2$

∴ Net force $= mg + (-kv^2)$

$= mg - kv^2$ ——— (1)

$\therefore$ as, Net force = mass x accelaration

$= m \times a$

$= m \times \frac{dv}{dt}$

$= m \times \frac{dv}{dx} \times \frac{dx}{dt}$

$= m \times \frac{dv}{dx} \times v$

$= mv\frac{dv}{dx}$

$\therefore mg - kv^2 = mv\frac{dv}{dx}$ from ①

$\therefore ka^2 - kv^2 = mv\frac{dv}{dx}$ given, $mg = ka^2$

$\therefore k(a^2 - v^2) = mv\frac{dv}{dx}$

$\therefore \frac{k}{m}(a^2 - v^2)\,dx = v\,dv$

$\therefore \frac{k}{m}dx = \frac{v\,dv}{a^2 - v^2}$

$\therefore \int \frac{k}{m}dx = \int \frac{v\,dv}{a^2 - v^2}$ $\cancel{+c}$

$\therefore \frac{kx}{m} + c = \frac{-1}{2} \times \int \frac{-2v\,dv}{a^2 - v^2}$ $\cancel{+c}$

$\therefore \frac{kx}{m} + c = \frac{-1}{2}\log(a^2 - v^2)$ $\cancel{+c}$ —①

as, $x = 0$, $V = 0$ put in ①

$\therefore 0 + c = \frac{-1}{2}\log(a^2 - 0)$ $\cancel{+c}$

$\therefore c = \frac{-1}{2}\log a^2$

put in ①

$$\frac{kx}{m} + \left(\frac{-1}{2}\log a^2\right) = \frac{-1}{2}\log(a^2-v^2)$$

$$\therefore \frac{2kx}{m} - \log a^2 = -\log(a^2-v^2)$$

$$\therefore \frac{2kx}{m} = -\log(a^2-v^2) + \log a^2$$

$$= \log a^2 - \log(a^2-v^2)$$

$$\therefore \frac{2kx}{m} = \log\left(\frac{a^2}{a^2-v^2}\right)$$

4) A body starts moving from rest is opposed by a force per unit mass of value 'cx' and resistance per unit mass of value 'bv^2', where x and v are the displacement and velocity of the body at that instant. Show that the velocity of body is,

$$V^2 = \frac{c}{2b^2}\left(1-e^{-2bx}\right) - \frac{cx}{b} \quad \text{(Dec-09, 05, Nov-15, 17)}$$

Net forces, $-cx$ and $-bv^2$

→ as, net force = mass × accelaration

$$\therefore -cx - bv^2 = \cancel{m\times}\frac{dv}{dt}$$

$$\therefore -cx - bv^2 = \cancel{m\times} v\frac{dv}{dx} \qquad \left[\because \frac{dv}{dt} = \frac{dv}{dx}\times\frac{dx}{dt} = v\frac{dv}{dx}\right]$$

$$\therefore \cancel{v\frac{dv}{dx}} + \cancel{cx} \cancel{=} \cancel{bv^2}$$

$$V\frac{dV}{dx} + bv^2 = -cx$$

$$\frac{dV}{dx} + \frac{bV^2}{V} = \frac{-cx}{V}$$

$$\frac{dV}{dx} + \underline{\underline{bv}} = \underline{\underline{-cxV^{-1}}} \qquad \left[\frac{dv}{dx} + P(\overset{\text{constant}}{x})V = Q(x)V^{-1}\right]$$

Bernaulli's diff in V

$\therefore \quad \frac{1}{v^{-1}}\frac{dv}{dx} + \frac{bv}{v^{-1}} = \frac{-cx\cancel{v^{-1}}}{\cancel{v^{-1}}}$

$\therefore \quad v\frac{dv}{dx} + (bv)(v) = -cx.$

$\therefore \quad v\frac{dv}{dx} + bv^2 = -cx \quad \text{--- (1)}$

put, $v^2 = u$

& diff. w.r.t. x.

$\therefore 2v\frac{dv}{dx} = \frac{du}{dx}$

$\therefore v\frac{dv}{dx} = \frac{1}{2}\frac{du}{dx}.$

put in (1).

