Fundamentals of Electric Circuit

2020. 05-06

Chapter 9

Sinusoidal Steady-State Analysis

Chapter 9 Sinusoidal Steady-State Analysis

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Chapter 9 Sinusoidal Steady-State Analysis

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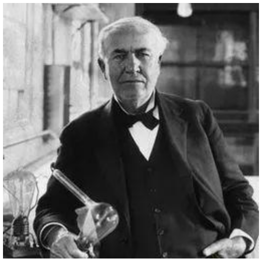
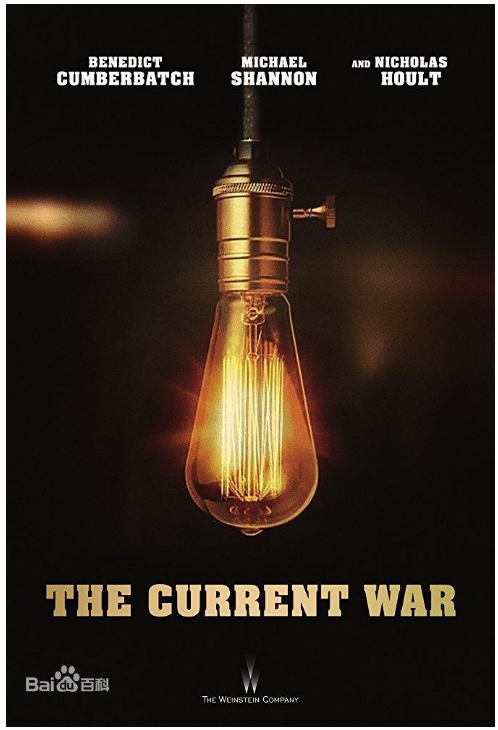
DC or AC ?



Thomas Edison
(1847.02.11-1931.10.18)

DC



Nikola Tesla
(1856.07.10-1943.01.07)

AC

Transmission √
Transformer √

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Tesla Coil

Wardenclyffe Tower

Magnetic flux density
1 Tesla= 10000 Gs

Chapter 9 Sinusoidal Steady-State Analysis (正弦稳态分析)

AC Analysis

DC Analysis: voltage and current are constant with respect to time.

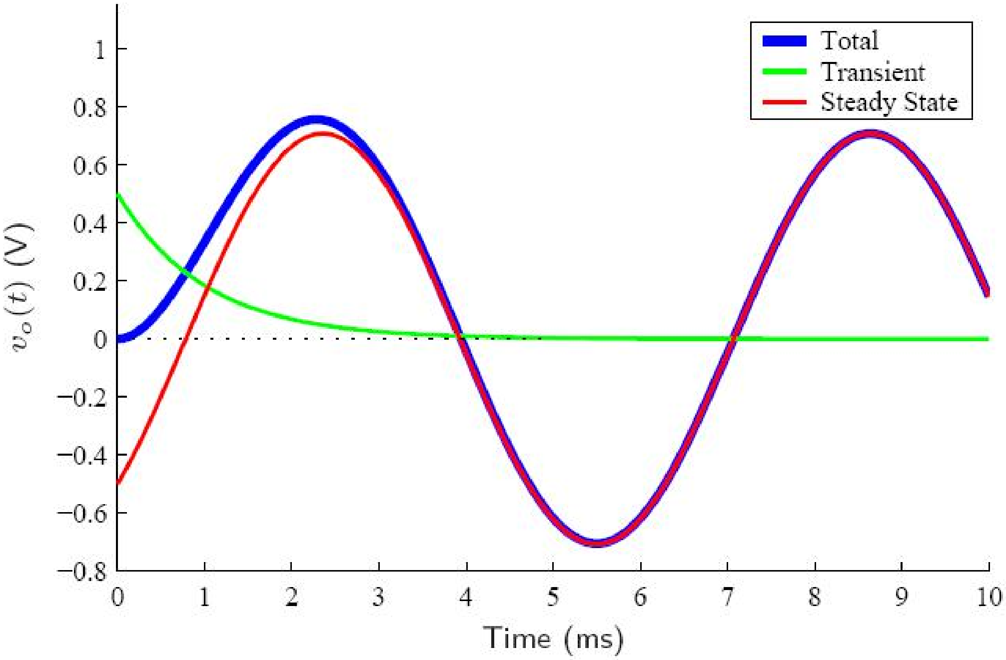
AC Analysis: voltage and current vary with time.

AC can be sinusoidal, square waves, or arbitrary periodic waveforms.

