CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1. The topics to be discussed

It seems useful to begin by outlining the range of subjects covered in this book, to give a broad picture of the way in which the several parts of the field have developed, and at the same time some explanation of the theme which has been used to connect them. The phenomena studied all depend on gravity acting on small density differences in a non-rotating fluid. Often the undisturbed fluid has a density distribution which varies in the vertical but is constant in horizontal planes; this will be called a stratified system whether the density changes smoothly or discontinuously. Special attention will be given to the problems of buoyant convection (arising from an unstable density distribution) and to various mechanisms of mixing when the stratification is stable.

Chapters 2 and 3 summarize relevant results on internal waves, and these were also historically the first phenomena to be studied. The original applications of the methods of perfect fluid theory to motion under gravity were to the problems of small amplitude surface waves and tides (subjects which will not be discussed here). These were soon extended to the case of two layers of uniform density with a density discontinuity between them. Some of the basic results had already been obtained by 1850 (notably by Stokes 1847), and they were applied to phenomena such as the drag experienced by a ship when it creates a wave on an interface close to the surface (Ekman 1904), and to internal seiches in lakes. The later developments in the theory of internal waves owe more to meteorology than to the study of the sea or lakes, probably because of the directly visible and often spectacular effects which are caused by gravity waves in the atmosphere. The studies of waves in a density gradient, waves in the lee of obstacles and the effect on these of wind velocity changes with height, all originated in this context. Some of

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these results have been extended to the case where the waves have a large amplitude. The recent work on interactions between waves has again been discussed with the oceanographic application in mind (Phillips 1966*a*).

Various finite amplitude flow phenomena in a stratified fluid are also considered in chapter 3.† Many of these occur in the context of hydraulic engineering, for example in the prediction of the velocity of intrusion of saline water into a lock filled with fresh water when the gate separating the two is opened, or the conditions under which one layer can be withdrawn from a stratified fluid without removing an adjacent layer. An elementary discussion of small scale fronts in the atmosphere also comes under this heading. The phenomenon of blocking, and the jet-like motions which can arise in slow flows of a stratified fluid are mentioned briefly, with some discussion of the effects of viscosity and diffusion.

Instabilities of various shear flows of a stratified fluid are treated in chapter 4. Some parts of this subject have been well understood for a long time, but other results are of more recent origin. A classification of the mechanisms of generation of turbulence is also given here, as a logical prelude to the discussion of turbulent flow in a stratified medium. The subject of forced and free convection in a shear flow over a rough plane again owes much to the meteorological work, which will be used as the basis for chapter 5. The behaviour of turbulent wakes in a stratified fluid seems to fit in naturally here.

In the next two chapters we consider gravitationally unstable flows, i.e. the various mechanisms of buoyant convection. The historical order of development is reversed here; the models of convection which emphasize the buoyant elements themselves are treated first, followed by studies of convection between horizontal planes and some discussion of the relation between the two. The impetus for much of this work has come from the problem of heat transfer from the ground to the lower atmosphere, but some attention is also given to other geometries, and to current work on the numerical simulation of turbulent motions. The discussion of convection is extended in chapter 8 to the case where two properties

† A more thorough historical review of this part of the subject has been given recently by Hinwood (1970).

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with different molecular diffusivities are present simultaneously in a fluid. When these have opposing effects on the vertical density gradient, convection in well-mixed layers can be driven by an unstable buoyancy flux (for example by a flux of heat from below), while a net density difference is preserved across the interfaces between them. These effects are believed to have important implications for vertical mixing in the ocean.

In the last two chapters, many of the ideas developed earlier are used to discuss the processes responsible for mixing in large bodies of stratified fluid, particularly in the ocean and atmosphere. It is shown that there are many mechanisms which will cause a smoothly stratified fluid to break up into a series of steps, and so a basic problem is the understanding of mixing across a density interface. Such mixing can be driven either directly by a boundary source of mechanical energy, or by energy propagated into the interior by internal waves.