$\therefore \frac{1}{2}\frac{du}{dx} + bu = -cx.$

$\therefore \frac{du}{dx} + 2bu = -2cx. \quad , P = 2b, \; Q = -2cx.$

this is linear diff. eq^n. in u.

I.F. is, $e^{\int 2b\,dx} = e^{2bx}.$

G.S. is,

$u \cdot I.F. = \int Q \times I.F.\, dx + c$

$\therefore u \cdot e^{2bx} = \int -2cx \times e^{2bx} dx + c_1$

$\therefore v^2 \cdot e^{2bx} = -2c \int \underset{u \cdot v}{x e^{2bx}} dx + c_1$

$\therefore v^2 \times e^{2bx} = -2c\left\{ x\int e^{2bx}dx - \int\left(\frac{d}{dx}(x) \times \int e^{2bx}dx\right)dx \right\} + c_1$

$= -2c\left\{ x \times \frac{e^{2bx}}{2b} - \int \frac{e^{2bx}}{2b}dx \right\} + c_1$

$= -2c\left\{ x \times \frac{e^{2bx}}{2b} - \frac{1}{2b} \times \frac{e^{2bx}}{2b} \right\} + c_1$

$\therefore v^2 \times e^{2bx} = -2c\left\{ x \times \frac{e^{2bx}}{2b} - \frac{e^{2bx}}{4b^2} \right\} + c_1$

$$\cancel{v^2 \times e^{2bx} = -2c \times e^{2bx}\left\{\frac{x}{2b} - \frac{1}{4b^2}\right\}}$$

$$\therefore \cancel{v^2 = -2c\left(\frac{x}{2b} - \frac{1}{4b^2}\right)}$$

$$v^2 \times e^{2bx} = -\frac{\cancel{2}c \times x e^{2bx}}{\cancel{2}b} + \frac{\cancel{2}c e^{2bx}}{\cancel{4}_{2} b^2} + c_1$$

$$= \frac{-cx e^{2bx}}{b} + \frac{c e^{2bx}}{2b^2} + c_1$$

$$\therefore v^2 = \frac{-cx}{b} \times \cancel{e^{2bx}} \times \cancel{e^{-2bx}} + \frac{c \cancel{e^{2bx}}}{2b^2} \times \cancel{e^{-2bx}} + c_1 e^{-2bx}$$

$$\therefore v^2 = \frac{-cx}{b} + \frac{c}{2b^2} + c_1 e^{-2bx} \quad \text{--- (1)}$$

now, when $x = 0, v = 0$.

$$\therefore 0 = \frac{-c}{b}(0) + \frac{c}{2b^2} + c_1 e^{-2b(0)}$$

$$\therefore 0 = 0 + \frac{c}{2b^2} + c_1$$

$$\therefore c_1 = \frac{-c}{2b^2}$$

put in (1)

$$\therefore v^2 = \frac{-cx}{b} + \frac{c}{2b^2} - \frac{c}{2b^2} \times e^{-2bx}$$

$$\therefore v^2 = \frac{c}{2b^2}\left(1 - e^{-2bx}\right) - \frac{cx}{b}.$$

Q A particle is moving in a straight line with an accelaration $k\left[x + \frac{a^4}{x^3}\right]$ directed towards origin. If it starts from rest at a distance a from the origin, prove that it will arrive at origin at the end of time $\frac{\pi}{4\sqrt{k}}$

(May 04, 07, 08, 13, Dec 16)

→ given accelaration $= k\left[x + \frac{a^4}{x^3}\right] = a$

∴ eqⁿ of motion is,

$$v\frac{dv}{dx} = -k\left[x + \frac{a^4}{x^3}\right] \qquad \left[\because a = \frac{dv}{dt} = \frac{dv}{dx}\times\frac{dx}{dt} = v\frac{dv}{dx}\right]$$

$$\therefore v\,dv = -k\left[x + \frac{a^4}{x^3}\right]dx$$

$$\therefore \int v\,dv = \int -k\left[x + \frac{a^4}{x^3}\right]dx + c$$

$$\therefore \frac{v^2}{2} = -k\int\left[x + \frac{a^4}{x^3}\right]dx + c = -k\left[\int x\,dx + \int a^4 \times x^{-3}\,dx\right]$$

