Q1 :

The time required for servicing transmissions is normally distributed with mean = 45 minutes and SD= 8 minutes. The service manager plans to have work begin on the transmission of a customer's car 10 minutes after the car is dropped off and the customer is told that the car will be ready within 1 hour from drop-off. What is the probability that the service manager cannot meet his commitment?

ANSWER :

Option B

Let T be the time it takes to work on servicing transmission.

We are given that $T \sim N(45, 8)$ meaning time required for servicing transmissions is normally distributed with mean = 45 minutes and SD= 8 minutes.

As the work begins 10 minutes after drop-off of the car , we need the work to be completed in $t \leq 50$ minutes. Since T has a normal distribution,

$$Z = (T - \text{mean})/S.D$$

Therefore:

$P(T \leq 50) = P(Z \leq (50 - 45)/8) = P(Z \leq 0.62) = 0.7324$ (from the Z-table)

Hence

$P(T > 50) = 1 - P(T \leq 50) = 1 - 0.7324 = 0.2676$

Hence the probability that commitment is not met is 0.2676

Q2.

The current age (in years) of 400 clerical employees at an insurance claims processing center is normally distributed with mean = 38 and Standard deviation =6. For each statement below, please specify True/False. If false, briefly explain why.

a) More employees at the processing center are older than 44 than between 38 and 44.

b) A training program for employees under the age of 30 at the center would be expected to attract about 36 employees.

Answer 2 :

a) **False**. The probability of the region between 38 and 44 corresponds to $P(0 < Z < 1) \approx 1/3$ whereas the region above 44 corresponds to $P(Z > 1) \approx 1/6$. Both of these probabilities were calculated based on the empirical rule. Hence we can say that less employees at the processing center are older than 44 than between 38 and 44 yrs.

b) **True**. If $X \sim N(38,6)$, then $P(X < 30) = P(Z \leq (30 - 38)/6) = P(Z \leq -1.33) \approx 0.0912$. This multiplied by 400 gives us the expected count. As 0.912*400 = 36.48 hence we can say that program would be able to attract around 36 employees

Q3.

If $X1 \sim N(\mu, \sigma 2)$ and $X2 \sim N(\mu, \sigma 2)$ are iid normal random variables, then what is the difference between 2X1 and X1 + X2? Discuss both their distributions and parameters.

Answer

2*X1* is simply a scaled version of the random variable *X1*. So the distribution of 2*X1* will have a shape identical to that of *X1* i.e. it will have a normal distribution. Further we know that **$E[aX] = aE[X]$** & **$SD[aX] = |a|SD[X]$**. Therefore **$2X1 \sim N(2\mu, (2\sigma)^2)$**

The sum of two independent normally distributed random variables also has a normal distribution. Therefore $X1 + X2$ has a normal distribution.
Further, **$E(X1 + X2) = E(X1) + E(X2)$** and

$Var(X1 + X2) = Var(X1) + Var(X2)$ when $X1$ and $X2$ are independent.
Therefore $(X1 + X2) \sim N(2\mu, (2\sigma)^2)$.

Thus both will have normal distributions, but the second one has lower variance.

Q4.

Let X ~ N(100, 202). Find two values, a and b, symmetric about the mean, such that the probability of the random variable taking a value between them is 0.99.

A. 90.5, 105.9
B. 80.2, 119.8
C. 22, 78
D. 48.5, 151.5
E. 90.1, 109.9

Answer :

Option D

Based on the empirical rule, in order to include 99% of the probability, we need to go 2.58 standard deviations on either side of the mean. Therefore, the required range here is 100 $\pm$ 2.58*20 = [**48.4, 151.6**]

Q5. Consider a company that has two different divisions. The annual profits from the two divisions are independent and have distributions Profit1 ~ N(5, 3 2) and Profit2 ~ N(7, 42) respectively. Both the profits are in $ Million. Answer the following questions about the total profit of the company in Rupees. Assume that $1 = Rs. 45

A. Specify a Rupee range (centered on the mean) such that it contains 95% probability for the annual profit of the company.

B. Specify the 5th percentile of profit (in Rupees) for the company

C. Which of the two divisions has a larger probability of making a loss in a given year?

Answer:

Let X be the profit of the entire company in Rupees. Before answering the specific questions, let us specify the distribution of X. Since X is the sum of two normal random variables it also has a normal distribution. Further applying the formulae related to the means and SDs of functions of random variables we have

E[X] = E[45*(Profit1 + Profit2)]
= 45*(5 + 7)

= 540 Million Rupees

SD[X] = SD[Profit1 + Profit2)
= 45*(sqrt(Var(Profit1) + Var(Profit2)))
= 45*sqrt(9 + 16) = 45*5
= 225 Million Rupees

Therefore, X ~ N(540, 225^2)

a) Based on the empirical rule, we need to go 1.96 standard deviations on either side of the mean to get 95% probability. Therefore, the required range is 540 $\pm$1.96*225 = [99,981] Million Rupees.

b) We need the 5th percentile of X i.e. the point on the distribution of X, such that there is only 5% of the area to the left (i.e. 5% cumulative probability). We know from the standard normal table that this cumulative probability is associated with Z = -1.645. Therefore, the required value of X is 540 – 1.645*225 = 169.875 Million Rupees.

   Note that another way of thinking about this is to consider the 90% range based on the empirical rule. The required value of X here is the lower end of that range since a 90% area centered on the mean implies that there is 5% of the area on either side of that range.

c) This question concerns the original profit distributions. Let us calculate the Z-scores associated with zero for each of the divisions.

   For Division 1: Z-score for a profit of zero = (0 – 5)/3 = (- 1.67)
   For Division 2: Z-score for a profit of zero = (0 – 7)/4 = (- 1.75)

   The probability of loss for division 1 is the area under the standard normal distribution pdf to the left of (-1.67), and that for division 2 is the area under the standard normal distribution pdf to the left of (-1.75). Therefore the probability associated with the second one will be lower (as we are going farther into the tail). *Note that we can compare probabilities by comparing the associated Z-scores without having to compute actual probabilities*.