It is impossible to arrange a subject with so many different strands in an entirely satisfactory order, but I hope that enough cross-references have been given to allow readers to choose a different sequence to suit their own interests. Some basic ideas and approximations which are common to the whole book are outlined in the following sections of this chapter, but it will be obvious that neither this introduction nor the later chapters can be comprehensive or complete. Most theoretical results will be quoted without proof, and a background knowledge of fluid mechanics of homogeneous media is assumed. (See, for example, Batchelor 1967.)

1.2. Equilibrium and departures from it

The only external force field considered in this book is that of gravity, which exerts a body force $\rho\mathbf{g}$ per unit volume on each element of fluid (where ρ is the local density and $\mathbf{g}$ is the acceleration due to gravity). The effects to be described result from variations of ρ from point to point in the fluid, which will nearly always be regarded as incompressible. The nature of the fluid is unimportant; most of the ideas may be applied to liquids, in which density variations are due to differences of temperature or the concentrations of solutes or sediment, or to gases, in which there may be differences of

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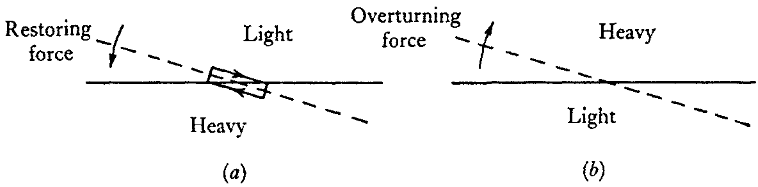

Fig. 1.1. Displacements from hydrostatic equilibrium: (*a*) stable, and (*b*) unstable density distributions.

temperature or composition. The compressibility of gases becomes significant in deep layers, but for many purposes these too can be treated as incompressible by using potential temperature and potential density (as defined below) in place of the actual density.

A body of homogeneous, inviscid incompressible fluid at rest is in a state of neutral equilibrium. At every point the weight of a fluid element is then exactly balanced by the pressure exerted on it by neighbouring fluid, and this continues to hold true if the elements are displaced to another position of rest. When ρ varies the hydrostatic equation

$$p = p_0 - g\int_0^z \rho\, dz \qquad (1.2.1)$$

shows that the fluid (in the absence of diffusion) is in equilibrium only when the density as well as the pressure is constant in every horizontal plane. This equilibrium stratification is stable when the heavier fluid lies below, since tilting of a density surface will produce a restoring force: the resulting motion can overshoot the equilibrium position and oscillate about it, thus giving rise to internal waves. When light fluid lies below heavier, the equilibrium is unstable and small displacements of density surfaces from the horizontal will grow and lead to convective motions (see fig. 1.1).

The corresponding state of neutral static stability in a compressible fluid is that for which the entropy is constant with depth. A change of pressure results in a change in temperature in an adiabatic process; and this must be taken into account when comparing displaced fluid with its surroundings. The potential temperature θ and potential density ρ_θ are defined to be the temperature and density when the fluid is compressed adiabatically to a standard pressure p_0. In a perfect gas for which $p = R\rho T$ (a good approximation for the atmosphere), R being the gas constant, they are related to the absolute temperature T and the local properties ρ and p by

$$T = \theta\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}, \quad \rho = \rho_\theta\left(\frac{p}{p_0}\right)^{1/\gamma}, \tag{1.2.2}$$

and it follows that

$$-\frac{1}{\rho_\theta}\frac{\partial \rho_\theta}{\partial z} = \frac{1}{\theta}\frac{\partial \theta}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z} - \frac{\gamma-1}{\gamma}\cdot\frac{1}{p}\frac{\partial p}{\partial z}, \tag{1.2.3}$$

where γ is the ratio of specific heats $\gamma = C_p/C_v = C_p/(C_p - R)$, and is about 1.4 for air. An isothermal atmosphere has T = constant, and it then follows from (1.2.1) and (1.2.3) that

$$p = p_0 e^{-gz/RT} \quad \text{and} \quad \rho = \rho_0 e^{-gz/RT}, \tag{1.2.4}$$

where ρ_0 is a reference density. The length $H_s = RT/g$ in which the density falls off by a factor e is called the scale height (it is about 8 km in the earth's atmosphere). In the *isothermal* atmosphere

$$\frac{1}{\rho_\theta}\frac{\partial \rho_\theta}{\partial z} = \frac{\gamma-1}{\gamma}\frac{1}{\rho}\frac{\partial \rho}{\partial z}. \tag{1.2.5}$$

An *adiabatic* atmosphere is one in which θ is constant, and the absolute temperature gradient is then

$$\frac{\partial T}{\partial z} = -\frac{\gamma-1}{\gamma}\frac{g}{R} = -\frac{g}{C_p} = -\Gamma \tag{1.2.6}$$

or $\partial \ln T/\partial \ln p = (\gamma-1)/\gamma$ using pressure as the vertical coordinate.

