Introduction to Mathematical Analysis
Problem Set #12

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December 4, 2019

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Problem 1. Find a function whose second derivative $f''(c)$ exists at some point c, but $f'''(c)$ does not.
Example: Consider the following function and its general derivatives (supposing they exist):

$$\begin{aligned} f(x) &= \frac{27}{440}x^{11/3} \\ f'(x) &= \frac{9}{40}x^{8/3} \\ f''(x) &= \frac{3}{5}x^{5/3} \\ f'''(x) &= x^{2/3}. \end{aligned}$$

Let $c = 0$. Consider the limit expression of $f''(c)$ to confirm its existence,

$$\lim_{x\to c} \frac{f''(x) - f''(c)}{x - c} = \lim_{x\to 0} \frac{\frac{3}{5}x^{5/3} - 0}{x - 0} = \lim_{x\to 0} x^{2/3} = 0 = f''(c).$$

Thus, the second derivative exists at c. Using the same procedure for the third derivative, we find

$$\lim_{x\to c} \frac{f'''(x) - f'''(c)}{x - c} = \lim_{x\to 0} \frac{x^{2/3} - 0}{x - 0} = \lim_{x\to 0} x^{-1/3} = \lim_{x\to 0} \frac{1}{\sqrt[3]{x}}.$$

We readily see that this limit does not exist (that is, $f'''(c) = \infty$). $\square$

Problem 2. Consider the function $f : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ in lowest terms, and} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ and } x = 0. \end{cases}$$

Claim: f is continuous at $x = 0$ and every irrational number.
Proof: The following argument has been adapted from the public notes on Thomae's Function,

from MAT371 at Arizona State University. Consider $x' \in \{\mathbb{R} \setminus \mathbb{Q}\} \cup \{0\}$, and some arbitrary $\varepsilon > 0$. By the Archimedian Property, we find $N \in \mathbb{N}$ such that $1 < \varepsilon N$, or equivalently, $\varepsilon > 1/N$. Then, we can define a finite set A comprised of lowest-term, nonzero rationals $p/q \in [x'-1, x'+1]$ such that $q \leq N$. Next, define $\delta = \min_{a \in A} |x - a|$. Now, consider what happens when $|x - x'| < \delta$. Either x must be irrational, zero, or $x = p/q$ with the property

$$\frac{1}{q} \leq \frac{1}{N} < \varepsilon.$$

Thus, for any arbitrary $\varepsilon > 0$, we have found that $|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon$. $\square$

Problem 3. *The Mean Value Theorem for Integrals.* Claim: Suppose that f is a continuous real-valued function on the interval $[a, b]$. There exists a point $c \in [a, b]$ such that

$$\int_a^b f = f(c)(b-a).$$

Proof: By FTC I, we can consider the antiderivative $F(b) = \int_a^b f$. Because f is continuous on (a, b), we know that F is differentiable on (a, b) by FTC I; notably, $F'(c) = f(c)$. Because F is differentiable on (a, b), we can employ the Mean Value Theorem for Derivatives to find a $c \in (a, b)$ such that

$$\begin{aligned} F'(c) &= \frac{F(b) - F(a)}{b-a} \\ F'(c)(b-a) &= F(b) - \int_a^a f \\ f(c)(b-a) &= F(b) = \int_a^b f \end{aligned}$$

as desired. $\square$

Problem 4. A recurring theme in this week's lectures was that a lot of reasonable properties of the functions in the sequence are passed to the limit function, but one must be careful. The following exercises investigate the "limits" of this philosophy.

(a) Suppose that $f_n \xrightarrow{U.C.} f$ and $g_n \xrightarrow{U.C.} g$.

(i) Claim: $\{f_n + g_n\} \xrightarrow{U.C.} f + g$.
Lemma Primo: We will verify that if $|a - b| < \varepsilon/2$ and $|c - d| < \varepsilon/2$, then $|a + c - (b + d)| < \varepsilon$. By the hypothesis, the know

$$\begin{gathered} -\varepsilon/2 < a - b < \varepsilon/2 \\ -\varepsilon/2 < c - d < \varepsilon/2 \\ -\varepsilon < a - b + c - d < \varepsilon \\ -\varepsilon < a + c - (b - d) < \varepsilon \\ |a + c - (b + d)| < \varepsilon. \end{gathered}$$

Lemma Secondo: We will verify that if F, G are bounded functions, then

$$\sup(F+G) \leq \sup(F) + \sup(G).$$

Given that $F \leq \sup(F)$ and $G \leq \sup(G)$, we know that

$$F + G \leq \sup(F) + \sup(G).$$

Thus, $F+G$ is bounded above by $\sup(F) + \sup(G)$, so the claim follows.
Proof: We know that $\forall \epsilon > 0$, there exists $N, M \in \mathbb{N}$ such that if $n \geq N$ we find