$$\therefore \frac{v^2}{2} = -k\left[\frac{x^2}{2} + a^4 \times \frac{x^{-3+1}}{-3+1}\right] + c$$

$$= -k\left[\frac{x^2}{2} + a^4 \times \frac{x^{-2}}{-2}\right] + c$$

$$\therefore \frac{v^2}{2} = -k\left[\frac{x^2}{2} - \frac{a^4}{2x^2}\right] + c \quad \text{--- (1)}$$

when, $x = a$, $v = 0$, put in (1)

$$\therefore 0 = -k\left[\frac{a^2}{2} - \frac{a^4}{2a^2}\right] + c$$

$$= -k\left[\frac{a^2}{2} - \frac{a^2}{2}\right] + c$$

$$0 = -k(0) + c$$

$$\therefore \boxed{c = 0}$$

put in (1)

$$\therefore \frac{v^2}{2} = -k\left[\frac{x^2}{2} - \frac{a^4}{2x^2}\right] + 0$$

$$\therefore \frac{v^2}{2} = -k\left[\frac{x^2}{2} - \frac{a^4}{2x^2}\right]$$

$$\therefore \frac{v^2}{\cancel{2}} = \frac{-k}{\cancel{2}}\left[x^2 - \frac{a^4}{x^2}\right] = -k\left[\frac{x^4 - a^4}{x^2}\right] = -k\left[\frac{-(a^4 - x^4)}{x^2}\right]$$

$$\therefore v^2 = \frac{k(a^4 - x^4)}{x^2}$$

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$$\therefore V = \pm \frac{\sqrt{k}}{x}\sqrt{a^4-x^4}$$

$$\therefore V = -\frac{\sqrt{k}}{x}\sqrt{a^4-x^4}$$

considering negative sign, because particle is directed towards origin.

$$\therefore V = \frac{dx}{dt} = -\sqrt{k} \times \frac{\sqrt{a^4-x^4}}{x}$$

$$\therefore \frac{x}{\sqrt{a^4-x^4}}\,dx = -\sqrt{k}\,dt \quad \text{--- (ii)}$$

put, $x^2 = a^2u \quad \Rightarrow x^4 = a^4u^2$

$$2x \cancel{dx} = a^2\frac{du}{dx}$$

$$\therefore 2x\,dx = a^2\,du$$

$$\therefore x\,dx = \frac{a^2}{2}\,du$$

put in (ii)

$$\therefore \cancel{\frac{\frac{a^2}{2}du}{\sqrt{a^4-x^4}}} \int \frac{\frac{a^2}{2}du}{\sqrt{a^4-a^4u^2}} = -\sqrt{k}\,t + c_1$$

$$\therefore \int \frac{\frac{a^2}{2}du}{\sqrt{a^4(1-u^2)}} = -\sqrt{k}\,t + c_1$$

$$\therefore \int \frac{\frac{\cancel{a^2}}{2}du}{\cancel{a^2}\sqrt{1-u^2}} = -\sqrt{k}\,t + c_1$$

$$\therefore \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = -\sqrt{k}\,t + c_1$$

$$\therefore \frac{1}{2}\sin^{-1} u = -\sqrt{k}\,t + c_1$$

$$\therefore \frac{1}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right) = -\sqrt{k}\,t + c_1 \quad \text{--- (iii)}$$

we have, $t=0$ then $x=a$

put in (iii)

$\therefore \frac{1}{2}\sin^{-1}\left(\frac{a^2}{a^2}\right) = -\sqrt{k}(0) + c_1$

$\therefore \frac{1}{2}\sin^{-1}(1) = 0 + c_1$

$\therefore \frac{1}{2} \times \frac{\pi}{2} = c_1$

$\therefore c_1 = \frac{\pi}{4}$

put in (iii)

$$\therefore \boxed{\frac{1}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right) = -\sqrt{k}\,t + \frac{\pi}{4}} \quad \text{--- (iv)}$$