Sinusoidal is particularly important

- ✓ Commonly used, e.g., power systems, communications, etc.
- ✓ Simple periodic function (e.g., derivative and anti-derivative of a sinusoidal is also a sinusoidal)
- ✓ Any periodic function can be represented as the sum of sinusoidal function **=> Fourier Series**

Steady-State Analysis



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9.1 The Sinusoidal Source

Sinusoidal Sources: voltage or current sources that vary sinusoidally with time.

The sinusoidal voltage may be written as

phase angle

$u(t)=U_m\cos(\omega t+\varphi_u)$

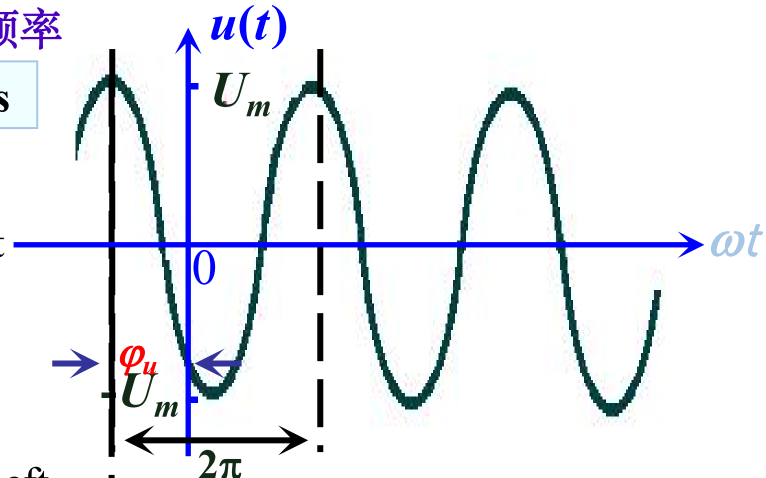
*amplitude*幅值

*initial phase angle*初相位

*angular frequency*角频率

$\omega = 2\pi f = 2\pi/T$ **Unit: rad/s**

Initial phase angle φ_u $\Longrightarrow$ time shift



$\cos(\omega t_1+\varphi_u) = 1$

$\omega t_1+\varphi_u = 0$

$\varphi_u = -\omega t_1$

if $\varphi_u > 0$, function shifts to the left.

if $\varphi_u < 0$, function shifts to the right.

$\varphi_u > 0$ or $\varphi_u < 0$??

$|\varphi_u| \le \pi$

Unit: rad

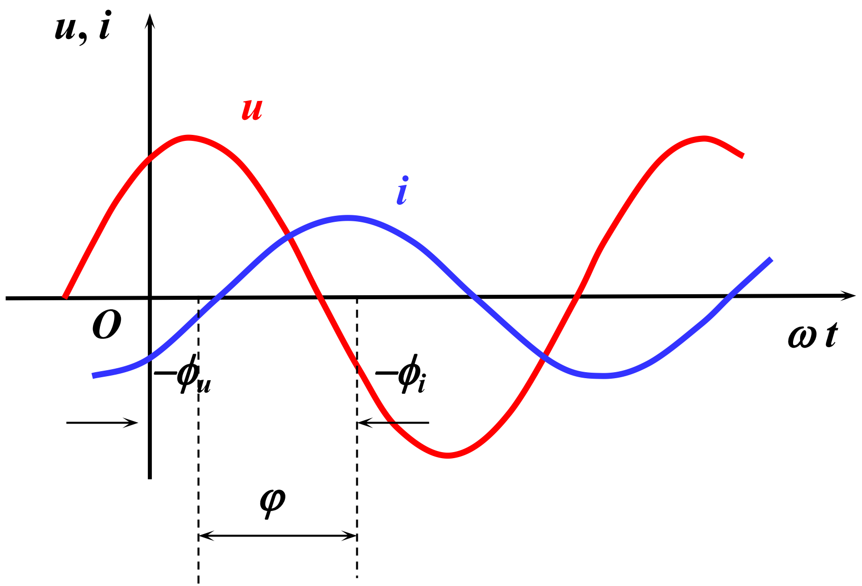
*

Phase angle difference（相位差）

$$u(t)=U_m\cos(\omega t+\varphi_u),\ \ i(t)=I_m\cos(\omega t+\varphi_i)$$

$$\varphi=(\omega t+\varphi_u)-(\omega t+\varphi_i)=\varphi_u-\varphi_i \qquad |\varphi|\le\pi$$

