Fundamentals of Electric Circuits 2020.11

Chapter 15
The Laplace Transform

CHAPTER 15 THE LAPLACE TRANSFORM

The Laplace transform is significant:

1. It can be applied to a *wider variety of inputs*.

2. It provides an easy way to solve circuit problems *involving initial conditions.*

3. It can provide us the complete response *comprising both the natural and forced responses.*

A very useful tool !

CHAPTER 15 THE LAPLACE TRANSFORM

15.1 Definition of the Laplace Transform

Given a function f(t), its Laplace transformation, denoted by F(s), is given by

$$\boxed{\begin{aligned} F(s) &= \int_{0^-}^{+\infty} f(t)e^{-st}dt \\ &= \int_{0^-}^{0^+} f(t)e^{-st}dt + \int_{0^+}^{+\infty} f(t)e^{-st}dt \end{aligned}}$$

Where s is a complex variable given by $s=\sigma+j\omega$

1. One_sided Laplace Transform

$$\begin{cases} F(s) = \int_{0^-}^{+\infty} f(t)e^{-st}dt & \text{the direct Laplace transform} \\ f(t) = \dfrac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds & \text{the inverse Laplace transform} \end{cases}$$

2. Convergence criterion:

$$\int_{0^-}^{\infty} \left|f(t)e^{-st}\right|dt < \infty \qquad \int_{0^-}^{\infty} \left|f(t)e^{-\sigma t}\right|dt < \infty$$

3. A Laplace transform pair:

f(t) ⟺ F(S) $\qquad \boxed{F(s) = L[f(t)]}$

4.The Laplace Transform of some typical functions:

$$F(S) = \int_{0^-}^{+\infty} f(t)e^{-st}dt$$

(1)The exponential function(指数函数):

$$L[e^{-at}1(t)] = \int_{0^-}^{\infty} e^{-at}e^{-st}dt = -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a}$$

(2)The unit step function(单位阶跃函数):

$$L[1(t)] = \int_{0^-}^{\infty} 1(t)e^{-st}dt = \int_{0^+}^{\infty} e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty} = \frac{1}{s}$$

当$a=0$时 $e^{-at}1(t) = 1(t)$

(3)The impulse function(冲激函数):

$$L[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = \int_{0^-}^{0^+} \delta(t)e^{-s0}dt = 1$$

15.2 Properties of The Laplace Transform:

1. Linearity: $F(S) = \int_{0^-}^{+\infty} f(t)e^{-st}dt$

$$if\, L[f_1(t)] = F_1(S),\ L[f_2(t)] = F_2(S)$$

$$then\, L[af_1(t) \pm bf_2(t)] = aF_1(S) \pm bF_2(S)$$

example1: $L[U1(t)] = \frac{U}{S}$

example2: $L[\sin\omega t 1(t)] = L[\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})1(t)]$

$$= \frac{1}{2j}[\frac{1}{S - j\omega} - \frac{1}{S + j\omega}] = \frac{\omega}{S^2 + \omega^2}$$

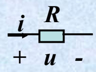
***Kirchhoff's laws in s domain:**

$KCL \quad \Sigma i = 0 \qquad \Sigma I(S) = 0$

$KVL \quad \Sigma u = 0 \qquad \Sigma U(S) = 0$

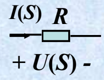
***Circuit elements in s domain:**

***R*:**



$u = Ri \qquad U(S) = RI(S)$

$I(S) = GU(S)$



$Z(s) = R$

$Y(s) = G$

2. scaling: $F(S) = \int_{0^-}^{+\infty} f(t)e^{-st}dt$

$$L[f(at)] = \frac{1}{a}F(\frac{s}{a})$$

Example:

$$L[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

$$L[\sin 2\omega t] = \frac{1}{2}\frac{\omega}{(\frac{s}{2})^2 + \omega^2}$$

3. Differentiation $\quad If\, L[f(t)] = F(s)$

(1). Time Differentiation $\quad L[\frac{df(t)}{dt}] = SF(S) - f(0^-)$

example1: $L[\cos\omega t 1(t)] = L[\frac{1}{\omega}\frac{d}{dt}(\sin\omega t 1(t))]$

$$= \frac{s}{\omega}\frac{\omega}{s^2 + \omega^2} - 0 = \frac{s}{s^2 + \omega^2}$$

example2: $L[\delta(t)] = L[\frac{d}{dt}1(t)] = S\frac{1}{S} = 1$

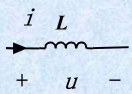
$$L[\frac{df(t)}{dt}] = SF(S) - f(0^-)$$

