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On flows induced by diffusion in a stably stratified fluid

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Abstract—It is shown theoretically and observed experimentally that spontaneous motions are set up in a stably stratified diffusive fluid in a container whose side-walls are not vertical. The fluid streams up or down the inclined walls in a kind of boundary layer whose Reynolds number is inversely proportional to the Prandtl number. Distributions of velocity and density are found, together with the mass transport in the layer. The same phenomenon results in convective transport along narrow fissures inclined to the vertical that can greatly exceed the diffusive transport associated with the overall gradient. This is particularly so when the fissure is almost horizontal; the convective mass transport is found to be proportional to the cube of the average density gradient along the fissure and to the ninth power of the fissure width.

1. INTRODUCTION

IT IS not difficult to produce in the laboratory a stably stratified fluid in hydrostatic equilibrium. A container with vertical sidewalls can be filled with water in which the concentration of dissolved salt decreases with height so that the density in the interior is, say, a linear function of height. At the bottom and the free surface, however, the condition of zero salt flux across the boundaries requires that the normal density gradient vanishes and a region of more-or-less constant density forms that gradually becomes thicker with the passage of time. The salt flux in the interior continues down the gradient of salinity while it vanishes at the upper and lower surfaces, so that the surface water becomes more saline and the bottom less so as the system approaches diffusive equilibrium. During the entire process there is no bulk motion of the water; the dynamical equilibrium is stable.

If, however, at least one of the impermeable sidewalls is not vertical, a rather surprising effect occurs. There can be no flux of salt from the wall so that the *normal* gradient of salinity (and density) vanishes; the lines of constant density which are horizontal in the body of the fluid, must curve to meet the inclined surface at right angles, rather as shown on the left of Fig. 1. Quite clearly, this is no longer a state of hydrostatic equilibrium; the hydrostatic pressure at a point on the inclined wall is *less* than it is at the same depth in the body of the fluid because of the turning down of the isopycnals and the decreased weight of fluid above. The fluid near the wall will tend to creep upwards along it until the viscous forces balance the hydrostatic pressure imbalance. The finite diffusivity will induce a bulk motion in the stably stratified fluid.

The necessity for such a motion can be seen alternatively by a consideration of the diffusive fluxes in the fluid. The density gradient in the interior of the fluid is constant so that the diffusive flux at A on the right of Fig. 1 is equal to that at B. The flux at A

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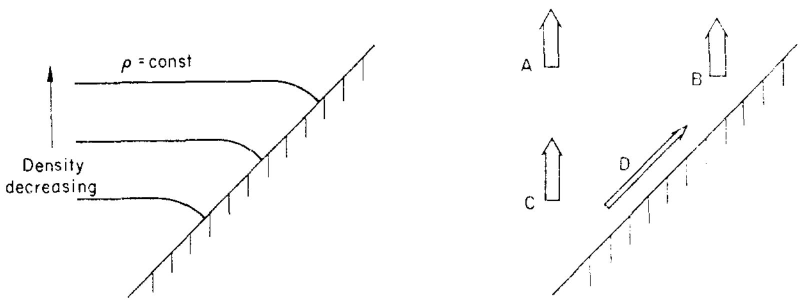

Fig. 1. The isopycnals near a sloping wall in a diffusive fluid are shown on the left. On the right, the fluxes of diffusing material are shown schematically. The flux at A down the gradient is supported by an equal flux at C, while the diffusive flux at B is maintained by the *convective* flux D along the wall.

is maintained by an equal flux C coming up from the region near the bottom of the tank, but what maintains the flux at B? Salt cannot diffuse horizontally across the gradient; the only possibility is that there is a *convective* flux D up the inclined boundary that continually leaks out to provide the vertical diffusive flux in the body of the fluid above the sloping wall.

