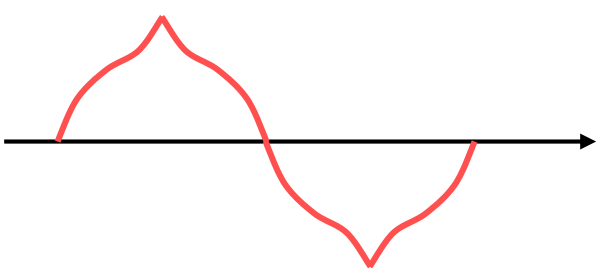
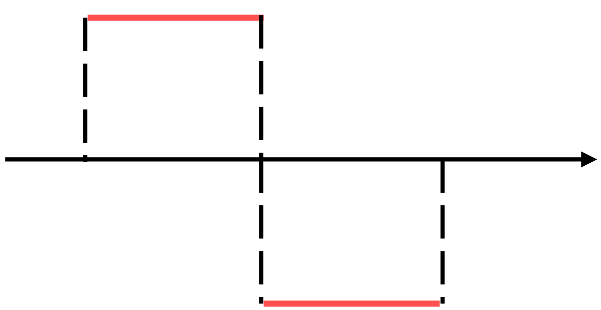
CHAPTER 17 THE FOURIER SERIES

17.1 INTRODUCTION

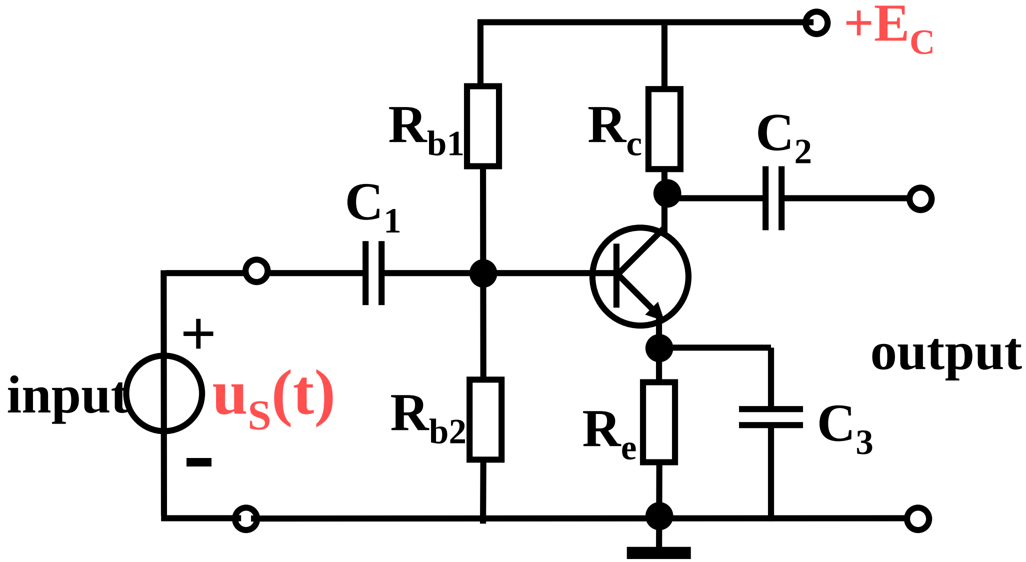
- **This chapter is concerned with a means of analyzing circuits with periodic, nonsinusoidal excitations**

（1） Many sources provide nonsinusoidal excitation in practice .

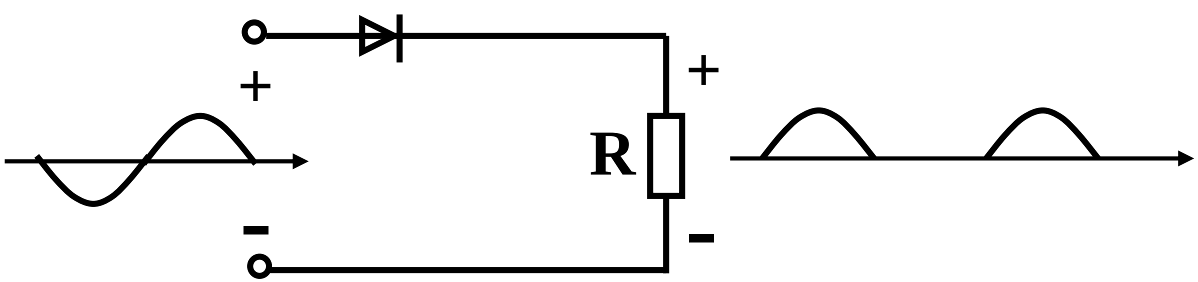


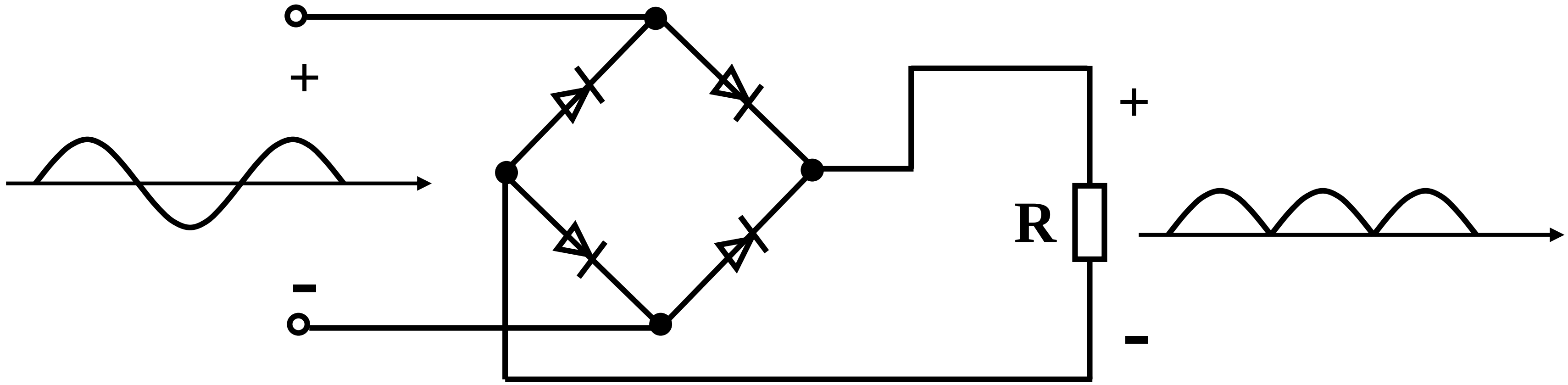


(2) Sources with different frequencies stimulate the circuits.



(3) Nonlinear circuit elements exist in the circuit.





- **Fourier series, a technique for expressing a periodic in terms of sinusoids.**

Periodic,nonlinear sources ⇒ **Linear time_invariant circuits** **(steady_state)**

(1) $$f(t+kT)=A_0+\sum_{k=1}^{\infty} A_{km}\sin(k\omega t+\varphi_k)$$

(2) Linear time_invariant circuits—— superposition therom can be applied.

CHAPTER 17 THE FOURIER SERIES

17.1 Trigonometric Fourier Series

17.2 Symmetry Considerations

17.3 Circuit Applications

17.4 Average Power and RMS Values

17.5 Balanced Three-Phase Circuits Excited By Nonsinusoidal periodic Functions

CHAPTER 17 THE FOURIER SERIES

Jean Baptiste Joseph Fourier
(1768-1830)

17.1 Trigonometric Fourier Series

The Fourier series of a periodic function f(t) is a representation that resolves f(t) into a dc component and an ac component comprising an infinite series of harmonic sinusoids.

傅里叶分析方法在数学、自然科学和工程界有着广泛的应用。它本身就是在热学研究中发现的。

傅里叶生平


1768—1830

1768年3月21日生于法国欧塞尔。9岁父母双亡，由教堂收养。12岁被送入地方军事学校读书。17岁回乡教数学，1794到巴黎，成为高等师范学校的首批学员，次年到巴黎综合工科学校执教。1798年随拿破仑远征埃及时任军中文书和埃及研究院秘书，1801年回国后任伊泽尔省地方长官。1817年当选为科学院院士，1822年任该院终身秘书，后又任法兰西学院终身秘书和理工科大学校务委员会主席。1830年5月16日逝于巴黎。

历史的回顾 （A Historical Perspective）

- 1748年欧拉研究振动弦时，认为振荡模式均为正弦函数，并成谐波关系。
- 1759年拉格朗日明确批评利用三角级数研究振动弦的主张，认为不可能用三角级数来表示一个具有间断点的函数。
- 1807年傅立叶向巴黎科学院递交“热的传播”论文，认为“任何周期信号都可以用正弦函数的级数来表示”，拉格朗日反对发表。
- 直到1822年，才以另外的形式出现在著作“热的分析理论”
- 1829年狄里赫利给出精确的收敛条件。
- 1965年，快速傅立叶变换被引入。

傅里叶的两个最重要的贡献——

- “周期信号都可以表示为成谐波关系的正弦信号的加权和”——傅里叶的第一个主要论点
- “非周期信号都可以用正弦信号的加权积分来表示”——傅里叶的第二个主要论点

Dirichlet conditions：

f(t) is single-valued everywhere.

f(t) has a finite number of finite discontinuities in any one period.

f(t) has a finite number of maxima and minima in any one period.

The integral $\int_{t_0}^{t_0+T} |f(t)| dt < \infty$ for any t_0.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_1 t + b_n \sin n\omega_1 t)$$

$\boldsymbol{a_0}$、$\boldsymbol{a_n}$、$\boldsymbol{b_n}$ **are the Fourier coefficients.**

$$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt = \frac{2}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos n\omega_1 t dt = \frac{2}{T}\int_0^T f(t)\cos n\omega_1 t dt$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin n\omega_1 t dt = \frac{2}{T}\int_0^T f(t)\sin n\omega_1 t dt$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} B_{nm} \sin(n\omega_1 t + \theta_n) = \frac{a_0}{2} + \sum_{n=1}^{\infty} B_{nm} (\sin n\omega_1 t \cos\theta_n + \cos n\omega_1 t \sin\theta_n)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_{nm} \cos(n\omega_1 + \psi_n)$$

$$a_n = B_{nm} \sin\theta_n \ , \quad b_n = B_{nm} \cos\theta_n$$

$$B_{nm} = \sqrt{a_n^2 + b_n^2} \ , \ \theta_n = tg^{-1} \frac{a_n}{b_n}$$

$$A_{nm} = \sqrt{a_n^2 + b_n^2} \ , \ \psi_n = tg^{-1} \frac{-b_n}{a_n}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} B_{nm} \sin(n\omega_1 t + \theta_n)$$

$\frac{a_0}{2}$： dc component.

$B_{1m}\sin(\omega_1 t+\theta_1)$： fundamental wave.

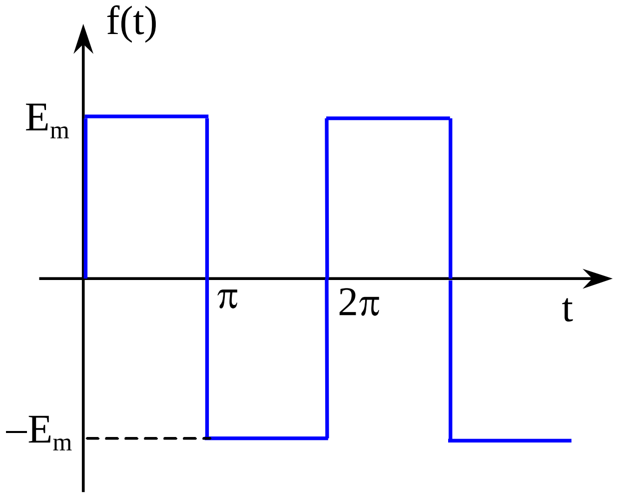
$B_{2m}\sin(2\omega_1 t+\theta_2)$： 2-nd harmonic

$B_{nm}\sin(n\omega_1 t+\theta_n)$： n-th harmonic.

The process of determining the coefficients is called Fourier analysis.

- The plot of the amplitude of the harmonics versus $n\omega_0$ is the amplitude spectrum of f(t). (幅值频谱)
- The plot of the phase θ_n versus $n\omega_0$ is the phase spectrum of f(t). (初相频谱)
- Both the amplitude and phase spectra form the frequency spectrum of f(t).

$$\begin{cases} f(t)=E_m & 0\le t\le \dfrac{T}{2} \\ f(t)=-E_m & \dfrac{T}{2}\le t\le T \end{cases}$$



$$a_0=\frac{1}{T}\int_0^T f(t)dt=0$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(t)\cos(n\omega_1 t)d(\omega_1 t)=\frac{1}{\pi}[\int_0^{\pi} E_m\cos(n\omega_1 t)d(\omega_1 t)$$

$$-\int_{\pi}^{2\pi} E_m\cos(n\omega_1 t)d(\omega_1 t)]=\frac{2E_m}{\pi}\int_0^{\pi}\cos(n\omega_1 t)d(\omega_1 t)=0$$

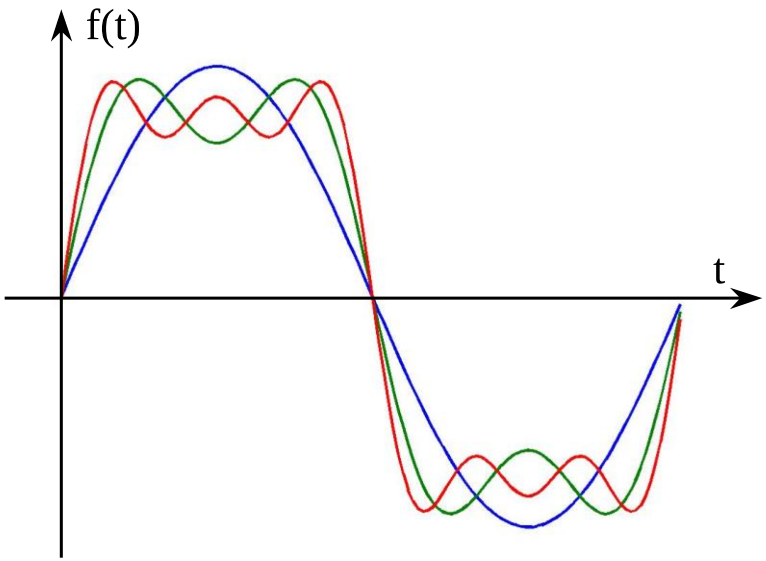
$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(t)\sin(n\omega_1 t)d(\omega_1 t)=\frac{1}{\pi}[\int_0^{\pi} E_m\sin(n\omega_1 t)d(\omega_1 t)-\int_{\pi}^{2\pi} E_m\sin(n\omega_1 t)d(\omega_1 t)]$$

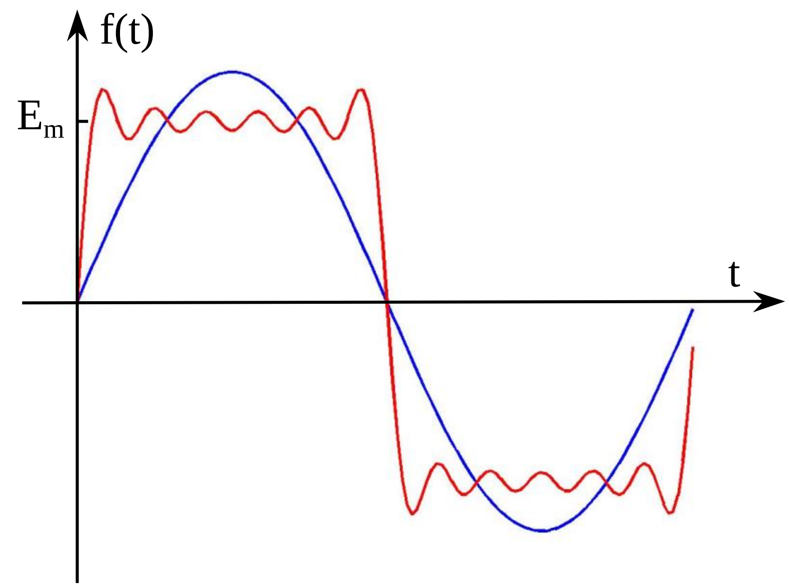
$$=\frac{2E_m}{\pi}\int_0^{\pi}\sin(n\omega_1 t)d(\omega_1 t)=\frac{2E_m}{\pi}\left[-\frac{1}{n}\cos(n\omega_1 t)\right]_0^{\pi}=\frac{2E_m}{n\pi}[1-\cos(n\pi)]$$

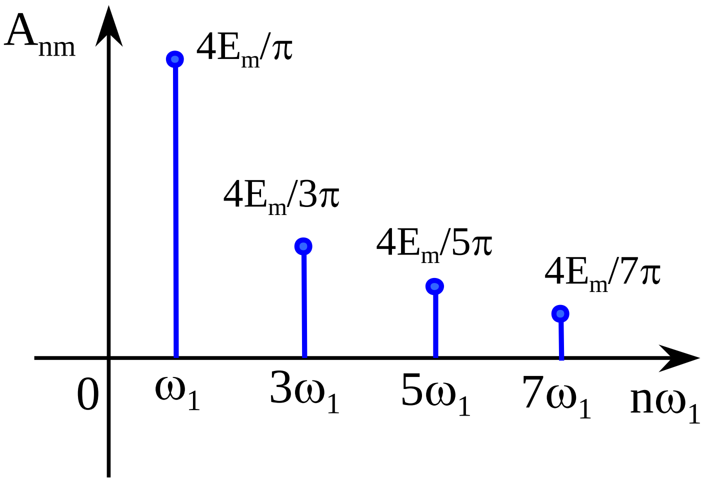
$$\cos(n\pi) = 1, \quad b_n = 0 \qquad \text{If n is even.}$$

$$\cos(n\pi) = -1, \quad b_n = \frac{4E_m}{n\pi} \qquad \text{If n is odd.}$$

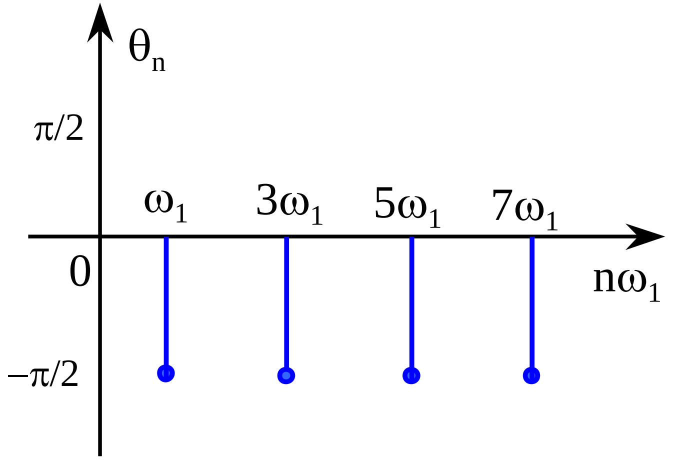
$$\therefore \ f(t) = \frac{4E_m}{\pi}[\sin(\omega_1 t) + \frac{1}{3}\sin(3\omega_1 t) + \frac{1}{5}\sin(5\omega_1 t) + \cdots]$$







amplitude spectrum

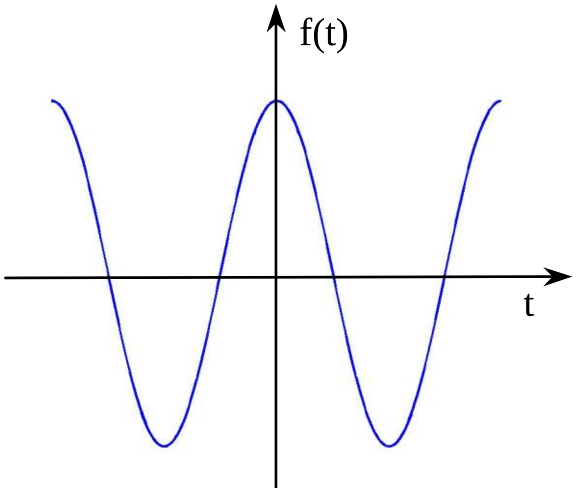


phase spectrum

Frequency spectrum of f(t)

