Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Comput. Methods Appl. Mech. Engrg. 195 (2006) 2393–2410
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Edge stabilization for the generalized Stokes problem: A continuous interior penalty method

Erik Burman [a,*], Peter Hansbo [b]

[a] *Ecole Polytechnique Fédérale de Lausanne, Institute of Analysis and Scientific Computing, CH-1015 Lausanne, Switzerland*
[b] *Department of Applied Mechanics, Chalmers University of Technology, SE-41296 Göteborg, Sweden*

Received 5 July 2004; received in revised form 20 December 2004; accepted 15 May 2005

Abstract

In this note we introduce and analyze a stabilized finite element method for the generalized Stokes equation. Stability is obtained by adding a least squares penalization of the gradient jumps across element boundaries. The method can be seen as a higher order version of the Brezzi–Pitkäranta penalty stabilization [F. Brezzi, J. Pitkäranta, On the stabilization of finite element approximations of the Stokes equations, in: W. Hackbusch (Ed.), Efficient Solution of Elliptic Systems, Vieweg, 1984], but gives better resolution on the boundary for the Stokes equation than does classical Galerkin least-squares formulation. We prove optimal and quasi-optimal convergence properties for Stokes' problem and for the porous media models of Darcy and Brinkman. Some numerical examples are given.

Keywords: Generalized Stokes' equation; Stabilized methods; Finite element; Interior penalty method; Gradient jumps; Inf–sup condition

1. Introduction

The use of equal order interpolation of the pressure and the velocities for the Stokes problem is not stable if implemented without stabilization. Over the years many stabilization methods have been proposed and stabilization is by now a well established discipline with different well explored methods like the SUPG/SD-method [18], the residual free bubbles [6,16] and more recent contributions like (local) projection methods [2,12] for Stokes problem. The relation between the different approaches is also well understood in most cases. In this paper we present a method which stabilizes both Stokes problem and Darcy's

* Corresponding author. Fax: +21 693 4246.
E-mail address: erik.burman@epfl.ch (E. Burman).

0045-7825/$ - see front matter
doi:10.1016/j.cma.2005.05.009

problem by adding a least-squares term based on the jump in the gradient over element boundaries. The method has many of the advantages of the above methods, but no additional degrees of freedom are added, no hierarchical meshes are needed, the formulation remains symmetric, and the mass can be lumped for efficient time marching and treatment of stiff source terms. The price to pay is an increased number of non-zero elements in the Jacobian due to the fact that the gradient jump term couple neighboring elements. The key observation that allows for satisfaction of the inf–sup condition is that the gradient jump operator controls the part of the gradient which is not in the finite element space. In this sense the stabilization represents a minimal stabilized method as proposed in [4] and is related to the projection stabilization method of [12]. The edge stabilization method has been successfully applied to the problem of convection–diffusion in [9] and it was noted that the stabilization parameter was independent of the diffusion parameter, hence making the method very well suited also for degenerate diffusion problems. For the Stokes problem the behavior is somewhat different and, depending on how the stabilization parameter scales with respect to the meshsize h, the analysis gives different results. Using the optimal choice yields the following a priori error estimates for the Stokes' and the Darcy's problems respectively,

- Stokes

$$\|u-u_h\|_{0,\Omega}+h(\|\nabla(u-u_h)\|_{0,\Omega}+\|p-p_h\|_{0,\Omega})\leqslant Ch^2(\|u\|_{2,\Omega}+\|p\|_{1,\Omega}).$$

- Darcy

$$\|u-u_h\|_{0,\Omega}+\|p-p_h\|_{0,\Omega}+h^{\frac{1}{2}}(\|\nabla\cdot(u-u_h)\|_{0,\Omega}+\|\nabla(p-p_h)\|_{0,\Omega})\leqslant Ch^{\frac{3}{2}}(\|u\|_{2,\Omega}+\|p\|_{2,\Omega}).$$

We observe that this is optimal for the case of Stokes equation. For Darcy's equation we have optimality for the divergence of the velocities and the gradient of the pressure and suboptimality with a gap of half a power of h for the pressures and the velocities in the L^2-norm. This result for the vanishing viscosity case is very similar to the corresponding convection–diffusion result. The possible use of gradient jump stabilization on P_1-iso-P_2 patches has been commented on in [2,22] however as possible variations of other stabilized methods and without neither analysis nor numerical examples for the generalized Stokes problem. Another recent method for pressure stabilization was proposed in [13], where it is proposed to add a term penalizing the difference between p and the local projection of p onto a space of lower polynomial order. This method can be seen as an alternative way of generalizing the Brezzi–Pitkäranta stabilization to higher order elements. In the P1/P1 case their stabilizing term is essentially equivalent to the standard Brezzi–Pitkäranta term. Compared to the approach of [13], the advantages of the present method lie in is its better consistency properties and its generality. We also give a numerical example showing better convergence of the pressure on the boundary.

As was shown in [9] the jump of the gradient over element edges stabilizes convection dominated flow and in this paper we consider the stabilization of the pressure and the incompressibility condition in the same fashion. This will be exploited in a forthcoming work for the stabilization of the incompressible Navier–Stokes equations (see [10]). The increased complexity of the coding is to a certain extent compensated by the generality: although an additional term has to be coded and a table of nearest neighbours is needed, this same term may be used to stabilize all instabilities induced by first order terms.

For work on stabilized methods for Darcy's equation we refer to [20] and for other methods that are stable in the Darcy limit for the generalized Stokes problem see [7,19].

2. Generalized Stokes' problem

We propose to study a generalized Stokes problem, with two parameters σ and ν including the Darcy's equation as a special case. We consider the problem of solving the partial differential equation

$$\begin{aligned} &\sigma u - \nu\Delta u + \nabla p = f \quad \text{in } \Omega,\\ &\nabla\cdot u = g \quad \text{in } \Omega,\\ &u\cdot n = 0 \quad \text{on } \partial\Omega,\\ &\nu u\times n = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is bounded polygonal domain in $\mathbb{R}^d$ with boundary $\partial\Omega$, $d = 2, 3$ and σ and ν are two positive parameters, that may not vanish simultaneously, $f \in [L^2(\Omega)]^d$ and $g \in L^2(\Omega)$, such that $\int_\Omega g\,\mathrm{d}x = 0$. This problem can be written in weak form as follows: Find $u \in V = \{v \in [H^1(\Omega)]^d : v|_{\partial\Omega} = 0\}$ when $\nu > 0$ ($u \in V = \{v \in [L^2(\Omega)]^d,\ \nabla\cdot v \in L^2(\Omega) : v\cdot n|_{\partial\Omega} = 0\}$ for $\nu = 0$) and $p \in Q = L_2(\Omega)/\mathbb{R}$ when $\nu > 0$ ($p \in H^1(\Omega)$ for $\nu = 0$) such that

$$B[(u,p),(v,q)] = L(v,q) \quad \forall (v,q) \in V\times Q, \tag{2}$$

where

$$B[(u,p),(v,q)] := a(u,v) + b(p,v) - b(q,u)$$

and

$$a(u,v) := \int_\Omega \sum_{i=1}^d \sigma u_i v_i + \nu\nabla u_i\cdot\nabla v_i\,\mathrm{d}x, \quad b(p,v) := -\int_\Omega p\nabla\cdot v\,\mathrm{d}x$$

and

$$L(v,q) := \int_\Omega f\cdot v\,\mathrm{d}x + \int_\Omega gq\,\mathrm{d}x.$$

The finite element method consists of seeking piecewise polynomial approximations u_h of u and p_h of p, where $u_h \in V^h \subset V$ and $p_h \in Q^h \subset Q$, with V^h and Q^h built from continuous functions. Consider a partitioning of Ω into a conforming, locally quasi-uniform, triangulation T_h of affine shape regular simplicies K. The diameter of a triangle K will be denoted h_K. We let $(\cdot,\cdot)$ denote the L^2-scalar product and $\|u\| = (u,u)^{\frac{1}{2}}$ the associated norm. We shall be concerned with the approximation

$$V^h = \{v \in [V\cap C^0(\Omega)]^d : v|_K \in [P^1(K)]^d\ \forall K \in T_h\}$$

and a continuous pressure space,

$$Q^h = \{q \in Q\cap C^0(\Omega) :\ q|_K \in P^1(K)\ \forall K \in T_h\}.$$

It is well known that the combination $V^h \times Q^h$ is unstable (see, e.g., [5]).

