Fundamentals of Electric Circuits 2020.10

Chapter 18 Two-Port Networks

Chapter 18 Two Port Networks

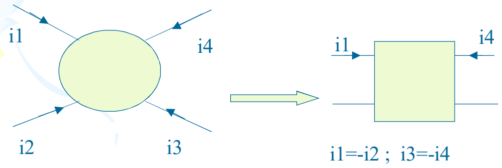
18.1 Introduction

What is a port?

It is a pair of terminals through which a *current* may enter or leave a network.

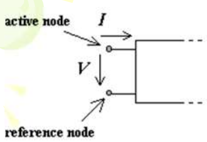
four-terminal network → the four terminals have been paired into ports (KCL) → two-port network

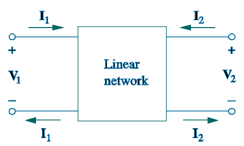
At all times, the instantaneous current flowing into one terminal is equal to the instantaneous current flowing out the other.



i1+i2+i3+i4=0(KCL)

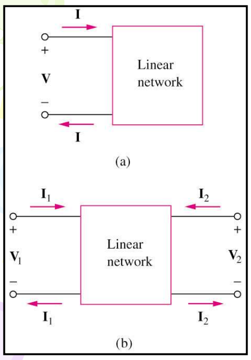
i1=-i2 ; i3=-i4





$V_1 = z_{11}I_1 + z_{12}I_2$ $V_2 = z_{21}I_1 + z_{22}I_2$	$V_1 = AV_2 - BI_2$ $I_1 = CV_2 - DI_2$	$V_1 = h_{11}I_1 + h_{12}V_2$ $I_2 = h_{21}I_1 + h_{22}V_2$
$I_1 = y_{11}V_1 + y_{12}V_2$ $I_2 = y_{21}V_1 + y_{22}V_2$	$V_2 = aV_1 - bI_1$ $I_2 = cV_1 - dI_1$	$I_1 = g_{11}V_1 + g_{12}I_2$ $V_2 = g_{21}V_1 + g_{22}I_2$

The network is linear(without independent sources).

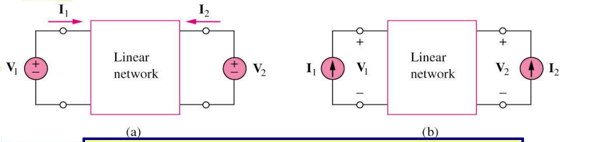


One port or two terminal circuit

Two port or four terminal circuit

- **It is an electrical network with two separate ports for input and output.**
- **No independent sources.**

18.2 Impedance parameters (1)



Assume no independent source in the network

$$V_1 = z_{11}I_1 + z_{12}I_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [z]\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

where the z terms are called the ***impedance parameters***, or simply z parameters, and have units of ohms.

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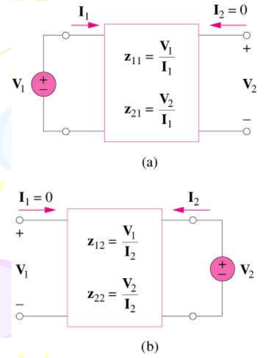
18.2 Impedance parameters (2a)



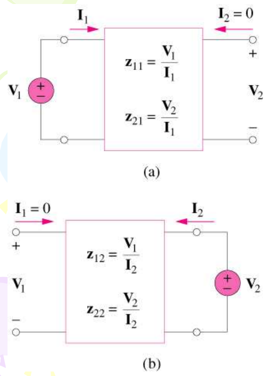
$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \quad \text{and} \quad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

z_{11} = Open-circuit input impedance
z_{21} = Open-circuit transfer impedance from port 1 to port 2

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \quad \text{and} \quad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

z_{12} = Open-circuit transfer impedance from port 2 to port 1
z_{22} = Open-circuit output impedance

18.2 Impedance parameters (2b)



$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \quad \text{and} \quad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

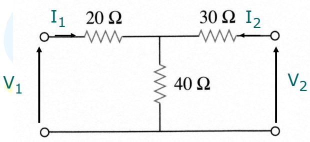
$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \quad \text{and} \quad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

- When $z_{11} = z_{22}$, the two-port network is said to be **symmetrical**.
- When the two-port network is **linear** and has **no dependent sources**, the transfer impedances are equal ($z_{12} = z_{21}$), and the two-port is said to be **reciprocal**.