It is evident that, in a container with sloping sidewalls, there is no state of static equilibrium of the stably stratified fluid and that a convective motion is necessarily generated. The velocities induced in this way are very slow, at least when the water is stratified with salt, because of the smallness of the appropriate diffusion coefficient. If, however, the fluid is stratified thermally, being warm on top and cold on the bottom, and the sloping walls are perfect insulators, it will be seen that the velocities induced are considerably larger; if liquid metals are used instead of water, the Reynolds number of the motion may be so large that there is the real possibility that the flow will become turbulent—all this produced spontaneously in a stably stratified fluid!

Motions of the same nature are set up in liquid filled cracks that are not vertical and along which a density gradient is maintained either thermally or by dissolved salts. Particularly when the fissure is nearly horizontal, the transport of salt along the fissure will be shown in some cases to be considerably greater than one would estimate from simple diffusion down the overall concentration gradient. This process may well have important implications in geological problems, where narrow fissures are involved and the time available for transport processes is very great.

2. THE DYNAMICS OF THE MOTION

Let us consider the steady state situation in which the bulk of the fluid is stratified stably and uniformly, so that the diffusive flux there is in the vertical direction and constant. For the moment we will neglect the growing region near the upper and lower surfaces where the flux is decreased, regarding these as either sources or sinks of the fluid moving along the sloping walls. If the thickness of the boundary region is of order δ, then the flux of mass is $\rho u\delta$, where u is a characteristic velocity. (For thermal stratification, the corresponding heat flux is proportional to $\theta u\delta$, where θ is the temperature). The gradient of this flux along the wall is constant, being needed to supply the constant vertical diffusive flux above the sloping wall, and since the vertical density gradient is constant, it follows that the volume flux $u\delta$ must likewise be constant in the layer. Moreover, the lateral spreading of the layer is inhibited by the stable density gradient in the interior. It is therefore natural to seek solutions to the governing equations in which the thickness δ is independent of position along the sloping wall and the velocity is a function of only the normal co-ordinate.

Accordingly, let us take the ξ-axis along the wall and the η-axis perpendicular to it, and assume that the velocity $\boldsymbol{u} = (u(\eta), 0)$. The equations of motion reduce to

$$-\frac{\partial p}{\partial \eta} - \rho g \cos \alpha = 0, \tag{1}$$

$$-\frac{\partial p}{\partial \xi} + \mu \frac{\partial^2 u}{\partial \eta^2} - \rho g \sin \alpha = 0, \tag{2}$$

where α is the angle of slope of the wall and μ the molecular viscosity. The density field is governed by the convection and diffusion of either salt or heat, so that

$$u \frac{\partial \rho}{\partial \xi} = \kappa \left(\frac{\partial^2 \rho}{\partial \xi^2} + \frac{\partial^2 \rho}{\partial \eta^2} \right), \tag{3}$$

where κ is the appropriate diffusivity. At the surface, the velocity and normal flux of density vanish:

$$u = 0, \quad \frac{\partial \rho}{\partial \eta} = 0 \quad \text{at } \eta = 0, \tag{4}$$

while far away, the velocity must vanish and the density distribution reduces to the undisturbed state

$$\left.\begin{aligned} u &\to 0, \\ \rho &\to \rho_0 - \frac{\rho_0}{g} N^2 (\xi \sin \alpha + \eta \cos \alpha), \end{aligned}\right\} \quad \text{as } \eta \to \infty, \tag{5}$$

where

$$N^2 = -\frac{g}{\rho_0} \frac{\partial \rho}{\partial z},$$

the stability frequency in the interior, and ρ_0 is the density in the interior at the level of the origin of co-ordinates.