- $\varphi>0$， u **leads** （超前） i **by** φ， 或i **lags**（滞后） u **by** φ (u 比 i 先到达最大值)；



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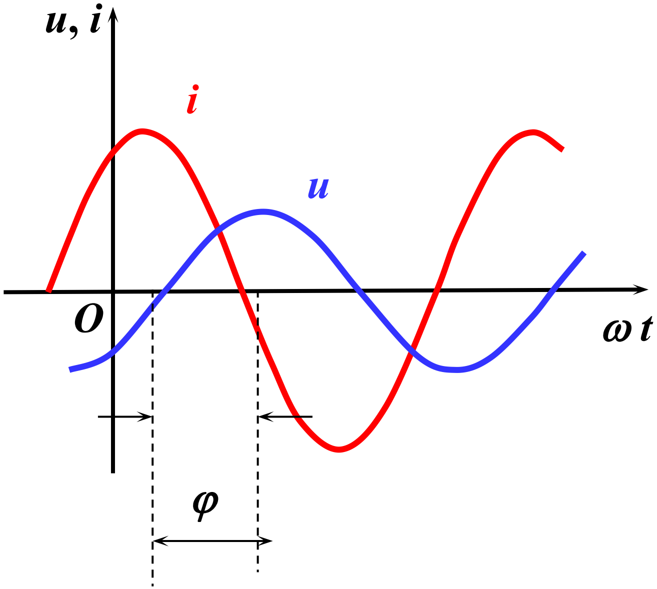
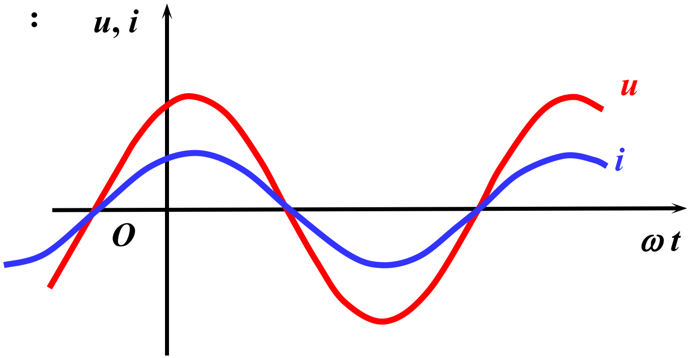
$\varphi = \varphi_u - \varphi_i$

- $\varphi < 0$， u **lags（滞后）** i by φ， 或 i **leads（超前）** u by φ

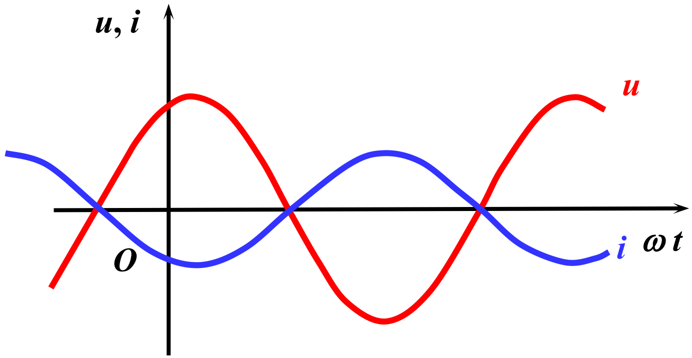


$\varphi = \varphi_u - \varphi_i$

$\varphi = 0$, in phase （同相）：

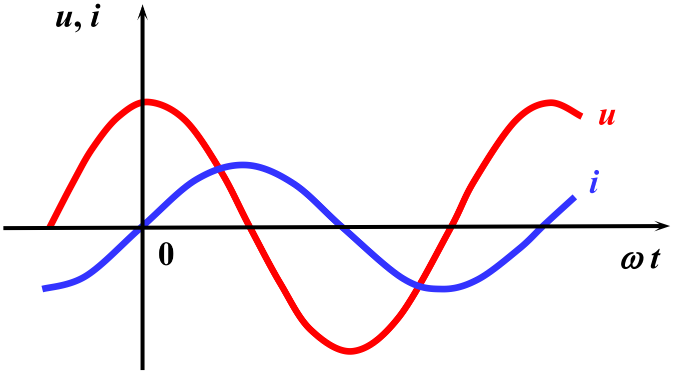


$\varphi = \pi$ ， out of phase （反相）：



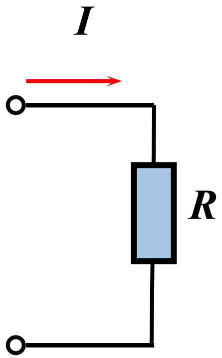
$\varphi = \varphi_u - \varphi_i$

$\varphi = \pi/2$ （正交）

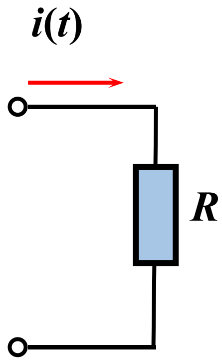


$|\varphi| \leq \pi$

Root mean square, RMS (有效值/方均根值)