The edge stabilization method can be formulated as follows: Find $(u_h, p_h) \in V^h \times Q^h$ such that

$$B[(u_h,p_h),(v,q)] + J[(u_h,p_h),(v,q)] = L(v,q) \tag{3}$$

for all $(v,q) \in V^h \times Q^h$, where $J[(u_h,p_h),(v,q)] := j(p_h,q) + \tilde{j}(u_h,v)$ and

$$j(p_h,q) := \sum_K \frac{1}{2}\int_{\partial K} \gamma h_K^{s+1}[n\cdot\nabla p_h][n\cdot\nabla q]\,\mathrm{d}s \tag{4}$$

and

$$\tilde{j}(u_h,v) := \sum_K \frac{1}{2}\int_{\partial K} \gamma h_K^{s+1}[\nabla\cdot u_h][\nabla\cdot v]\,\mathrm{d}s, \tag{5}$$

where $[x]$ denotes the jump of quantity x over edge ∂K when $\partial K \cap \partial\Omega = \emptyset$ else $[x] = 0$. The coefficient s takes the values $s = 2$ in the case $\nu \geqslant h$ and $s = 1$ in the case $\nu < h$.

Remark 2.1. The change of the order of the parameter when passing from the viscous case to the non-viscous case resembles the behavior of the SUPG method for convection–diffusion problems. A standard way of handling this for problems where the viscosity is non-uniform in the domain is to use the ν-weighted parameter $\gamma h_K^2(1+\frac{\nu}{h_K})^{-1}$ (assuming for simplicity that $\sigma = 1$).

Remark 2.2. On a uniform mesh, the jump term $j(p,q)$ (with $s = 2$) can be seen as the only remaining contribution from a discretization of $h^4\Delta^2 p$ when applying the discontinuous method proposed by Baker [1] to piecewise linear approximations p_h of p. In this sense the method is related to that of Brezzi–Pitkäranta [8], where the corresponding stabilization term can be seen as an approximation of $h^2\Delta p$.

Remark 2.3. The term penalizing the incompressibility condition is necessary only in the case where $\nu < h$. This term is needed to give a $\|h^{\frac{1}{2}}\nabla\cdot u\|$ contribution to the triple norm necessary to obtain optimal order estimates in the case of Darcy flow. However it should be noted that for $u \in H^2(\Omega)$ we may use the same jump operator for the pressure and the velocity, hence stabilizing the jumps of the gradient (component wise for the velocities). This will not affect the order of the a priori estimates but gives increased control of the gradients and lower matrix bandwidth at the cost of larger constants in the error estimate.

Remark 2.4. The stabilized Galerkin/least-squares (GLS) method in different guises has been used extensively; for pioneering work in this direction, see, e.g., [8,14,18]. However, there is in GLS a decrease of accuracy close to the boundaries due to artificial pressure boundary conditions, for which a number of remedies have been proposed, cf. [3,12,15]. Following the edge stabilization method, there is less degradation of accuracy close to the boundary (see Figs. 1 and 2).

3. The inf–sup condition

For Stokes equations the essential feature of a stabilized method is the satisfaction of the *inf–sup* condition. We introduce the triple norm

$$|||(u_h,p_h)|||_s^2 := \sigma\|u_h\|^2 + \nu\|\nabla u_h\|^2 + c_d\|h^{\frac{s}{2}}\nabla\cdot u_h\|^2 + c_g\|h^{\frac{s}{2}}\nabla p_h\|^2 + c_p\|p_h\|^2,$$

where c_d, c_g, and c_p are constants, depending on the material data, which will be defined in the stability analysis below (cf. Remark 3.1). We also define the bilinear form

$$A[(u,p),(v,q)] := B[(u,p),(v,q)] + J[(u,p),(v,q)].$$

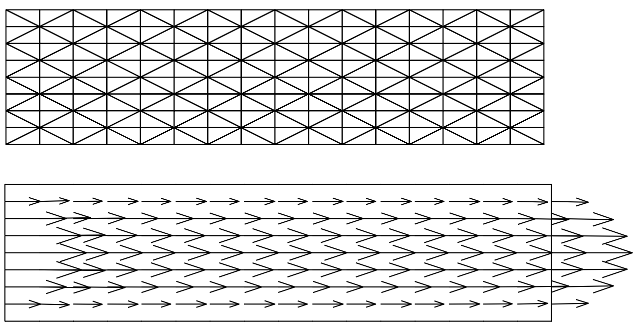

Fig. 1. Mesh and computed velocity for Poiseuille flow.

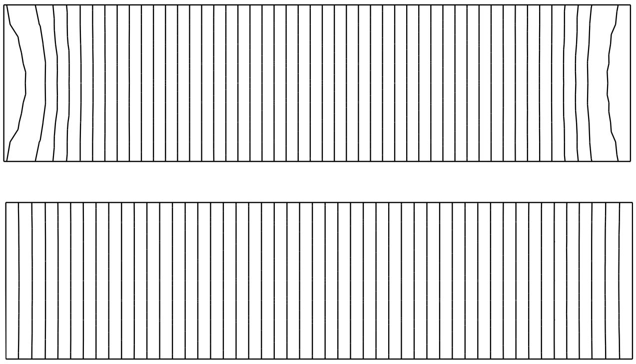

Fig. 2. Brezzi–Pitkäranta (top) and edge stabilization (bottom) pressure isolines.

The stability of the method is obtained by the fact that the edge operator controls the projection error of $h^s\nabla p_h$. This allows us to control $\|h^{\frac{s}{2}}\nabla p_h\|$, which in its turn leads to satisfaction of the inf–sup condition. We will for simplicity assume that h_K is uniform so that $\pi_h h^s \nabla p_h = h^s \pi_h \nabla p_h$, where π_h is an interpolation operator to be defined later, and that $h_K < 1$ for all K. By $\{\varphi_i\}$ we denote the set of finite element basis functions spanning the space V_h. Let $\mathscr{N}_i$ be the set of all triangles K^i containing node i and assume that the cardinality of $\mathscr{N}_i$ is bounded uniformly in i. Let $\mathscr{F}_K$ be the set of all test functions φ_i such that $K \subset \operatorname{supp}\varphi_i$ and $\Omega_i = \bigcup_{\mathscr{N}_i} K^i$. We will consider a function $y \in [P_0(K)]^d$, and its element wise representation in the finite element basis $\tilde{y}$ defined by

$$\tilde{y}|_K = y|_K \sum_{i \in \mathscr{F}_K} \varphi_i. \tag{6}$$

It follows that $\tilde{p} = p$ everywhere except on elements adjacent to Dirichlet boundaries where the boundary nodes are not included in the finite element space. We note that, with $y := \nabla p_h$, we wish to choose as our testfunction $v = h^s \pi_h y$ to obtain after an integration by parts

$$b(p_h, v) = \|h^{\frac{s}{2}} y\|^2 + (y, h^s(\pi_h y - y)) \tag{7}$$

and we wish to bound the projection error using the jump term. This cannot be done exactly since $\pi_h y$ must obey the boundary conditions, unlike y. However, (7) can equally well be written $(h^s y, \tilde{y}) + (h^s y, \pi_h y - \tilde{y})$, and if we can show that $c_b\|y\|^2 \leqslant (y, \tilde{y})$ we have

$$c_b\|h^{\frac{s}{2}} y\|^2 + (y, h^s(\pi_h y - \tilde{y})) \leqslant (y, h^s\tilde{y}) + (y, h^s(\pi_h y - \tilde{y}))$$

and we can proceed to bound the second term on the left-hand side in terms of the first together with the jumps. Thus, we need:

Lemma 1. *Suppose that K is an element with at least one node on a Dirichlet boundary then*

$$\|y\|_K^2 = \frac{d+1}{n_i}(y, \tilde{y}), \tag{8}$$

where n_i denotes the number of interior nodes of the element.