18.2 Impedance parameters (3)

Example 1

Determine the Z-parameters of the following circuit.



$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \quad \text{and} \quad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \quad \text{and} \quad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

$$z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}\Omega$$

Answer: $z = \begin{bmatrix} 60 & 40 \\ 40 & 70 \end{bmatrix}\Omega$

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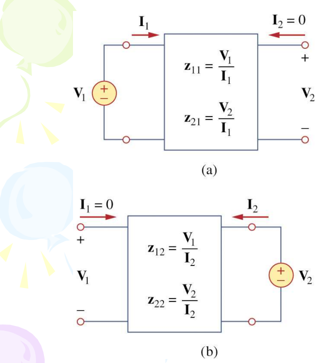

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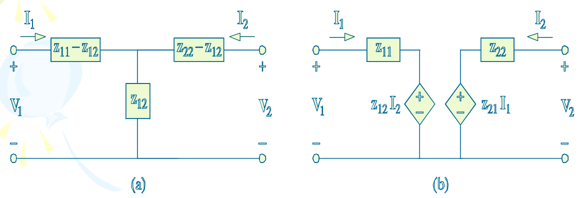
(a) T equivalent circuit (for reciprocal case only), (b) general equivalent circuit



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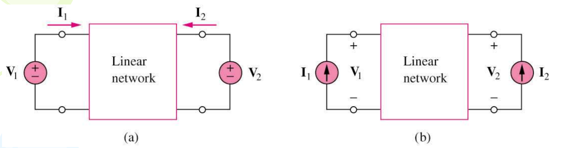
18.3 Admittance parameters (1)



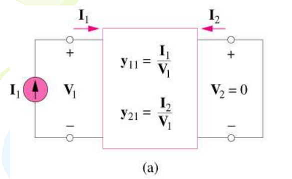
Assume no independent source in the network

$$I_1 = y_{11}V_1 + y_{12}V_2$$
$$I_2 = y_{21}V_1 + y_{22}V_2$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = [y] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where the **y** terms are called the ***admittance parameters***, or simply y parameters, and they have units of **Siemens**.

18.3 Admittance parameters (2)



$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \quad \text{and} \quad y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

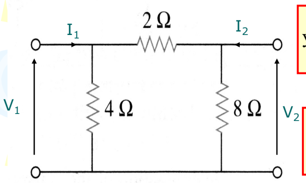
y_{11} = Short-circuit input admittance
y_{21} = Short-circuit transfer admittance from port 1 to port 2

$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} \quad \text{and} \quad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

y_{12} = Short-circuit transfer admittance from port 2 to port 1
y_{22} = Short-circuit output admittance

Example 2

Determine the y-parameters of the following circuit.



$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \quad \text{and} \quad y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

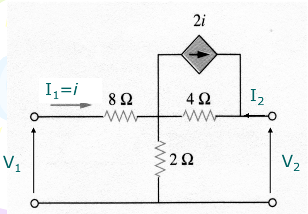
$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} \quad \text{and} \quad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

$$y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} S$$

Answer: $y = \begin{bmatrix} 0.75 & -0.5 \\ -0.5 & 0.625 \end{bmatrix} S$

Example 3

Determine the y-parameters of the following circuit.



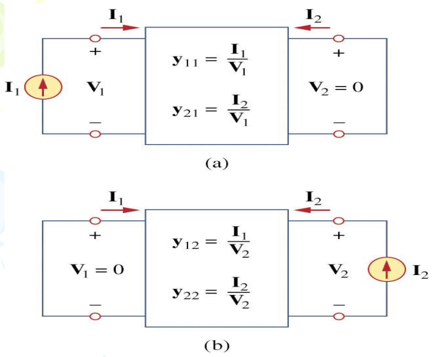
$$I_1 = y_{11}V_1 + y_{12}V_2$$
$$I_2 = y_{21}V_1 + y_{22}V_2$$