In seeking a solution to these equations, let

$$\rho = \rho_0 - \frac{\rho_0}{g} N^2 (\xi \sin \alpha + \eta \cos \alpha) + \rho_0 f(\eta), \tag{6}$$

where the boundary disturbance $\rho_0 f(\eta) \to 0$ as $\eta \to \infty$. The substitution of this form into (1) and (2) leads to

$$u''(\eta) - \frac{g \sin \alpha}{\nu} f(\eta) = 0,$$

while from (3),

$$u(\eta) + \frac{\kappa g}{N^2 \sin \alpha} f''(\eta) = 0, \tag{7}$$

whence

$$f^{IV}(\eta) + \left(\frac{N^2 \sin^2 \alpha}{\nu \kappa} \right) f(\eta) = 0. \tag{8}$$

Now, from (5) and (7),

$$f, f'' \to 0 \quad \text{as } \eta \to \infty,$$

while from (4) and (6),

$$f'(0) = \frac{N^2}{g} \cos \alpha, \quad f''(0) = 0.$$

The solution subject to these boundary conditions is

$$f(\eta) = -\frac{N^2 \cos \alpha}{\gamma g} e^{-\gamma\eta} \cos \gamma\eta, \tag{9}$$

where

$$\gamma = \left(\frac{N^2 \sin^2 \alpha}{4\nu\kappa}\right)^{\frac{1}{4}}. \tag{10}$$

The distribution of velocity up the inclined surface is given by equation (7).

$$u(\eta) = 2\kappa\gamma \cot \alpha \, e^{-\gamma\eta} \sin \gamma\eta. \tag{11}$$

A solution of this type was obtained independently by WUNSCH (1970). A closely related problem involving convective flow up a heated sloping surface bounding an unstratified fluid is described by PRANDTL (1952) at the end of his book *The Essentials of Fluid Dynamics.*

3. PROPERTIES OF THE SOLUTION

The volume flux in the boundary region is, from (7),

$$Q = \int_0^\infty u \, d\eta = \frac{\kappa g}{N^2 \sin \alpha} f'(0),$$

$$= \kappa \cot \alpha, \tag{12}$$

which is, rather surprisingly, independent of the basic stratification. The flux of mass is likewise found from (6), (9) and (11) to be

$$M = \int_0^\infty \rho u \, d\eta = \rho_0 \kappa \cot \alpha - \tfrac{5}{4} \rho_0 \kappa \frac{N^2 \cos^2 \alpha}{g\gamma \sin \alpha} - \rho_0 \kappa \frac{N^2}{g} \xi \cos \alpha, \tag{13}$$

whose gradient up the slope,

$$\frac{dM}{d\xi} = -\rho_0 \kappa \frac{N^2}{g} \cos \alpha,$$

$$= \kappa \frac{\partial \rho}{\partial z} \cos \alpha, \tag{14}$$

just balances the diffusive density flux in the interior.

The corresponding solutions for the motion generated when the impermeable sloping boundary is above the stratified fluid, are found by replacing α in the expressions (9) and (11) by $\pi - \alpha$. In a tilted container whose cross-section is as shown in Fig. 2, the velocity and density fields near the boundary AB are then

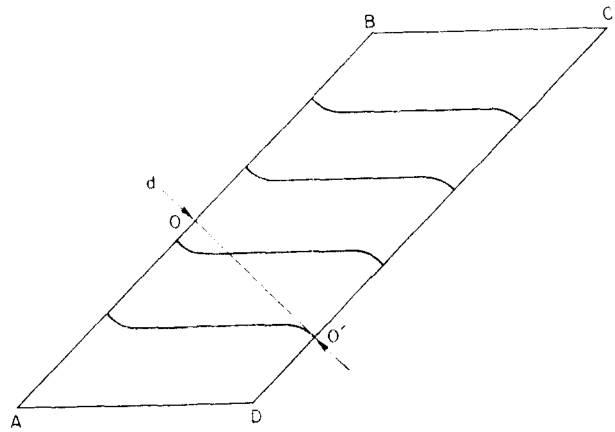

Fig. 2. Isopycnals in an inclined vessel.