17.2 Symmetry Considerations

Even symmetry $f(t)=f(-t)$



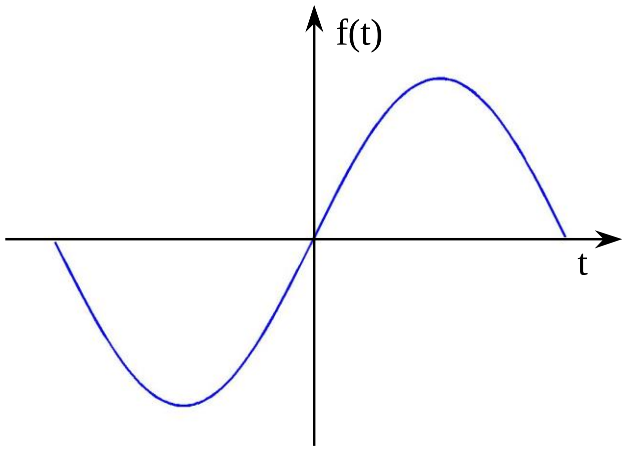
$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega_1 t+b_n\sin n\omega_1 t)$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos n\omega_1(-t)+b_n\sin n\omega_1(-t)\right]$$

$$\Rightarrow f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\omega_1 t \quad (b_n=0)$$

$$a_n=\frac{2}{T}\int_0^T f(t)\cos n\omega_1 t dt=\frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos n\omega_1 t dt$$

Odd symmetry $f(-t) = -f(t)$



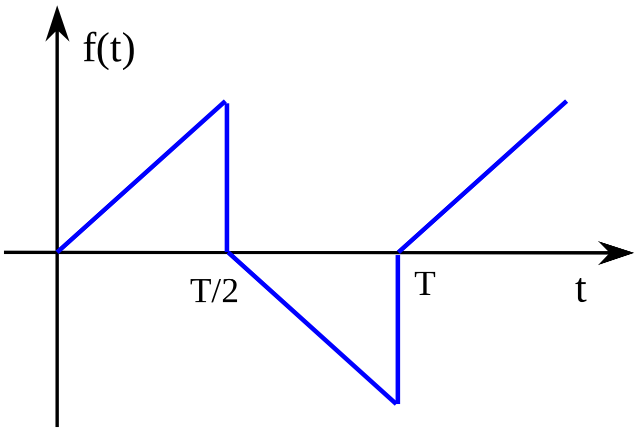
$$f(-t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_n t - b_n \sin n\omega_n t)$$

$$= -\frac{a_0}{2} - \sum_{n=1}^{\infty} (a_n \cos n\omega_n t + b_n \sin n\omega_n t)$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_1 t \qquad (a_0 = 0\,,\ a_n = 0)$$

$$b_n = \frac{4}{T} \int_0^{\frac{T}{2}} f(t) \sin n\omega_1 t dt$$

Half-wave symmetry $f(t) = -f(t + \frac{T}{2})$



The Fourier coefficients:

$$a_0 = 0$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_n t\,dt,\ for\ n\ odd$$

$$a_n = 0,\ for\ n\ even$$

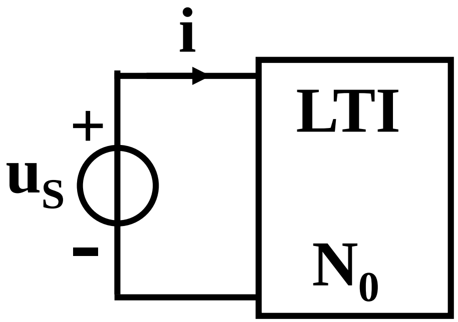
$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_n t\,dt,\ for\ n\ odd$$

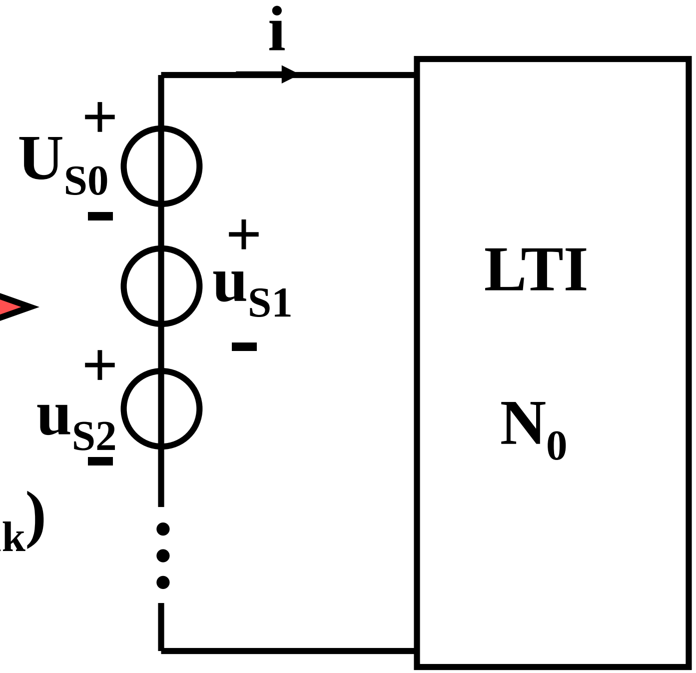
$$b_n = 0,\ for\ n\ even$$

17.3 Circuit Applications

Steps for applying Fourier series：

1. Express the excitation as a Fourier series.
2. Find the response of each term in the Fourier series.
3. Add the individual responses using the superposition principle (time domain).

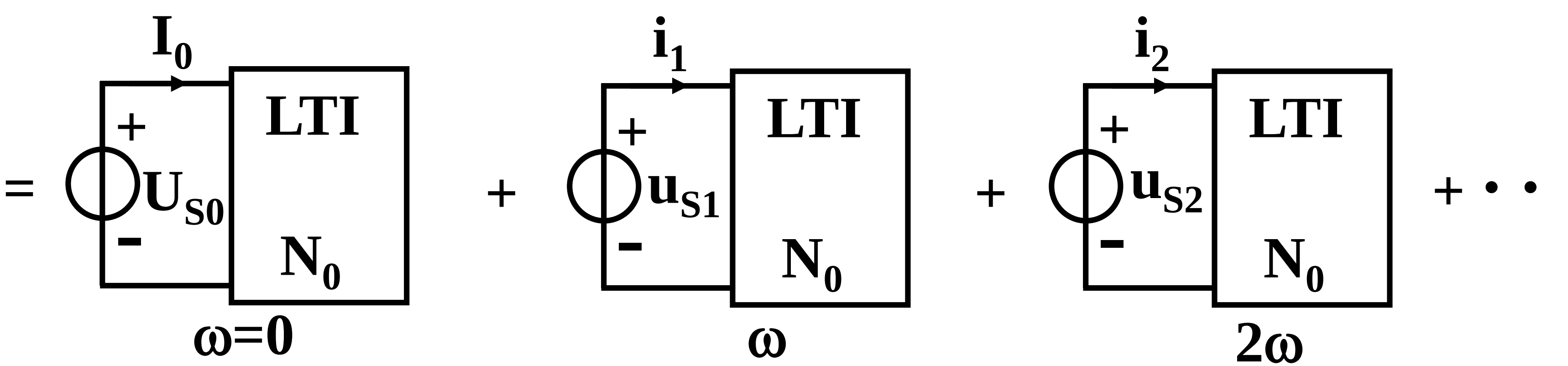




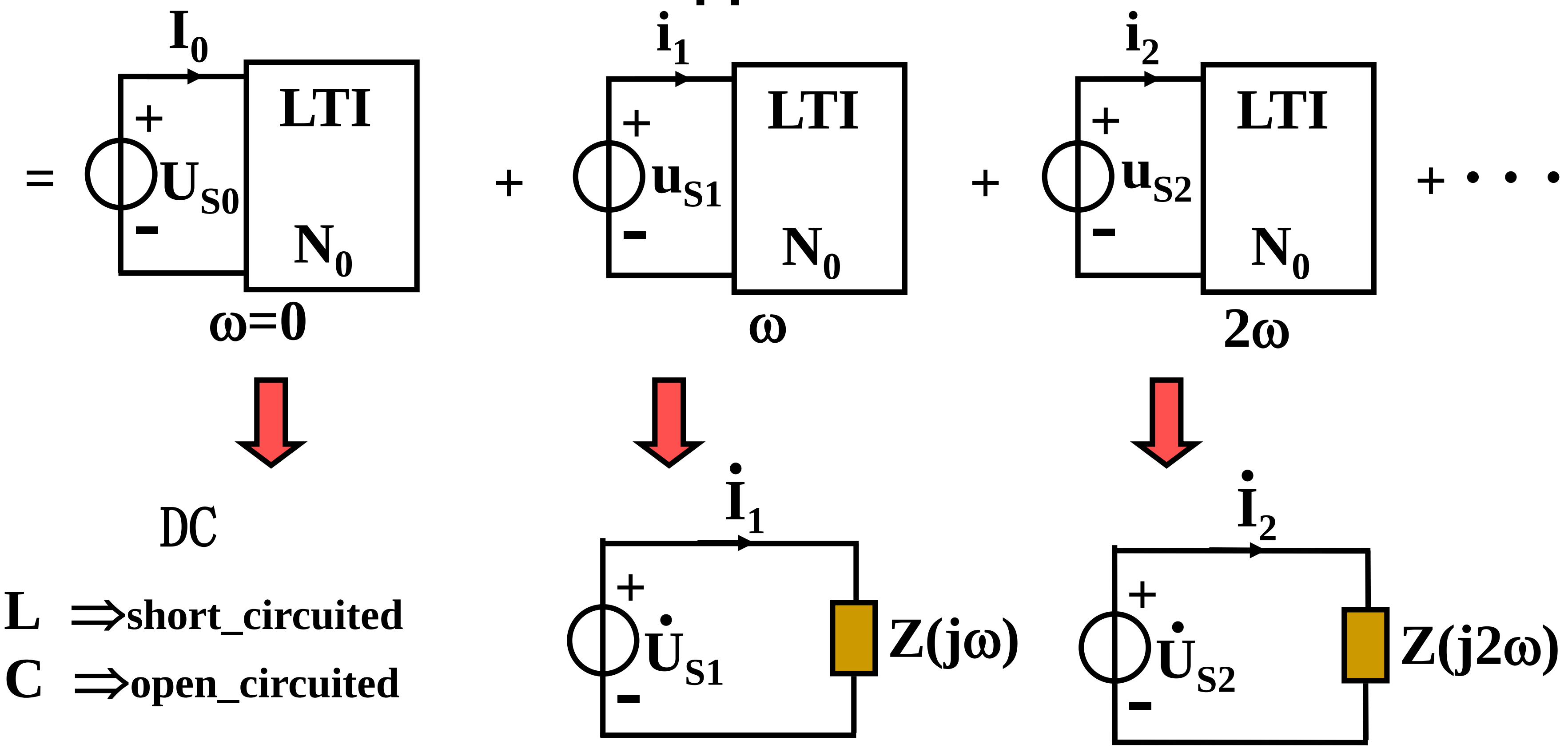
$$u_S(t)=U_{S0}+\sum_{k=1}^{\infty}\sqrt{2}\,U_{Sk}\sin(k\omega t+\varphi_{uk})$$

$$=U_{S0}+u_{S1}+u_{S2}+\cdots$$

superposition



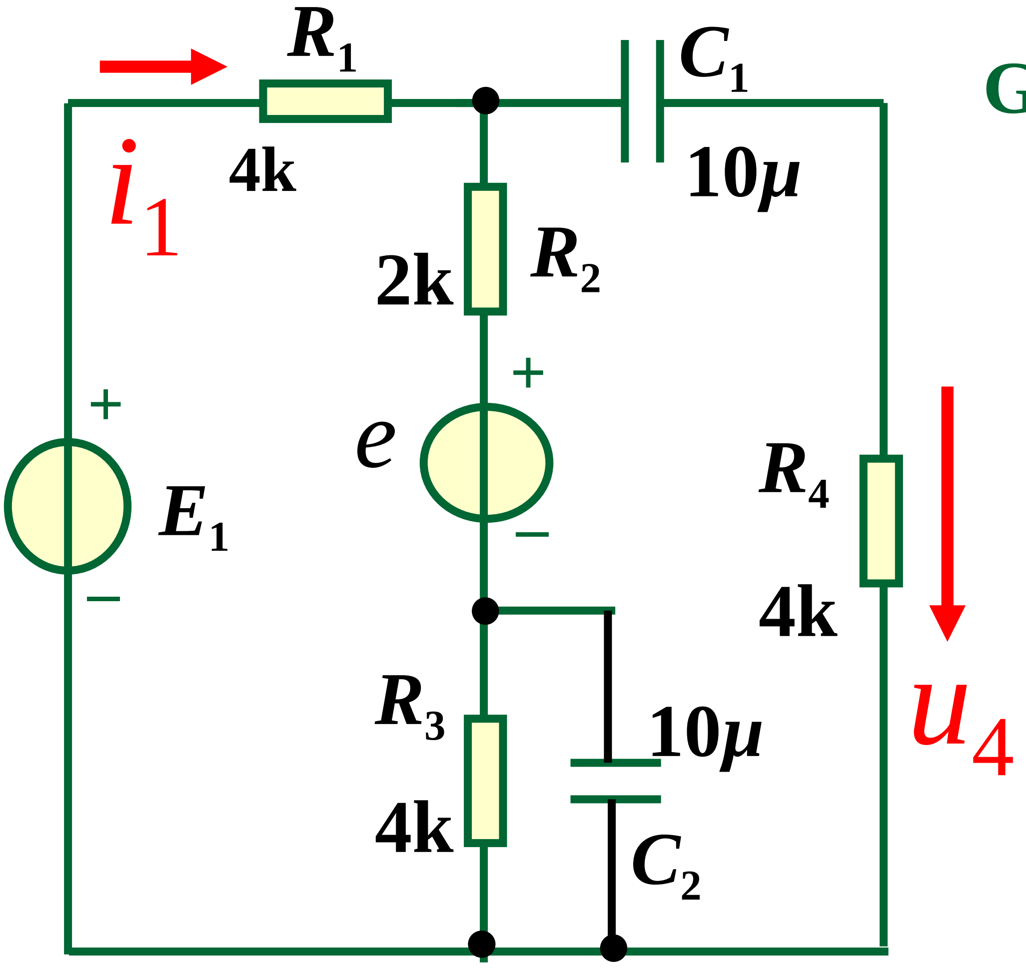
17.3 Circuit Applications



$$i(t)=I_0+i_1(t)+i_2(t)+\cdots\cdots \qquad I=\sqrt{I_0^2+I_1^2+I_2^2+\cdots\cdots}$$

$$P=P_0+P_1+P_2+\cdots\cdots$$

Example:



Given :

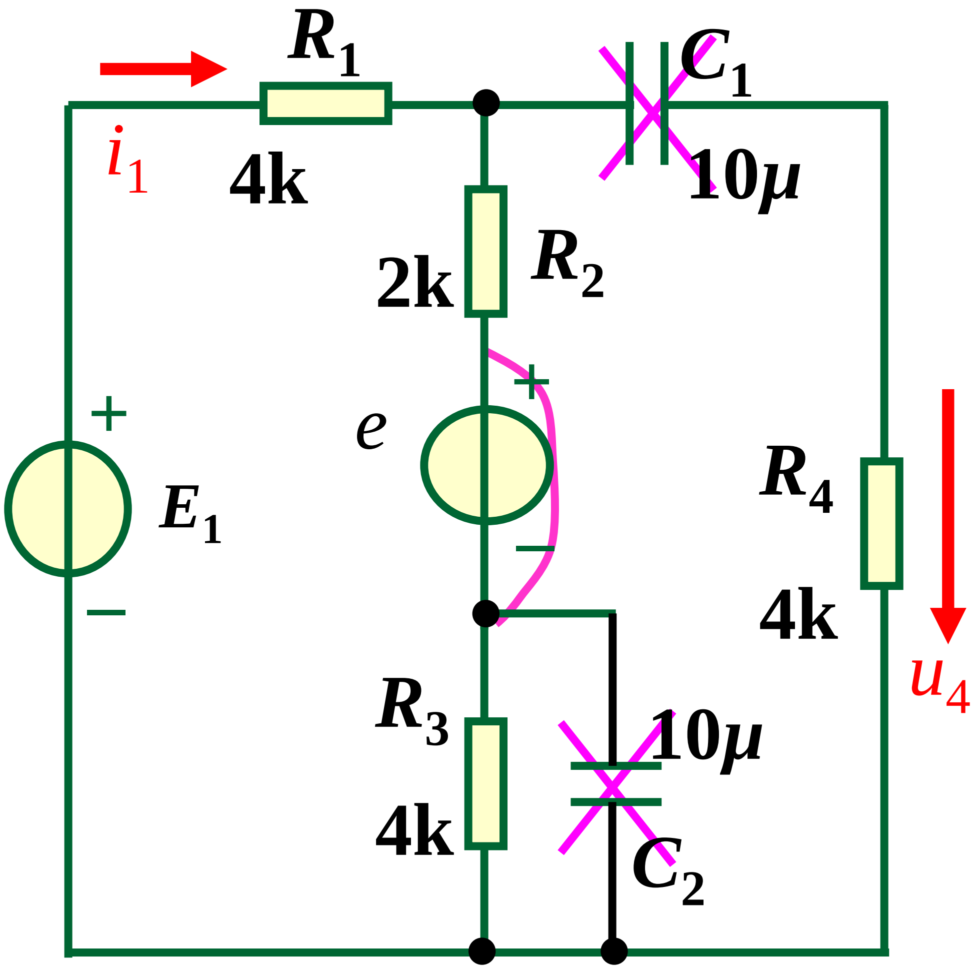
E_1=12 V

$$e = 40\sqrt{2}\sin 1000t \ \text{mV}$$

$$i_1 = ?$$

$$u_4 = ?$$

(1) E_1 acts alone:

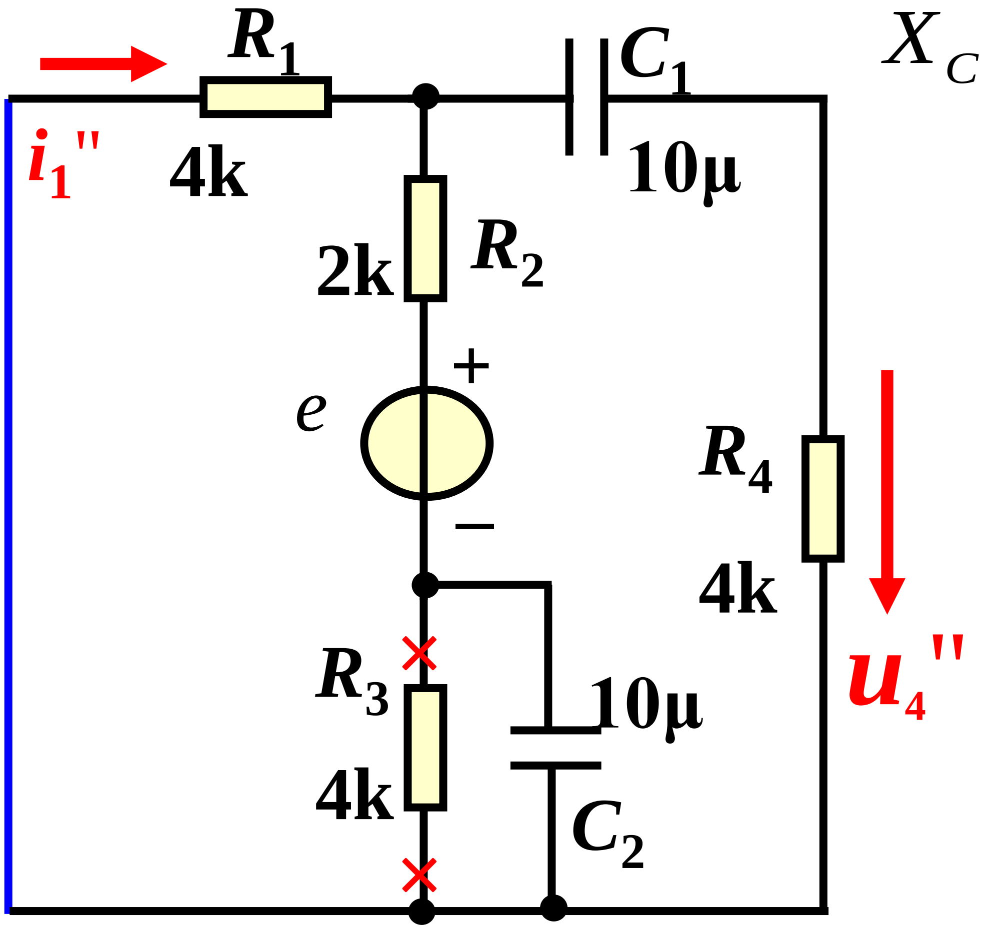


$$I'_1 = \frac{E_1}{R_1 + R_2 + R_3}$$

$$= \frac{12}{10} = 1.2\,(\text{mA})$$

$$\boxed{U'_4 = 0\text{V}}$$

(2) e acts alone:

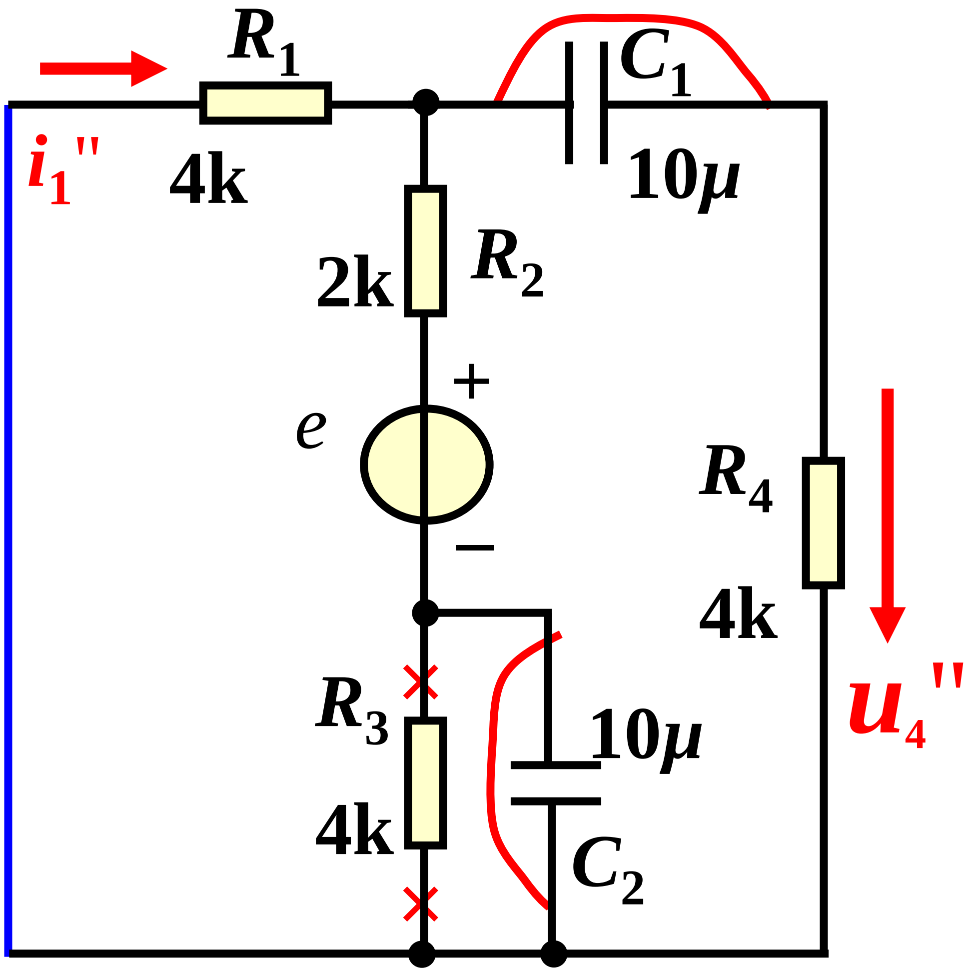


$$X_{C1} = X_{C2} = \frac{1}{1000 \times 10 \times 10^{-6}}$$

$$= 100\ \Omega$$

$\because \quad R_3 >> X_{C2}$

$$R_3 // (-jX_{C2}) \approx -jX_{C2}$$

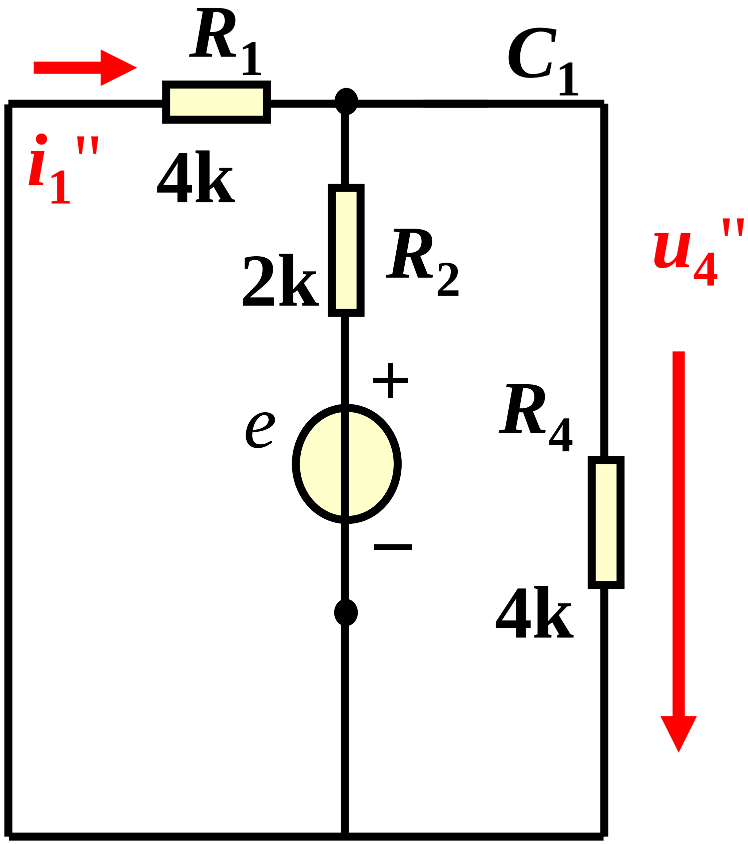


$\because$ $R_4 >> X_{C1}$

$$-jX_{C1} + R_4 \approx R_4$$

$\because$ $R_2 >> X_{C2}$

C_2 is short circuit



Simplified circuit

$$u_4'' = \frac{1}{2}e$$

$$= 20\sqrt{2}\sin 1000t \ \ \mathrm{mV}$$

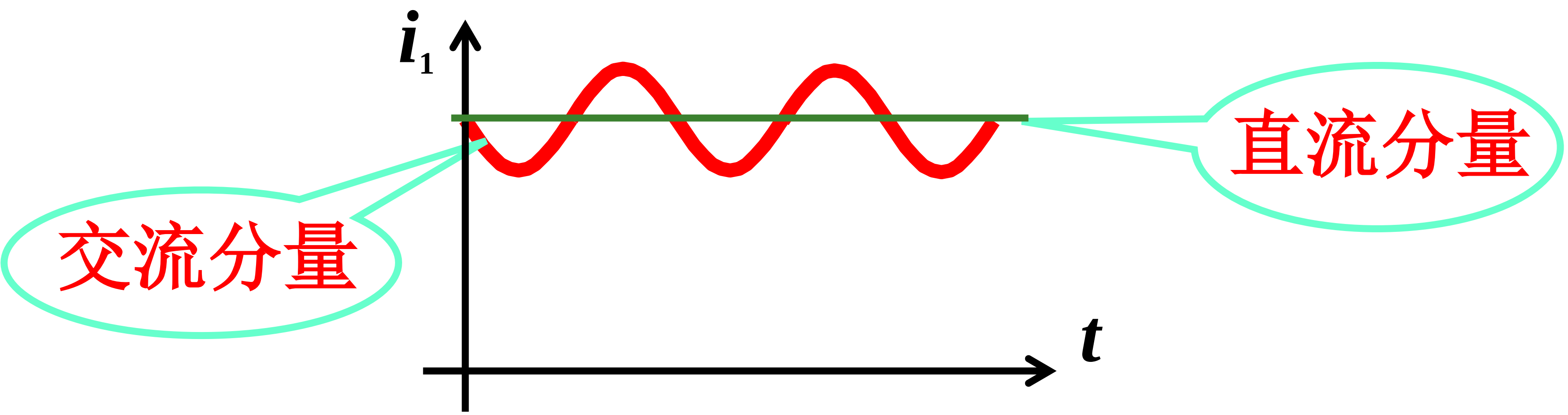
$$i_1'' = -\frac{u_4''}{R_1}$$

$$= -5\sqrt{2}\sin 1000t \quad \mu\mathrm{A}$$

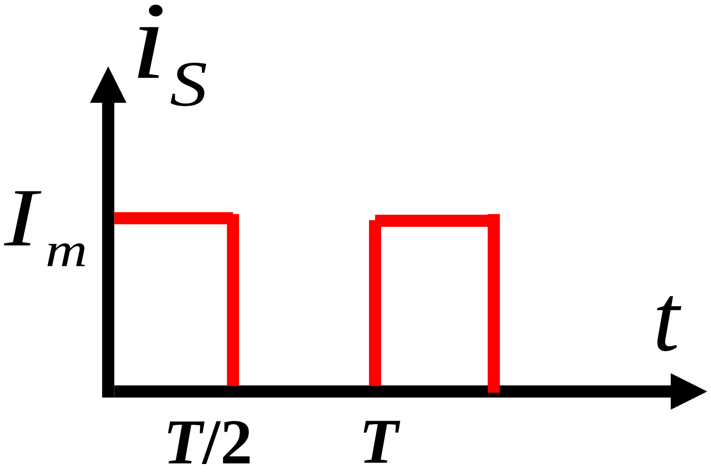
Use superposition principle:

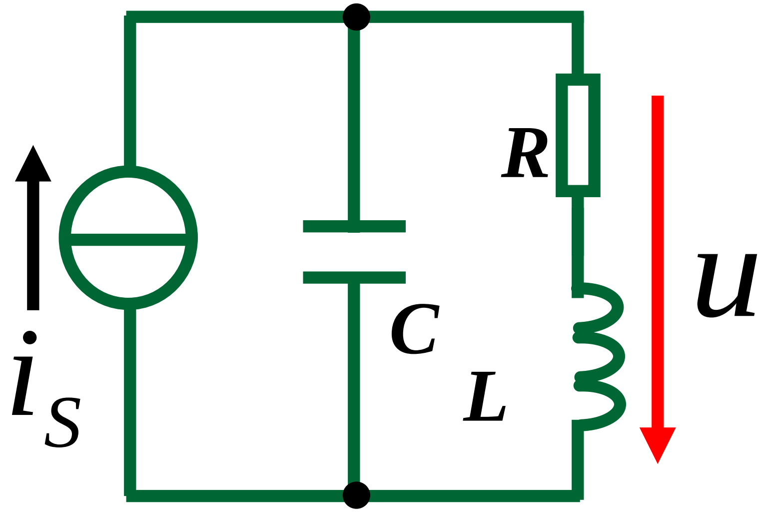
$$u_4 = u'_4 + u''_4$$
$$= 20\sqrt{2}\sin 1000t \quad \text{mV}$$

$$i_1 = i'_1 + i''_1$$
$$= 1200 - 5\sqrt{2}\sin 1000t \quad \mu\text{A}$$



Example：





Given: $R = 20\Omega$、 $L = 1\text{mH}$、 $C = 1000\text{pF}$

$I_m = 157\mu\text{A}$、 $T = 6.28\mu S$

calculate: u

1.Express the excitation as a Fourier series.

Dc component:

$$I_O = \frac{1}{T}\int_0^T i(t)\,dt = \frac{1}{T}\int_0^{T/2} I_m dt = \frac{I_m}{2}$$

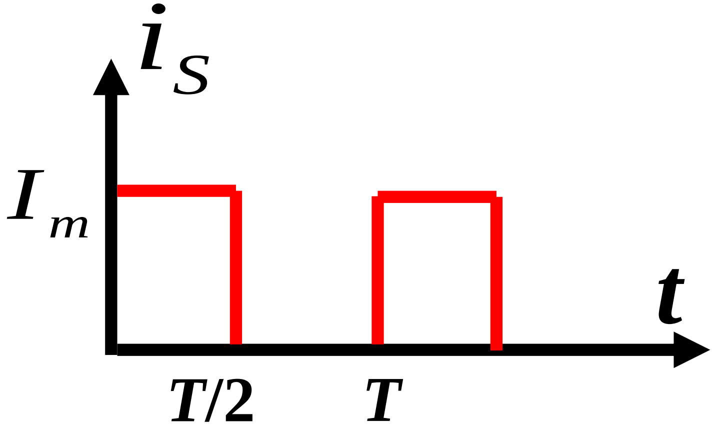
harmonics:

$$B_{Km} = \frac{1}{\pi}\int_0^{2\pi} i(\omega t)\sin k\omega t d(\omega t)$$

$$= \frac{I_m}{\pi}\left(-\frac{1}{k}\cos k\omega t\right)\Big|_0^{\pi} = \begin{cases} 0 & K\text{为偶数} \\ \dfrac{2I_m}{k\pi} & K\text{为奇数} \end{cases}$$

$$C_{km} = \frac{2}{\pi}\int_0^{2\pi} i(\omega t)\cos k\omega t d(\omega t)$$

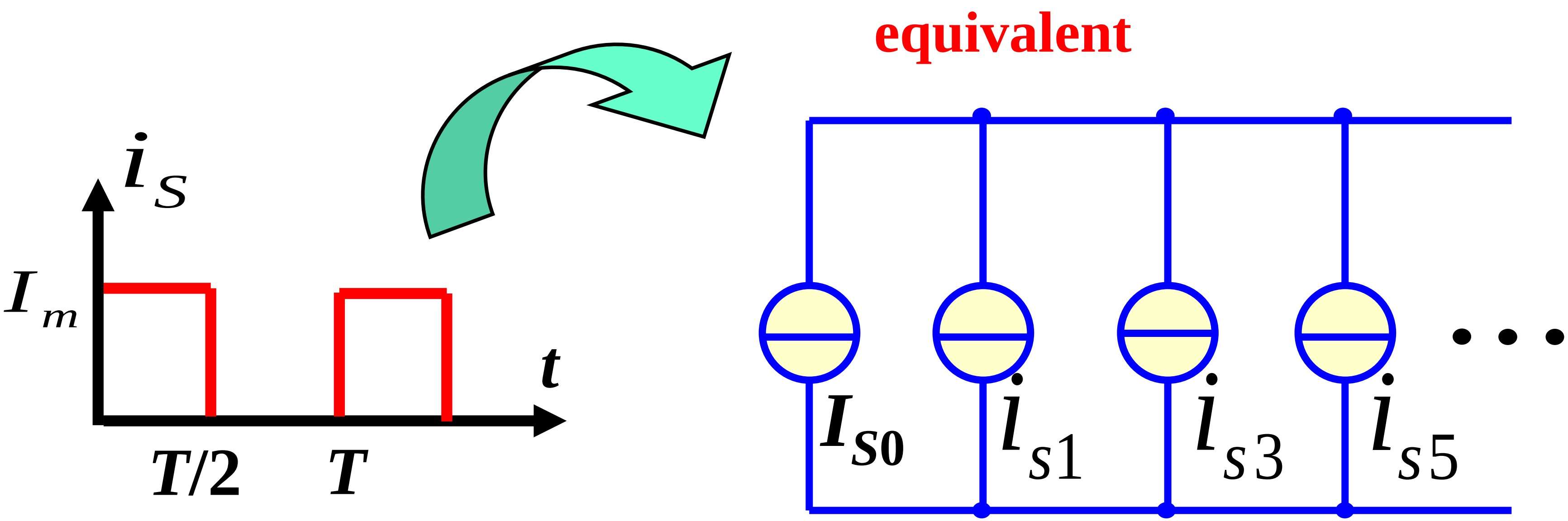
$$= \frac{2I_m}{\pi}\cdot\frac{1}{k}\sin k\omega t\Big|_0^{\pi} = 0$$



$$i_S = I_0 + \sum_{K=1}^{\infty} A_{Km} \sin(k\omega t + \varphi_K)$$

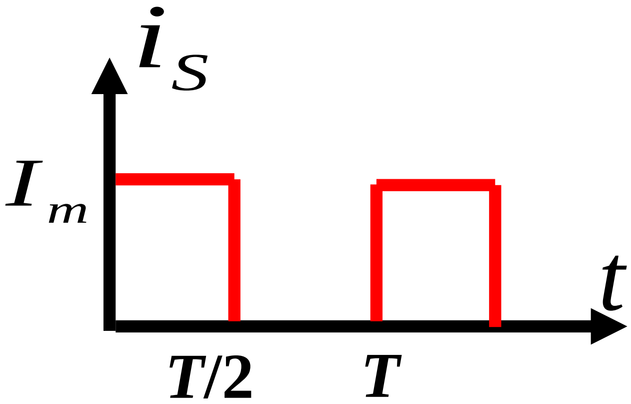
$$= \frac{I_m}{2} + \frac{2I_m}{\pi}(\sin \omega t + \frac{1}{3}\sin 3\omega t$$

$$+ \frac{1}{5}\sin 5\omega t + \ldots)$$



$$i_S = \frac{I_m}{2} + \frac{2I_m}{\pi}\left(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega\, t + \ldots\right)$$

$\boxed{I_{s0}}$ $\boxed{\dot{i}_{s1}}$ $\boxed{\dot{i}_{s3}}$ $\boxed{\dot{i}_{s5}}$



The Fourier series：

$$I_{s0} = 78.5 \quad \mu\text{A}$$

$$i_{s1} = 100 \sin 10^6 t \quad \mu\text{A}$$

$$i_{s3} = \frac{100}{3} \sin 3 \cdot 10^6 t \quad \mu\text{A}$$

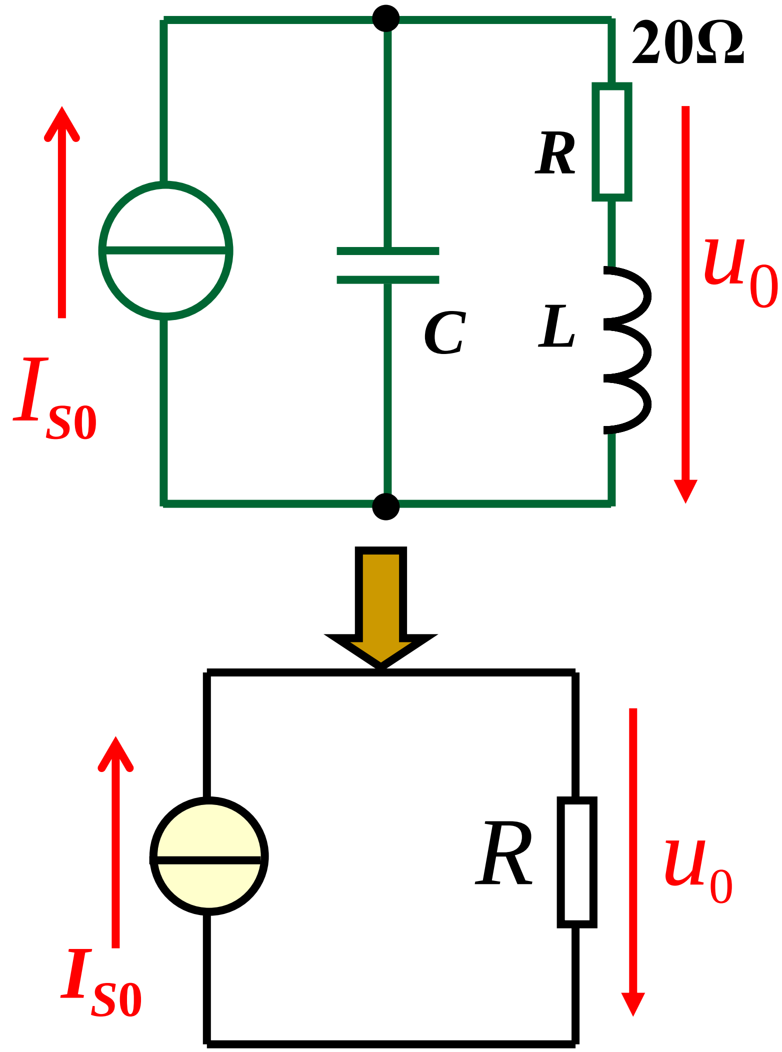
$$i_{s5} = \frac{100}{5} \sin 5 \cdot 10^6 t \quad \mu\text{A}$$