Apply KVL

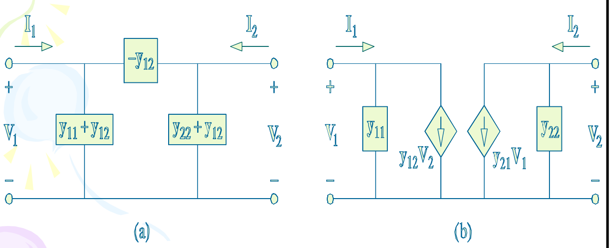
$$V_1 = 8I_1 + 2(I_1 + I_2)$$
$$V_2 = 4(2i + I_2) + 2(I_1 + I_2)$$

$$I_1 = 0.15V_1 - 0.05V_2$$
$$I_2 = -0.25V_1 + 0.25V_2$$

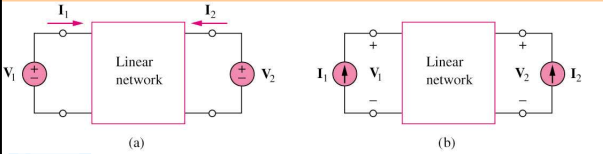
Answer: $y = \begin{bmatrix} 0.15 & -0.05 \\ -0.25 & 0.25 \end{bmatrix} S$



(a) π-equivalent circuit (for reciprocal case only), (b) general equivalent circuit.



18.4 Hybrid parameters (1)



Assume no independent source in the network

$$V_1 = h_{11}I_1 + h_{12}V_2 \qquad I_2 = h_{21}I_1 + h_{22}V_2$$

$$\Rightarrow \begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = [h] \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

where the **h** terms are called the ***hybrid parameters***, or simply h parameters, and each parameter has **different units**, refer above.

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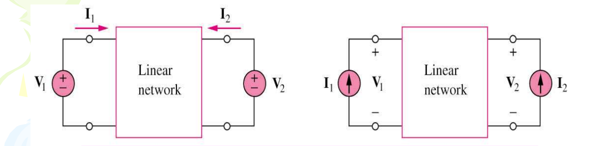
18.4 Hybrid parameters (2)



Assume no independent source in the network

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} \qquad h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

h_{11} = short-circuit input impedance (Ω)

H_2 = short-circuit forward current gain

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} \qquad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

H_{12} = open-circuit reverse voltage-gain

H_{22} = open-circuit output admittance (S)

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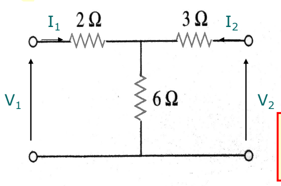
Example 4

Determine the h-parameters of the following circuit.



$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} \quad \text{and} \quad h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} \quad \text{and} \quad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

$$h = \begin{bmatrix} h_{11}\,\Omega & h_{12} \\ h_{21} & h_{22}\,S \end{bmatrix}$$

Answer: $h = \begin{bmatrix} 4\Omega & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{9}S \end{bmatrix}$

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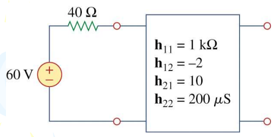
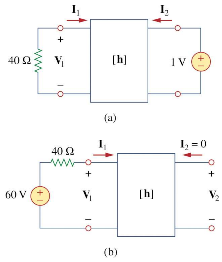
Example 5

Determine the Thevenin equivalent at the output port of the circuit shown in the following Fig.





$$V_1 = h_{11}I_1 + h_{12}V_2 \qquad I_2 = h_{21}I_1 + h_{22}V_2$$

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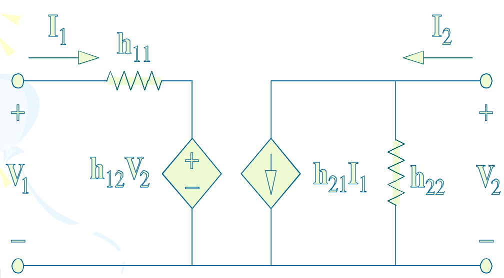
The h-parameter equivalent network of a two-port network



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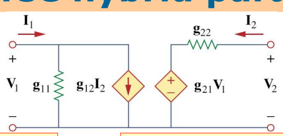
Inverse hybrid parameters



$$I_1 = g_{11}V_1 + g_{12}I_2 \qquad V_2 = g_{21}V_1 + g_{22}I_2$$

$$\Rightarrow \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = [g] \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} \qquad h_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0}$$

g_{11}=open-circuit input admittance(S)

g_{21} = open- circuit forward voltage gain

$$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0} \qquad g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0}$$

g_{12} = Short-circuit reverse current-gain

g_{22} = short-circuit output impedance (Ω)