$$\left.\begin{aligned} u &= -2\kappa\gamma \cot\alpha\, e^{-\gamma\eta} \sin\gamma\eta, \\ \rho &= \rho_1 - \frac{\rho_1}{g} N^2 (\xi \sin\alpha - \eta\cos\alpha) + \rho_1 \frac{N^2 \cos\alpha}{\gamma g} e^{-\gamma\eta} \cos\gamma\eta, \end{aligned}\right\} \tag{15}$$

where η is measured normal to the surface into the fluid, ξ upwards along the sloping boundary and ρ_1 is the undisturbed density at the level of the origin of co-ordinates. The flux of mass along this surface is then

$$\begin{aligned} M_1 &= \int_0^\infty \rho u \, d\eta, \\ &= -\rho_1 \kappa \cot\alpha - \tfrac{5}{4} \frac{\rho_1 \kappa N^2 \cos^2\alpha}{g\gamma \sin\alpha} + \frac{\rho_1 \kappa N^2}{g} \xi \cos\alpha. \end{aligned} \tag{16}$$

From these expressions, the net mass flux across the section 00′ (Fig. 2) can be found readily. If the perpendicular distance between the two sloping walls is d and the average density in the section is $\bar{\rho}$, then

$$\rho_0 = \bar{\rho} - \tfrac{1}{2} \left(\frac{\partial\rho}{\partial z}\right) d\cos\alpha,$$

$$\rho_1 = \bar{\rho} + \tfrac{1}{2} \left(\frac{\partial\rho}{\partial z}\right) d\cos\alpha,$$

and the net convective mass flux is, from (13) and (16),

$$M_c = -\kappa d \left(\frac{\partial\rho}{\partial z}\right) \left(1 - \frac{5}{2\gamma d}\right) \frac{\cos^2\alpha}{\sin\alpha}, \tag{17}$$

which augments the diffusive flux down the density gradient.

4. SOME OBSERVATIONS OF THE EFFECT

A simple experiment was conducted to demonstrate this flow. The small plexiglas tank shown in Fig. 3, was filled with a stably stratified salt (sodium chloride) solution formed from three initial layers that were allowed to diffuse for twelve hours. The concentration of the solution at the bottom was maintained by spreading salt crystals over the floor of the tank; these gradually dissolved. The stability frequency N was estimated to be approximately 4 rad/sec, and except near the bottom and the free surface, was nearly constant with height. Early attempts to demonstrate the flow were confused by weak circulations in the body of the fluid that were probably induced by evaporative cooling at the surface. To prevent these, a lid was fitted and the surface warmed a little by means of heating wires that maintained the lid at a temperature of 42°C. The tank was placed on a large aluminium girder to stabilize the temperature fluctuations, the ambient varying somewhat about an average of 20°C.

The inclined surface was also made of plexiglas, milled to fit the tank closely, so that the edges were sealed. This, together with the syringe mounted on the lid, was inserted twelve hours after the initial filling, causing a transient disturbance that appeared to die away within a few seconds. Three hours later, crystals of Blue Dextran dye were dropped into the reservoir at the top of the syringe and dissolved, the dyed fluid seeping downwards. The molecular weight of the dye is of order 40,000 so that its diffusivity in water is very much less than that of salt; it follows the convective motion without diffusing as does the salt itself. In concentrated solution it is rather denser than water, so that some ran down the inclined face and collected in a pool at the bottom of the tank. A significant amount, however, was sufficiently diluted to remain at the level of injection and it is evident from the sequence of photographs of Fig. 3, that the dye very close to the wall gradually moved upwards as the theory predicts. It was not possible to identify with certainty the leading edge of the dye stream so that a quantitative comparison with the theory could not be made. Nevertheless, these experiments did serve to demonstrate the effect quite clearly.

In water stratified with salt, the motion is of course a feeble one, the Reynolds number being, from (12),

$$R = \frac{Q}{\nu} = \frac{\kappa \cot \alpha}{\nu},$$

which, unless α is very small, is of the order of the inverse Prandtl number. In water, for example, the Reynolds number of the motion in a thermally stratified fluid is about 10^{-1}, and for a fluid stratified with salt, approximately 10^{-3}. On the other hand, the Prandtl numbers for liquid metals are of the order 10^{-2}–5×10^{-3}, so that the Reynolds number for such a flow in mercury will be of the order 200. Note that in the parallel flow (11) the inertia terms of the momentum equation vanish identically so that the solution remains valid for any Reynolds number. But under these conditions, the velocity profile (11) with its inflexion points, may well become dynamically unstable and the motion turbulent. It is indeed strange that an insulating container of mercury, with a ‘ stable ’ thermal stratification may spontaneously generate a turbulent flow.