2. Find the response of each term in the Fourier series.

(1). I_{S0} acts alone:

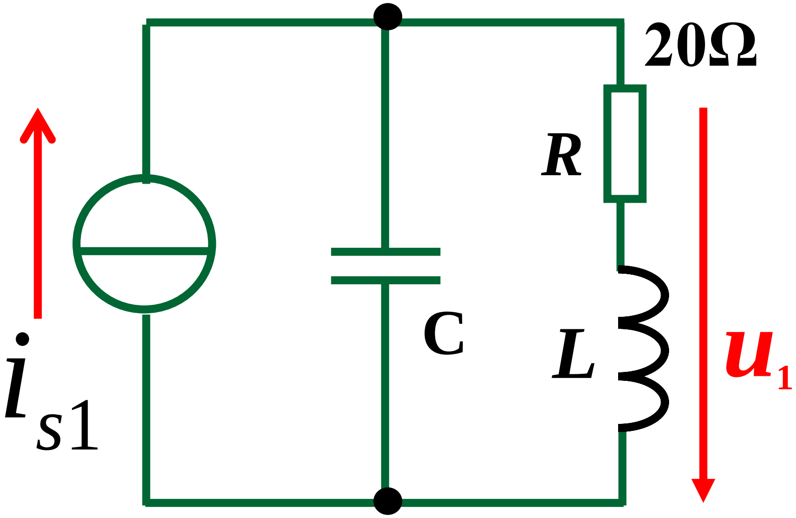
$$I_{S0} = 78.5\mu\text{A}$$

$$\boxed{\begin{aligned} U_0 &= RI_{S0} \\ &= 20 \times 78.5 \times 10^{-6} \\ &= 1.57\ \text{mV} \end{aligned}}$$



(2). **I_{s1} acts alone:**

$$i_{s1} = 100 \sin 10^6 t \ \ \mu\text{A}$$



$$\frac{1}{\omega_1 C} = \frac{1}{10^6 \times 1000 \times 10^{-12}} = 1\text{k}\Omega$$

$$\omega_1 L = 10^6 \times 10^{-3} = 1\text{k}\Omega$$

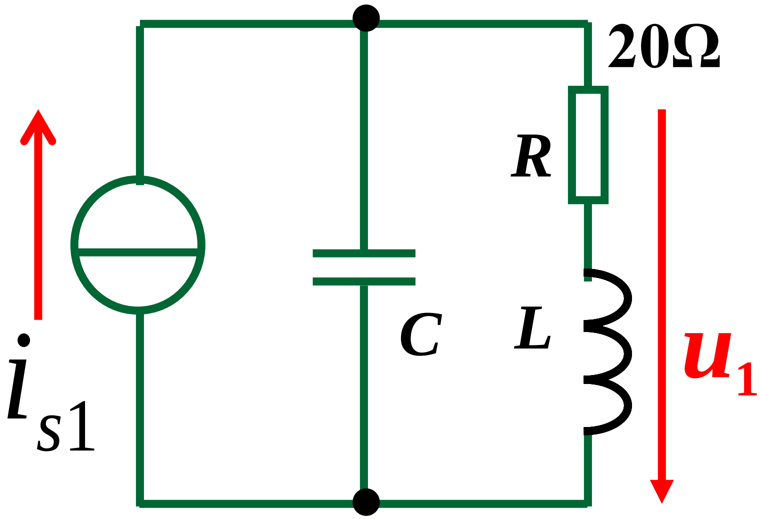
$$L = 1\text{mH}$$

$$C = 1000\text{pF}$$

$$\omega = 10^6 \text{rad/s}$$

$$Z(\omega_1) = \frac{(R + jX_L)\cdot(-jX_C)}{R + j(X_L - X_C)} \approx \frac{X_L X_C}{R} = \frac{L}{RC} = 50\text{k}\Omega$$

$$\boxed{X_L >> R}$$



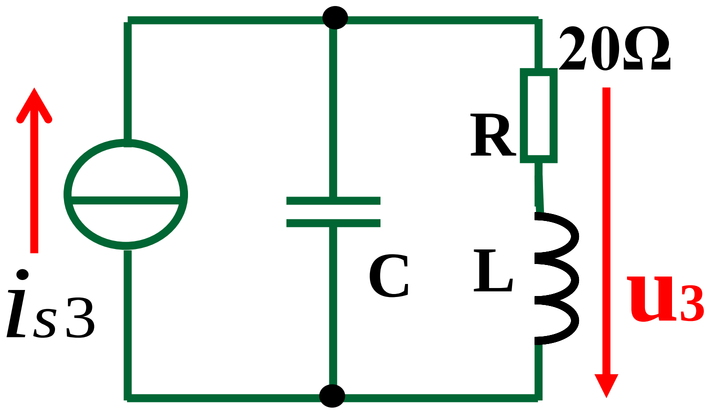
$$Z(\omega_1) = 50\text{K}\Omega$$

$$i_{s1} = 100\sin 10^6 t \ \ \mu\text{A}$$

$$\boxed{\begin{aligned} \dot{U}_1 &= \dot{I}_1 \cdot Z(\omega_1) = \frac{100\times 10^{-6}}{\sqrt{2}} \cdot 50 \\ &= \frac{5000}{\sqrt{2}}\ \text{mV} \end{aligned}}$$

(3). **I_{s3} acts alone:**

$$i_{s3} = \frac{100}{3}\sin 3\cdot 10^6 t \quad \mu A$$



$$\frac{1}{\omega_3 C} = \frac{1}{3\times 10^6 \times 1000 \times 10^{-12}}$$

$$= 0.33K\Omega$$

$$\omega_3 L = 3\times 10^6 \times 10^{-3} = 3k\Omega$$

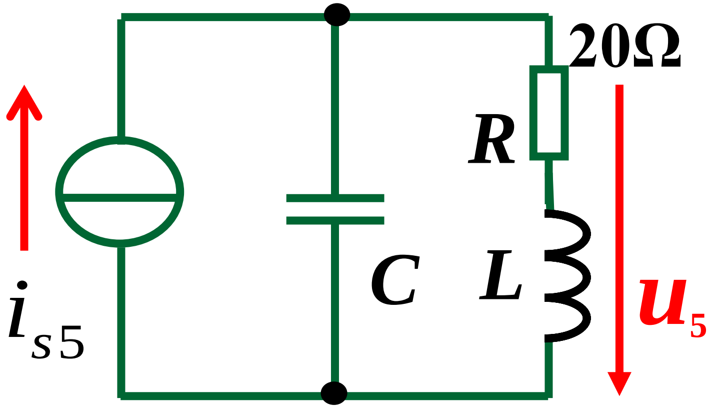
$$Z(3\omega_1) = \frac{(R + jX_{L3})(-jX_{C3})}{R + j(X_{L3} - X_{C3})}$$

$$= 374.5\angle -89.19^\circ\,\Omega$$

$$\begin{cases} i_{s3} = \dfrac{100}{3}\sin 3\cdot 10^{6} t \ \mu\mathrm{A} \\ Z(3\omega_1) = 374.5\angle -89.19^\circ\,\Omega \end{cases}$$

$$\boxed{\begin{aligned} \dot{U}_3 &= \dot{I}_{S3}\cdot Z(3\omega_1) \\ &= 33.3\times\frac{10^{-6}}{\sqrt{2}}\times 374.5\angle -89.19^\circ \\ &= \frac{12.47}{\sqrt{2}}\angle -89.2^\circ\,\mathrm{mV} \end{aligned}}$$

(4). I_{s5} acts alone:



$$i_{s5} = \frac{100}{5}\sin 5\cdot 10^6 t \quad \mu A$$

$$\frac{1}{\omega_5 C} = \frac{1}{5\times 10^6 \times 1000 \times 10^{-12}}$$

$$= 0.2(\mathrm{K}\Omega)$$

$$\omega_5 L = 5\times 10^6 \times 10^{-3} = 5\mathrm{k}\Omega$$

$$Z(5\omega_1) = \frac{(R + jX_{L5})(-jX_{C5})}{R + j(5X_{L5} - X_{C5})}$$

$$= 208.3\angle -89.53^\circ \Omega$$

$$\begin{cases} i_{s5} = \dfrac{100}{5}\sin 5\cdot 10^{6} t \ \mu\text{A} \\ Z(5\omega_1) = 208.3\angle -89.53^{\circ}\ \Omega \end{cases}$$

$$\boxed{\begin{aligned} \dot{U}_5 &= \dot{I}_{5s}\cdot Z(5\omega_1) \\ &= 20\times 10^{-6}\Big/\sqrt{2}\cdot 208.3\angle -89.53^{\circ} \\ &= \frac{4.166}{\sqrt{2}}\angle -89.53^{\circ}\,\text{mV} \end{aligned}}$$

3. Add the individual responses using the superposition principle (time domain).

$$U_0 = 1.57\ \text{mV}$$

$$\dot{U}_3 = \frac{12.47}{\sqrt{2}} \angle -89.2^\circ\ \text{mV}$$

$$\dot{U}_1 = \frac{5000}{\sqrt{2}} \text{mV}$$

$$\dot{U}_5 = \frac{4.166}{\sqrt{2}} \angle -89.53^\circ\ \text{mV}$$

$$\boxed{\begin{aligned} u &= U_0 + u_1 + u_3 + u_5 \\ &\approx 1.57 + 5000 \sin \omega t \\ &\quad + 12.47 \sin(3\omega t - 89.2^\circ) \\ &\quad + 4.166 \sin(5\omega t - 89.53^\circ)\ \text{mV} \end{aligned}}$$

Notice:

1. use the superposition principle in time domain, not in phasor domain.

$$\dot{U} \cancel{=} U_0 + \dot{U}_{\omega 1} + \dot{U}_{\omega 3} + \dot{U}_{\omega 5} + \cdots$$

2. X_C、 X_L changes with your harmonic frequencies.

17.4 Average Power and RMS Values

17.4.1 RMS Values

$$U=\sqrt{\frac{1}{T}\int_0^T u^2(t)dt}$$ （ periodic function)

$$u(t)=U_0+\sum_{k=1}^{\infty}\sqrt{2}\,U_k\sin(k\omega t+\varphi_k)$$

$u^2(t)$ {
各次谐波的平方：$U^2{}_0$，$u^2{}_k(t)$

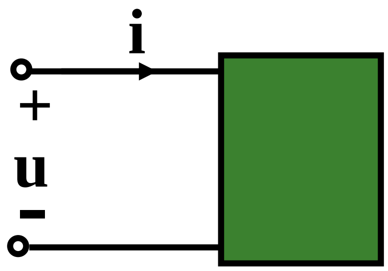
不同次谐波的乘积：

$$U_{km}\sin(k\omega t+\varphi_k)U_{qm}\sin(q\omega t+\varphi_q)$$

$$U=\sqrt{U^2{}_0+U^2{}_1+U^2{}_2+\cdots}$$

17.4.2 Average Power

$$P=\frac{1}{T}\int_0^T u(t)i(t)dt$$



$$u(t)=U_0+\sum_{k=1}^{\infty}\sqrt{2}U_k\sin(k\omega t+\varphi_{uk})$$

$$i(t)=I_0+\sum_{k=1}^{\infty}\sqrt{2}I_k\sin(k\omega t+\varphi_{ik})$$

$u(t)i(t)\begin{cases}\text{同次谐波电压与电流的乘积} & u_k(t)i_k(t)\\ \text{不同次谐波电压与电流的乘积} & u_k(t)i_q(t)\end{cases}$

17.4.2 Average Power

$$P=U_0I_0+\sum_{k=1}^{\infty}\frac{1}{T}\int_0^T u_k(t)i_k(t)dt$$

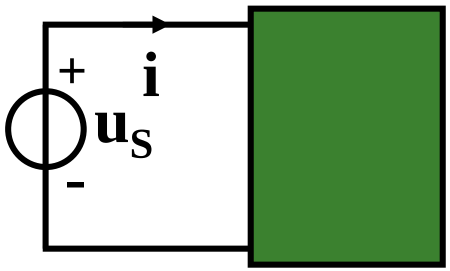
➢The total average power is the sum of the average powers in each harmonically related voltage and current.

$$P=U_0I_0+U_1I_1\cos\varphi_1+U_2I_2\cos\varphi_2+U_3I_3\cos\varphi_3+\cdots$$

17.4.3 Apparent Power and Power Factor

$$S \triangleq UI=\sqrt{U_0^2+U_1^2+U_3^2+\cdots}\sqrt{I_0^2+I_1^2+I_3^2+\cdots}$$

$$\cos\theta \triangleq P/S$$

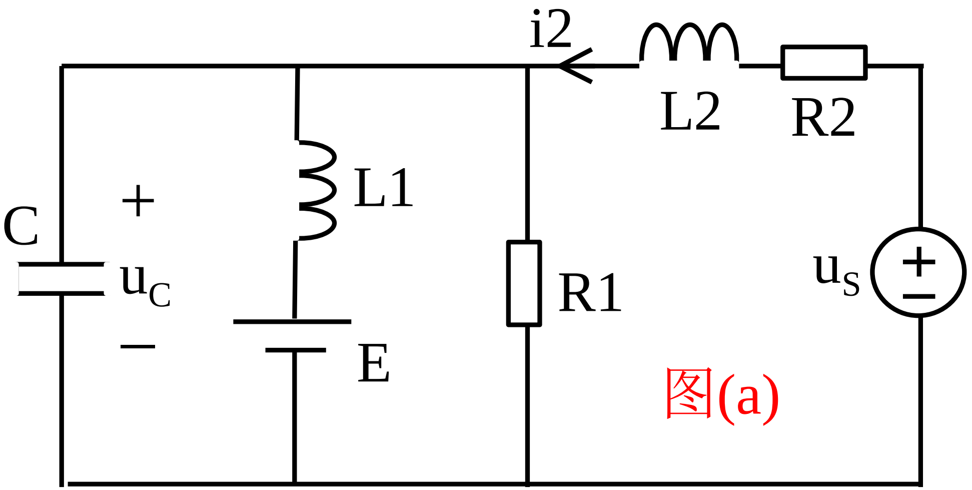


$$u_S=\sqrt{2}\,U\sin\omega t$$

$$i=I_0+\sqrt{2}\,I_1\sin(\omega t+\varphi_1)+\sqrt{2}\,I_2\sin(2\omega t+\varphi_2)+\cdots$$

$$\cos\theta=\frac{P}{S}=\frac{UI_1\cos\varphi_1}{UI}=\frac{I_1\cos\varphi_1}{I}<\cos\varphi_1$$

Example:



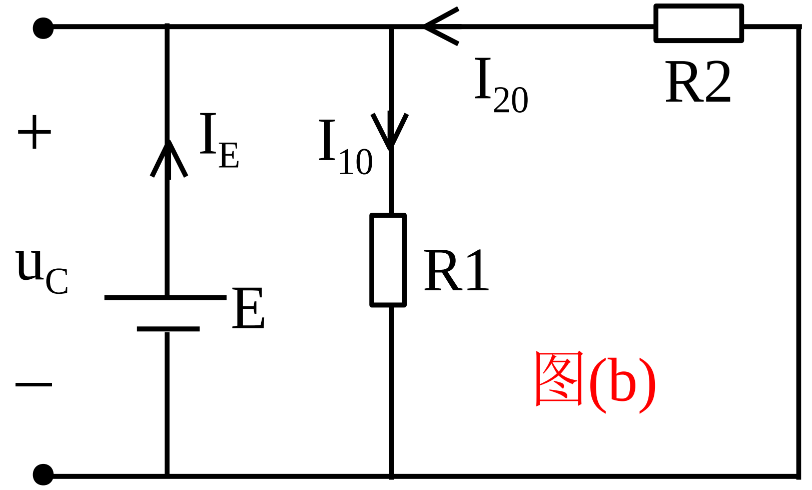
Given: $R_1=1\Omega$, $R_2=2\Omega$, $L_1=1H$, $L_2=2H$,

$C=\frac{1}{4}F$, $E=4V$, $u_S=10\sqrt{2}\sin 2t\ V$,

Find: i_2, I_2; u_C, P_{u_S}, P_E, P_R。

Solution:

(1) E acts alone： 图(b)

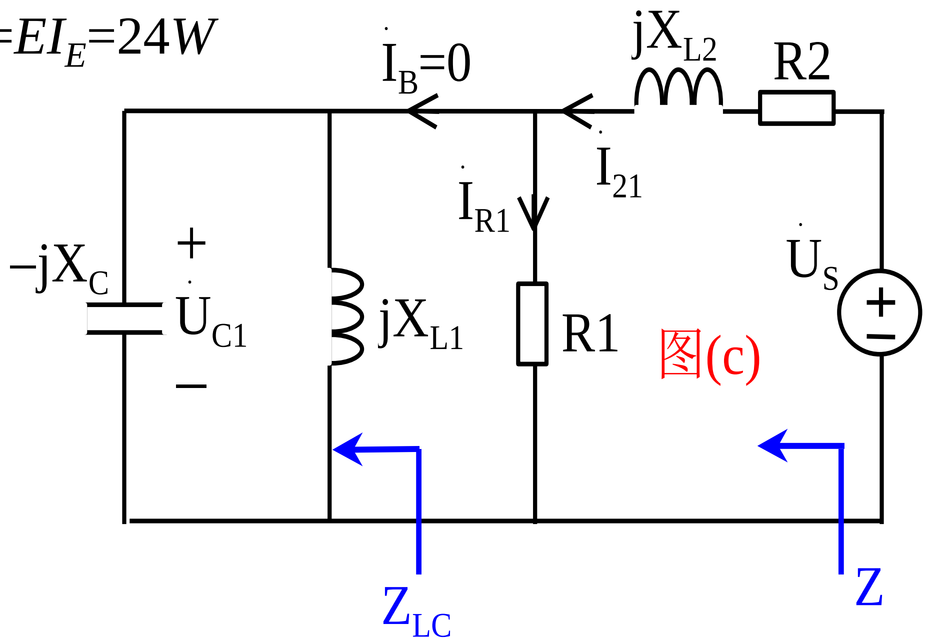


$$I_{20}=-\frac{E}{R_2}=-2A,\ \ I_{10}=\frac{E}{R_1}=4A$$

$$I_E=I_{10}-I_{20}=6A,\quad u_{C0}=E=4V$$

$P_{10}=R_1I_{10}^2=16\mathrm{W}$， $P_{20}=R_2I_{20}^2=8W$； $P_E=EI_E=24W$

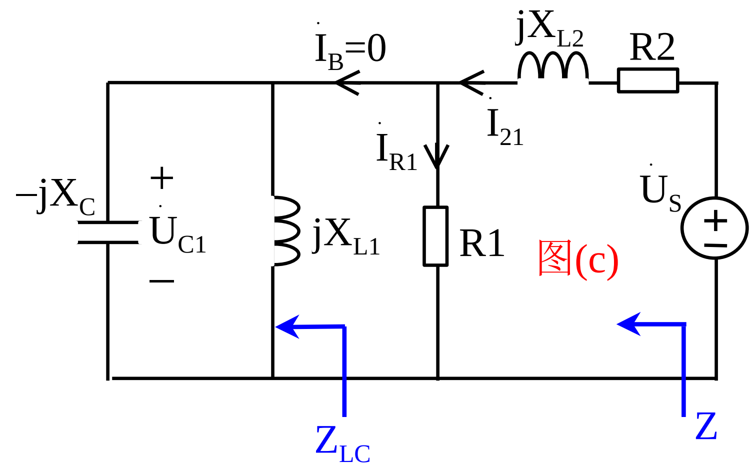
(2) u_S acts alone ： 图(c)



$$\omega L_1=2\Omega,\ \frac{1}{\omega C}=2\Omega,\ \omega L_2=4\Omega$$

$Z_{LC}=\infty$， parallal resonance， $\dot{I}_B=0$

$$Z = R_1 + R_2 + j\omega L_2 = 3 + j4 = 5\angle 53.1^{\circ}\,\Omega$$



$$\dot{I}_{21} = \dot{I}_{R1} = \frac{\dot{U}_S}{Z} = 2\angle -53.1^{\circ}\ A$$

$$\dot{U}_{C1} = R_1 \dot{I}_{R1} = 2\angle -53.1^{\circ}\ V$$

$$i_{21} = 2\sqrt{2}\sin(2t - 53.1^{\circ})A,\ u_{C1} = 2\sqrt{2}\sin(2t - 53.1^{\circ})\,V$$