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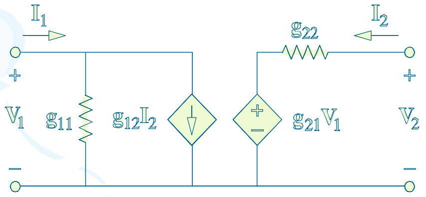
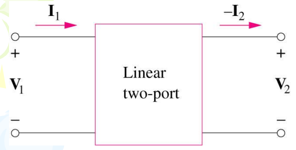
Inverse hybrid parameters (g parameters)

$$I_1 = g_{11}V_1 + g_{12}I_2$$
$$V_2 = g_{21}V_1 + g_{22}I_2$$



The g-parameter model of a two-port network

18.5 Transmission parameters (1)



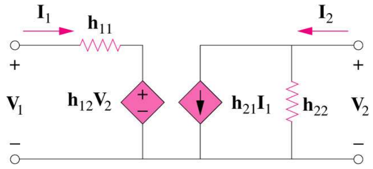
Assume no independent source in the network

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = [T]\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

where the **T** terms are called the ***transmission parameters***, or simply T or ***ABCD parameters***, and each parameter has different units.

18.5 Transmission parameters (2)



$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} \qquad C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$

A=open-circuit voltage ratio

C= open-circuit transfer admittance (S)

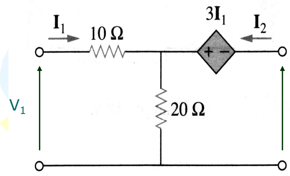
$$B = -\left.\frac{V_1}{I_2}\right|_{V_2=0} \qquad D = -\left.\frac{I_1}{I_2}\right|_{V_2=0}$$

B= negative short-circuit transfer impedance (Ω)

D=negative short-circuit current ratio

Example 6

Determine the T-parameters of the following circuit.



$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

Apply KVL

$$V_1 = 10I_1 + 20(I_1 + I_2)$$
$$V_2 = -3I_1 + 20(I_1 + I_2)$$

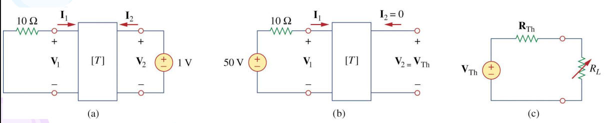
$$V_1 = \frac{30}{17}V_2 - \frac{260}{17}I_2$$
$$I_1 = \frac{1}{17}V_2 - \frac{20}{17}I_2$$

Answer: $T = \begin{bmatrix} 1.765 & 15.294\Omega \\ 0.059S & 1.176 \end{bmatrix}$

Example 7 The ABCD parameters of the two-port network in following Fig. are

$$\begin{bmatrix} 4 & 20\Omega \\ 0.1S & 2 \end{bmatrix}$$

The output port is connected to a variable load for maximum power transfer. Find R_L and the maximum power transferred



Answer: V_{TH} = 10V V; R_L = 8Ω; Pm = 3.125W.

Inverse Transmission parameters

Inverse transmission parameters are defined by expressing the variables at the output port in terms of the variables at the input port.

$$V_2 = aV_1 - bI_1$$
$$I_2 = cV_1 - dI_1$$

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} V_1 \\ -I_1 \end{bmatrix} = [t]\begin{bmatrix} V_1 \\ -I_1 \end{bmatrix}$$

$$a = \left.\frac{V_2}{V_1}\right|_{I_1=0} \qquad c = \left.\frac{I_2}{V_1}\right|_{I_1=0}$$

a=open-circuit voltage gian

c= open-circuit transfer admittance (S)

$$b = -\left.\frac{V_2}{I_1}\right|_{V_1=0} \qquad d = -\left.\frac{I_2}{I_1}\right|_{V_1=0}$$

b= negative short-circuit transfer impedance (Ω)

d=negative short-circuit current gain

In terms of the transmission or inverse transmission parameters, a network is reciprocal if

AD-BC=1 ad-bc=1

Reciprocal Two-Port Circuits

------------- **linear and has no dependent source**

If a two-port circuit is reciprocal, the following relationships exist among the port parameters:

$$z_{12} = z_{21}$$
$$y_{12} = y_{21}$$
$$h_{12} = -h_{21}$$
$$g_{12} = -g_{21}$$
$$\Delta T = AD - BC = 1$$
$$\Delta T' = ad - bc = 1$$

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Symmetric Two-Port Circuit

A reciprocal two-port circuit is **symmetric** if its ports can be interchanged without disturbing the values of the terminal currents and voltages.
If a two-port circuit is symmetric, the following relationships exist among the port parameters: (besides those exist in reciprocal)

$$z_{11} = z_{22}$$
$$y_{11} = y_{22}$$
$$\Delta h = h_{11}h_{22} - h_{12}h_{21} = 1$$
$$\Delta g = g_{11}g_{22} - g_{12}g_{21} = 1$$
$$A = D$$
$$a = d$$

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Question: How many calculations or measurements are needed to determine a set of parameters of a two-port circuit?