If the actual 'lapse rate', or decrease of temperature with height, equals this (Γ = 10 °C/km in the earth's atmosphere), a displaced fluid element will always have the same density as its new surroundings and the equilibrium will be neutral; if the temperature decreases less rapidly than Γ the situation will be stable. An isothermal atmosphere will be very stable in this sense, and an *inversion*, in which the absolute temperature increases with height over some interval, even more so. Thorough mixing of an arbitrarily stratified compressible fluid results in the formation of an adiabatic atmosphere. When the air is moist and lifting produces saturation, the release of latent heat of condensation will heat the rising air, and the rate of cooling of an unmixed parcel will be reduced to the *saturated adiabatic* lapse rate. An atmosphere stable to dry convection may be unstable when there is moisture present, and one must be careful to specify which adiabatic process is to be used for reference when assessing the static stability. Strictly speaking, the potential density should be used also for liquids. The compressibility of small volumes of water, say in the laboratory, may be neglected, but it becomes significant for the deep ocean (see Phillips 1966*a*, p. 13). More restrictions must be put on the velocity and amplitudes of the *motion* in a compressible fluid before it can be regarded as incompressible, and these will be considered in the next section.

1.3. The equations of motion, and various approximations

The characteristic differences between the motion of a homogeneous and a heterogeneous fluid can most easily be explained by writing down the equations of motion in several forms: and it will also be useful to have these here for reference in later chapters. (For a detailed derivation of these equations, see for example Yih (1965).) From now on the fluid will be assumed incompressible and non-diffusive unless it is explicitly stated otherwise, and this means that

$$\frac{D\rho}{Dt} = 0, \tag{1.3.1}$$

where D/Dt denotes differentiation following the motion. The continuity equation in vector notation is

$$\nabla \cdot \mathbf{u} = 0, \tag{1.3.2}$$

where $\mathbf{u} = (u, v, w)$ is the velocity. The momentum (Navier–Stokes) equations with the force of gravity included can be written (with $\mathbf{g} = (0, 0, -g)$, the x and y axes being in the horizontal plane and z vertically upwards) as

$$\rho \frac{D\mathbf{u}}{Dt} = \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla\mathbf{u})\right) = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{u}. \tag{1.3.3}$$

The last term is the result of molecular viscosity μ (assumed constant here), and if this is neglected one obtains the Euler equations of motion.

If p and ρ are now expanded about the values p_0 and ρ_0 in a reference state of hydrostatic equilibrium for which $\nabla p_0 = \rho_0 \mathbf{g}$

(i.e. one sets $p = p_0+p'$ and $\rho = \rho_0+\rho'$), the Euler equations can be written in terms of the deviations p' and ρ' from this state as

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p' + \rho'\mathbf{g}. \tag{1.3.4}$$

Thus as already implied in the elementary discussion of static stability, only differences of density ρ' from some standard value are relevant in determining the effect of gravity. In a two-layer system, for example, the layer with the standard density ρ_0 may be regarded as weightless, and that with density $\rho = \rho_0+\rho'$ as if it were acted on by a reduced gravitational acceleration $g\rho'/\rho$. (See Prandtl 1952, p. 368.)