$P_{11}=R_1I_{R1}{}^2=4W$，$P_{21}=R_2I_{21}{}^2=8W$，$P_{us}=U_SI_{21}\cos(\theta_{us}-\theta_{i21})=12W$

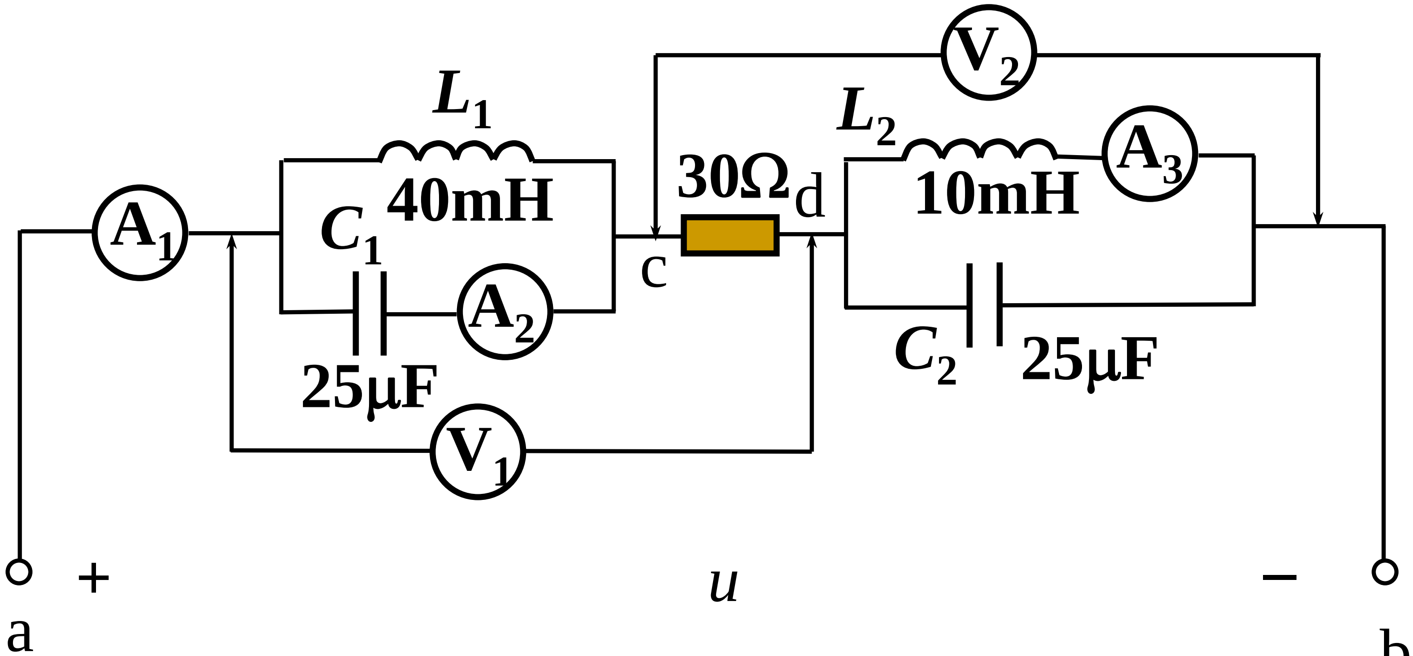
叠加：$P_{\mathrm{E}}=24\mathrm{W}$，$P_{\mathrm{us}}=12\mathrm{W}$，$P_R=P_{10}+P_{20}+P_{11}+P_{21}=36W$

$$i_2 = I_{20} + i_{21} = -2 + 2\sqrt{2}\sin(2t - 53.1^{\circ})\ A,\ I_2 = \sqrt{2^2 + 2^2} = \sqrt{8} = 2.83A$$

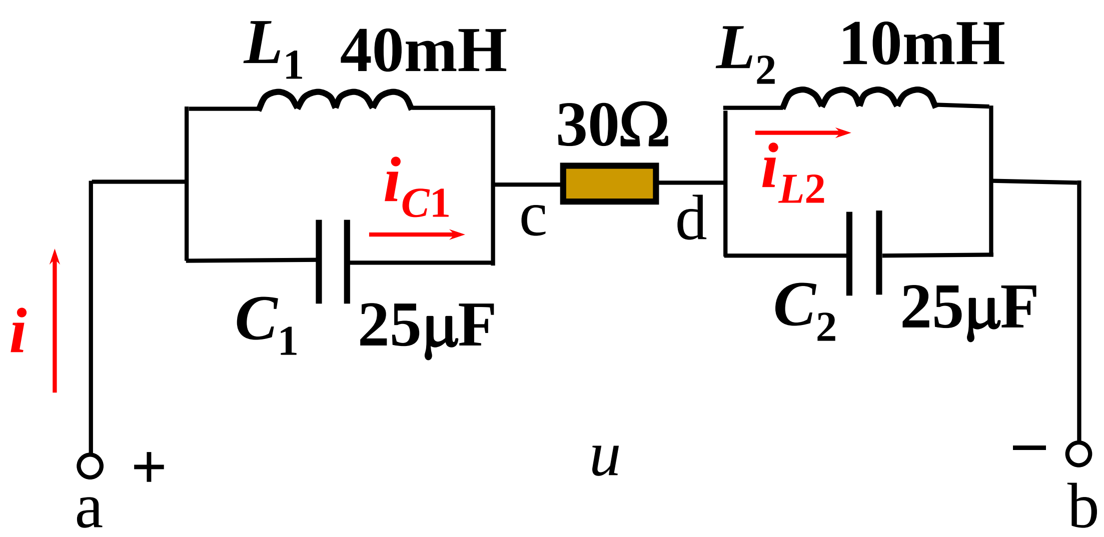
$$u_C = u_{C0} + u_{C1} = 4 + 2\sqrt{2}\sin(2t - 53.1^{\circ})\ A$$

$$u = 30 + 120\cos 1000t + 60\cos(2000t + \frac{\pi}{4})\ \text{V}.$$

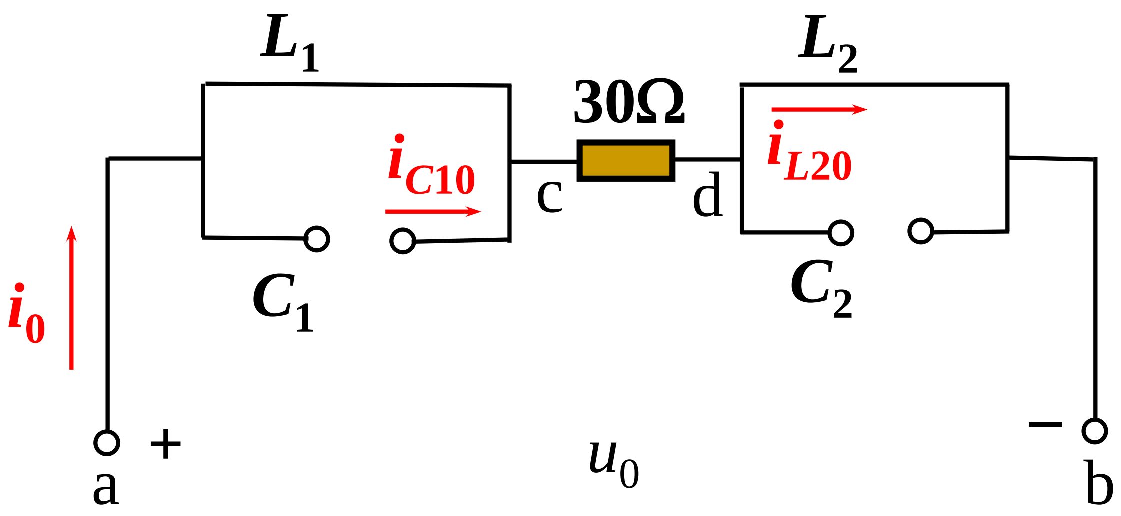
Calculate the reading of each meter and the total average power



Solution:



(1) u_0=30V acts alone，L_1、 L_2 is short circuit， C_1 、 C_2 is open circuit。



$i_0 = i_{L20} = u_0/R$ =30/30=1A, i_{C10}=0, $u_{ad0} = u_{cb0} = u_0$ =30V

(2) u_1=120cos1000t V **acts alone**

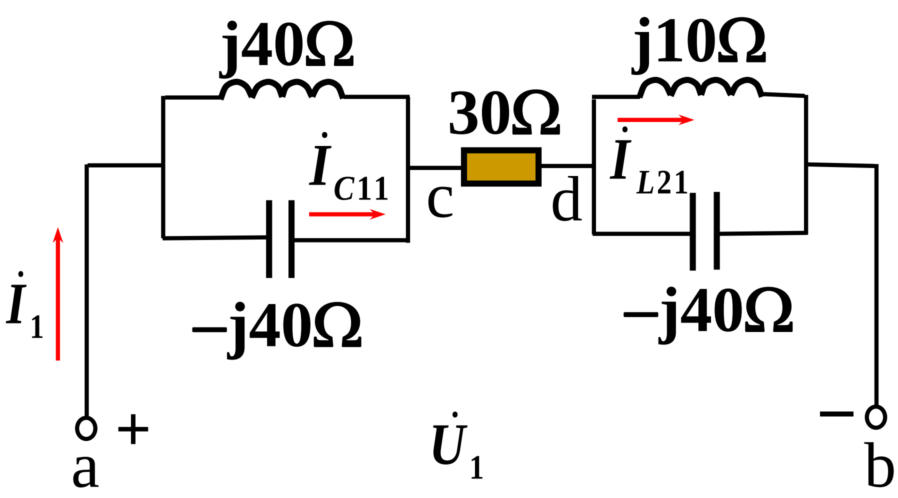
$$\omega L_1 = 1000\times 40\times 10^{-3} = 40\Omega,\ \omega L_2 = 1000\times 10\times 10^{-3} = 10\Omega$$

$$\frac{1}{\omega C_1} = \frac{1}{\omega C_2} = \frac{1}{1000\times 25\times 10^{-6}} = 40\Omega$$

L_1、C_1 发生并联谐振。

$$\dot{U}_1 = 120\angle 0^\circ \text{ V}$$

$$\dot{I}_1 = \dot{I}_{L21} = 0,\quad \dot{U}_{cb1} = 0,\quad \dot{U}_{ad1} = \dot{U}_1 = 120\angle 0^\circ \text{ V}$$



$$\dot{I}_{C11} = \mathrm{j}\omega C_1 \dot{U}_1 = \frac{120\angle 0^\circ}{-\mathrm{j}40} = 3\angle 90^\circ \text{ A}$$

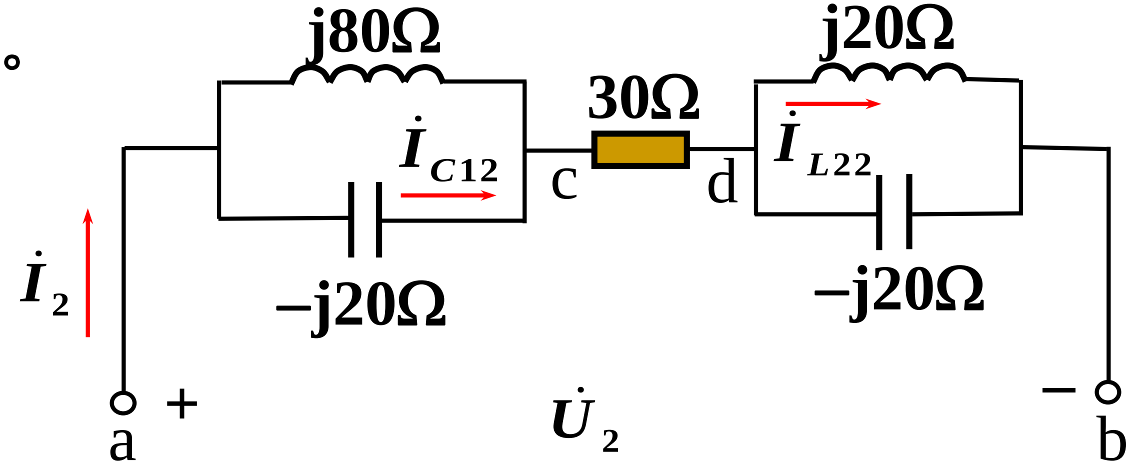
(3) u_2=60cos(2000t+π /4)V **acts alone**

$$2\omega L_1 = 2000 \times 40 \times 10^{-3} = 80\Omega,\quad 2\omega L_2 = 2000 \times 10 \times 10^{-3} = 20\Omega$$

$$\frac{1}{2\omega C_1} = \frac{1}{2\omega C_2} = \frac{1}{2000 \times 25 \times 10^{-6}} = 20\Omega$$

L_2、C_2 发生并联谐振。

$$\dot{U}_2 = 60\angle 45^\circ \text{ V}$$



$$\dot{I}_2 = \dot{I}_{C12} = 0,\quad \dot{U}_{ad2} = 0,\quad \dot{U}_{cb2} = \dot{U}_2 = 60\angle 45^\circ \text{ V}$$

$$\dot{I}_{L22} = \frac{\dot{U}_1}{\text{j}2\omega L_2} = \frac{60\angle 45^\circ}{\text{j}20} = 3\angle -45^\circ \text{ A}$$

所求的电压、电流的瞬时值为：

$$i=i_0+i_1+i_2=1\text{A}$$

$$i_{C1}=i_{C10}+i_{C11}+i_{C12}=3\cos(1000t+90°)\ \text{A}$$

$$i_{L2}=i_{L20}+i_{L21}+i_{L22}=3+3\cos(2000t-45°)\ \text{A}$$

$$u_{ad}=u_{ad0}+u_{ad1}+u_{ad2}=30+120\cos 1000t\ \text{V}$$

$$u_{cb}=u_{cb0}+u_{cb1}+u_{cb2}=30+60\cos(2000t+45°)\ \text{V}$$

电流表A_1的读数： $I=1\ \text{A}$

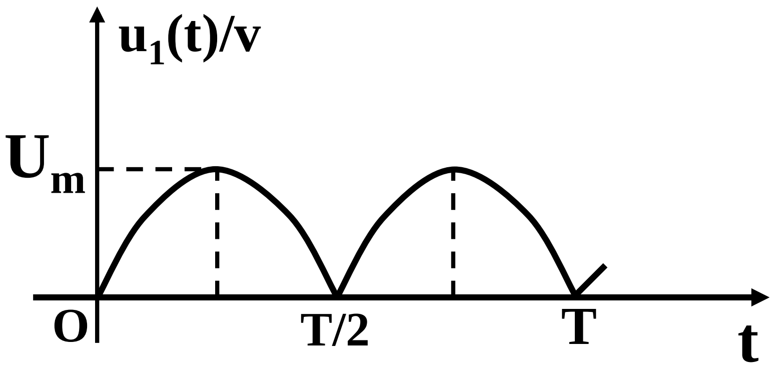
电流表A_2的读数： $3/\sqrt{2}=2.12\text{A}$

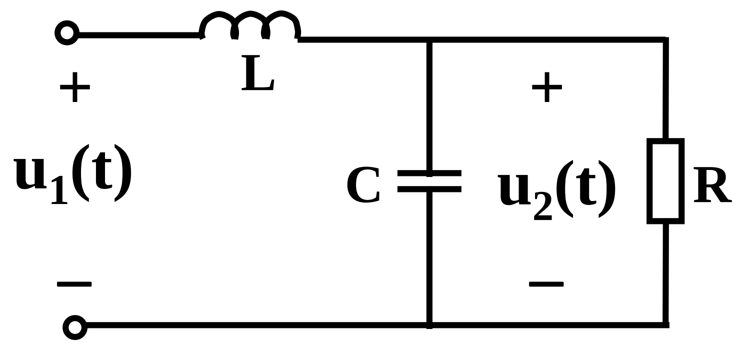
电流表A_3的读数： $\sqrt{1^2+(3/\sqrt{2})^2}=2.35\text{A}$

电压表V_1的读数： $\sqrt{30^2+(120/\sqrt{2})^2}=90\text{V}$

电压表V_2的读数： $\sqrt{30^2+(60/\sqrt{2})^2}=52.0\text{V}$

1、图示全波整流器的输出电压$u_1(t)$，U_m=157V，T=0.02s，通过LC滤波电路作用于负载R，L=5H，C=10μF，R= 2kΩ。求负载两端电压$u_2(t)$及其有效值。谐波电压考虑到4次谐波。



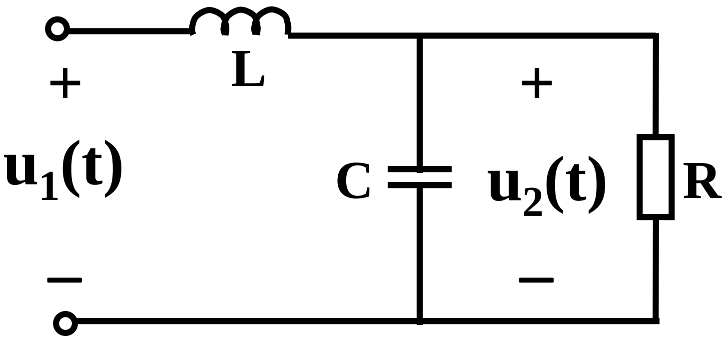


由查表

$$u_1(t)=\frac{4\times 157}{\pi}\left(\frac{1}{2}-\frac{1}{3}\cos 2\omega t-\frac{1}{15}\cos 4\omega t\right)$$

$$=100-66.7\cos 2\omega t-13.3\cos 4\omega t$$

$\omega=2\pi/T=314$ rad/s

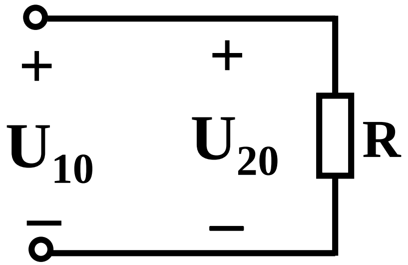


$u_1(t) = 100 - 66.7\cos 2\omega t - 13.3\cos 4\omega t$

$L=5H$，$C=10\mu F$，$R=2k\Omega$

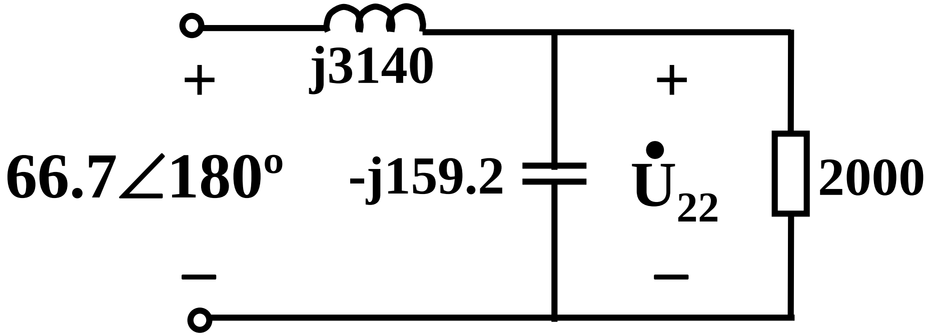
$\omega = 2\pi/T = 314\ rad/s$

直流分量单独作用：

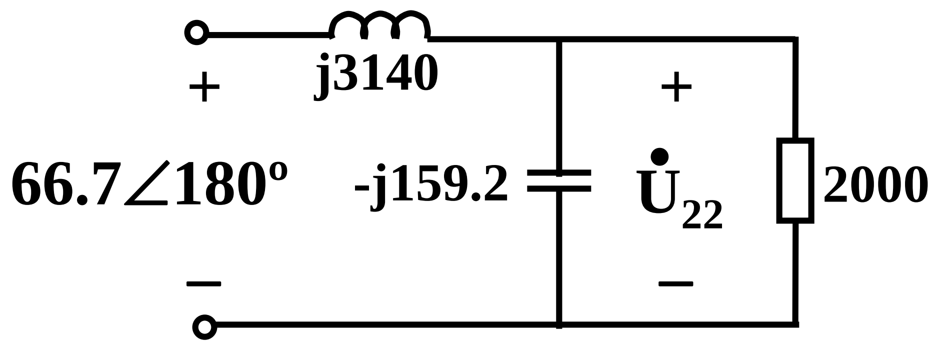


$U_{20}=100V$

二次谐波单独作用：$2\omega L=3140\Omega$，$\dfrac{1}{2\omega C}=159.2\Omega$



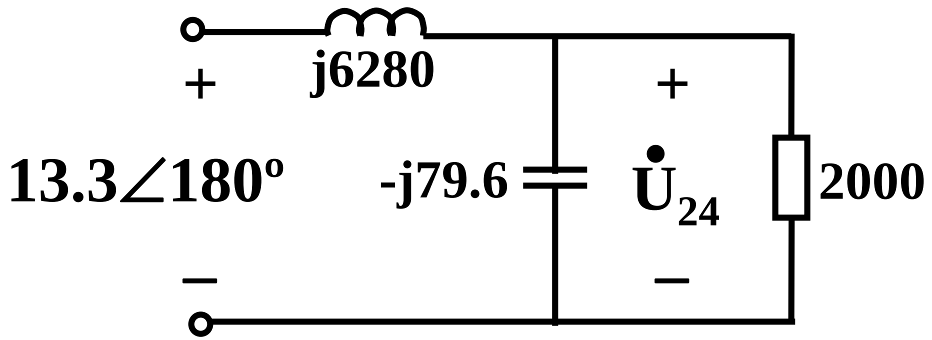
二次谐波单独作用：



$$\dot{U}_{22}=\frac{66.7\angle 180^{\circ}}{j3140+\dfrac{2\times 10^{3}(-j159.2)}{2\times 10^{3}-j159.2}}\cdot\frac{2\times 10^{3}(-j159.2)}{2\times 10^{3}-j159.2}$$

$$=\frac{21237280\angle 90^{\circ}}{5982521\angle 85.2^{\circ}}=3.55\angle 4.8^{\circ}$$