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| < \frac{\varepsilon}{4}, \tag{α}$$

and if $m \geq M$ we find

$$\sup_{x\in\mathbb{R}} |g_m(x) - g(x)| < \frac{\varepsilon}{4}. \tag{β}$$

Applying Lemma Secondo, we find

$$\sup(|f_n(x) - f(x)| + |g_m(x) - g(x)|) \leq \alpha + \beta = \frac{\varepsilon}{2}.$$

Because this is the sum of two positive terms, we know

$$\sup|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \quad \sup|g_m(x) - g(x)| < \frac{\varepsilon}{2}.$$

Thus, applying Lemma Primo, we find

$$\sup|f_n(x) + g_m(x) - (f(x) + g(x))| < \varepsilon.$$

Lastly, denote $p = \max\{n.m\}$. Then,

$$\sup|f_p(x) + g_p(x) - (f(x) + g(x))| < \varepsilon,$$

as desired. $\square$

(ii) <u>Claim:</u> If $\{f_n\}$ and $\{g_n\}$ are bounded, then $\{f_n g_n\} \xrightarrow{U.C.} fg$.
Boundedness Lemma: If $|f_n(x)| < B$ and $f_n \xrightarrow{U.C.} f$, then $|f| < B$. Suppose toward a contradiction that f were not bounded by B; then, $\exists x' \in S$ such that $f(x') > B$. Because we have uniform convergence, we know that $f(x')$ and $f_n(x')$ are arbitrarily close to one another as n increases. That means that for some $f(x') = C$ where $C \gg B$, we can find an n such that $f_n(x')$ is arbitrarily close to C. Yet, this C is far larger than B, which contradicts the bound on $f_n(x)$. Hence, the lemma holds.
Proof: By our Boundedness Lemma, we can let $|f_n(x)|, |f(x)| < B$ and $|g_n(x)|, |g(x)| < C$, where C, B are fixed, positive (nonzero) real values. Then, we know by uniform convergence that

$$\begin{aligned} \sup|g_n - g| &< \frac{\varepsilon}{2B} \quad &\Rightarrow B\sup|g_n - g| < \frac{\varepsilon}{2} \\ \sup|f_n - f| &< \frac{\varepsilon}{2C} \quad &\Rightarrow C\sup|f_n - f| < \frac{\varepsilon}{2}, \end{aligned}$$

which yields $B \sup |g_n - g| + C \sup |f_n - f| < \varepsilon$. Now, consider

$$\begin{aligned}\sup |f_n g_n - fg| &= \sup |f_n g_n - f_n g + f_n g - fg| = \sup |f_n(g_n - g) + g(f_n - f)| \\ &\leq \sup |f_n(g_n - g)| + \sup |g(f_n - f)| \leq \sup(|f_n||(g_n - g)|) + \sup(|g||(f_n - f)|) \\ &< B \sup |g_n - g| + C \sup |f_n - f| < \varepsilon,\end{aligned}$$

as desired. □

(iii) The boundedness restriction in the above statement is necessary. There are functions f_n, g_n that converge uniformly to f and g respectively, but whose product does not converge uniformly to fg.
Counterexample: Define

$$f_n(x) = \frac{1}{n}, \quad g_n(x) = x$$
$$f(x) = 0, \quad g(x) = x.$$

Suppose toward a contradiction that $\{f_n g_n\} \xrightarrow{U.C.} fg$; that is, $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\sup_{x \in \mathbb{R}} \left| \frac{x}{n} - (x \cdot 0) \right| < \varepsilon.$$

Thus, $\sup \left| \frac{x}{n} \right| < \varepsilon$. However, we know that $\forall n \in \mathbb{N}$, there exists an $x \in \mathbb{R}$ such that $x = 17n\varepsilon$. Thus, we found

$$\left| \frac{17n\varepsilon}{n} \right| < \varepsilon,$$

which implies that $17 < 1$. As our class-long meme has recurrently emphasized, that is a profound contradiction! $\Rightarrow\Leftarrow$

(b) Claim: We shall verify the Cauchy Criterion for Uniform Convergence:
Theorem: *The sequence $\{f_n\}$ converges uniformly on S iff $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n, m > N$ and $x \in S$ implies*

$$|f_n(x) - f_m(x)| < \varepsilon.$$

($\Rightarrow$) Suppose we know that $\{f_n\} \xrightarrow{U.C.} f$ on S. Then, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n > N \Rightarrow \sup_{x \in S} \left| f(x) - f_n(x) \right| < \frac{\varepsilon}{2}$$
$$m > N \Rightarrow \sup_{x \in S} \left| f_m(x) - f(x) \right| < \frac{\varepsilon}{2}.$$