Taking the curl of the Navier–Stokes equation (1.3.3) leads to an equation for the vorticity $\boldsymbol{\zeta} = \nabla \times \mathbf{u}$, namely

$$\frac{\mathrm{D}\boldsymbol{\zeta}}{\mathrm{D}t} = \boldsymbol{\zeta}.\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\zeta} + \nabla p \times \nabla\left(\frac{1}{\rho}\right), \tag{1.3.5}$$

where $\nu = \mu/\rho_0$ is the kinematic viscosity (which is also taken to be a constant, implying the neglect of ρ' compared to ρ_0 in this term). The first three terms are the same as for a fluid of constant density; a change of vorticity of a fluid element can again be brought about by a stretching of vortex lines, or by the diffusion of vorticity from boundaries. The last term contains the essential difference between a stratified fluid and a uniform one. Vorticity will be created whenever a non-homogeneous fluid is displaced from a state in which ∇p and $\nabla \rho$ are parallel (the only condition for which the vector product is zero). In the simplest case where p effectively depends on gravity alone (and it is often true that contributions to p' due to other accelerations are negligible compared to the hydrostatic part) displacements of density surfaces away from the horizontal will produce vorticity. This will oscillate in magnitude and direction in stable stratification (so that internal waves are *rotational* phenomena), and it will tend to increase monotonically during the development of convection.

The creation of vorticity also implies the creation of circulation defined by

$$\Gamma = \int_C \mathbf{u}\cdot\mathrm{d}\mathbf{l} = \iint_S \boldsymbol{\zeta}\cdot\mathrm{d}\mathbf{A}, \tag{1.3.6}$$

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where $d\mathbf{l}$ is a line element of a closed curve C and $d\mathbf{A}$ is an element of a surface S bounded by C. The result for an inviscid fluid, obtained originally by Bjerknes, can be written in the two forms

$$\begin{aligned}\frac{d\Gamma}{dt} &= -\int_C \frac{1}{\rho}\nabla p \cdot d\mathbf{l} \\ &= \iint_S \left(\nabla p \times \nabla\left(\frac{1}{\rho}\right)\right)\cdot d\mathbf{A}.\end{aligned} \tag{1.3.7}$$

This is a generalization of Kelvin's theorem, which states that in an inviscid fluid of constant density the circulation round a closed curve moving with the fluid remains constant. When ρ is variable along the path of integration, circulation is generated unless density and pressure surfaces coincide. For example, a circuit taken just below and just above an interface separating layers of different density shows that a shear must be developed at the interface when it is tilted from a state of rest, so that the pressure varies along the path (fig. 1.1). If the surface S is chosen to be one of constant density, the integrand of (1.3.7) vanishes, and a motion started from rest can be treated as a two-dimensional irrotational flow within each density surface, the vortex lines being imbedded in these surfaces. This last property can be exploited to derive an integrated form of the Euler equations, which correspond to Bernoulli's equation in a uniform fluid, and in which the Bernoulli constant is replaced by a function of density alone. These are given and used explicitly in chapter 3.

Two widely used approximate forms of the Euler equations (1.3.4) must also be introduced. The first simplification is that of *linearization*, the neglect of the non-linear convection terms like $u\,\partial u/\partial x$ in comparison with $\partial u/\partial t$. This procedure is justified when the motions and velocities are of small amplitude. The parameter ϵ which must be kept small is different in different situations (see chapter 3), but typically it is a ratio of a vertical displacement to a horizontal lengthscale. The product terms are of order ϵ^2 and can be omitted to give the first order equations

$$\rho\frac{\partial \mathbf{u}}{\partial t} = -\nabla p' + \rho'\mathbf{g}. \tag{1.3.8}$$

The importance of the linear equations lies in the fact that any small oscillation described by them can be resolved into a set of 'normal modes', in each of which the particle motions are simple harmonic and independent of all other modes. (See Lamb 1932, ch. 8.) Note that no assumption has been made here about the magnitude of the density variation, and some linear problems can be solved without further approximation (see § 2.2.1).

In the second approximation to be mentioned here, on the other hand, the density variation ρ' is assumed to be small compared to ρ_0. Rewriting (1.3.4) in the form

$$\left(1+\frac{\rho'}{\rho_0}\right)\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0}\nabla p' + \frac{\rho'}{\rho_0}\mathbf{g}, \qquad (1.3.9)$$

we see that the density ratio ρ'/ρ_0 appears twice, in the first (inertia) term and in the buoyancy term. When ρ'/ρ_0 is small, it produces only a small correction to the inertia compared to a fluid of density ρ_0, but it is of primary importance in the buoyancy term. The approximation introduced by Boussinesq (1903) consists essentially of neglecting variations of density in so far as they affect inertia, but retaining them in the buoyancy terms, where they occur in the combination $g' = g\rho'/\rho_0$. When viscosity and diffusion are included, variations of fluid properties are also neglected in this approximation.