4次谐波单独作用：$4\omega L=6280\Omega$， $\dfrac{1}{4\omega C}=79.6\Omega$



$$\dot{U}_{24}=\frac{13.3\angle 180^\circ}{j6280+\dfrac{2\times 10^3(-j79.6)}{2\times 10^3-j79.6}}\cdot\frac{2\times 10^3(-j79.6)}{2\times 10^3-j79.6}=\frac{2117360\angle 90^\circ}{12410871\angle 87.7^\circ}$$

$$=0.17\angle 2.3^\circ$$

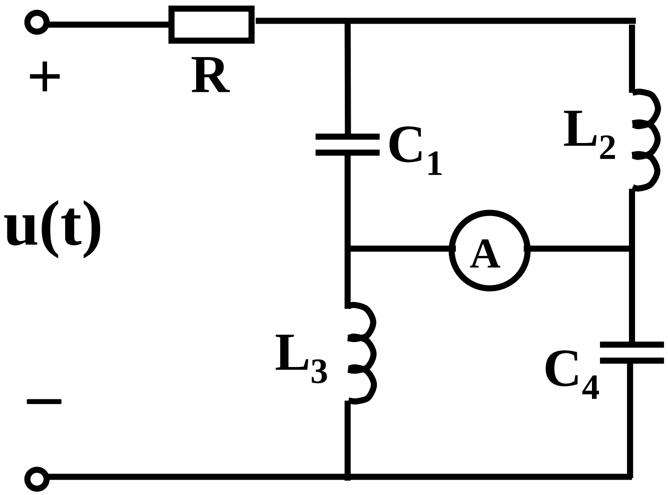
$$u_2(t)=100+3.55\cos(2\omega t+4.8^\circ)+0.17\cos(4\omega t+2.3^\circ)$$

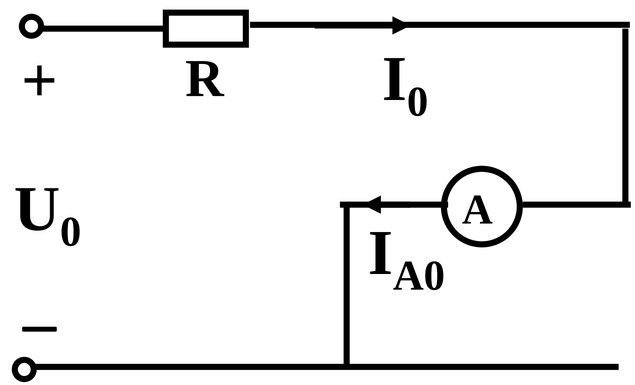
$$U_2=\sqrt{100^2+\frac{3.55^2}{2}+\frac{0.17^2}{2}}\approx 100\text{V}$$

小结： 谐波阻抗 瞬时值叠加

2、$u(t)=60+282\sin\omega t+169\sin(2\omega t-22.5^{\circ})$V,

$R=10\Omega$, $\frac{1}{\omega C_1}=40\Omega$, $\omega L_2=20\Omega$, $\omega L_3=20\Omega$, $\frac{1}{\omega C_4}=20\Omega$，calculate: the reading of the ammeter and the average power of the source.



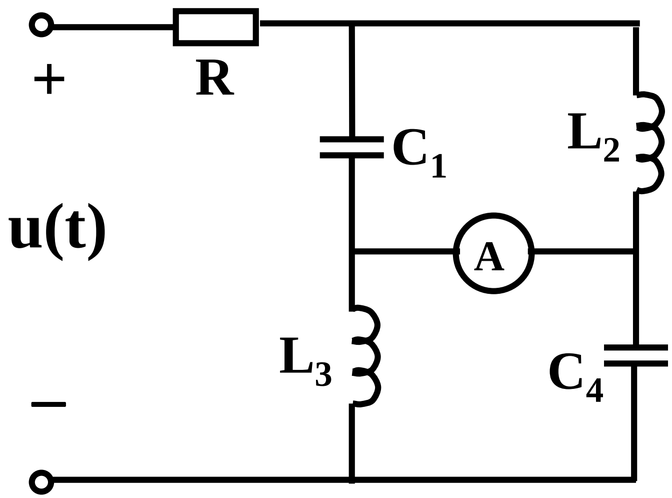


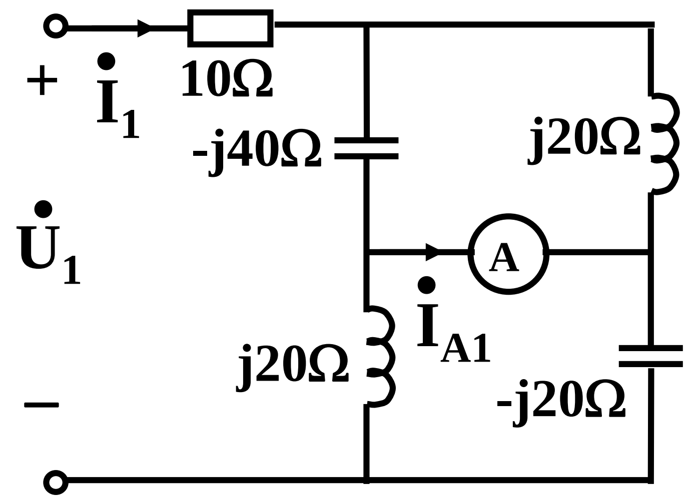
直流分量单独作用：

$$I_0=I_{A0}=60/10=6\text{ A}$$

$$P_0=60\times 6=360\text{ W}$$

$u(t)=60+282\sin\omega t+169\sin(2\omega t-22.5^{\circ})\,\mathrm{V}$





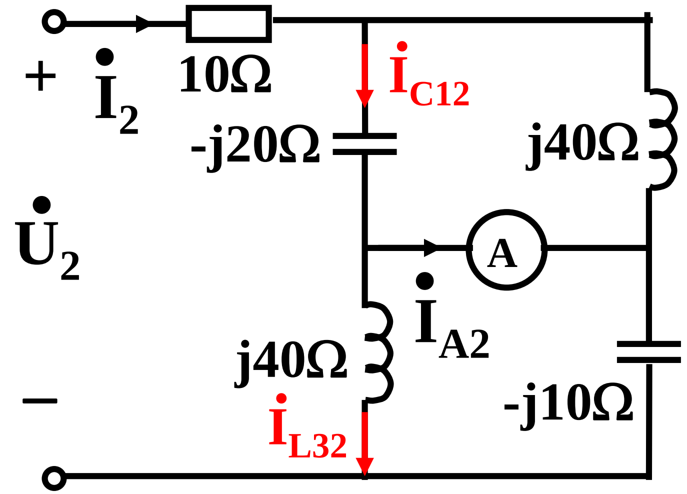
基波分量单独作用：

二次谐波分量单独作用：

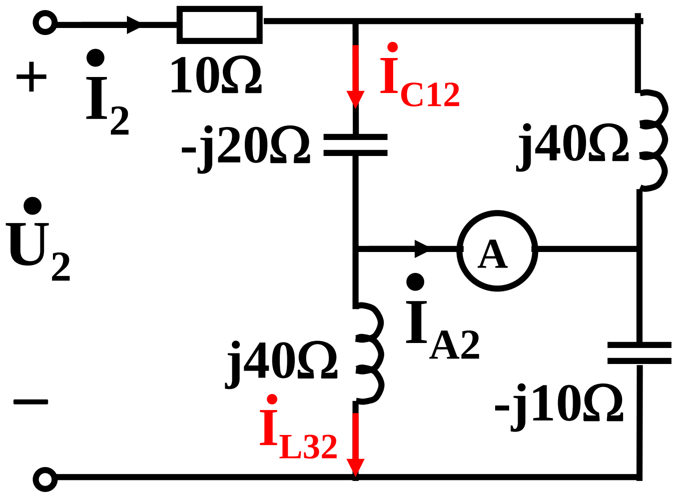
$\dot{I}_1=0$

$\dot{I}_{A1}=\dfrac{200\angle 0^{\circ}}{-j20}=10\angle 90^{\circ}$

$P_1=0$



二次谐波分量单独作用：



$$Z=10+\frac{j40(-j20)}{j20}+\frac{j40(-j10)}{j30}$$

$$=54\angle-79.4^\circ$$

$$\dot{I}_2=\frac{120\angle-22.5^\circ}{54\angle-79.4^\circ}=2.22\angle56.9^\circ$$

$$P_2=120\times2.22\cos(-79.4^\circ)=49\text{W}$$

$$\dot{I}_{C12}=\frac{j40}{j20}\times2.22\angle56.9^\circ$$

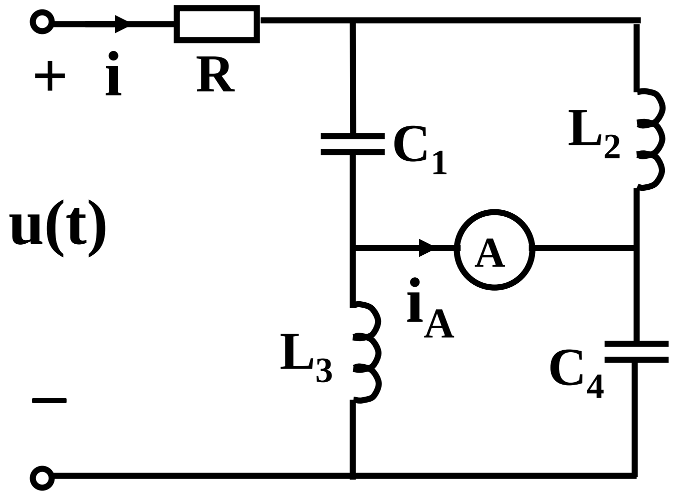
$$=2.42+j3.71$$

$$\dot{I}_{A2}=\dot{I}_{C12}-\dot{I}_{L32}=2.82+j4.33$$

$$=5.17\angle56.9^\circ$$

$$\dot{I}_{L32}=\frac{-j10}{j30}\times2.22\angle56.9^\circ$$

$$=-0.403-j0.618$$



$I_0=I_{A0}=6\ A$ $\quad P_0=360\ W$

$\dot{I}_1=0$ $\quad P_1=0$

$\dot{I}_{A1}=10\angle 90°$

$\dot{I}_2= 2.22\angle 56.9°$ $\quad P_2=49W$

$\dot{I}_{A2} =5.17\angle 56.9°$

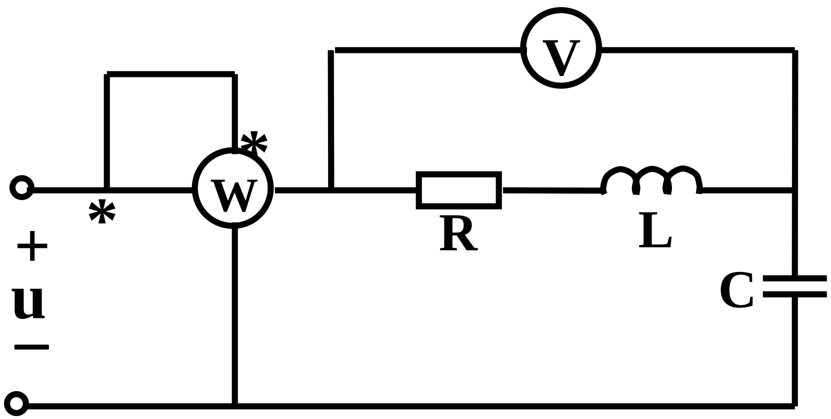
$$I_A=\sqrt{6^2+10^2+5.17^2}=12.8\ A$$

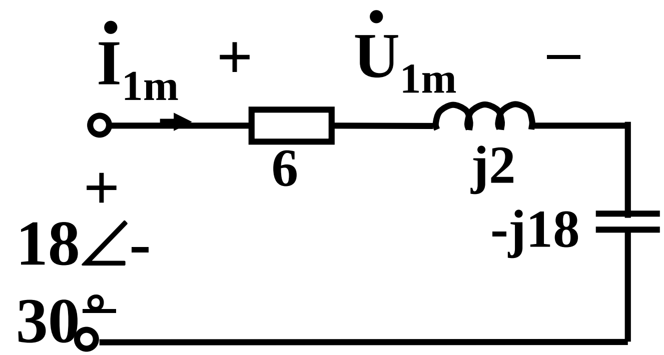
$$P=P_0+P_1+P_2=409\ \ W$$

或 $P=10I^2=10\times(6^2+2.22^2)=409\ W$

3、

R=6Ω， ωL=2Ω，1/ωC=18Ω，u=[18sin(ωt–30º)+18sin3ωt+9sin(5ωt+90º)]V，求电压表和功率表的读数。





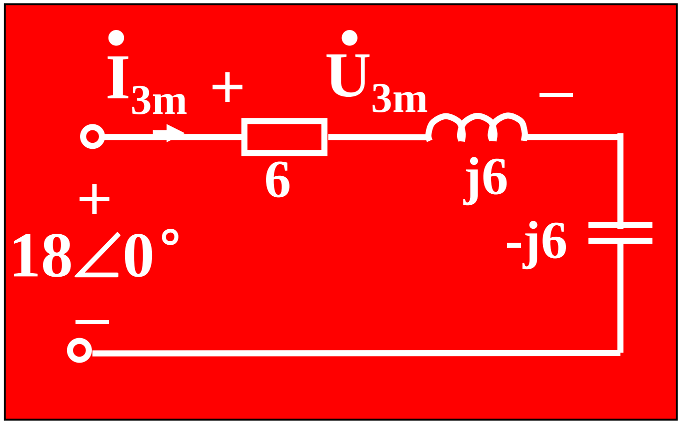
基波电源单独作用：

$$\dot{I}_{1m}=\frac{18\angle-30°}{6-j16}=1.05\angle 39.4°$$

$$\dot{U}_{1m}=(6+j2)\times 1.05\angle 39.4°=6.64\angle 57.8°$$

$$P_1=0.5\times 18\times 1.05\cos(-69.4°)=3.32\text{W}$$

三次谐波电源单独作用：

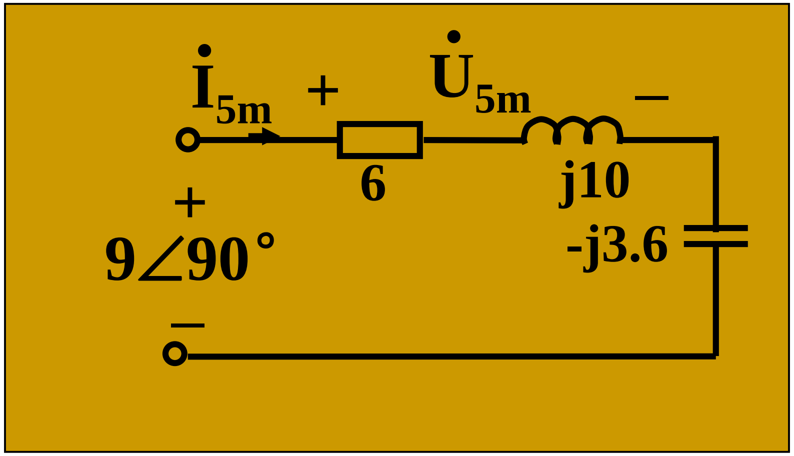


$$\dot{I}_{3m}=\frac{18\angle 0°}{6}=3\angle 0°$$

$$\dot{U}_{3m}=(6+j6)\times 3\angle 0°$$
$$=25.5\ \angle 45°$$

$$P_3=0.5\times 18\times 3=27W$$

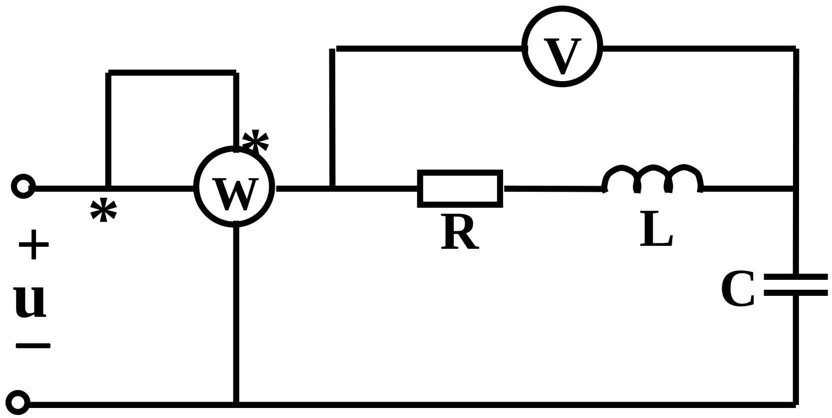
五次谐波电源单独作用：



$$\dot{I}_{5m}=\frac{9\angle 90°}{6+j6.4}=1.03\angle 43.2°$$

$$\dot{U}_{5m}=(6+j10)\times 1.03\angle 43.2°$$
$$=12.1\ \angle 102.2°$$

$$P_5=0.5\times 9\times 1.03\cos 46.8°=3.32W$$



$\dot{U}_{1m}= 6.64\angle 57.8°$ $P_1=3.32W$

$\dot{U}_{3m}=25.5\angle 45°$ $P_3=27W$

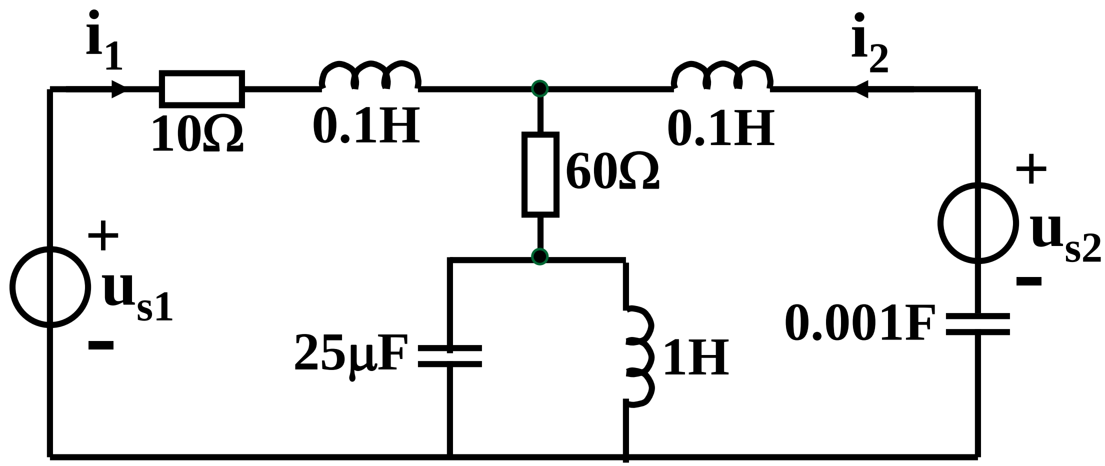
$\dot{U}_{5m}=12.1\angle 102.2°$ $P_5=3.32W$

$$U=\sqrt{\frac{6.64^2}{2}+\frac{25.5^2}{2}+\frac{12.1^2}{2}}$$

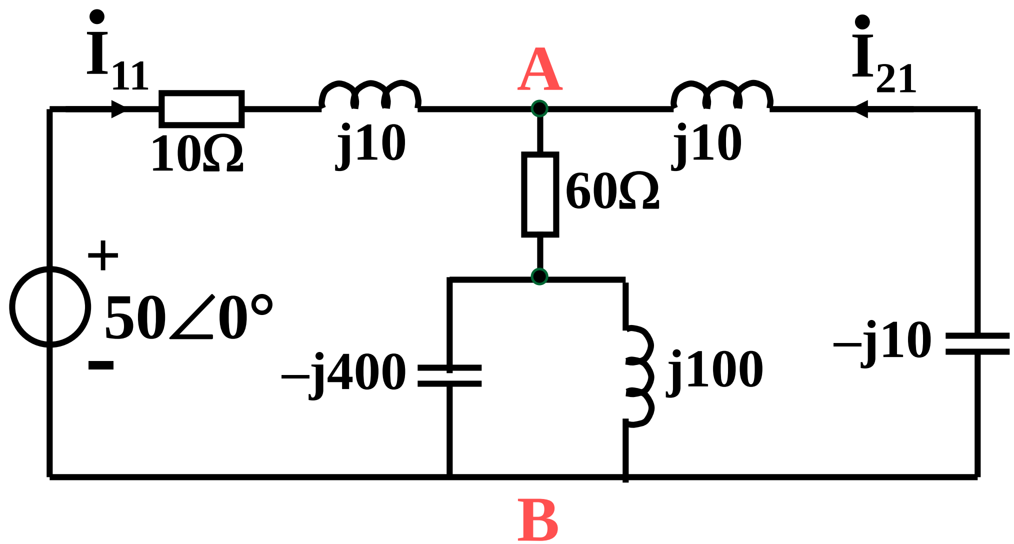
$$=20.5V$$

$$P=P_1+P_3+P_5=33.5W$$

4、图示电路中，$u_{s1}=50\sqrt{2}\sin100t+25\sqrt{2}\sin200t$ V，$u_{s2}=50\sqrt{2}\sin200t$ V。求稳态电流i_1、i_2和各电源提供的功率。