For a general two-port with sources:	6
For a general linear two-port:	4
For a reciprocal two-port:	3
For a symmetric two-port:	2

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Relationships between parameters

Example: $\boxed{z \text{ parameters}}$ $\Longrightarrow$ $\boxed{y \text{ parameters}}$

$$V_1 = z_{11}I_1 + z_{12}I_2 \qquad I_1 = y_{11}V_1 + y_{12}V_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2 \qquad I_2 = y_{21}V_1 + y_{22}V_2$$

$$\therefore y_{11} = \frac{z_{22}}{\Delta z}, \quad y_{12} = -\frac{z_{12}}{\Delta z},$$
$$y_{21} = -\frac{z_{21}}{\Delta z}, \quad y_{22} = \frac{z_{11}}{\Delta z}$$

where
$$\Delta z = z_{11}z_{22} - z_{12}z_{21}$$

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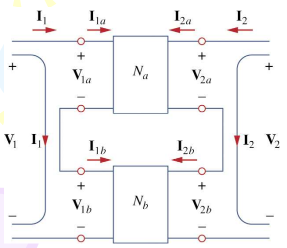
18.7 Interconnection of Networks(1)

1. The series connection of two-port networks

The series connection of two-port networks is shown in following Fig.. For the series connection, the input currents of the ports are the same and their voltage add.



For network N_a,

$$\boxed{\begin{aligned} V_{1a} &= z_{11a}I_{1a} + z_{12a}I_{2a} \\ V_{2a} &= z_{21a}I_{1a} + z_{22a}I_{2a} \end{aligned}}$$

For network N_b,

$$\boxed{\begin{aligned} V_{1b} &= z_{11b}I_{1b} + z_{12b}I_{2b} \\ V_{2b} &= z_{21b}I_{1b} + z_{22b}I_{2b} \end{aligned}}$$

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18.7 Interconnection of Networks(2)

We notice that from above Fig.

$$I_1 = I_{1a} = I_{1b}, I_2 = I_{2a} = I_{2b}$$

and that

$$V_1 = V_{1a} + V_{1b} = (Z_{11a} + Z_{11b})I_1 + (Z_{12a} + Z_{12b})I_2$$
$$V_2 = V_{2a} + V_{2b} = (Z_{21a} + Z_{21b})I_1 + (Z_{22a} + Z_{22b})I_2$$

Thus, the z parameters for the overall network are

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} z_{11a} + z_{11b} & z_{12a} + z_{12b} \\ z_{21a} + z_{21b} & z_{22a} + z_{22b} \end{bmatrix}$$

or

$$[z] = [z_a + z_b]$$

Showing that the z parameters for the overall network are the sum of the z parameters for the individual network.
This can be extended to n networks in series.

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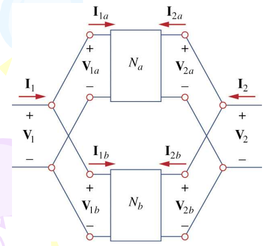
18.7 Interconnection of Networks(3)

2. The parallel connection of two-port networks

Two two-port networks are in parallel when their port voltages are equal and the port currents of the lager network are the sums of the individual port currents. The parallel connection of two two-port networks is shown in following Fig.