There are other restrictions necessary in compressible fluids which are often included by the name 'Boussinesq approximation'; these are discussed fully by Spiegel and Veronis (1960) and only the results will be quoted here. First one must replace density by potential density, as already shown in § 1.2. The limitation of small density deviations from a standard ρ_0 implies two things; the vertical scale of the *mean* motion must be much smaller than the scale height H_s (1.2.4), and the fluctuating density changes due to local pressure variations must also be negligible. The latter is the most important of the extra conditions; it implies that the fluid can be treated as incompressible (so that (1.3.2) is an adequate approximation to the continuity equation), and it therefore excludes sound and shock waves. Finally, the ratio of the length to the timescales of any variation in an unsteady flow should be much smaller than the velocity of

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sound, to ensure that information about pressure changes is transmitted effectively instantaneously, as it is in an incompressible fluid.

The Boussinesq approximation may be made either independently of the linear (small amplitude) assumption or in combination with it. The *linearized* Boussinesq equations for an inviscid liquid are

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p' + \frac{\rho'}{\rho_0}\mathbf{g}, \tag{1.3.10}$$

together with (1.3.1) and (1.3.2). The linearized form of (1.3.1) is just

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \rho_0}{\partial z} = 0, \tag{1.3.11}$$

expressing the fact that changes of density at a point are due to bodily displacement of the mean density structure. The second (product) term is not small here, since $\partial\rho_0/\partial z$ is a finite quantity.

It is important to note immediately, however, that there are some circumstances where it is inconsistent to use the Boussinesq approximation, even when the density differences are everywhere small. In the theory of internal solitary waves, for example, (§3.1.2) certain non-linear terms are retained which are of the same order as those neglected in obtaining (1.3.9), and the whole phenomenon depends on the consistent inclusion of all terms to this order.

Under other circumstances the Boussinesq approximation works surprisingly well in fluids with large density variations (even in the atmosphere), but this depends very much on the type of flow considered. Most meteorological flows with a vertical scale less than about 1 km (small compared to H_s) can be treated in this way, and this includes wave motions which extend to great heights, provided the vertical excursions of individual fluid particles are not too large. The assumptions are most seriously violated by convective motions extending through a height of the order H_s or more since fluid then can traverse the whole depth. However another simplifying result is available for incompressible fluids which can in some problems remove the inertial effect of the density variation entirely from the equations, and allow one to calculate it *afterwards* from the solution of the corresponding Boussinesq problem. The simplest example is a steady flow in which gravity effects are absent (Yih 1958). In this

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case a transformation of (1.3.4) to remove ρ from the convective terms $\rho u(\partial u/\partial x)$ shows that the actual velocity is a factor $(\rho_0/\rho)^{\frac{1}{2}}$ greater than that calculated using a constant ρ_0. With a fixed pressure gradient, this just implies that the kinetic energy per unit volume, ρu^2 say, is constant throughout such a flow. This same form of result is obtained in §2.2.1 for small amplitude waves of a particular form in an exponential density gradient (a problem which certainly involves gravity), and Drazin (1969) has shown that this remains true for internal gravity waves even when the amplitude is large. It has not been strictly proved for other cases, but scaling u with $\rho^{-\frac{1}{2}}$ will often give a good approximation for other incompressible flows in regions of large density variation. No such general simplification is possible for non-linear motions of compressible fluids over many scale heights. (See Claus (1964) and §3.1.4.)