$$W = I^2 R T$$



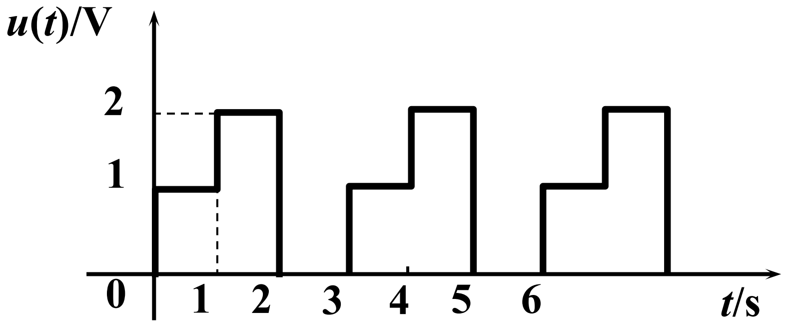
$$W_{交} = \int_0^T i^2(t) R \mathrm{d}t$$

$$I^2 R T = \int_0^T i^2(t) R \mathrm{d}t$$

$$I = \sqrt{\frac{1}{T}\int_0^T i^2(t)\mathrm{d}t}$$

$$U \overset{\text{def}}{=} \sqrt{\frac{1}{T}\int_0^T u^2(t)\mathrm{d}t}$$

Example: Calculate the RMS of the periodic function shown below



Solution: According to the definition of RMS,

$$U=\sqrt{\frac{1}{T}\int_0^T u^2(t)\mathrm{d}t}$$

$$=\sqrt{\frac{1}{3}\left(\int_0^1 1^2\,\mathrm{d}t+\int_1^2 2^2\,\mathrm{d}t+\int_2^3 0^2\,\mathrm{d}t\right)}=1.29\ \mathrm{V}$$

Root mean square for sinusoidal quantities

For $i(t)=I_m\cos(\omega t+\varphi_i)$

$$I=\sqrt{\frac{1}{T}\int_0^T I_m^2 cos^2(\omega t+\varphi_i)dt}$$

$$=\sqrt{\frac{1}{T}\int_0^T I_m^2 \frac{1+cos[2(\omega t+\varphi_i)]}{2}dt}$$

$$\therefore \quad I=\sqrt{\frac{1}{T}I_m^2\cdot\frac{T}{2}}=\frac{I_m}{\sqrt{2}}=0.707I_m$$

或 $\quad I_m=\sqrt{2}I$

$$u \quad i \quad U \quad I \quad U_m \quad I_m$$

Brief summary

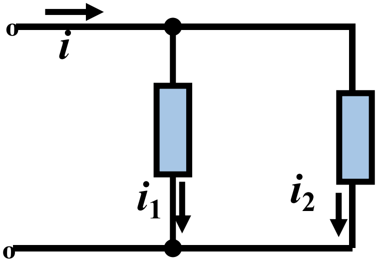
（1） amplitude(幅值),angular frequency(角频率), initial phase angle(初相位)

（2） Phase angle difference(相位差)

（3）Root mean square, RMS(有效值/方均根值)

$$u \quad i \quad U \quad I \quad U_m \quad I_m$$

9.3 Phasor（相量）



$$i_1 = 10\sqrt{2}\cos(314t - 60°)A$$

$$i_2 = 22\sqrt{2}\cos(314t - 150°)A$$

$$i = i_1 + i_2$$

$$i = 24.16\sqrt{2}\cos(314t - 125.55°)A$$

amplitude(幅值), initial phase angle(初相位)