Proof. The proof is immediate noting that

$$(y, \tilde{y}) = |y_K|^2 \int_K \sum_{i \in \mathscr{F}_K} \varphi_i \, \mathrm{d}x = \frac{n_i}{d+1} |y_K|^2 m(K). \qquad \square$$

We will now recall some results from [9] essential for the analysis. The stability argument is based on the fact that the projection error of the gradient is controlled by the edge stabilization term

$$\|h^{\frac{s}{2}}(\tilde{y} - \pi_h y)\|^2 \leqslant \widetilde{J}_s(y, y)$$

with

$$\widetilde{J}_s(y, y) = \sum_K \int_{\partial K} \gamma h^{s+1} [y]^2 \, \mathrm{d}s.$$

The operator $\pi_h : \nabla Q_h \to V_h$, which denotes the lowest order Clément operator is constructed as follows:

$$\pi_h y = \sum_i y_i \varphi_i \tag{9}$$

with

$$y_i = \frac{1}{m(\Omega_i)} \sum_{\mathcal{N}_i} y|_{K^i} m(K^i). \tag{10}$$

We shall frequently use the following inequalities, which we collect in a lemma.

Lemma 2. *For the Clément operator there holds*

$$\|\pi_h u\|_{s,\Omega} \leqslant C_c \|u\|_{s,\Omega} \quad \forall u \in H^s(\Omega) \tag{11}$$

for $s = 0, 1$. *Further,*

$$\|h_K \nabla \pi_h p_h\| \leqslant C_i \|p_h\| \quad \forall p_h \in Q_h. \tag{12}$$

Finally, we have the trace inequality

$$\|v\|^2_{0,\partial K} \leqslant C_t \left(h_K^{-1} \|v\|^2_{0,K} + h_K \|v\|^2_{1,K} \right) \quad \forall v \in H^1(K). \tag{13}$$

Proof. Inequality (11) follows from the interpolation estimate

$$\|u - \pi_h u\|_{s,\Omega} \leqslant c_i \|u\|_{s,\Omega}, \quad s = 0, 1,$$

cf. [11], and (12) follows from (11) and the well known inverse inequality

$$\|v\|_{1,K} \leqslant C h_K^{-1} \|v\|_{0,K} \quad \forall v \in V_h. \tag{14}$$

Finally, a proof of (13) is given in [23]. □

In [9] we proved the following lemma.

Lemma 3. *If y is some piecewise constant function,* $\tilde{y}$ *is defined by* (6) *and* π_h *is the Clément interpolant on* V_h, *then the edge stabilization term satisfies*

$$\|h^{\frac{s}{2}}(\pi_h y - \tilde{y})\|^2 \leqslant \gamma \widetilde{J}_s(y, y) \tag{15}$$

for some $\gamma \geqslant \gamma_0 > 0$ *independent of h but not of the mesh regularity.*

Finally, we shall also need the following lemma.

Lemma 4. *For* $v \in V_h$, *the jump operator fulfills*

$$\tilde{j}(v, v)^{\frac{1}{2}} \leqslant C_s \gamma^{\frac{1}{2}} \|h^s \nabla \cdot v\|. \tag{16}$$

Proof. For each edge E shared by two elements K_1 and K_2, we have

$$\int_E \gamma h^{s+1}[\nabla \cdot v]^2 \, \mathrm{d}s \leqslant \sum_{i=1}^{2} \int_E \gamma h^{s+1}(\nabla \cdot v|_{K_i})^2 \, \mathrm{d}s \leqslant C\gamma \sum_{i=1}^{2} \|h^s \nabla \cdot v\|_{K_i}^2$$

by the trace inequality (13). Summing over all edges gives the result. □

Theorem 5. *Suppose that either $\sigma \geqslant 1$ or $\nu \geqslant 1$ and that $s = 1$ when $\nu < h$ and $s = 2$ when $\nu > h$ then the formulation (3) satisfies the inf–sup condition*

$$|||(u_h, p_h)||| \leqslant c_0 \sup_{(v,q)\in V_h\times Q_h} \frac{A[(u_h,p_h),(v,q)]}{|||(v,q)|||}$$

Proof. First we take $(v,q) = (u_h, p_h)$ to obtain

$$A[(u_h,p_h),(u_h,p_h)] = \sigma\|u_h\|^2 + \nu\|\nabla u_h\|^2 + \tilde{j}(u_h,u_h) + j(p_h,p_h). \tag{17}$$

By taking $(v,q) = (\pi_h(h^s\nabla p_h), 0)$ we obtain the desired control of $\|h^{\frac{s}{2}}\nabla p_h\|$ in the following fashion. We have

$$\begin{aligned} A[(u_h,p_h),(\pi_h(h_K^s\nabla p_h),0)] &= \sigma(u_h, \pi_h(h^s\nabla p_h)) + \nu(\nabla u_h, \nabla\pi_h(h^s\nabla p_h)) \\ &\quad - (p_h, \nabla\cdot\pi_h(h^s\nabla p_h)) + \tilde{j}(u_h, \pi_h(h^s\nabla p_h)). \end{aligned}$$

Estimating termwise, we have

$$(\sigma u_h, \pi_h h^s \nabla p_h) \geqslant -C_c \sigma^{\frac{1}{2}} h^{\frac{s}{2}} \|\sigma^{\frac{1}{2}} u_h\| \|h^{\frac{s}{2}} \nabla p_h\| \tag{18}$$

and

$$(\nu\nabla u_h, \nabla\pi_h h^s\nabla p_h) \geqslant -\|\nu^{\frac{1}{2}}\nabla u_h\|\|\nu^{\frac{1}{2}}\nabla(\pi_h h^s \nabla p_h)\| \geqslant -\|\nu^{\frac{1}{2}}\nabla u_h\| C_i C_c \nu^{\frac{1}{2}} \|h^{s-1}\nabla p_h\|.$$

Thus, for $s = 2$ we find

$$(\nu\nabla u_h, \nabla\pi_h h^s\nabla p_h) \geqslant -\|\nu^{\frac{1}{2}}\nabla u_h\| C_i C_c \nu^{\frac{1}{2}} \|h^{\frac{s}{2}}\nabla p_h\| \tag{19}$$

and for $s = 1$, i.e., the case $\nu \leqslant h$,

$$(\nu\nabla u_h, \nabla\pi_h h^s\nabla p_h) \geqslant -\|\nu^{\frac{1}{2}}\nabla u_h\| C_i C_c \|\nu^{\frac{1}{2}}\nabla p_h\| \geqslant -\|\nu^{\frac{1}{2}}\nabla u_h\| C_i C_c \|h^{\frac{s}{2}}\nabla p_h\|. \tag{20}$$

Further,

$$\begin{aligned} (p_h, \nabla\cdot\pi_h h^s\nabla p_h) &= (h^s\nabla p_h, \pi_h\nabla p_h - \widetilde{\nabla p_h}) + (h^s\nabla p_h, \widetilde{\nabla p_h}) \\ &\geqslant -\|h^{\frac{s}{2}}\nabla p_h\|\|h^{\frac{s}{2}}\pi_h\nabla p_h - \widetilde{\nabla p_h}\| + \frac{1}{d+1}\|h^{\frac{s}{2}}\nabla p_h\|^2 \\ &\geqslant \frac{3}{4(d+1)}\|h^{\frac{s}{2}}\nabla p_h\|^2 - (d+1)\|h^{\frac{s}{2}}(\nabla p - \widetilde{\nabla p_h})\|^2, \end{aligned} \tag{21}$$

where we used that $ab \leqslant \frac{a^2}{4} + b^2$ for real numbers a, b. Finally, using Lemmas 2 and 4,

$$\begin{aligned}\tilde{j}(u_h, \pi_h h^s \nabla p_h) &= \sum_K \frac{1}{2} \int_{\partial K} \gamma h^{s+1} [\nabla \cdot u_h] \cdot [\nabla \cdot (\pi_h h^s \nabla p_h)] \, \mathrm{d}s \\ &\leqslant \tilde{j}(u_h, u_h)^{\frac{1}{2}} \left(\sum_K \frac{1}{2} \int_{\partial K} \gamma h^{s+1} [\nabla \cdot (\pi_h h^s \nabla p_h)]^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \\ &\leqslant \tilde{j}(u_h, u_h)^{\frac{1}{2}} C_s \| \gamma^{\frac{1}{2}} h^s \nabla \cdot (\pi_h h^s \nabla p_h) \| \\ &\leqslant \tilde{j}(u_h, u_h)^{\frac{1}{2}} C_s C_i \| \gamma^{\frac{1}{2}} h^{s-1} \pi_h h^s \nabla p_h \| \\ &\leqslant \tilde{j}(u_h, u_h)^{\frac{1}{2}} C_s C_i C_c \gamma^{\frac{1}{2}} h^{\frac{(3s-2)}{2}} \| h^{\frac{s}{2}} \nabla p_h \|. \end{aligned} \tag{22}$$