Let, $t = T$, partical reaches at $x = 0$ put in (iv)

$\therefore$ ~~$0 = -\sqrt{\ }$~~

$\therefore \frac{1}{2}\sin^{-1}\left(\frac{0}{a^2}\right) = -\sqrt{k}\,T + \frac{\pi}{4}$

$\therefore \frac{1}{2}\sin^{-1}(0) = -\sqrt{k}\,T + \frac{\pi}{4}$

$\therefore 0 = -\sqrt{k}\,T + \frac{\pi}{4}$

$\therefore -\frac{\pi}{4} = -\sqrt{k}\,T$

$\therefore \boxed{\frac{\pi}{4\sqrt{k}} = T}$ this is req^d time.

A bullet is fired into sand tank, its retardation is proportional to square root of its velocity. Show that in time $\frac{2\sqrt{V}}{k}$ the bullet will come to rest where V is the initial velocity [Dec-10]

$\longrightarrow$ we have, here, retarding force $\propto \sqrt{v}$

$\therefore$ ~~$ma = -k m\sqrt{v}$~~

$\cancel{m}a = -k\cancel{m}\sqrt{v}$

$\therefore \frac{dv}{dt} = -k\sqrt{v}$

$\therefore \frac{dv}{\sqrt{v}} = -k\,dt$

$\therefore \int \frac{dv}{\sqrt{v}} = \int -k\,dt$

$$\therefore \int \frac{dv}{v^{1/2}} = -kt + c_1$$

$$\therefore \int v^{-1/2} dv = -kt + c_1$$

$$\therefore \frac{v^{-1/2+1}}{-1/2+1} = -kt + c_1$$

$$\therefore \frac{v^{1/2}}{1/2} = -kt + c_1$$

$$\therefore 2\sqrt{v} = -kt + c_1 \quad \text{--- (1)}$$

at $t = 0$ $v = V$

$$\therefore 2\sqrt{V} = -k(0) + c_1$$

$$\therefore c_1 = 2\sqrt{V}$$

put in (1)

$$\therefore \cancel{2\sqrt{V} = -kt + 2}$$

$$2\sqrt{v} = -kt + 2\sqrt{V}$$

$$\therefore 2\sqrt{v} - 2\sqrt{V} = -kt$$

$$\therefore \frac{2(\sqrt{v} - \sqrt{V})}{-k} = t$$

$$\therefore \frac{2(\sqrt{V} - \sqrt{v})}{k} = t$$

we have, at $t = T$, $v = 0$

$$\therefore \frac{2(\sqrt{V} - 0)}{k} = T$$

$$\therefore \boxed{T = \frac{2\sqrt{V}}{k}}$$ this is req^d time.

A particle moves in a horizontal line OA with an accelaration $\frac{k}{r^3}$ at a distance r and directed towards O. Initially a particle was at rest at a distance 'a' from O. find the time when it will be at a distance $\frac{a}{2}$ from 'O' [Dec-04, 09]

→ we have, the eqn of motion,

$$v \frac{dv}{dr} = -\frac{k}{r^3} \qquad \left[\because a = \frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt} = v\frac{dv}{dr} \right]$$

$$\therefore v\,dv = \frac{-k}{r^3} dr$$

$$\therefore \int v\,dv = \int -k \times r^{-3} dr$$

$$\therefore \frac{v^2}{2} = -k \times \frac{r^{-3+1}}{-3+1} + c$$

$$\therefore \frac{v^2}{2} = -k \times \frac{r^{-2}}{-2} + c$$

$$\therefore \frac{v^2}{2} = \frac{k}{2r^2} + c \quad \text{—— (1)}$$

at $t=0$, $v=0$ and $r=a$

$\therefore$ put in (1)

$$\therefore 0 = \frac{k}{2a^2} + c$$

$$\therefore c = -\frac{k}{2a^2}$$

put in (1)

$$\therefore \frac{v^2}{2} = \frac{k}{2r^2} - \frac{k}{2a^2}$$

$$\therefore \frac{v^2}{\cancel{2}} = \frac{k}{\cancel{2}}\left(\frac{1}{r^2} - \frac{1}{a^2}\right)$$