Complex number

9.3.1 Representation of Complex Numbers

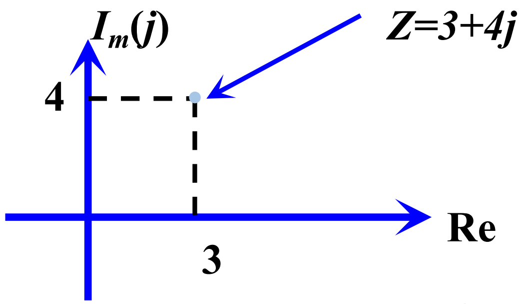
Complex number = real number + imaginary number

$$z = x + j\,y$$

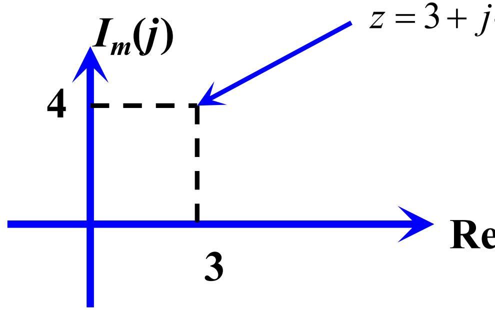
(1) Rectangular form

(2) Polar Form

$$z = x + jy = r\angle\theta \qquad z = |z|\angle\theta$$



$z = 3 + j4 = 5\angle 53.13^o$



where

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

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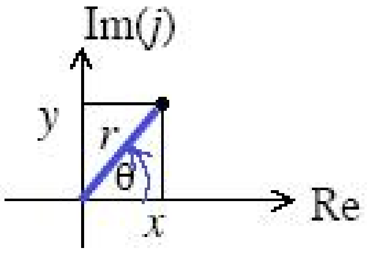
(3) Exponential form $z = re^{j\theta}$



$$z = r\angle\theta \quad \begin{cases} x = \mathrm{Re}\{z\} = r\cos(\theta) \\ y = \mathrm{Im}\{z\} = r\sin(\theta) \end{cases}$$

(projections)

$$z = x + j\,y = r\cos(\theta) + j\,r\sin(\theta) = r\,(\cos(\theta) + j\sin(\theta)) = re^{j\theta}$$

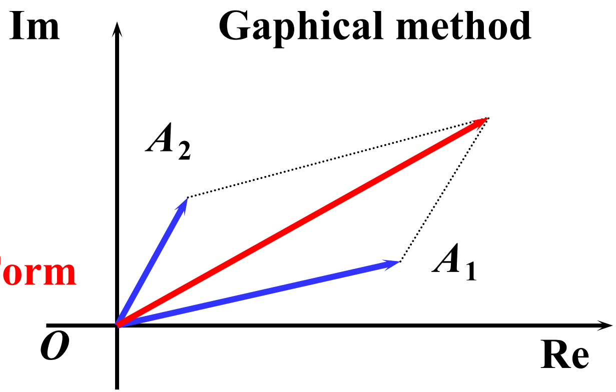
Euler identity: $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

*

(1) Addition and subtraction: Rectangular form

$A_1 = a_1 + jb_1, \quad A_2 = a_2 + jb_2$

$A_1 \pm A_2 = (a_1 \pm a_2) + j(b_1 \pm b_2)$



(2) Multiplication and division: Polar Form

$A_1 = |A_1| \angle \theta_1 \ , \quad A_2 = |A_2| \angle \theta_2$

$A_1 A_2 = |A_1| \, |A_2| \ \underline{/\theta_1 + \theta_2}$

$$\frac{A_1}{A2} = \frac{|A_1| \angle \theta_1}{|A2| \angle \theta_2} = \frac{|A_1| e^{j\theta_1}}{|A2| e^{j\theta 2}} = \frac{|A_1|}{|A2|} e^{j(\theta_1 - \theta_2)} = \frac{|A_1|}{|A2|} \underline{/\theta_1 - \theta_2}$$

*

Example

$$\frac{(10 + j6.28)(20 - j31.9)}{10 + j6.28 + 20 - j31.9}$$

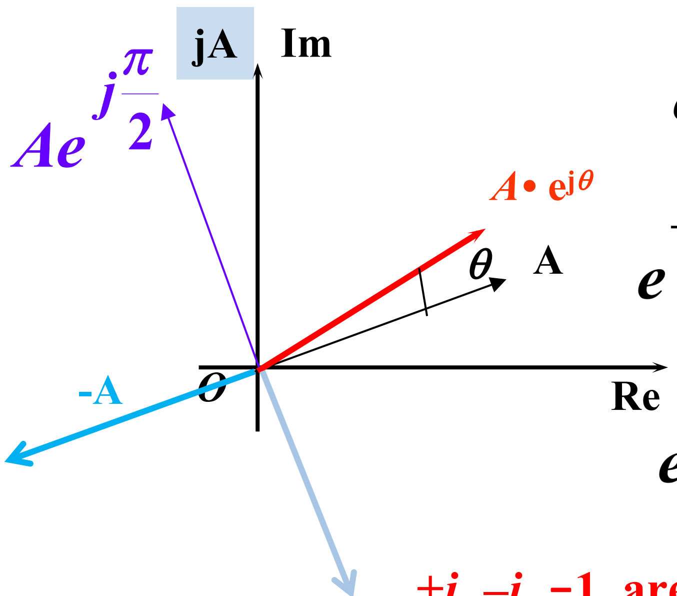
$$= \frac{11.81\angle 32.13^{o} \times 37.65\angle -57.61^{o}}{39.45\angle -40.5^{o}}$$