Using (18)–(22) we deduce

$$\begin{aligned} A[(u_h, p_h), (\pi_h(h_K^s \nabla p_h), 0)] \geqslant\; & c_b \| h^{\frac{s}{2}} \nabla p_h \|^2 + (h^{\frac{s}{2}} \nabla p_h, h^{\frac{s}{2}} (\pi_h \nabla p_h - \widetilde{\nabla p_h})) \\ & - A[(u_h, 0), (u_h, 0)]^{\frac{1}{2}} \alpha_1 \| h^{\frac{s}{2}} \nabla p_h \|, \end{aligned} \tag{23}$$

where $\alpha_1 = \max(C_s \sigma^{\frac{1}{2}} h^{\frac{s}{2}}, C_i C_c \nu^{\frac{1}{2}}, C_i C_c, C_i C_s C_i C_c \gamma^{\frac{1}{2}} h^{\frac{(3s-2)}{2}})$. We conclude that by Lemma 3 we have

$$A[(u_h, p_h), (\pi_h(h^s \nabla p_h), 0)] \geqslant c_b \big(1 - \epsilon_1 - \alpha_1^2 \epsilon_1\big) \| h^{\frac{s}{2}} \nabla p_h \|^2 - \frac{1}{4 c_b \epsilon_1} A[(u_h, p_h), (u_h, p_h)].$$

We now choose $\epsilon_1 = \frac{1}{2(1+\alpha_1^2)}$ and multiply by $c_b \epsilon_1$ to obtain

$$\frac{c_b^2 \epsilon_1}{2} \| h^{\frac{s}{2}} \nabla p_h \|^2 - \frac{1}{4} A[(u_h, p_h), (u_h, p_h)] \leqslant A[(u_h, p_h), (c_b \epsilon_1 \pi_h(h^s \nabla p_h), 0)]. \tag{24}$$

By the surjectivity of the divergence operator (see [17]) there exists $v_p \in [H_0^1(\Omega)]^d$ such that $\nabla \cdot v_p = p_h$ and $\|v_p\|_{1,\Omega} \leqslant C \|p_h\|$. We now choose $(v, q) = (\pi_h v_p, 0)$ and use that $\|p_h\|^2 - (p_h, \nabla \cdot v_p) = 0$ by the properties of v_p. This gives

$$\begin{aligned} & \|p_h\|^2 - (p_h, \nabla \cdot v_p) + A[(u_h, p_h), (\pi_h v_p, 0)] \\ &\quad = \|p_h\|^2 + (p_h, \nabla \cdot (\pi_h v_p - v_p)) + \sigma(u_h, \pi_h v_p) + \nu(\nabla u_h, \nabla \pi_h v_p) + \tilde{j}(u_h, \pi_h v_p) \\ &\quad \geqslant (1 - \alpha_2^2 \epsilon_2) \|p_h\|^2 + (p_h, \nabla \cdot (\pi_h v_p - v_p)) - \frac{1}{4\epsilon_2} A[(u_h, 0), (u_h, 0)] \end{aligned} \tag{25}$$

with $\alpha_2 = \max(\sigma^{\frac{1}{2}} C_c C_f, \nu^{\frac{1}{2}} C_c C_f, C_s C_c C_f h^{\frac{s}{2}} \gamma^{\frac{1}{2}})$, where we used the stability of the Clément interpolation operator: $\|\nabla \cdot \pi_h v_p\| \leqslant C \|v_p\|_{1,\Omega} \leqslant C_f \|p_h\|$. We have also used the following lower bound on the stabilizing term

$$\begin{aligned} \tilde{j}(u_h, \pi_h v_p) &\geqslant -\frac{1}{\epsilon_2} \tilde{j}(u_h, u_h) - \epsilon_2 \tilde{j}(\pi_h v_p, \pi_h v_p) \\ &\geqslant -\frac{1}{\epsilon_2} \tilde{j}(u_h, u_h) - \epsilon_2 C_s^2 \gamma \| h^{\frac{s}{2}} \nabla \cdot \pi_h v_p \|^2 \\ &\geqslant -\frac{1}{\epsilon_2} \tilde{j}(u_h, u_h) - \epsilon_2 C_s^2 C_c^2 C_f^2 h^s \gamma \|p_h\|^2 \end{aligned}$$

obtained by the trace inequality (13) and the stability of the Clément operator. Focusing now on the second term on the right-hand side we obtain by partial integration and the properties of v_p.

$$(p_h, \nabla \cdot (\pi_h v_p - v_p)) \leqslant \frac{c_b \epsilon_1}{4 \epsilon_2} \| h^{\frac{s}{2}} \nabla p_h \|^2 + \frac{\epsilon_2 c_i}{c_b \epsilon_1} \| h^{\frac{(2-s)}{2}} p_h \|^2.$$

This leads to the following inequality for p_h

$$\left(1-\alpha_2^2\epsilon_2-\frac{\epsilon_2 c_i h^{(2-s)}}{c_b\epsilon_1}\right)\|p_h\|^2-\frac{c_b\epsilon_1}{4\epsilon_2}\|h^{\frac{s}{2}}\nabla p_h\|^2-\frac{1}{4\epsilon_2}A[(u_h,0),(u_h,0)]\leqslant A[(u_h,p_h),(\pi_h v_p,0)]. \tag{26}$$

Choosing now $\epsilon_2=\frac{c_b\epsilon_1}{2(c_b\epsilon_1\alpha_2^2+c_i h^{2-s})}$ and multiplying through by ϵ_2 we have

$$\frac{\epsilon_2}{2}\|p_h\|^2-\frac{c_b\epsilon_1}{4}\|h^{\frac{s}{2}}\nabla p_h\|^2-\frac{1}{4}A[(u_h,0),(u_h,0)]\leqslant A[(u_h,p_h),(\epsilon_2\pi_h v_p,0)]. \tag{27}$$

It remains to control $\|h^{\frac{s}{2}}\nabla\cdot u_h\|$. We choose $v=0$, $q=\pi_h h^s\nabla\cdot u_h$ to obtain

$$\|h^{\frac{s}{2}}\nabla\cdot u_h\|^2+\left(h^{\frac{s}{2}}\nabla\cdot u_h,h^{\frac{s}{2}}(\pi_h\nabla\cdot u_h-\nabla\cdot u_h)\right)+j(p_h,\pi_h h^s\nabla\cdot u_h)=A[(u_h,p_h),(0,\pi_h h^s\nabla\cdot u_h)].$$

Arguing as before we find that

$$\begin{aligned}&\frac{3}{4}\|h^{\frac{s}{2}}\nabla\cdot u_h\|^2-\|h^{\frac{s}{2}}(\pi_h\nabla\cdot u_h-\nabla\cdot u_h)\|^2-\frac{1}{\epsilon_3}j(p_h,p_h)-\epsilon_3 j(\pi_h h^s\nabla\cdot u_h,\pi_h h^s\nabla\cdot u_h)\\&\quad\leqslant A[(u_h,p_h),(0,\pi_h\nabla\cdot u_h)].\end{aligned}$$

Using now Lemma 4 followed by (12) we find

$$j(\pi_h h^s\nabla\cdot u_h,\pi_h h^s\nabla\cdot u_h)\leqslant C_s^2C_i^2\gamma\|h^{\frac{3s-2}{2}}\nabla\cdot u_h\|^2$$

and we have (since $h<1$)

$$\left(\frac{3}{4}-\epsilon_3C_s^2C_i^2\gamma\right)\|h^{\frac{s}{2}}\nabla\cdot u_h\|^2-\tilde{j}(u_h,u_h)-\frac{1}{\epsilon_3}j(p_h,p_h)\leqslant A[(u_h,p_h),(0,\pi_h h^s\nabla\cdot u_h)].$$