When, in a container with at least one sloping side, the density increases with height, there is likewise no state of static equilibrium, no matter what the overall Rayleigh number may be. In this circumstance, however, the motion generated diffusively is necessarily unsteady and the layer thickness increases continually with time. A detailed analysis of this has some additional points of interest and will be presented elsewhere.

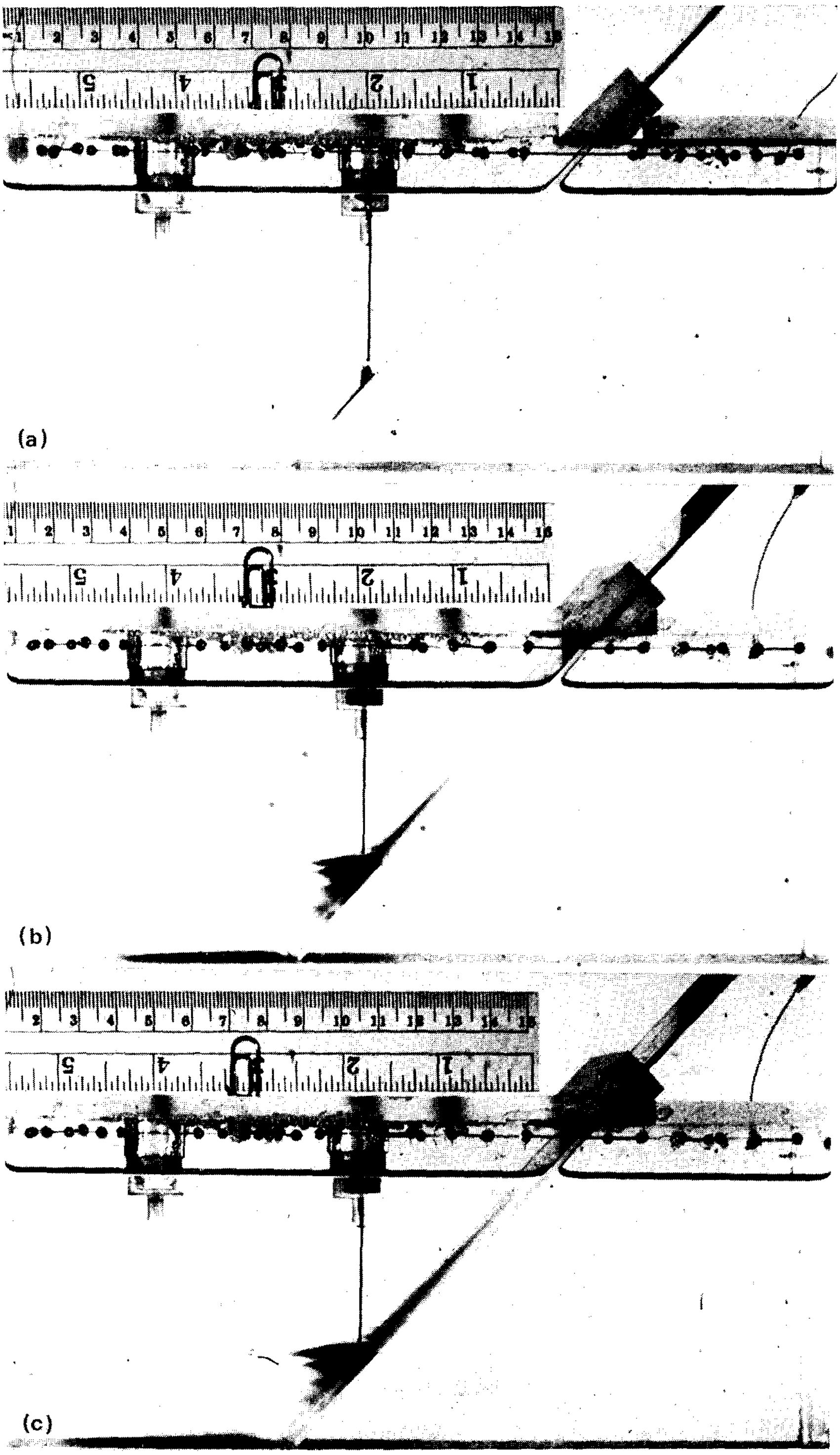

Fig. 3. Photographs of a simple experiment illustrating the diffusively driven flow. In the top picture, the dye has just been injected near the surface. Being denser than the ambient fluid, part of the dye is running down the face of the sloping wall. In the lower two photographs, the diffusive flow is progressively carrying the dye *up* the wall.

5. DIFFUSIVE FLOW IN A FISSURE

An important application of this effect is to geological problems involving mass transport along liquid filled fissures which may be inclined at only a small angle to the horizontal. If the width d of the crack (Fig. 2) is of order γ^{-1} or less, the interior region of horizontal stratification vanishes and the analysis must be modified somewhat. It is convenient to take the origin of co-ordinates at a point on the mid-plane of the fissure so that the boundary conditions (4) and (5) are replaced by

$$u = 0, \quad \frac{\partial \rho}{\partial \eta} = 0 \quad \text{at } \eta = \pm \tfrac{1}{2} d. \tag{18}$$

Let

$$\rho = \rho_0 - \Gamma \xi + \rho_0 f(\eta), \tag{19}$$

where Γ is the gradient of density along the fissure. If the fissure rises through a vertical height h over which the density decreases by the increment $\Delta\rho$, then

$$\Gamma = \frac{\Delta\rho}{h} \sin \alpha.$$

The substitution of (19) into (3) leads to