$$= 10.89 + j2.86$$

(3) Twiddle factor（旋转因子）：

$$e^{j\theta} = 1\angle\theta$$

Any of the trigonometric constant coefficients that are multiplied by the data in the course of the algorithm.



Euler's Identity

$$e^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = j$$

$$e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$e^{j\pi} = \cos\pi + j\sin\pi = -1$$

$+j$, $-j$, -1 are all twiddle factors.

9.3.2 Phasor Notation

Suppose a sinusoidal voltage source:

$u(t) = U_m \cos(\omega t + \varphi_u)$

$$A(t) = \sqrt{2} U e^{j(\omega t + \varphi_u)}$$

Euler's Identity

$$= \sqrt{2} U \cos(\omega t + \varphi_u) + j\sqrt{2} U \sin(\omega t + \varphi_u)$$

$||$

$$Re[A(t)] = \sqrt{2} U \cos(\omega t + \varphi_u)$$

$$A(t) = \sqrt{2} U e^{j(\omega t + \varphi_u)}$$

$$= \sqrt{2} \underbrace{U e^{j\varphi_u}}_{\text{Phasor}} e^{j\omega t} = \sqrt{2} \dot{U} e^{j\omega t}$$

$$\dot{U} = \underbrace{U \angle}_{\text{standard polar form}} \varphi_u$$

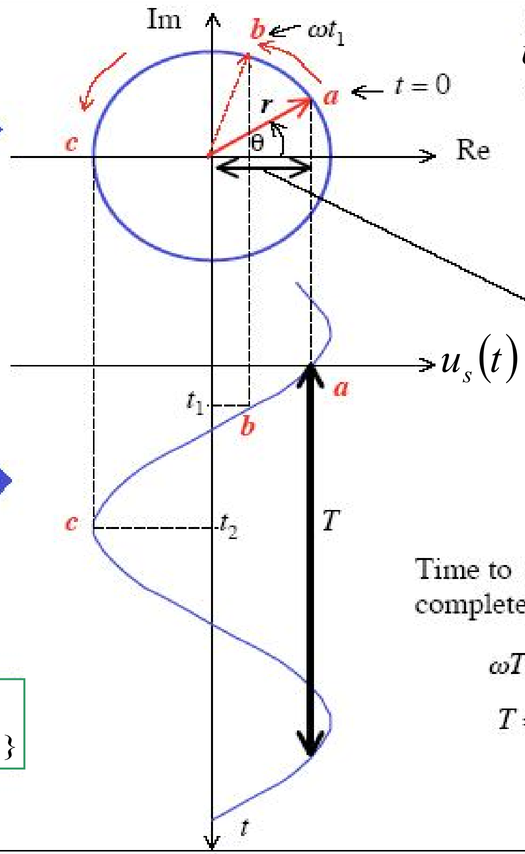
captures the two unknowns of the response signal (amplitude and phase angle)

standard polar form

9.3.2 Phasor Notation

Sinusoidal function in the complex-number domain (also called frequency domain)

Sinusoidal function in the time domain



graphically: at $t = 0$, $U_{sm}\angle\theta$ is a point in the complex plane (a)

$e^{j\omega t}$ describes the motion of a point in the complex plane

Re {} is the projection onto the real axis
ex.: Re $\{U_{sm}\angle\theta\}$

sinusoidal function is a cosine function

Time to go one cycle (i.e., complete one circle):

$$\omega T = 2\pi$$

$$T = (2\pi)/\omega$$

$$\boxed{\begin{aligned} u_s(t) &= U_{sm}\cos(\omega t + \theta) \\ &= \mathrm{Re}\{(U_{sm}\angle\theta)e^{j\omega t}\} \end{aligned}}$$

$$r = |U_{sm}|$$

*

Example

Find the phasors

$$i_1 = 14.14\cos(314t + 30^\circ)A$$
$$i_2 = -14.14\sin(10^3 t - 60^\circ)A$$

Solution

$$i_2 = -14.14\cos(10^3 t - 60^\circ - 90^\circ)A$$
$$= -14.14\cos(10^3 t - 150^\circ)A$$