For network N_a,

$$I_{1a} = y_{11a}V_{1a} + y_{12a}V_{2a}$$
$$I_{2a} = y_{21a}V_{1a} + y_{22a}V_{2a}$$

For network N_b,

$$I_{1b} = y_{11b}V_{1b} + y_{12b}V_{2b}$$
$$I_{2b} = y_{21b}V_{1b} + y_{22b}V_{2b}$$

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18.7 Interconnection of Networks(4)

But from above Fig., we can get

$$V_1 = V_{1a} = V_{1b},\quad V_2 = V_{2a} = V_{2b}$$
$$I_1 = I_{1a} + I_{1b},\quad I_2 = I_{2a} + I_{2b}$$

From these equations, we obtain

$$I_1 = (y_{11a} + y_{11b})V_{1a} + (y_{12a} + y_{12b})V_{2a}$$
$$I_2 = (y_{21a} + y_{21b})V_{1a} + (y_{22a} + y_{22b})V_{2a}$$

Thus, the y parameters for the overall network are

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} y_{11a} + y_{11b} & y_{12a} + y_{12b} \\ y_{21a} + y_{21b} & y_{22a} + y_{22b} \end{bmatrix}$$

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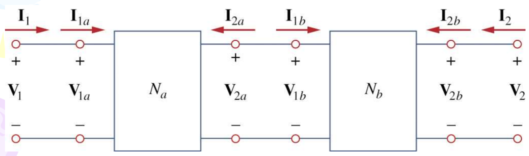
18.7 Interconnection of Networks(5)

3. The ***cascaded connection*** of two-port networks

Two networks are said to cascaded when the output of one is the input of the other. The connection of two two-port networks in cascade is shown in following Fig. For the two networks,

$$\begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix}$$

$$\begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix} = \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix}$$



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From above Fig.,

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix},\quad \begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix} = \begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix},\quad \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix} = \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

From these equations, we obtain

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

Thus, the transmission parameters for the overall network are the product the transmission parameters for the individual transmission parameters:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}$$

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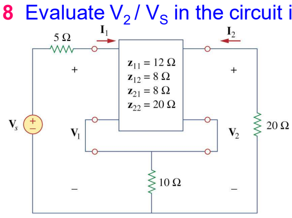
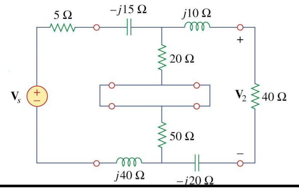
Example 8 Evaluate V_2 / V_S in the circuit in following Fig.



Practice Find V_2 / V_S in the circuit in following Fig.



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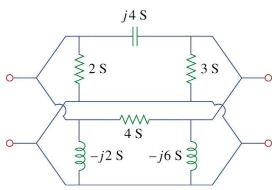
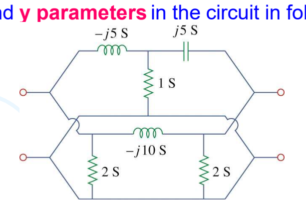
Example 9 Find the **y parameters** in the circuit in following Fig.



Practice Find **y parameters** in the circuit in following Fig.



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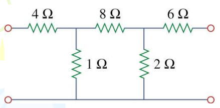
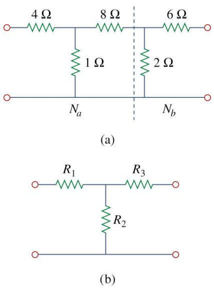
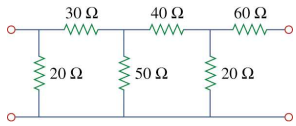
Example 10 Find the **transmission parameters** in the circuit in following Fig.





Practice Find the **ABCD parameters** in the circuit in following Fig.



Answer :

$$[T] = \begin{bmatrix} 29.5 & 2200\Omega \\ 0.425S & 32 \end{bmatrix}$$

SUMMARY(1)

- A two-port network is one with two ports (two pairs of access terminals), known as input and output ports.
- The six parameters used to model a two-port network are the impedance **[z],** admittance **[y],** hybrid **[h],** inverse hybrid **[g],** transmission **[T],** and inverse transmission **[t]** parameters.
- The parameters can be calculated or measured by short-circuiting or open-circuiting the appropriate input or output port.
- A two-port network is reciprocal if $z_{12}=z_{21}$, $y_{12}=y_{21}$, $h_{12}=-h_{21}$, $g_{12}=-g_{21}$, $\Delta_T=1$, $\Delta_t=1$. Network containing dependent sources are generally not reciprocal.

SUMMARY(2)

- Three important relationships are

  $[y]=[z]^{-1}$, $[g]=[h]^{-1}$, $[t]\neq[T]^{-1}$

- Two –port network may be connected in series, in parallel, or in cascade. In the series connection the z parameters are added, in the parallel connection the y parameters added, and in the cascade connection the transmission parameters are multiplied in the correct order.