ω=100rad/s的电源作用



$\dot{U}_{AB}=0$

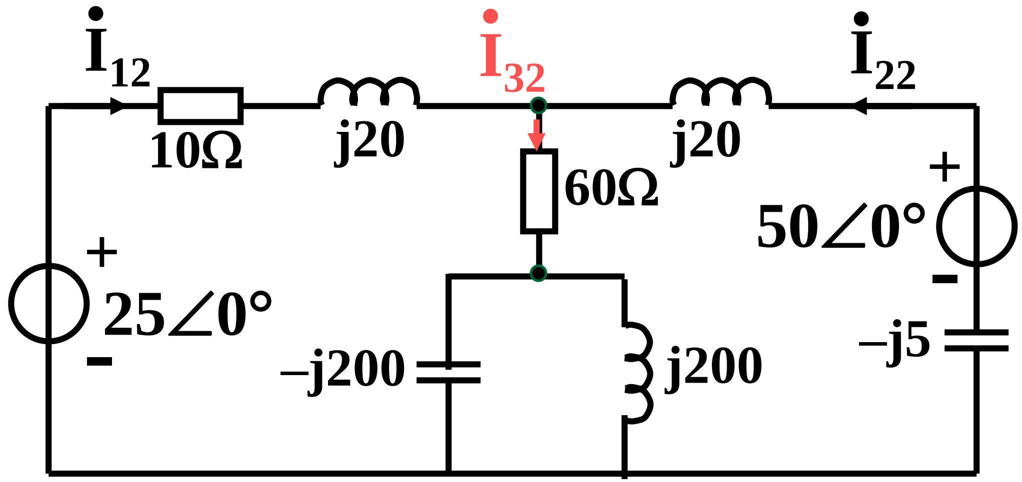
$$\dot{I}_{11}=-\dot{I}_{21}=\frac{50\angle0°}{10+j10}$$

$$=3.54\angle-45°$$

$P_{11}=50\times3.54\cos45°=125\text{W}$

$\dot{I}_{11} = -\dot{I}_{21} = 3.54\angle -45°$ $P_{11}=125W$

ω=200rad/s的电源作用



$\dot{I}_{32}=0$

$$\dot{I}_{12} = -\dot{I}_{22} = \frac{25\angle 0° - 50\angle 0°}{10+j35}$$

$$=0.687\angle 106°$$

$P_{12}=25\times 0.687\cos(-106°) = -4.73W$

$P_{22}=50\times 0.687\cos 74° = 9.47W$

$i_1 = -i_2 = 3.54\sqrt{2}\sin(100t-45°)+0.687\sqrt{2}\sin(200t+106°)$ A

$P_1=125-4.73=120.3W$ $P_2=9.47W$

Balanced loads

Balanced three phase sources

Same amplitude 同一相位点在时间上依次相差T/3

Same period

$$u_A(t)=f(t)$$

$$u_B(t)=f(t-T/3)$$

$$u_C(t)=f(t-2T/3)$$

Three phase voltages produced by AC generators are nonsinusoidal functions in practice.

$$f(t)= -f(t \pm T/2)$$

DC and even functions are not contained by f(t).

17.4 Balanced Three-Phase Circuits Excited By Nonsinusoidal Periodic Functions

Harmonics in balanced three-phase circuits

$$u_{Aph}=U_{ph1m}\sin(\omega_1 t+\theta_1)+U_{ph3m}\sin(3\omega_1 t+\theta_3)+U_{ph5m}\sin(5\omega_1 t+\theta_5)$$

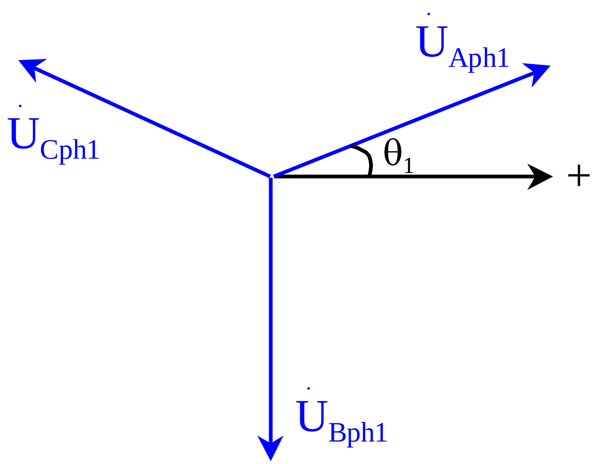
Fundamental frequency：

$$u_{Aph1}=U_{ph1m}\sin(\omega_1 t+\theta_1)$$

$$u_{Bph1}=u_{Aph1}(t-\frac{T}{3})=U_{ph1m}\sin(\omega_1 t+\theta_1-120^{\circ})$$

$$\omega_1 T=2\pi$$

$$u_{Cph1}=u_{Aph1}(t+\frac{T}{3})=U_{ph1m}\sin(\omega_1 t+\theta_1+120^{\circ})$$



The abc sequence or positive sequence（正序）

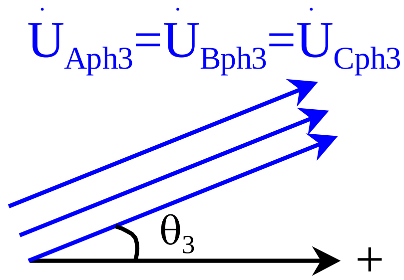
3-rd harmonic frequency:

$$u_{Aph3}(t)=U_{ph3m}\sin(3\omega_1 t+\theta_3)$$

$$u_{Bph3(t)} = u_{Aph3}(t-\frac{T}{3}) = U_{ph3m}\sin\left[3\omega_1(t-\frac{T}{3})+\theta_3\right]$$

$$= U_{ph3m}\sin(3\omega_1 t+\theta_3)$$

$$u_{Cph3}(t) = u_{Aph3}(t+\frac{T}{3}) = U_{ph3m}\sin(3\omega_1 t+\theta_3)$$



Zero sequence（零序）

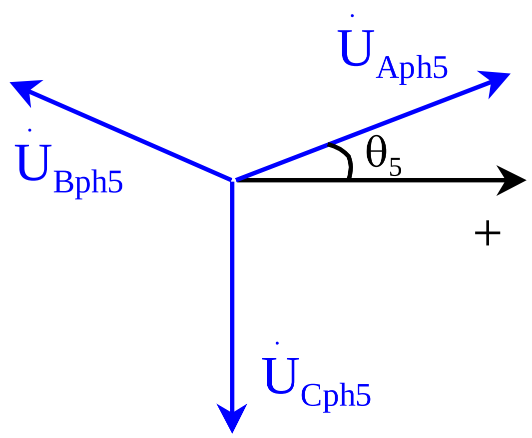
5-th harmonic frequency:

$$u_{Aph5}(t)=U_{ph5m}\sin(5\omega_1 t+\theta_5)$$

$$u_{Bph5(t)} = u_{Aph5}(t-\frac{T}{3}) = U_{ph5m}\sin\left[5\omega_1(t-\frac{T}{3})+\theta_5\right]$$

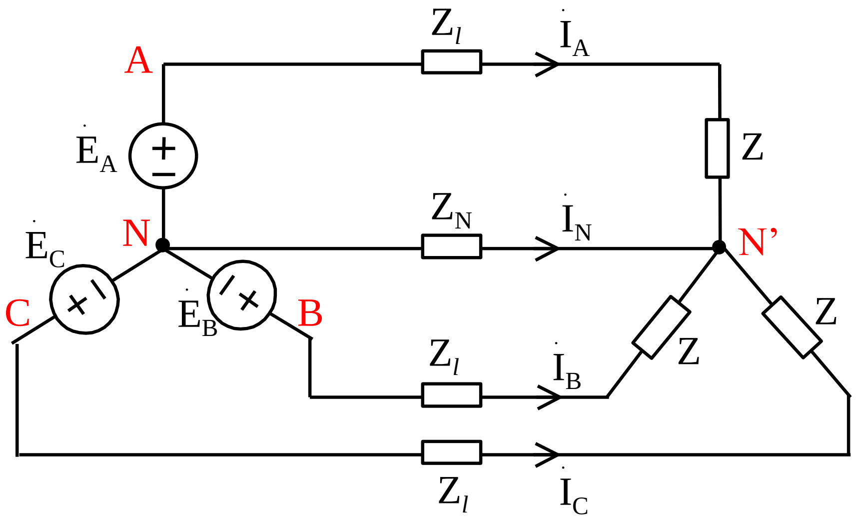
$$= U_{ph5m}\sin(5\omega_1 t+\theta_5+120^\circ)$$

$$u_{Cph5(t)} = u_{Cph5}(t+\frac{T}{3}) = U_{ph5m}\sin(5\omega_1 t+\theta_5-120^\circ)$$



The acb sequence or negative sequence（负序）

$$\dot{U}_{N'N}=\frac{\dfrac{\dot{E}_A}{Z_l+Z}+\dfrac{\dot{E}_B}{Z_l+Z}+\dfrac{\dot{E}_C}{Z_l+Z}}{\dfrac{1}{Z_l+Z}+\dfrac{1}{Z_l+Z}+\dfrac{1}{Z_l+Z}+\dfrac{1}{Z_N}}=0$$



$$\dot{I}_A=\frac{\dot{E}_A}{Z_l+Z},\quad \dot{I}_B=\dot{I}_A\angle-120^\circ,$$

$$\dot{I}_C=\dot{I}_A\angle 120^\circ,\quad \dot{I}_N=0$$

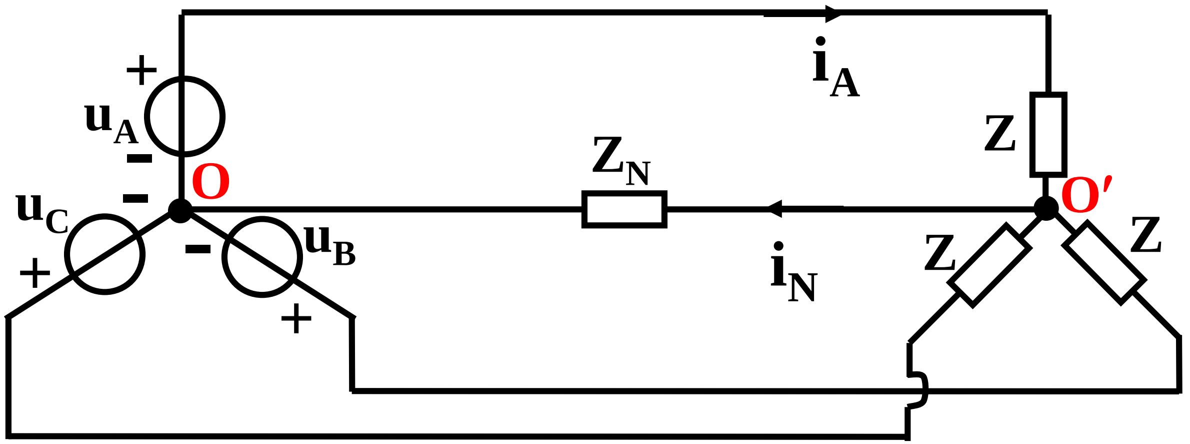
Phase voltages:

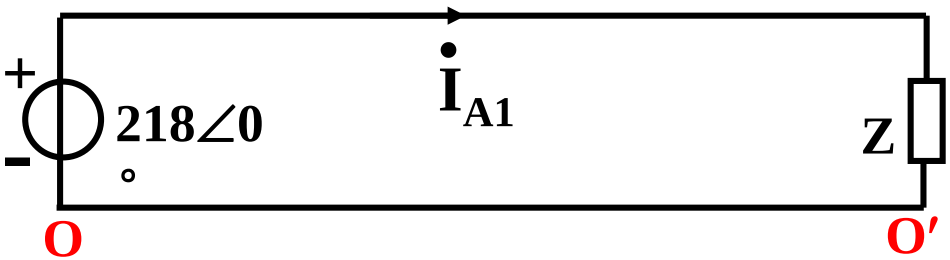
$$\dot{U}_A=\dot{I}_A Z,\quad \dot{U}_B=\dot{U}_A\angle-120^\circ,\quad \dot{U}_C=\dot{U}_A\angle 120^\circ$$

- The neutral line can be removed without affecting the system.
- The balanced three-phase circuit may be replaced with its single-phase equivalent circuit.

A balanced three phase circuit， $u_A=\dfrac{8U_m}{\pi^2}[\sin\omega t+\dfrac{1}{6}\sin 3\omega t+\dfrac{1}{35}\sin 5\omega t]\mathrm{V}$，

U_m=380V， ω =314rad/s，Z=R+jωL=(3+j6)Ω，$Z_N=R_N+j\omega L_N=(1+j2)\Omega$，
calculate:effective value of neutral line current and phase currents





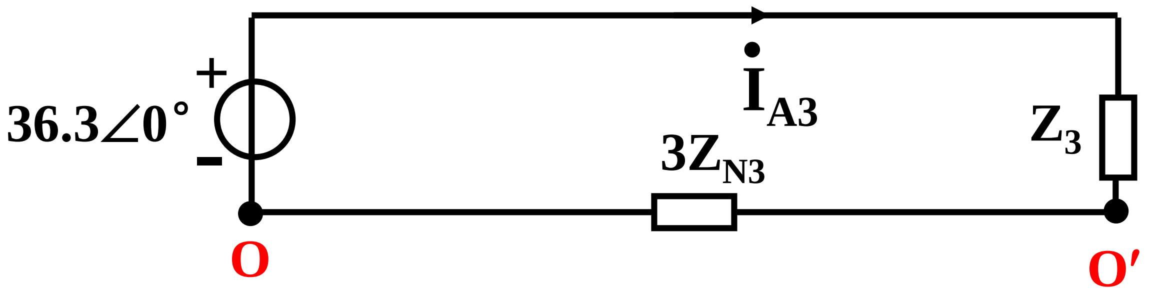
$\dot{I}_{N1}=0$

$$\dot{I}_{A1}=\frac{218\angle 0^\circ}{3+j6}$$

$$=32.5\angle -63.4^\circ$$

$$u_A=\frac{8U_m}{\pi^2}[\sin\omega t+\frac{1}{6}\sin 3\omega t+\frac{1}{35}\sin 5\omega t]\text{V} \qquad U_m=380\text{V}$$

$$Z=R+j\omega L=(3+j6)\Omega \qquad Z_N=R_N+j\omega L_N=(1+j2)\Omega$$

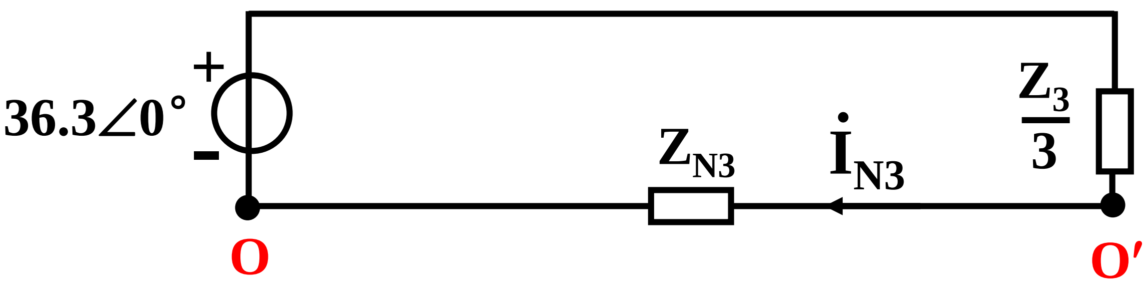


$$Z_3=R+j3\omega L=(3+j18)\Omega$$

$$Z_{N3}=R_N+j3\omega L_N=(1+j6)\Omega$$

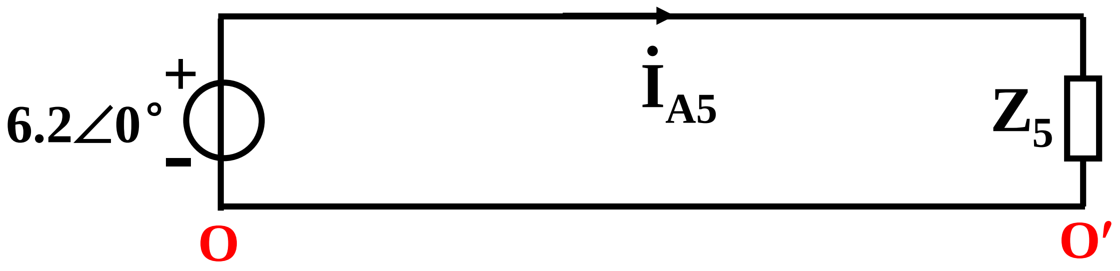
$$\dot{I}_{A3}=\frac{36.3\angle 0°}{6+j36}=1\angle -80.5°$$

$$\dot{I}_{N3}=3\dot{I}_{A3}=3\angle -80.5°$$



$$u_A=\frac{8U_m}{\pi^2}[\sin\omega t+\frac{1}{6}\sin3\omega t+\frac{1}{35}\sin5\omega t]V \qquad U_m=380V$$