We fix $\epsilon_3=\frac{1}{4C_s^2C_i^2\gamma}$ and then multiply both sides with $\epsilon_4=(4\max(1,\frac{1}{\epsilon_3}))^{-1}$ resulting in

$$\frac{\epsilon_4}{2}\|h^{\frac{s}{2}}\nabla\cdot u_h\|^2-\frac{1}{4}A[(u_h,p_h),(u_h,p_h)]\leqslant A[(u_h,p_h),(0,\epsilon_4\pi_h h^s\nabla\cdot u_h)]. \tag{28}$$

Summing Eqs. (17), (24), (27) and (28) yields

$$\begin{aligned}&\frac{1}{4}A[(u_h,p_h),(u_h,p_h)]+\frac{\epsilon_4}{2}\|h^{\frac{s}{2}}\nabla\cdot u_h\|^2+\frac{c_b\epsilon_1}{2}\|h^{\frac{s}{2}}\nabla p_h\|^2+\frac{\epsilon_2}{2}\|p_h\|^2\\&\quad\leqslant A[(u_h,p_h),(u_h+c_b\epsilon_1\pi_h(h^s\nabla p_h)+\epsilon_2\pi_h v_p,p_h+\epsilon_4\pi_h h^s\nabla\cdot u_h)].\end{aligned}$$

Setting now, $c_d=2\epsilon_4$, $c_g=2c_b\epsilon_1$ and $c_p=2\epsilon_2$ we may write

$$\frac{1}{4}|||(u_h,p_h)|||^2\leqslant A[(u_h,p_h),(u_h+c_b\epsilon_1\pi_h(h^s\nabla p_h)+\epsilon_2\pi_h v_p,p_h+\epsilon_4\pi_h h^s\nabla\cdot u_h)].$$

The thesis follows by noting that there exists some constant c such that $|||(v,q)|||\leqslant c\,|||(u_h,p_h)|||$. By similar arguments as above there follows:

$$|||(c_b\epsilon_1\pi_h(h^s\nabla p_h),0)|||^2\leqslant C\|h^{\frac{s}{2}}\nabla p_h\|^2,$$
$$|||(\epsilon_2\pi_h v_p,0)|||^2\leqslant C\|p_h\|^2$$

and

$$|||(0,\epsilon_4\pi_h h^s\nabla\cdot u_h)|||^2\leqslant C\|h^{\frac{s}{2}}\nabla\cdot u_h\|^2,$$

where the constants C depend on material data but not on h. The constant c is in fact of order unity under the condition $\sigma\geqslant 1$ or $\nu\geqslant 1$. $\square$

Remark 3.1. The essential dependencies of the constants c_g, c_d and c_p are

$$c_g, c_p \approx \mathrm{O}\left(\frac{1}{\max(\sigma, \nu)}\right),$$
$$c_d \approx \mathrm{O}(1).$$

4. A priori error estimates

A priori estimates are obtained in the standard fashion using

- stability,
- consistency,
- approximation.

The first point was handled in the previous section and we will now take care of the other two, before proving our error estimates.

By definition of our method, we have the consistency condition.

Lemma 6. *For $(u,p) \in [H^2(\Omega)]^{d+1}$ there holds*

$$A[(u - u_h, p - p_h), (v, q)] = 0$$

for all $(v,q) \in V^h \times Q^h$.

In addition we have the following approximation property.

Lemma 7. *Let $(u,p) \in [H^2(\Omega)]^3 \times H^2(\Omega)$. Then we have*

$$|||(u - \pi_h u, p - \pi_h p)||| \leqslant Ch\left((h^{\frac{s}{2}}(c_d + \gamma^{\frac{1}{2}}) + \sigma^{\frac{1}{2}}h + \nu^{\frac{1}{2}})\|u\|_{2,\Omega} + h^{\frac{s}{2}} \max(c_g^{\frac{1}{2}}, c_p^{\frac{1}{2}}, \gamma^{\frac{1}{2}})\|p\|_{2,\Omega}\right).$$

Proof. Using standard interpolation we obtain for the velocities

$$\|u - \pi_h u\|_{0,\Omega} \leqslant Ch^2\|u\|_{2,\Omega},$$
$$\|\nabla(u - \pi_h u)\|_{0,\Omega} \leqslant Ch\|u\|_{2,\Omega},$$
$$\|h^{\frac{s}{2}}\nabla \cdot (u - \pi_h u)\|_{0,\Omega} \leqslant Ch^{1+\frac{s}{2}}\|u\|_{2,\Omega}$$

and equivalently for the pressure

$$\|p - \pi_h p\|_{0,\Omega} \leqslant Ch^2\|p\|_{2,\Omega},$$
$$\|h^{\frac{s}{2}}\nabla(p - \pi_h p)\|_{0,\Omega} \leqslant Ch^{1+\frac{s}{2}}\|p\|_{2,\Omega}.$$

Further, we have, using (13),

$$\|n \cdot \nabla(p - \pi_h p)\|^2_{0,\partial K} \leqslant C\left(h_K^{-1}\|\nabla(p - \pi_h p)\|^2_{0,K} + \|\nabla(p - \pi_h p)\|_{0,K}\|\nabla(p - \pi_h p)\|_{1,K}\right) \leqslant Ch_K\|p\|^2_{2,K}$$

and it follows by summation that $j(p - \pi_h p, p - \pi_h p)^{\frac{1}{2}} \leqslant Ch^{1+\frac{s}{2}}\|p\|_{2,\Omega}$. In the same fashion clearly $\tilde{j}(u - \pi_h u, u - \pi_h u)^{\frac{1}{2}} \leqslant Ch^{1+\frac{s}{2}}\|u\|_{2,\Omega}$. □

Theorem 8. *If $u \in [H^2(\Omega)]^d$ and $p \in H^2(\Omega)$ then the solution (u_h, p_h) to (3) satisfies*

$$|||(u - u_h, p - p_h)||| \leqslant Ch\left(\max(c_d + \gamma^{\frac{1}{2}}, c_g^{-\frac{1}{2}})H_s + \sigma^{\frac{1}{2}}h + \nu^{\frac{1}{2}}\|u\|_{2,\Omega} + \max(c_d^{-\frac{1}{2}}, c_g^{\frac{1}{2}}, c_p^{\frac{1}{2}}, \gamma^{\frac{1}{2}})H_s\|p\|_{2,\Omega}\right)$$

with $H_s = \max(h^{\frac{s}{2}}, h^{\frac{2-s}{2}})$.

Proof. First of all we note that $|||(u-u_h,p-p_h)||| \leqslant |||(u-\pi_h u,p-\pi_h p)||| + |||\ (\pi_h u-u_h,\pi_h p-p_h)|||$. By Theorem 5 we have

$$|||(\pi_h u-u_h,\pi_h p-p_h)||| \leqslant c_0 \sup_{(v,q)\in V_h\times Q_h} \frac{A[(\pi_h u-u_h,\pi_h p-p_h),(v,q)]}{|||(v,q)|||}$$

and, by Lemma 6,

$$|||(\pi_h u-u_h,\pi_h p-p_h)||| \leqslant \sup_{(v,q)\in V_h\times Q_h} \frac{A[(\pi_h u-u,\pi_h p-p),(v,q)]}{|||(v,q)|||}. \tag{29}$$

Writing out the terms in $A[(\pi_h u-u,\pi_h p-p),(v,q)]$ we obtain

$$\begin{aligned} A[(\pi_h u-u,\pi_h p-p),(v,q)] &= a(\pi_h u-u,v)+b(\pi_h p-p,v)-b(q,\pi_h u-u)\\ &\quad + j(\pi_h p-p,q)+\tilde{j}(\pi_h u-u,v) = \mathrm{i}+\mathrm{ii}+\mathrm{iii}+\mathrm{iv}+\mathrm{v}. \end{aligned}$$