$$u = -(\kappa\rho_0/\Gamma) f'', \tag{20}$$

while from (1) and (2),

$$u''' - \frac{g\,\rho_0 \sin\alpha}{\mu} f' = \frac{\Gamma g \cos\alpha}{\mu}. \tag{21}$$

The solutions to these equations, subject to the boundary conditions (18) are found in a straightforward manner. In this context,

$$\gamma = \left(\frac{g\Gamma \sin\alpha}{4\mu\kappa}\right)^{\frac{1}{4}}, \tag{22}$$

and it is found that

$$u(\eta) = \frac{\Gamma g \cos\alpha}{2\gamma^3 \mu} \{-F(\tfrac{1}{2}\gamma d) \cos\gamma\eta \sinh\gamma\eta + G(\tfrac{1}{2}\gamma d) \sin\gamma\eta \cosh\gamma\eta\}, \tag{23}$$

and

$$\rho = \rho_0 - \Gamma\xi - \Gamma\eta \cot\alpha + \frac{\Gamma \cot\alpha}{\gamma} \{F(\tfrac{1}{2}\gamma d) \sin\gamma\eta \cosh\gamma\eta + G(\tfrac{1}{2}\gamma d) \cos\gamma\eta \sinh\gamma\eta\}, \tag{24}$$

where

$$F(X) = \frac{\sin X \cosh X}{\sin X \cos X + \sinh X \cosh X},$$

$$G(X) = \frac{\sinh X \cos X}{\sin X \cos X + \sinh X \cosh X}.$$

From these solutions, the convective mass flux along the fissure

$$M_c = \int_{-\frac{1}{2}d}^{\frac{1}{2}d} \rho u \, d\eta$$

is found after some algebra to be

$$M_c = \kappa \Gamma d \cot^2 \alpha \left\{ 1 + \frac{\sin \gamma d \sinh \gamma d}{(\sin \gamma d + \sinh \gamma d)^2} + \frac{5}{2\gamma d} \frac{\cos \gamma d - \cosh \gamma d}{\sin \gamma d + \sinh \gamma d} \right\}. \quad (25)$$

The numerical value of the function in the curly brackets is very small when γd is small (less than 10^{-2} when $\gamma d = 2$) and increases monotonically as γd increases. In the limit, as $\gamma d \to \infty$,

$$M_c \to \kappa \Gamma d \cot^2 \alpha \left\{ 1 - \frac{5}{2\gamma d} \right\} \quad (26)$$

in accordance with (17), since $\Gamma = -\partial\rho/\partial\xi = -(\partial\rho/\partial z) \sin \alpha$.