$$Z=R+j\omega L=(3+j6)\Omega \qquad Z_N=R_N+j\omega L_N=(1+j2)\Omega$$



$$Z_5=R+j5\omega L$$
$$=(3+j30)\Omega$$

$$\dot{I}_{N5}=0$$

$$\dot{I}_{A5}=\frac{6.2\angle0°}{3+j30}=0.2\angle-84.3°$$

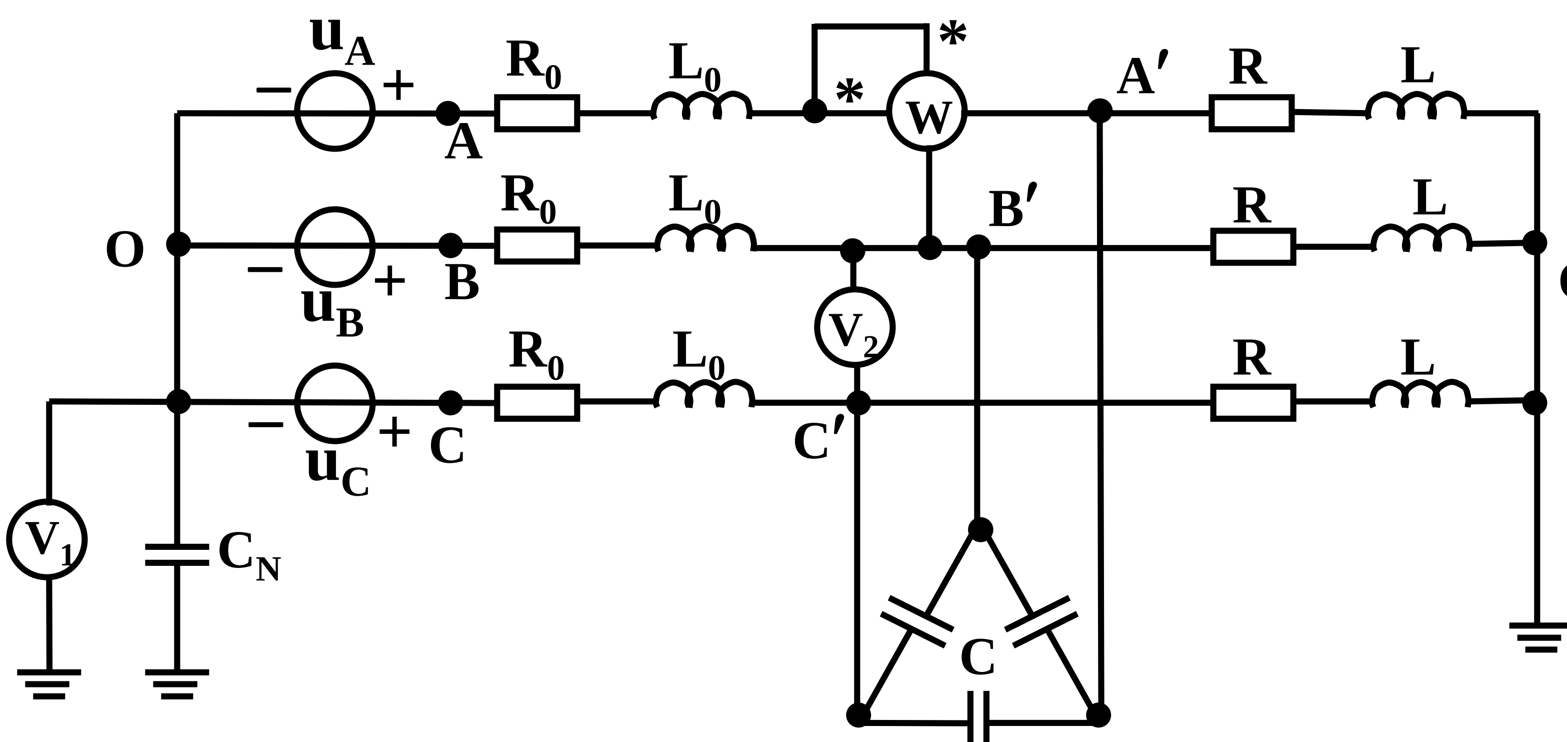
$$I_N=3\text{ A}$$

$$I_{ph}=(32.5^2+1^2+0.2^2)^{0.5}=32.5\text{ A}$$

A balanced three phase circuit，

$u_A(t) = 15+60\sqrt{2}\sin\omega t+10\sqrt{2}\cos(3\omega t+60^o)V$,

to fundamental frequency，$R=2\Omega$，$X_L=2\Omega$，$X_C=6\Omega$，$X_{L0}=2\Omega$，$R_0=4\Omega$，$X_{CN}=12\Omega$，calculate: the reading of each AC meter.



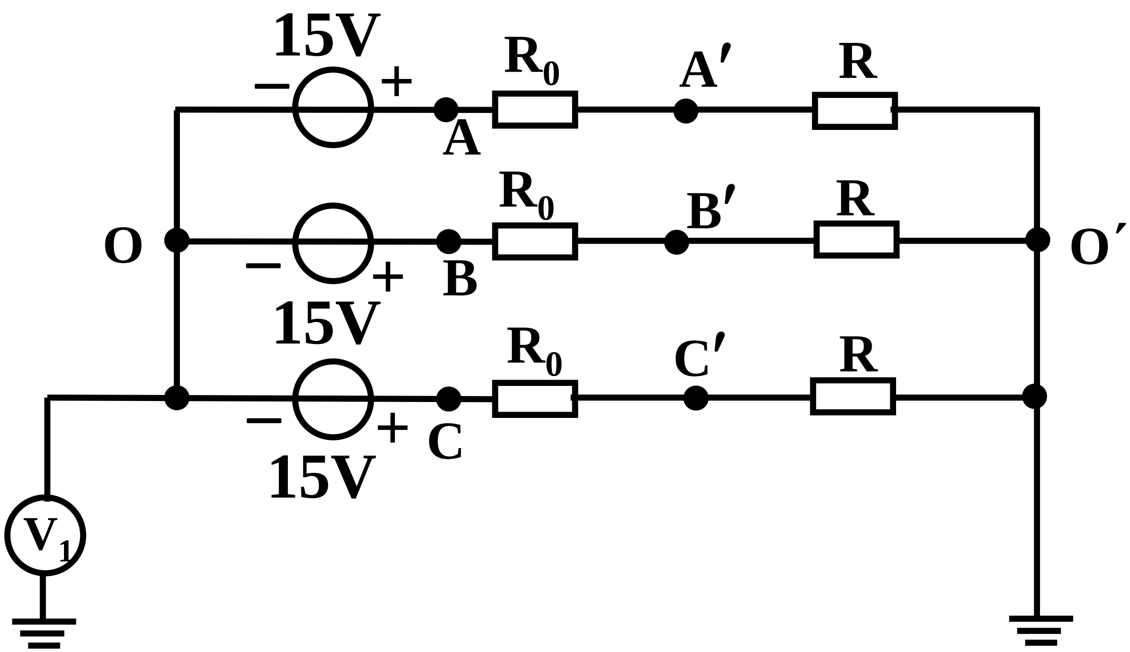
DC acts alone, zero sequence, without neutral line.

Fundamental wave acts alone, positive sequence, with neutral line.

Third harmonics acts alone, zero sequence, with neutral line and neutral impedance.

$u_A(t) = 15+60\sqrt{2}\sin\omega t+10\sqrt{2}\cos(3\omega t+60^\circ)\mathrm{V}$，对于基波，$R=2\Omega$，$X_L=2\Omega$，$X_C=6\Omega$，$X_{L0}=2\Omega$，$R_0=4\Omega$，$X_{CN}=12\Omega$，求各电压表与功率表的读数。

直流分量单独作用：电路工作于零序对称，无中线



各相电流为零

$P_0=0$

线电压无零序分量

$U_{20}=0$

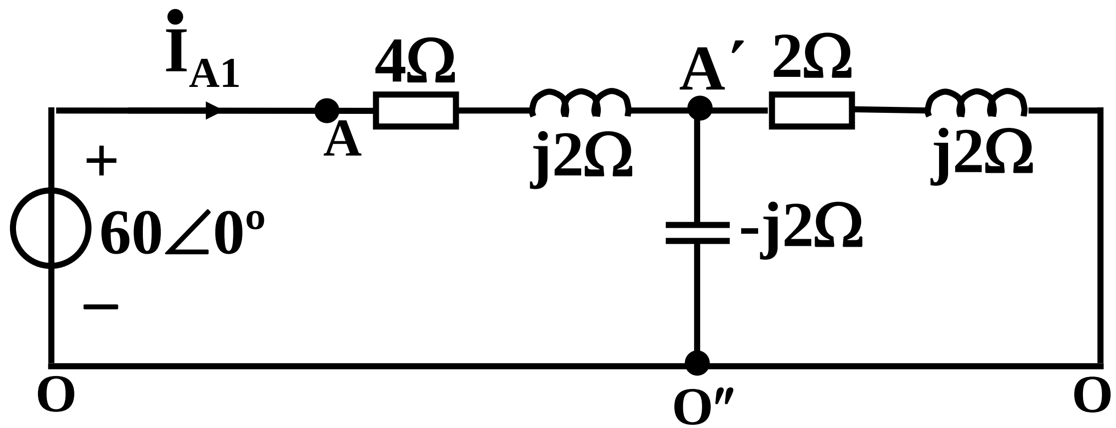
$U_{O'O0}=15\mathrm{V}$

$U_{10}=15\mathrm{V}$

$u_A(t) = 15+60\sqrt{2}\sin\omega t+10\sqrt{2}\cos(3\omega t+60^o)V$，对于基波，
$R=2\Omega$，$X_L=2\Omega$，$X_C=6\Omega$，$X_{L0}=2\Omega$，$R_0=4\Omega$，$X_{CN}=12\Omega$，
求各电压表与功率表的读数。

基波分量单独作用：正序对称

$U_{11}=0$



$$\dot{I}_{A1}=\frac{60\angle 0^o}{4+j2+\frac{(2+j2)(-j2)}{2}}$$

$=10\angle 0^o$

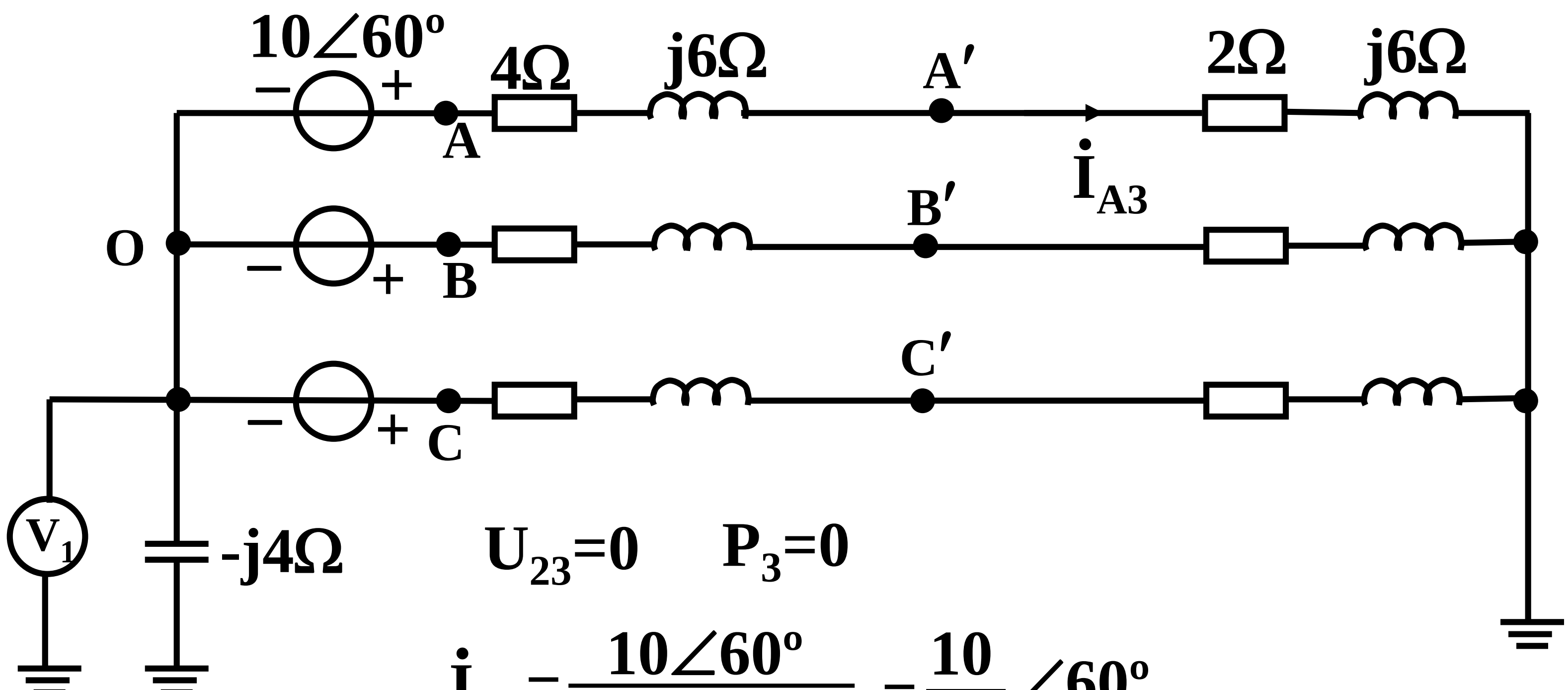
$$\dot{U}_{A'O'1}=\dot{I}_{A1}\frac{(2+j2)(-j2)}{2}=28.3\angle -45^o$$

$$\dot{U}_{A'B'1}=\sqrt{3}\,\dot{U}_{A'O'1}\angle 30^o=49\angle -15^o$$

$U_{21}=49V$ $\quad$ $P_1=49\times 10\cos 15^o=473W$

$u_A(t) = 15+60\sqrt{2}\sin\omega t+10\sqrt{2}\cos(3\omega t+60^o)V$，对于基波，$R=2\Omega$，$X_L=2\Omega$， $X_C=6\Omega$，$X_{L0}=2\Omega$，$R_0=4\Omega$，$X_{CN}=12\Omega$，求各电压表与功率表的读数。

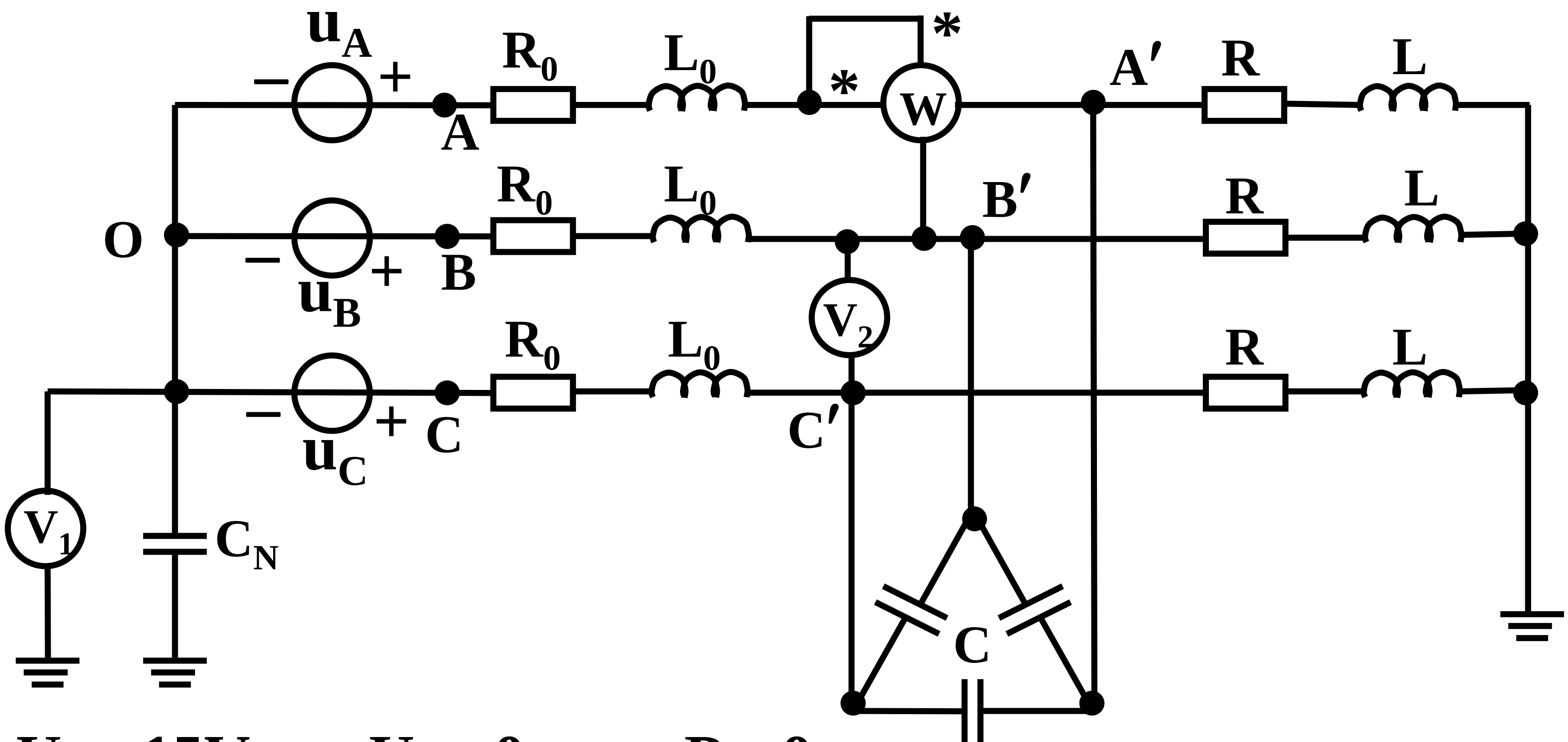
三次谐波电源单独作用：零序对称，有中线



$U_{23}=0$ $P_3=0$

$$\dot{I}_{A3}=\frac{10\angle 60^o}{6+j12-j12}=\frac{10}{6}\angle 60^o$$

$$U_{13}=4I_{N3}=4\times 3\times 10/6=20V$$



U_{10}=15V　　U_{20}=0　　P_0=0

U_{11}=0　　U_{21}=49V　　P_1=473W

U_{13}=20V　　U_{23}=0　　P_3=0

电压表V_1的读数：　　$U_1=\sqrt{15^2+20^2}=25V$

电压表V_2的读数：　　U_2=49V

功率表的读数：　　P=473W

——End

17-4 Exponential Fourier Series

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega_1 t+b_n\sin n\omega_1 t)$$

Euler's identity: $\cos n\omega_1 t=\dfrac{e^{jn\omega_1 t}+e^{-jn\omega_1 t}}{2}$, $\sin n\omega_1 t=\dfrac{e^{jn\omega_1 t}-e^{-jn\omega_1 t}}{2j}$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n(\frac{e^{jn\omega_1 t}+e^{-jn\omega_1 t}}{2})+b_n(\frac{e^{jn\omega_1 t}-e^{-jn\omega_1 t}}{2j})\right]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[(\frac{a_n-jb_n}{2})e^{jn\omega_1 t}+(\frac{a_n+jb_n}{2})e^{-jn\omega_1 t}\right]$$

$$b_n=\frac{2}{T}\int_0^T f(t)\sin n\omega_1 t dt$$

$$\sum_{n=1}^{\infty}\frac{a_n+jb_n}{2}e^{-jn\omega_1 t}=\sum_{n=-1}^{-\infty}\frac{a_n-jb_n}{2}e^{jn\omega_1 t}$$

$$f(t)=\sum_{n=-\infty}^{+\infty}\frac{a_n-jb_n}{2}e^{jn\omega_1 t}=\sum_{n=-\infty}^{+\infty}\dot{F}_n e^{jn\omega_1 t}$$

$$\dot{F}_n=\frac{a_n-jb_n}{2}=\frac{1}{T}[\int_0^T f(t)(\cos n\omega_1 t-j\sin n\omega_1 t)dt]=\frac{1}{T}[\int_0^T f(t)e^{-jn\omega_1 t}dt]$$

The exponential Fourier series of a periodic function f(t) describes the spectrum of f(t) in terms of the amplitude and phase angle of ac components at positive and negative harmonic frequencies.

$$i=[5+6\cos(\omega_1 t+\frac{\pi}{2})+2\cos(3\omega_1 t-\frac{\pi}{2})]$$