$$\therefore v^2 = k\left(\frac{a^2 - r^2}{a^2 r^2}\right)$$

$$\therefore v = \frac{\sqrt{k}}{a} \times \frac{\sqrt{a^2 - r^2}}{r}$$

$$\therefore \frac{dr}{dt} = \frac{\sqrt{k}}{a} \times \frac{\sqrt{a^2 - r^2}}{r}$$

$$\therefore \int \frac{r\,dr}{\sqrt{a^2-r^2}} = \frac{\sqrt{k}}{a}\int dt$$

$$\therefore -\sqrt{a^2-r^2} = \frac{\sqrt{k}}{a}t + c_1 \quad \text{--- (11)}$$

at $t=0$, $r=a$

$\therefore$ eqn (11) becomes,

$$-\sqrt{a^2-a^2} = \frac{\sqrt{k}}{a}(0) + c_1$$

$$\therefore c_1 = 0$$

put in (11)

$$\therefore -\sqrt{a^2-r^2} = \frac{\sqrt{k}\,t}{a} + 0$$

$$\therefore -\sqrt{a^2-r^2} = \frac{\sqrt{k}\,t}{a}$$

squaring both side,

$$\therefore \left(-\sqrt{a^2-r^2}\right)^2 = \left(\frac{\sqrt{k}\,t}{a}\right)^2$$

$$\therefore a^2 - r^2 = \frac{kt^2}{a^2}$$

now we find t at $r = \frac{a}{2}$

$$\therefore a^2 - \frac{a^2}{4} = \frac{kt^2}{a^2}$$

$$\therefore \frac{4a^2-a^2}{4} = \frac{kt^2}{a^2}$$

$$\therefore \frac{3a^2}{4} = \frac{kt^2}{a^2}$$

$$\therefore \frac{3a^4}{4k} = t^2$$

$\therefore t = \frac{a^2}{2}\sqrt{\frac{3}{k}}$ this is reqd time.

$$a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v\frac{dv}{dx}.$$

(3) $acc^n \rightarrow k\left(x + \frac{a^4}{x^3}\right)$

$\because$ Net Force = mass $\times$ acc^n

$$= m \times a$$

$$\cancel{m} \times \underline{\underline{a}} = \cancel{m} \times -k\left(x + \frac{a^4}{x^3}\right)$$

$-$ sign $\because$ ~~x~~ acc^n towards origin.

$$\therefore v\frac{dv}{dx} = -k\left(x + \frac{a^4}{x^3}\right)$$

(4) same as above,

$$v\frac{dv}{dx} = \frac{-k}{x^3}$$

(5) Net force = mass $\times$ acc^n.

~~$m \times a$~~ =

$$mg - kv^2 = m \times \underline{\underline{a}}$$

$$\therefore mg - kv^2 = mv\frac{dv}{dx}$$

(6) Net force = mass $\times$ acc^n

$$\therefore -g - kv^2 = v\frac{dv}{dx}$$

consider force per unit mass i.e. $\underline{\underline{m = 1}}$.

(7) Net force = mass $\times$ acc^n.

$$-cx - bv^2 = m \times v\frac{dv}{dx}$$

$$\therefore -cx - bv^2 = v\frac{dv}{dx} \quad [\because m = 1]$$

Here given in ex. force per unit mass $\therefore m = 1$

(8) Net force = mass $\times$ acc^n.

~~$-mkv + mg = mv\frac{dv}{dx}$~~

~~$\therefore -kv + g = v\frac{dv}{dx}$~~

~~$\therefore v\frac{dv}{dx} = g - kv$~~

~~$\therefore \frac{dv}{dt} =$~~

$$\Rightarrow -mkv + mg = ma$$

$$\Rightarrow -mkv + mg = m\frac{dv}{dt}$$

$$\therefore \frac{dv}{dt} = g - kv.$$

[as retarding force is mk times velocity]