We bound the five terms as follows:

$$\begin{aligned} &\mathrm{i} \leqslant |||(u-\pi_h u,0)|||\cdot|||(v,0)|||,\\ &\mathrm{ii} \leqslant \|h^{-\frac{s}{2}}(\pi_h p-p)\|\|h^{\frac{s}{2}}\nabla\cdot v\| \leqslant Ch^{\frac{4-s}{2}}c_d^{-\frac{1}{2}}\|p\|_{2,\Omega}|||(v,0)||| \end{aligned}$$

and using integration by parts

$$\begin{aligned} &\mathrm{iii} = (h^{\frac{s}{2}}\nabla q, h^{-\frac{s}{2}}(u-\pi_h u)) \leqslant Ch^{\frac{4-s}{2}}c_g^{-\frac{1}{2}}\|u\|_{2,\Omega}|||(0,q)|||,\\ &\mathrm{iv} \leqslant |||(0,p-\pi_h p)|||\cdot|||(0,q)||| \end{aligned}$$

and

$$\mathrm{v} \leqslant |||(u-\pi_h u,0)|||\cdot|||(v,0)|||.$$

The theorem follows by Lemma 7. □

Remark 4.1. Observe that the stabilizing terms $j(p_h,p_h)$ and $\tilde{j}(u_h,u_h)$ may be included in the triple norm. This yields the following convergences of the jump terms

$$(j(p_h,p_h)+j(u_h,u_h))^{\frac{1}{2}} \leqslant Ch\Big((H_s+\sigma^{\frac{1}{2}}h+\nu^{\frac{1}{2}})\|u\|_{2,\Omega}+H_s\|p\|_{2,\Omega}\Big).$$

Let us comment briefly on the dependence on the constants in the above estimate. The important point to notice is that there is no factor ν^{-1} or σ^{-1} in the estimate. This is what allows us to treat all viscous regimes. The main dependence are on $\max(\sigma^{\frac{1}{2}},\nu^{\frac{1}{2}})$. It is worthwhile to notice that when the viscosity becomes small the optimal choice is $s=1$ giving $\mathrm{O}(h^{\frac{3}{2}})$ convergence of the error in the L^2-norm for both the velocities and the pressure and $\mathrm{O}(h)$ convergence of the pressure in the H^1-norm and of the velocities in the H_{div}-norm.

4.1. The Stokes problem

For the Stokes system it is unnatural to assume that p belongs to $H^2(\Omega)$. Thus, below we prove that some classical finite element results for the Stokes problem hold also for our method, namely,

- convergence in the triple norm, assuming only $p\in H^1(\Omega)$,
- optimal convergence in the L^2-norm for the velocities.

Below we always take $s=2$ and we omit the jump term $\tilde{j}(u_h,v)$ stabilizing the incompressibility condition.

Corollary 4.2. *If the solution to the continuous problem has the regularity $u \in [H^2(\Omega)]^d$ and $p \in H^1(\Omega)$ then the solution (u_h, p_h) to (3) satisfies*

$$|||(u - u_h, p - p_h)||| \leqslant Ch\Big(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}\Big). \tag{30}$$

Proof. The proof is very similar to the proof of Theorem 8, but has to be modified to account for the loss of Galerkin orthogonality. Eq. (29) now becomes

$$|||(\pi_h u - u_h, \pi_h p - p_h)||| \leqslant \sup_{(v,q)\in V_h\times Q_h} \frac{B[(\pi_h u - u, \pi_h p - p), (v, q)] + j(\pi_h p, q)}{|||(v, q)|||}. \tag{31}$$

We have that

$$b(\pi_h p - p, v) \leqslant \|\pi_h p - p\|\|v\|_{1,\Omega} \leqslant \|\pi_h p - p\||||(v, 0)||| \leqslant h\|p\|_{1,\Omega}|||(v, 0)|||$$

and

$$b(q, \pi_h u - u) \leqslant |||(0, q)|||\|\nabla \cdot (\pi_h u - u)\| \leqslant |||(0, q)|||h\|u\|_{2,\Omega}.$$

We end the proof by noting that

$$j(\pi_h p, q) \leqslant Cj(\pi_h p, \pi_h p)^{\frac{1}{2}}|||(0, q)||| \leqslant C\sum_K \int_{\partial K} \gamma h^3 [\nabla \pi_h p]^2 \,\mathrm{d}s|||(0, q)|||$$

and that, by the trace inequality (13) and the stability of the Clément interpolant

$$\sum_K \int_{\partial K} h^3 [\nabla \pi_h p]^2 \,\mathrm{d}s \leqslant C\|h^2 \nabla \pi_h p\|^2 \leqslant C\|h^2 \nabla p\|^2. \qquad \square \tag{32}$$

Corollary 4.3. *If $u \in [H^2(\Omega)]^d$ and $p \in H^1(\Omega)$ then the solution (u_h, p_h) to (3) satisfies*

$$j(p_h, p_h) \leqslant c|||(\pi_h u - u_h, \pi_h p - p_h)|||^2 + Ch^2\|p\|^2_{1,\Omega}.$$

Proof. The proof is immediate noting that

$$j(p_h, p_h) = j(p_h - \pi_h p + \pi_h p, p_h - \pi_h p + \pi_h p) \leqslant j(p_h - \pi_h p, p_h - \pi_h p) + j(\pi_h p, \pi_h p),$$

where we now apply the trace inequality and an inverse inequality in the first term to obtain

$$j(p_h - \pi_h p, p_h - \pi_h p) \leqslant c|||(\pi_h u - u_h, \pi_h p - p_h)|||^2$$

and we conclude using Eq. (32). $\square$

We now proceed to prove an L^2-error estimates for the velocities in the case of the Stokes equations. We introduce the following dual problem, find $(\varphi, r) \in V \times Q$ such that

$$a(v, \varphi) + b(q, \varphi) - b(r, v) = (\eta, v)_\Omega \quad \forall (v, q) \in V \times Q \tag{33}$$

and assume that the solution enjoys the additional regularity

$$\|\varphi\|^2_{2,\Omega} + \|r\|^2_{1,\Omega} \leqslant C\|\eta\|^2 \tag{34}$$

valid if the boundary is sufficiently smooth, cf. [17]. We now prove the L_2-error estimate in the case when $\nu > h$.

Theorem 9. *If $u \in [H^2(\Omega)]^d$ and $p \in H^1(\Omega)$ is the solution to the Stokes problem and (u_h, p_h) the solution to (3), then we have*

$$\|u - u_h\| \leqslant Ch^2(\|u\|_{2,\Omega} + \|p\|_{1,\Omega}).$$

Proof. Choosing $\eta = v = u - u_h$, $q = 0$ in (33) gives

$$\|\eta\|^2 = a(\eta, \varphi) - b(r, \eta)$$

and by Galerkin orthogonality, setting $\zeta = p - p_h$,

$$\|\eta\|^2 = a(\eta, \varphi - \pi_h\varphi) + b(r - \pi_h r, \eta) - b(\zeta, \varphi - \pi_h\varphi) + j(p_h, \pi_h r) = \mathrm{i} + \mathrm{ii} + \mathrm{iii} + \mathrm{iv}.$$

The terms are bounded in the following fashion:

$$\begin{aligned} &\mathrm{i} \leqslant |||(\eta, 0)||| \, |||(\varphi - \pi_h\varphi, 0)||| \leqslant Ch^2\|\varphi\|_{2,\Omega}, \\ &\mathrm{ii} \leqslant |||(\eta, 0)||| \, \|r - \pi_h r\| \leqslant Ch^2\|r\|_{1,\Omega}, \\ &\mathrm{iii} \leqslant |||(0, \zeta)||| \, \|\nabla \cdot (\varphi - \pi_h\varphi)\| \leqslant Ch^2\|\varphi\|_{2,\Omega} \end{aligned}$$

and finally we bound the residual part using Corollary 4.3 and Eq. (32)

$$j(p_h, \pi_h r) \leqslant j(p_h, p_h)^{\frac{1}{2}} j(\pi_h r, \pi_h r)^{\frac{1}{2}} \leqslant Ch^2\|r\|_{1,\Omega}.$$

We conclude using the regularity assumption on the dual problem (34). □

5. Numerical examples

In this section we will show the performance of our method on two academic examples with known solution. Since we are dealing with the generalized Stokes' problem we consider the classical Stokes' equations on the one hand with $\nu = 1$ and $\sigma = 0$ and, on the other hand, the Darcy's equations, with $\nu = 0$ and $\sigma = 1$.