Seeing as the supremum is an upper bound, we know for any $x \in S$,

$$|f(x) - f_n(x)| + |f_m(x) - f(x)| < \varepsilon,$$

and by Lemma Primo (from Part a, i) we find

$$|f(x) - f_n(x) + f_m(x) - f(x)| = |f_m(x) - f_n(x)| < \varepsilon,$$

as desired. $\square$

($\Leftarrow$) Suppose we have the Cauchy Criterion; that is, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $n, m > N$, $x \in S$ implies $|f_n(x) - f_m(x)| < \varepsilon$. Notice, $\{f_n(x)\}$ is a Cauchy sequence on $\mathbb{R}$, and therefore must converge to $f(x) \in \mathbb{R}$; that is,

$$\lim_{m\to\infty} f_m(x) = f(x).$$

Because our condition on m requires only that it be a natural number greater than N, we can take an arbitrarily large m: say, $m \to \infty$. Thus,

$$\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)| < \varepsilon,$$

which still holds from the supremum of $x \in S$. $\square$

Problem 5. Claim: Derivatives are not preserved by uniform convergence. For example, the sequence of functions $f_n(x) = \frac{x}{1+n^2x^2}$ converges uniformly to some function f, where $f, f_n : [-1, 1] \mapsto \mathbb{R}$, but

$$f'(0) \neq \lim f_n'(0).$$

(a) *Proof of Unequal Derivatives:* First, we shall verify that $f'(0) \neq \lim f_n'(0)$. Assume from context that these functions are differentiable at zero. (We also readily see that this function is differentiable for all $x \in \mathbb{R}$ by considering it as the quotient of two polynomials, where the denominator cannot be zero.) By inspection, we can confidently let $f(x) = 0$. It is trivial to see that the derivative of the zero function, $f(x)$, is the zero function, $f'(x)$; hence, $f'(0) = 0$. Toward the product rule, let $f_n(x) = l(x) \cdot h(x)$ where

$$\begin{aligned} l(x) &= x, \quad & h(x) &= (n^2x^2+1)^{-1} \\ l'(x) &= 1, \quad & h'(x) &= \frac{-2n^2x}{(n^2x^2+1)^2}. \end{aligned}$$

Hence, the product rule applies

$$\begin{aligned} f_n'(x) &= l(x)h'(x) + l'(x)h(x) \\ &= \frac{-2n^2x^2}{(n^2x^2+1)^2} + (n^2x^2+1)^{-1} \\ f_n'(x) &= \frac{1-n^2x^2}{(n^2x^2+1)^2}. \end{aligned}$$

Evaluating at $x = 0$ yields

$$f_n'(0) = 1,$$

for all $n \in \mathbb{N}$. Thus, $\lim f_n' = 1$, which does not equal zero. $\square$

(b) *Proof of Uniform Convergence:* Because f_n is continuous (it is differentiable) and maps from a compact set to $\mathbb{R}$, we know that f_n achieves a maximum/minimum value at some point on its domain, $[-1, 1]$. Furthermore, we know that if f_n is differentiable on $(-1, 1)$ and achieves a max/min at $c \in (-1, 1)$, then $f_n'(c) = 0$. Thus, there are two cases to consider: either the extremum occurs at the boundary $x = \pm 1$, or at $c \in (-1, 1)$ where $f_n'(c) = 0$. Let's find the candidate extrema by setting $f_n'(x) = 0$:

$$0 = f_n'(x) = \frac{1 - n^2x^2}{(n^2x^2 + 1)^2}$$
$$x = \pm\frac{1}{n}.$$

Thus, the extrema candidates for the interior case are:

$$f_n\left(\frac{1}{n}\right) = \frac{1}{2n}, \quad f_n\left(\frac{-1}{n}\right) = \frac{-1}{2n}.$$

Let's see how the boundary values compare:

$$f_n(1) = \frac{1}{1 + n^2}, \quad f_n(-1) = \frac{-1}{1 + n^2}.$$

Noticing that squares grow faster than factors of two, we find that as n increases, the maximum value must be $f_n(1/n) = 1/2n$. Similarly, the minimum value must be $f_n(-1/n) = -1/2n$. Hence,

$$\sup_{x \in [-1,1]} \left| f_n(x) \right| = \frac{1}{2n}.$$

Now, we can prove uniform convergence. Pick some $\varepsilon > 0$. By the Archimedian Property, we can find an $n \in \mathbb{N}$ such that $1/2\varepsilon < n$, or equivalently that

$$\frac{1}{2n} < \varepsilon.$$

Now, consider

$$\begin{aligned}
\sup_{x \in [-1,1]} \left| f_n(x) - f(x) \right| &= \sup_{x \in [-1,1]} \left| f_n(x) - 0 \right| \\
&= \sup_{x \in [-1,1]} \left| f_n(x) \right| \\
&= \frac{1}{2n} \\
&< \varepsilon.
\end{aligned}$$

Given the arbitrary nature of ε, we have shown that $f_n(x)$ converges uniformly to $f(x)$. $\square$