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(9) Net force = mass × accelaration

$$-mg - kv = m \times \frac{dv}{dt}$$

$$\therefore m\frac{dv}{dt} = -mg - kv$$

(10) Net force = mass × accel^m

$$-(a^2 + b^2v^2) = m\frac{dv}{dt}$$

$$\therefore m\frac{dv}{dt} = -(a^2 + b^2v^2)$$

(11) given,

$$\frac{dv}{dt} = g - kv$$

$$\frac{dv}{dt} + kv = g \quad \text{LDE}$$

$$\therefore I.F. = e^{\int k\,dt} = e^{kt}$$

$\therefore$ G.S. is,

$$v.\, I.F. = \int Q \times I.F.\, dt + c$$

$$v.e^{kt} = \int g \times e^{kt}\, dt + c$$

$$\therefore v.e^{kt} = g \times \frac{e^{kt}}{k} + c$$

$$\therefore v.e^{kt} = \frac{g}{k} e^{kt} + c \quad \text{—(1)}$$

when $t=0, v=0$ put in (1)

$$\therefore 0 = \frac{g}{k} + c$$

$$\therefore c = -g/k$$

put in (1)

$$\therefore v.e^{kt} = \frac{g}{k}e^{kt} - \frac{g}{k}$$

$$\therefore v.e^{kt} - g/k\, e^{kt} = -g/k$$

$$\therefore e^{kt}(v - g/k) = -g/k$$

$$\therefore e^{kt} = \frac{-g/k}{v - g/k}$$

$$= \frac{-g/\cancel{k}}{\frac{vk - g}{\cancel{k}}}$$

$$= \frac{-g}{vk - g}$$

$$= \frac{\cancel{-}g}{\cancel{-}(g - vk)}$$

$$= \frac{g}{g - vk}$$

$$\therefore \log e^{kt} = \log\left(\frac{g}{g - kv}\right)$$

$$\therefore kt = \log\left(\frac{g}{g - kv}\right)$$

$$\therefore t = \frac{1}{k}\log\left(\frac{g}{g - kv}\right)$$

(12) given, ~~mv~~

$$mv\frac{dv}{dx} = k(a^2-v^2)$$

$$\frac{v\,dv}{(a^2-v^2)} = \frac{k}{m}dx$$

$$-\frac{1}{2}\int\frac{-2v\,dv}{a^2-v^2} = \int\frac{k}{m}dx + c$$

$$\therefore \frac{-1}{2}\log|a^2-v^2| = \frac{k}{m}x + c \quad \text{--- (1)}$$

at $x=0$, $v=0$

$$\therefore \frac{-1}{2}\log|a^2-0| = \frac{k}{m}(0)+c$$

$$\therefore c = \frac{-1}{2}\log a^2$$

put in (1).

$$\therefore \frac{-1}{2}\log(a^2-v^2) = \frac{k}{m}x - \frac{1}{2}\log a^2$$

$$\therefore -\log(a^2-v^2) = \frac{2kx}{m} - \log a^2$$

$$\therefore \log a^2 - \log(a^2-v^2) = \frac{2kx}{m}$$

$$\therefore \boxed{\frac{2kx}{m} = \log\left(\frac{a^2}{a^2-v^2}\right)}$$

(13) given,

$$\frac{dv}{dt} = k(1-t/T)$$

$$dv = k(1-t/T)dt$$

$$\therefore \int dv = \int k(1-t/T)dt + c$$

$$\therefore v = k\int(1-t/T)dt + c$$

$$= k\left[t - \frac{1}{T}\times\frac{t^2}{2}\right] + c$$

$$\therefore v = k\left[t - t^2/2T\right] + c$$

14 given, $v\frac{dv}{dr} = \frac{-k}{r^3}$

$\therefore v\,dv = -kr^{-3}dr$

$\therefore \int v\,dv = \int -kr^{-3}dr + c$

$\therefore \frac{v^2}{2} = -k\int r^{-3}dr + c$

$= -k \times \frac{r^{-3+1}}{-3+1} + c$

$= -k \times \frac{r^{-2}}{-2} + c$

$\therefore \frac{v^2}{2} = \frac{k}{2r^2} + c$ —— (1)

now, $r = a$ when $v = 0$

$\therefore 0 = \frac{k}{2a^2} + c$

$\therefore c = \frac{-k}{2a^2}$ put in (1)