5.1. Stokes' problem

We consider the unit square with exact flow solution (from [21]) given by $u = (20xy^3, 5x^4 - 5y^4)$ and $p = 60x^2y - 20y^3 + C$. Imposing zero mean pressure ($C = -5$), we obtain the convergence shown in Fig. 3; second order for the velocity and the pressure in L_2-norm.

Pressure isolines and velocity vectors on the final mesh in the sequence used to obtain the convergence plot are shown in Fig. 4.

5.2. Darcy's problem

The second numerical example, taken from [20], is a study of convergence rates for Darcy flow. The domain under consideration is the unit square with a given exact pressure solution $p = \sin 2\pi x \sin 2\pi y$. The exact velocity field is then computed from Darcy's law to give boundary conditions and a source term for the divergence. In order to create a unique pressure field we also impose zero mean pressure.

In Fig. 5, we show the approximate velocities and pressures on the final mesh in a sequence. In Fig. 6, we show the convergence of the method in the L_2-norm, which yields second order accuracy for the velocities and the pressure.

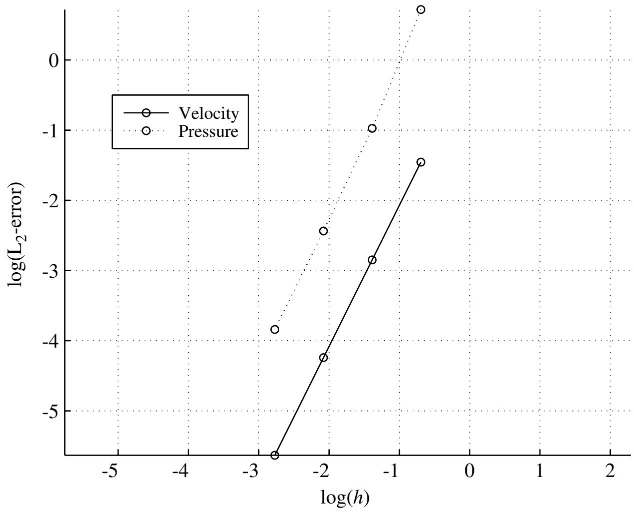

Fig. 3. L_2-norm convergence of the velocity and of the pressure for Stokes.

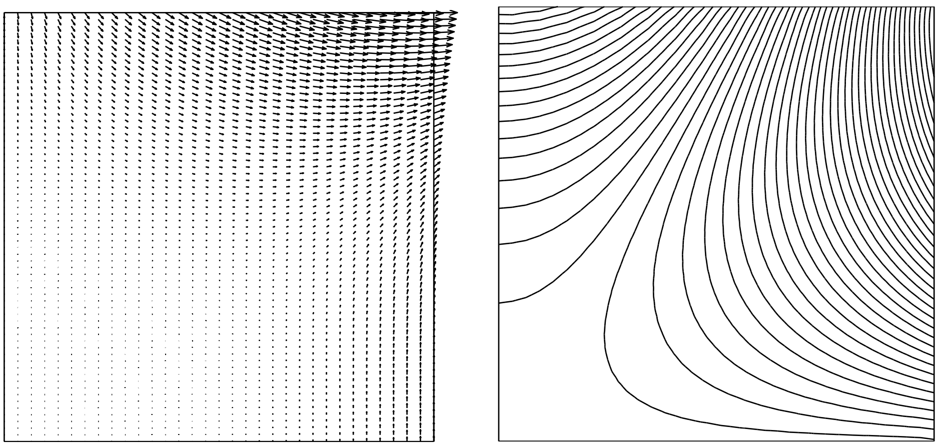

Fig. 4. Approximate velocity field and pressure on the final mesh in a sequence.

Numerical experimentation indicates that even on this simple example another choice than $s = 1$ will give poorer convergence properties. In particular if the stabilization of the incompressibility condition is left out the convergence of the error in the velocities is of order $h^{\frac{3}{2}}$. If, on the other hand, this term becomes too dominant ($s = 0$) the convergence of the error in the pressure is of order $h^{\frac{3}{2}}$.

5.3. A comparison with the Brezzi–Pitkäranta/Dohrmann–Bochev method

For the Stokes problem, Brezzi and Pitkäranta [8] suggested modifying a P1–P1 approximation by adding a discrete diffusion term to the divergence zero condition so that

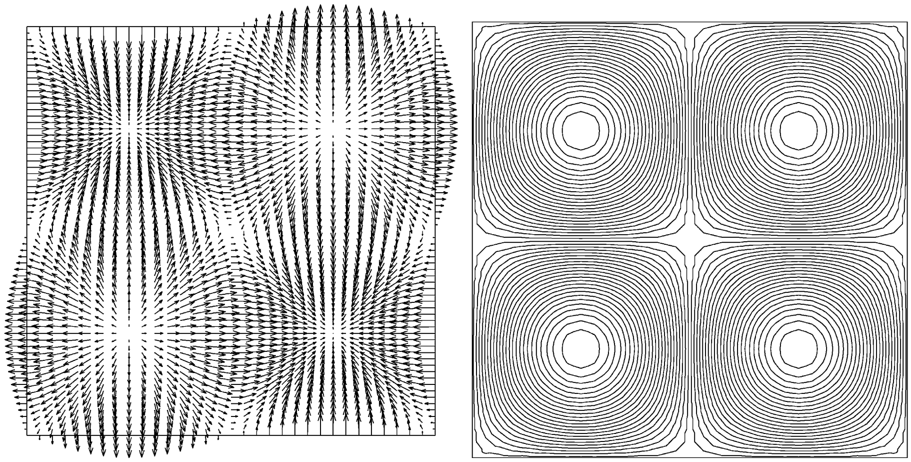

Fig. 5. Approximate velocity field and pressure on the final mesh in a sequence.

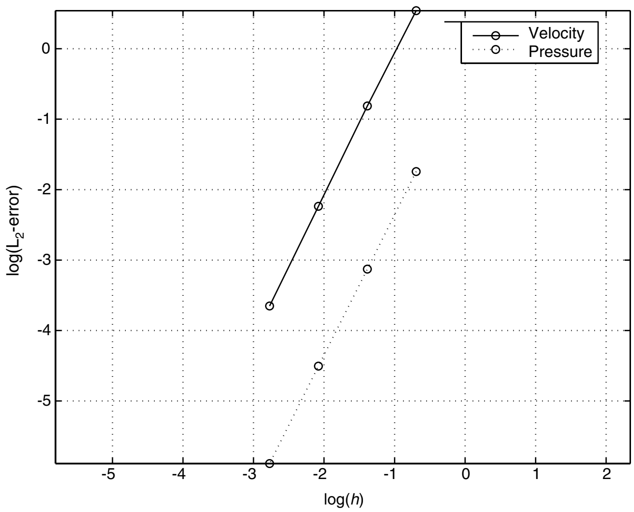

Fig. 6. L_2-norm convergence of the velocity and of the pressure for Darcy.

$$\int_\Omega \nabla \cdot u^h q \,\mathrm{d}x + C \int_\Omega h_K^2 \nabla p^h \cdot \nabla q \,\mathrm{d}x = 0 \quad \forall q \in Q^h \tag{35}$$

with C a constant of O(1). The method of Dohrmann and Bochev [13] can be seen as a generalization of this approach. Here, the same equation is, in the P1–P1 case, instead modified to

$$\int_\Omega \nabla \cdot u^h q \,\mathrm{d}x + \frac{1}{v} \int_\Omega (p^h - \pi_0 p^h)(q - \pi_0 q) \,\mathrm{d}x = 0 \quad \forall q \in Q^h, \tag{36}$$

where π_0 is the projection onto piecewise constants. By the Bramble–Hilbert lemma, this is essentially the same as (35). The optimal choice of the stabilizing term must involve deciding a multiplicative constant, as in the Brezzi–Pitkäranta scheme. In [13] this constant is somewhat arbitrarily set to $1/v$, as in (36). By using a projection onto a polynomial space one order lower than that used for the approximation of the pressure, the method generalizes to arbitrary degree polynomials.