According to the definition of phasor

$$\dot{I}_1 = \frac{14.14}{\sqrt{2}}\angle 30^\circ A = 10\angle 30^\circ A$$

$$\dot{I}_2 = \frac{-14.14}{\sqrt{2}}\angle -150^\circ A = -10/\underline{-150^\circ}A = 10\angle 30^\circ A$$

Example

Find the sinusoidal quantities, *f* = 50 Hz

$$\dot{U}_1 = 6\underline{/50^\circ}V \quad \dot{U}_2 = 3\underline{/-60^\circ}V$$

Solution

$$U_1 = 6\sqrt{2}\cos(2\pi f t + 50^\circ) = 6\sqrt{2}\cos(314t + 50^\circ)V$$

$$U_2 = 3\sqrt{2}\cos(314t - 60^\circ)V$$

9.3.3 Definition of Phasor

Phasor for sinusoidal function

$$u_s(t) = U_{sm}\cos(\omega t + \theta)$$
$$= U_{sm}\,\mathrm{Re}\{e^{j\theta}e^{j\omega t}\}$$

phasor transform (transfers the sinusoidal function from the *time domain* to the complex number or *frequency domain*).

$$\dot{U}_s = P\{U_{sm}\cos(\omega t + \theta)\} = U_{sm}e^{j\theta}$$

or $\dot{U}_s = U_{sm}\angle\theta$ or $\dot{U}_s = U_{sm}\cos\theta + jU_{sm}\sin\theta$

Inverse phasor transform (transfers the sinusoidal function from the frequency/complex domain to the time domain) (multiply by e $^{j\omega t}$ and extract the real part).

$$P^{-1}\{U_{sm}e^{j\theta}\} = U_{sm}\cos(\omega t + \theta)$$

Phasor Operation (at the same frequecy)

(1) Addition and subtraction:

$$i_1 = \sqrt{2}I_1\cos(\omega t+\varphi_1) \quad i_2 = \sqrt{2}I_2\cos(\omega t+\varphi_2)$$

$$i = i_1 + i_2 = \sqrt{2}I_1\cos(\omega t+\varphi_1) + \sqrt{2}I_2\cos(\omega t+\varphi_2)$$

$$= Re[\sqrt{2}\dot{I}_1 e^{j\omega t}] + Re[\sqrt{2}\dot{I}_2 e^{j\omega t}]$$

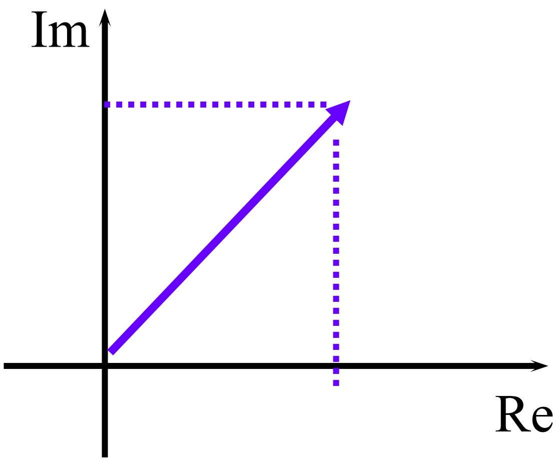
$$= Re[\sqrt{2}(\dot{I}_1 + \dot{I}_2)e^{j\omega t}]$$

$$i = Re[\sqrt{2}\dot{I}e^{j\omega t}]$$

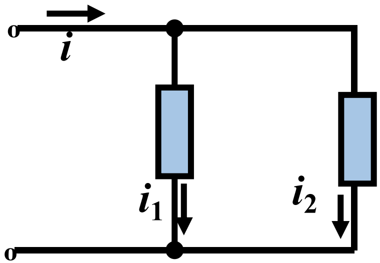
$$\dot{I} = \dot{I}_1 + \dot{I}_2$$

$$\boldsymbol{i_1 \pm i_2 = i_3}$$

$$\dot{I}_1 \pm \dot{I}_2 = \dot{I}_3$$



Example



$$i_1 = 10\sqrt{2}\cos(314t - 60^\circ)A$$

$$i_2 = 22\sqrt{2}\cos(314t - 150^\circ)A$$

$$\boldsymbol{i = i_1 + i_2}$$

Solution

$$\dot{I}_1 = 10/\underline{-60^\circ}A \quad \dot{I}_2 = 22/\underline{-150^\circ}A$$

$$\dot{I} = \dot{I}_1 + \dot{I} = 10/\underline{-60^\circ} + 22/\underline{-150^\circ}A$$