$\therefore \frac{v^2}{2} = \frac{k}{2r^2} - \frac{k}{2a^2}$

$\therefore \frac{v^2}{\cancel{2}} = \left(\frac{1}{r^2} - \frac{1}{a^2}\right)\frac{k}{\cancel{2}}$

$= k\left(\frac{1}{r^2} - \frac{1}{a^2}\right)$

15) $m\frac{dv}{dt} = -mg - kv$

$\therefore \cancel{m\frac{dv}{t}}$

$\therefore \frac{dv}{dt} = \frac{-mg - kv}{m}$

$\therefore \frac{dv}{dt} = \cancel{\frac{-g}{}} -g - \frac{k}{m}v$

$\therefore \frac{dv}{dt} + \frac{k}{m}v = -g$

$\therefore$ I.F. is, $e^{\int \frac{k}{m} dt} = e^{k/m \times t} = e^{kt/m}$

$\therefore$ G.S. is,

$$v \cdot I.F. = \int -g \times e^{kt/m} dt + c$$

$$= -g \int e^{kt/m} dt + c$$

$$= -g \times \frac{e^{kt/m}}{k/m} + c$$

$$\therefore v \times e^{kt/m} = e^{kt/m} \times \frac{-mg}{k} + c$$

$$\therefore v e^{kt/m} + \frac{mg}{k} e^{kt/m} = c$$

$$\therefore \left(v + \frac{mg}{k}\right) e^{kt/m} = c \quad \text{—— (1)}$$

at $t=0$, $v=0$

$$\therefore \left(0 + \frac{mg}{k}\right) e^{0} = c$$

$$\therefore c = \frac{mg}{k}$$

put in (1)

$$\therefore \left(v + \frac{mg}{k}\right) e^{kt/m} = \frac{mg}{k}$$

$$\therefore \frac{v + \frac{mg}{k}}{mg/k} = e^{kt/m}$$

$$\therefore \frac{\frac{kv + mg}{\cancel{k}}}{\frac{mg}{\cancel{k}}} = e^{kt/m}$$

$$\therefore \log\left(\frac{kv + mg}{mg}\right) = \log e^{kt/m}$$

$$\therefore \log\left(\frac{kv + mg}{mg}\right) = kt/m \Rightarrow t = \frac{m}{k} \log\left(\frac{kv + mg}{mg}\right)$$

(16) $\frac{dv}{dt} = g - kv \Rightarrow \frac{dv}{dt} + kv = g$

$\therefore$ I.F. is, $e^{\int k\,dt} = e^{kt}$

G.S. is,

$v \cdot I.F. = \int g \cdot I.F.\,dt + c$

$\therefore v \cdot e^{kt} = \int g \times e^{kt}\,dt + c$

$\therefore v \cdot e^{kt} = g \times \frac{e^{kt}}{k} + c$ — (1)

at $v = 0$, $t = 0$

$\therefore 0 = g \times \frac{1}{k} + c$

$\therefore c = -g/k$

put in (1)

$\therefore v \cdot e^{kt} = \frac{ge^{kt}}{k} - \frac{g}{k}$

$\therefore$ terminal velocity

17) $\frac{dv}{dt} = -k\sqrt{v}$

$\therefore \frac{dv}{\sqrt{v}} = -k\,dt$

$\therefore v^{-1/2}\,dv = -k\,dt$

$\therefore \int v^{-1/2}\,dv = \int -k\,dt + c_1$

$\therefore \frac{v^{-1/2+1}}{-1/2+1} = -kt + c_1$

$\therefore \frac{v^{1/2}}{1/2} = -kt + c_1$

$\therefore 2\sqrt{v} = -kt + c_1$ — (1)

initially $v = v_0$ and $t = 0$

put in (1)

$\therefore 2\sqrt{v_0} = -k(0) + c_1$

$\therefore c_1 = 2\sqrt{v_0}$

put in (1)

$\therefore 2\sqrt{v} = -kt + 2\sqrt{v_0}$

$\therefore \cancel{2\sqrt{v} - 2\sqrt{v_0} = -kt}$