Both the Brezzi–Pitkäranta method and the Dohrmann–Bochev method suffer from spurious boundary layers in the pressure (which can be interpreted as suppressed Neumann boundary conditions). In Fig. 7 we show the convergence in $L_2(\partial\Omega)$ of the different methods applied to the problem of Section 5.1, using $C = 1$

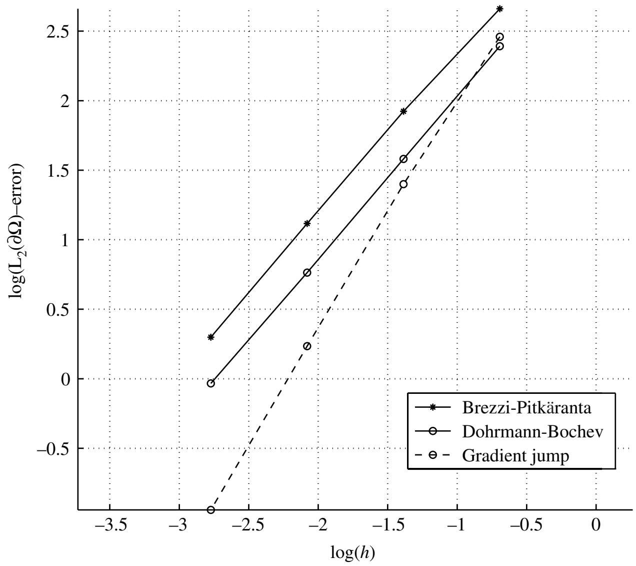

Fig. 7. Comparison of the $L_2(\partial\Omega)$-norm convergence of the pressure for Stokes.

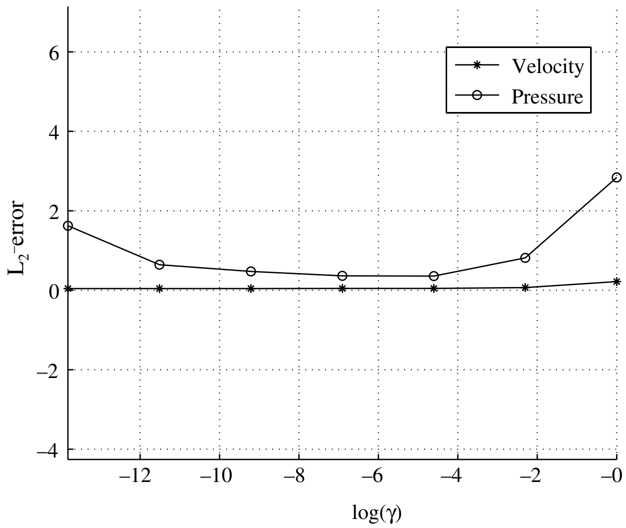

Fig. 8. Effect of the choice of γ upon the L_2-error on a fixed mesh.

in the Brezzi–Pitkäranta scheme and $\gamma = 1/100$ in the edge stabilization scheme. We note that the edge stabilization method gives half a power of h better convergence than the other two.

5.4. Influence of the size of the stabilization parameter

Finally, in Fig. 8, we give an example showing the influence of the choice of γ upon the convergence for the Stokes problem of Section 5.1. Note that γ can be chosen very small compared to the choice in a Brezzi–Pitkäranta type scheme. Note also that the method is robust in that it will give small error for a wide range of γ's. In particular, the velocity is unaffected by γ.

6. Concluding remarks

We have suggested the use of derivative jump stabilization for P1–P1 approximations of the generalized Stokes problem. However work in progress shows that the extension to finite elements of any polynomial order is feasible. We show optimal convergence for the pure Stokes case and near optimal (with a loss of one half power of h) in the case of the pure Darcy problem.

Our method has some decisive benefits: mass lumping is possible (unlike in the case of SUPG-type schemes) which is useful for extensions involving time stepping and stiff source terms. No additional unknowns are added, no special structure on the mesh is assumed. Finally, numerical evidence shows that no boundary layers appear in the pressures.

References

[1] G.A. Baker, Finite element methods for elliptic equations using nonconforming elements, Math. Comp. 31 (1977) 45–59.
[2] R. Becker, M. Braack, A finite element pressure gradient stabilization for the Stokes equation based on local projection, Calcolo 38 (2001) 173–199.
[3] H. Blank, M. Rudgyard, A. Wathen, Stabilised finite element methods for steady incompressible flow, Comput. Methods Appl. Mech. Engrg. 174 (1999) 91–105.
[4] F. Brezzi, M. Fortin, A minimal stabilisation procedure for mixed finite element methods, Numer. Math. 89 (2001) 457–491.
[5] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer, Berlin, 1991.
[6] F. Brezzi, T.J.R. Hughes, L.D. Marini, A. Russo, E.A. Süli, A priori error analysis of residual-free bubbles for advection–diffusion problems, SIAM J. Numer. Anal. 36 (1999) 1933–1948.
[7] E. Burman, P. Hansbo, Stabilized Crouzeix–Raviart element for the Darcy–Stokes Problem, PREPRINT Chalmers Finite Element Center, 2003-15.
[8] F. Brezzi, J. Pitkäranta, On the stabilization of finite element approximations of the Stokes equations, in: W. Hackbusch (Ed.), Efficient Solution of Elliptic Systems, Vieweg, 1984.
[9] E. Burman, P. Hansbo, Edge stabilization for Galerkin approximations of convection–diffusion problems, Comput. Methods Appl. Mech. Engrg. 193 (2004) 1437–1453.
[10] E. Burman, M. Fernàndez, P. Hansbo, Edge stabilization for the incompressible Navier–Stokes equations: a continuous interior penalty method, PREPRINT EPFL 23.2004.
[11] P. Clément, Approximation by finite element functions using local regularization, RAIRO Anal. Numer. 9 R-2 (1975) 77–84.
[12] R. Codina, J. Blasco, Analysis of a pressure-stabilized finite element approximation of the stationary Navier–Stokes equations, Numer. Math. 8 (2000) 59–81.
[13] C. Dohrmann, P. Bochev, A stabilized finite element method for the Stokes problem based on polynomial pressure projections, Int. J. Numer. Methods Fluids 46 (2004) 183–201.
[14] J. Douglas, J.P. Wang, An absolutely stabilized finite-element method for the Stokes problem, Math. Comp. 52 (1989) 495–508.
[15] J.J. Droux, T.J.R. Hughes, A boundary integral modification of the Galerkin least-squares formulation for the Stokes problem, Comput. Methods Appl. Mech. Engrg. 113 (1994) 173–182.
[16] L.P. Franca, A. Russo, Approximation of the Stokes problem by residual-free macro bubbles, East-West J. Numer. Math. 4 (4) (1996) 265–278.

[17] V. Girault, P.-A. Raviart, Finite Element Methods for Navier–Stokes Equations: Theory and Algorithms, Springer-Verlag, Berlin, 1986.

[18] T.J.R. Hughes, L.P. Franca, M. Balestra, A new finite element formulation for computational fluid dynamics. V. Circumventing the Babuška–Brezzi condition: a stable Petrov–Galerkin formulation of the Stokes problem accomodating equal-order interpolations, Comput. Methods Appl. Mech. Engrg. 59 (1986) 85–99.

[19] K.A. Mardal, X.C. Tai, R. Winther, A robust finite element method for Darcy–Stokes flow, SIAM J. Numer. Anal. 40 (2002) 1605–1631.

[20] A. Masud, T.J.R. Hughes, A stabilized mixed finite element method for Darcy flow, Comput. Methods Appl. Mech. Engrg. 191 (2002) 4341–4370.

[21] S. Norburn, D. Silvester, Stabilised vs stable mixed methods for incompressible flow, Comput. Methods Appl. Mech. Engrg. 166 (1998) 131–141.

[22] D. Silvester, Optimal low order finite element methods for incompressible flow, Comput. Methods Appl. Mech. Engrg. 111 (1994) 357–368.

[23] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems, Springer-Verlag, Berlin, 1984.