科学计算器

$$= 24.16/\underline{-122.55^\circ}A$$

$$i = 24.16\sqrt{2}\cos(314t - 125.55^\circ)A$$

Phasor Operation (at the same frequecy)

(2) Differential

$$i = \sqrt{2}I\cos(\omega t + \varphi)$$

$$\frac{di}{dt} = \frac{d}{dt}[\sqrt{2}I\cos(\omega t + \varphi)] = \frac{d}{dt}\{Re[\sqrt{2}\dot{I}e^{j\omega t}]\}$$

$$= Re\{\frac{d}{dt}[\sqrt{2}\dot{I}e^{j\omega t}]\} = Re[\sqrt{2}\dot{I}\cdot j\omega e^{j\omega t}]\}$$

$$= Re[\sqrt{2}(j\omega\dot{I})e^{j\omega t}]\}$$

$$\frac{di}{dt} \leftrightarrow j\omega\dot{I}$$

Phasor Operation (at the same frequecy)

(3) Integral

$$i = \sqrt{2}I\cos(\omega t + \varphi)$$

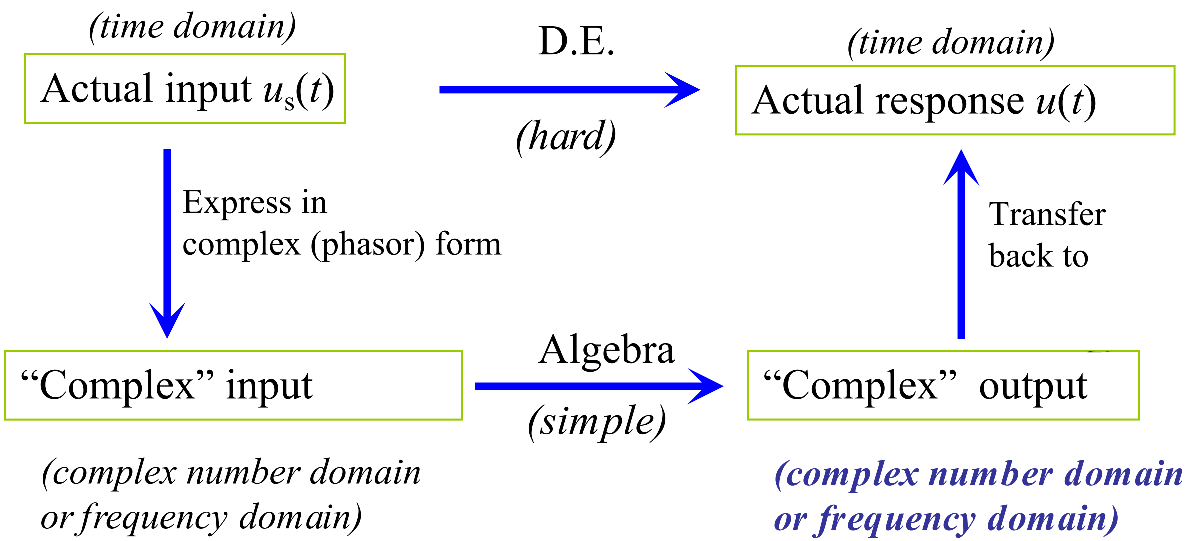
$$\int i dt = \int Re[\sqrt{2}I\cos(\omega t + \varphi)]dt = \int Re[\sqrt{2}\dot{I}e^{j\omega t}]dt$$

$$= Re[\int \sqrt{2}\dot{I}e^{j\omega t}dt] \quad = Re[\sqrt{2}\dot{I}\cdot\frac{1}{j\omega}e^{j\omega t}]\}$$

$$= Re[\sqrt{2}\frac{\dot{I}}{j\omega}e^{j\omega t}]\}$$

$$\int i dt \leftrightarrow \frac{\dot{I}}{j\omega}$$

Conceptually:



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Brief summary

（1） **Representation of Complex Numbers （复数)**

（2） **Definition of Phasor(正弦量的相量表示)**

（3） **Phasor Operation (at the same frequecy) (同频相量的加减、微积分运算)**

$$u \quad i \quad U \quad I \quad U_m \quad I_m \quad \dot{U} \quad \dot{I}$$