The diffusive mass flux along the fissure is simply $\kappa \Gamma d$, so that when α is small, the convective flux may be many times greater than one would expect by diffusion down the overall gradient. However, as $\alpha \to 0$, $\gamma d \to 0$ also so that the limiting expression for small fissure slope must be obtained from the more general solutions (23) and (24). It is readily shown, by expanding the trigonometric and hyperbolic functions in power series that

$$u(\eta) = \frac{\Gamma g \cos \alpha}{6\mu} \eta (\eta^2 - \tfrac{1}{4} d^2), \quad (27)$$

and

$$\rho = \rho_0 - \Gamma\xi - \frac{g\Gamma^2 \cos \alpha}{4\mu\kappa} \left(\frac{48\eta^5 - 40\eta^3 d^2 + 15\eta d^4}{1440} \right), \quad (28)$$

as illustrated in Fig. 4. The convective mass flux is then

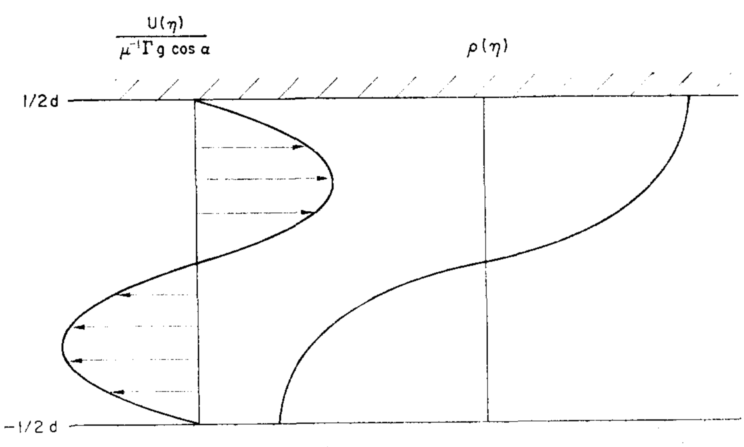

Fig. 4. Velocity and density profiles in a narrow, almost horizontal fissure.

$$M_c = \frac{1}{362,880} \frac{g^2 \Gamma^3 \cos^2 \alpha \, d^9}{\mu^2 \kappa}, \quad (29)$$

and the ratio β of the convective flux to the diffusive flux, $\kappa \Gamma d$, is

$$\beta = \frac{g^2 \Gamma^2 \cos^2 \alpha \, d^8}{362,880 \, \mu^2 \kappa^2}. \quad (30)$$

The rapid increase of this ratio with the thickness d of the crack is remarkable. If the fissure is nearly horizontal so that $\alpha \approx 0$ and $\gamma d \ll 1$, the convective flux dominates if

$$d > d_c = 4{\cdot}8 \left(\frac{\mu\kappa}{g\Gamma}\right)^{\frac{1}{4}}, \tag{31}$$

and in view of the very rapid dependence on M_c of d, if the width is only slightly greater than the value (31), the convective flux is increased enormously. For example, if $d = 2d_c$, the convective flux is about 300 times the diffusive value.

The numerical value of d_c is of interest. Suppose, for example, that a fissure extends from the side of a lake or stream and that dissolved salts produce a gradient along the fissure of, say, 0·1 g/cm^3 over 10 cm. Since for water, $\mu \sim 10^{-2}$, $k \sim 10^{-5}$, c.g.s. units, then

$$d_c \sim 4{\cdot}8 \times 10^{-2}\,\text{cm}.$$

In a crack thinner than this value, the flux is essentially diffusive, but in a wider one, the transport can be augmented enormously by convective flow.

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