1.4. Basic parameters of heterogeneous flows

Several quantities and concepts which recur in different contexts throughout this subject will now be introduced in an elementary way (leaving a more detailed discussion of their physical significance to the appropriate later chapter). Consider first the motion of an element of inviscid fluid displaced a small distance η vertically from its equilibrium position in a stable environment. The vertical component of (1.3.10) (neglecting the small pressure fluctuation), together with (1.3.11) gives

$$\frac{\partial^2\eta}{\partial t^2} = \frac{g}{\rho}\frac{\partial\rho_0}{\partial z}\eta. \tag{1.4.1}$$

The element will thus oscillate in simple harmonic motion with angular frequency

$$N = \left(-\frac{g}{\rho}\frac{\partial\rho_0}{\partial z}\right)^{\frac{1}{2}}. \tag{1.4.2}$$

This is the frequency associated with the names of Brunt (in meteorology, where of course the potential density gradient must be used) and Väisälä (especially in oceanography), but it will be referred to here by the less cumbersome and more descriptive name of *buoyancy frequency*. The corresponding periods $2\pi/N$ are typically

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a few minutes in the atmosphere and the oceanic thermocline, and up to many hours in the deep ocean. Notice that N is constant when the density varies exponentially with height (if $\rho = \rho_0 e^{-z/H}$ then $N^2 = g/H$), and a variation of this kind is often assumed to simplify the analysis. In the limit of small density differences the corresponding gradient is linear, so the latter assumption is useful when the Boussinesq equations are appropriate.

In a shear flow the vertical gradient of the horizontal velocity also has the dimensions of frequency, and the non-dimensional ratio

$$Ri = N^2 \Big/ \left(\frac{\partial u}{\partial z}\right)^2 = -g\frac{\partial \rho}{\partial z}\Big/\rho\left(\frac{\partial u}{\partial z}\right)^2 \tag{1.4.3}$$

is called the *gradient Richardson number*. (This was named in honour of L. F. Richardson, though it is not exactly the form he used – see Brunt 1952.) There are other related ratios called by this name, for example, the flux Richardson number Rf defined in chapter 5, which is the ratio of the rate of removal of energy by buoyancy forces to its production by the shear, and the particular definition implied in each case must be kept clearly in mind. Another ratio of this kind has a fundamental significance as an *overall* parameter describing a whole flow (Batchelor 1953*a*). When the Boussinesq and hydrostatic approximations are made and the motion is steady, the ratio of the buoyancy to the inertia terms in (1.3.9) is the *only* dimensionless number needed to specify an inviscid flow. Using the scales of velocity U and length L imposed by the boundary conditions, this can be written

$$Ri_0 = g'L/U^2. \tag{1.4.4}$$

A subscript will always be used to denote the overall or finite difference form of the Richardson number.

In hydraulic engineering it is more common to use, instead of Ri_0, the inverse square root of Ri_0 i.e.

$$F = U/(g'L)^{\frac{1}{2}}, \tag{1.4.5}$$

which is called the internal or densimetric *Froude number*. (Sometimes this is written with a subscript i to denote 'internal', but this will be dropped here.) This usage has arisen because of the correspondence with the ordinary Froude number (defined with g

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replacing g' in the above), which compares a characteristic flow velocity with the velocity of long waves on a free surface. An entirely analogous interpretation of F involving internal waves is given in chapter 3. The use of the corresponding gradient parameter $(\partial u/\partial z)/N(=F_g$ say) instead of Ri (cf. (1.4.3)) has much to recommend it, but both Ri and F will be retained since common usage seems to demand them.

Other parameters of course can become important when extra physical effects are taken into account. The most obvious is the *Reynolds number* $Re = UL/\nu$ (where ν is the kinematic viscosity) a measure of the balance between inertial and viscous terms in (1.3.3). When the length and velocity scales, and therefore Re, are large, it is justifiable in a homogeneous fluid to treat the motion as inviscid, except perhaps near boundaries. Even in laboratory experiments, especially the kind of free turbulent flows considered in chapter 6, it is often possible to ignore the viscous effects and still to make reliable comparisons between a model and the prototype in nature. In a strongly stratified fluid, however, there is another possibility which must be kept in mind: the density gradient can suppress vertical motions, and so introduce a much smaller internal lengthscale into the problem. The Reynolds number defined using this new scale need no longer be large, and viscosity (and also the effects of molecular diffusion) again become relevant. (See § 10.2.3.)

When molecular diffusion is taken into account explicitly, various non-dimensional ratios may be defined, the simplest of which is the *Prandtl number* $Pr = \nu/\kappa$. The other parameters depend on the ones already defined, and they are not so fundamental to the whole range of problems under discussion, so their introduction will be left to the appropriate place in later chapters.

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