$$L[\frac{d^2 f(t)}{dt^2}] = S[SF(S) - f(0^-)] - f'(0^-)$$

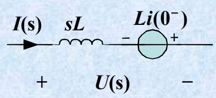
$$= S^2F(S) - Sf(0^-) - f'(0^-)$$

$$L[\frac{d^n f(t)}{dt^n}] = S^nF(S) - S^{n-1}f(0^-) - \cdots - f^{n-1}(0^-)$$

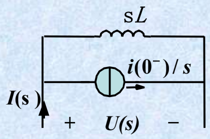
***L*:**



$$u = L\frac{di}{dt}$$



$$U(S) = L(SI(S) - i(0^-))$$
$$= SLI(S) - Li(0^-)$$

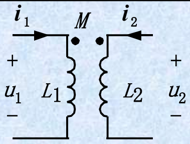


$$I(S) = \frac{U(S)}{SL} + \frac{i(0^-)}{S}$$

$Z(s) = sL$

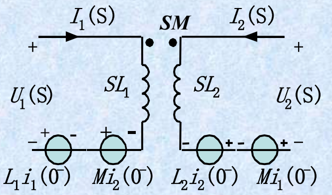
$Y(s) = 1/sL$

M :



$$\begin{cases} u_1 = L_1 \dfrac{di_1}{dt} + M\dfrac{di_2}{dt} \\ u_2 = L_2 \dfrac{di_2}{dt} + M\dfrac{di_1}{dt} \end{cases}$$

$$\begin{cases} U_1(S) = SL_1I_1(S) - L_1i_1(0^-) + SMI_2(S) - Mi_2(0^-) \\ U_2(S) = SL_2I_2(S) - L_2i_2(0^-) + SMI_1(S) - Mi_1(0^-) \end{cases}$$



$$Z_M(s) = sM$$
$$Y_M(s) = 1/sM$$

(2).Frequency Differentiation:

$$If L[f(t)] = F(S)$$

$$L[-t\,f(t)] = \frac{dF(S)}{dS}$$

$$1:\ L[t1(t)] = -\frac{d}{ds}(\frac{1}{S}) = (\frac{1}{S^2})$$

$$2:\ L[t^n 1(t)] = (-1)^n \frac{d^n}{ds^n}(\frac{1}{S}) = (\frac{n!}{S^{n+1}})$$

$$3:\ L[te^{-\alpha t}] = -\frac{d}{ds}(\frac{1}{S+\alpha}) = \frac{1}{(S+\alpha)^2}$$

4. Integration:

(1) Time Integration:

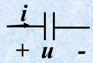
$$if \qquad L[f(t)] = F(S)$$

$$L[\int_{0^-}^{t} f(\tau)d\tau] = \frac{1}{S}F(S)$$

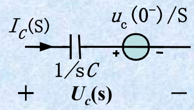
$$example1:\ L[t1(t)] = L[\int_{0^-}^{t} 1(t)dt] = \frac{1}{S} \times \frac{1}{S}$$

$$example2:\ L[t^2 1(t)] = \frac{2}{S^3} \qquad \because [t^2 1(t)] = 2\int_{0^-}^{t} tdt$$

C :

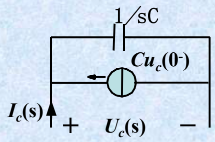


$$u_c = u_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i_c dt$$



$$U_c(S) = \frac{1}{SC}I_c(S) + \frac{u_c(0^-)}{S}$$

$$I_C(S) = SCU_C(S) - Cu_c(0^-)$$



$$Z(s) = 1/sC$$
$$Y(s) = sC$$

4. Shift

(1).Frequency shift $\qquad if:\ L[f(t)] = F(S)$

$$L[e^{-\alpha t} f(t)] = F(S+\alpha) \qquad F(S+\alpha) = L[e^{-\alpha t} f(t)]$$

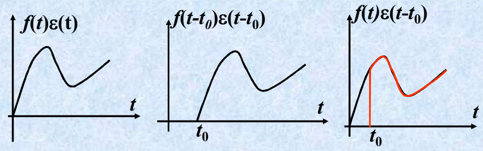
$$1:\ L[te^{-\alpha t}1(t)] = \frac{1}{(S+\alpha)^2} \qquad L[t1(t)] = \frac{1}{S^2}$$

$$L[\cos\omega t] = \frac{s}{s^2+\omega^2}$$

$$2:\ L[e^{-\alpha t}\cos\omega t 1(t)] = \frac{S+\alpha}{(S+\alpha)^2+\omega^2}$$

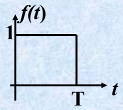
4. Shift 设：$L[f(t)] = F(S)$

(2).Time shift (延迟定理)

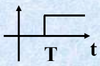


$$L[f(t-t_0)1(t-t_0)] = e^{-st_0}F(S)$$

e.g 1:

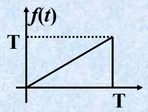


$$f(t) = \varepsilon(t) - \varepsilon(t-T)$$



$$F(S) = \frac{1}{S} - \frac{1}{S}e^{-ST}$$

e.g 2:



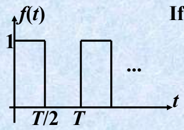
$$f(t) = t[\varepsilon(t) - \varepsilon(t-T)]$$

$$F(S) \overset{?}{=} \frac{1}{S^2} - \frac{e^{-ST}}{S^2}$$

$$f(t) = t\varepsilon(t) - (t-T)\varepsilon(t-T) - T\varepsilon(t-T)$$

$$F(S) = \frac{1}{S^2} - \frac{1}{S^2}e^{-ST} - \frac{T}{S}e^{-ST}$$

5. Time periodicity



If $f(t)(f_1(t) \in (0, T))$ is a periodic function

$$L[f_1(t)] = F_1(S)$$

$$then:\ L[f(t)] = \frac{1}{1-e^{-ST}}F_1(S)$$

证: $f(t) = f_1(t) + f_1(t-T)\varepsilon(t-T) + f_1(t-2T)\varepsilon(t-2T) + \cdots$

$$L[f(t)] = F_1(S) + e^{-ST}F_1(S) + e^{-2ST}F_1(S) + \cdots$$

$$= F_1(S)[e^{-ST} + e^{-2ST} + e^{-3ST} + \cdots]$$

$$= \frac{1}{1-e^{-ST}}F_1(S)$$

6.Initial and Final Values:

Initial_value therom:

$f(t)$在$t=0$处无冲激则

$$f(0^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} SF(S)$$

Final_value theorem:

$\lim_{t\to\infty} f(t)$存在时

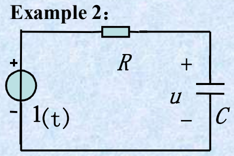
$$\lim_{s\to 0} SF(S) = \lim_{t\to\infty} f(t) = f(\infty)$$

$Example.1$: $Given\ F(S) = \frac{3S^2+4S+5}{S(S^2+2S+3)}$, $Find\ \ f(0^+)$

$$f(0^+) = \lim_{s\to\infty}\frac{3S^2+4S+5}{(S^2+2S+3)} = 3$$

$$f(0^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} SF(S)$$

Example 2:



$$u_c(0^-) = 0$$

$$RC\frac{du}{dt} + u = 1(t)$$

$$SRCU(S) + U(S) = \frac{1}{S}$$

Then:

$$U(S) = \frac{1}{S(1+SRC)}$$

$$u(0^+) = \lim_{s\to\infty} \cancel{S}\frac{1}{\cancel{S}(1+SRC)} = \lim_{s\to\infty}\frac{1}{(1+SRC)} = 0$$

$$u(\infty) = \lim_{s\to 0}\frac{1}{(1+SRC)} = 1$$

15.3. The Inverse Laplace Transform

Given F(s), how to obtain the corresponding f(t):

(1)The inverse Laplace transform

$$\boxed{f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(S)e^{st}ds}$$

(2)Matching entries in Table 15.2:

$$F(S) = F_1(S) + F_2(S) + \cdots + F_n(S)$$

$$f(t) = f_1(t) + f_2(t) + \cdots + f_n(t)$$

(3)Partial fraction expansion to break F(s) down into simple terms whose inverse transform can be got from Table 15.2

$$F(S)=\frac{F_1(S)}{F_2(S)}=\frac{a_0S^m+a_1S^{m-1}+\cdots+a_m}{b_0S^n+b_1S^{n-1}+\cdots+b_n}\ (n>m)$$

1. Simple poles: 1. $F_2(S)=0$的根为不等实根 $S_1\cdots S_n$

Use partial fraction expansion to decompose F(s):

$$(S-S_1)F(S)=\frac{\cancel{(S-S_1)}k_1}{\cancel{S-S_1}}+\frac{(S-S_1)k_2}{S-S_2}+\cdots+\frac{(S-S_1)k_n}{S-S_n}$$

$$f(t)=k_1e^{s_1t}+k_2e^{s_2t}+\cdots k_ne^{s_nt}$$

$$k_1=(S-S_1)F(S)\big|_{S=S_1}$$

$$k_2=(S-S_2)F(S)\big|_{S=S_2}$$

$$\cdots k_n=(S-S_n)F(S)\big|_{S=S_n}$$

例：$F(S)=\frac{4S+5}{S^2+5S+6}=\frac{4S+5}{(S+2)(S+3)}=\frac{K_1}{S+2}+\frac{K_2}{S+3}$

$S_1=-2,\ S_2=-3$

$K_1=\frac{4S+5}{S+3}\big|_{S=-2}=-3 \qquad K_2=\frac{4S+5}{S+2}\big|_{S=-3}=7$

$f(t)=-3e^{-2t}\varepsilon(t)+7e^{-3t}\varepsilon(t)$

（洛比达法则）

$$k_i=\lim_{s\to s_i}\frac{F_1(S)(S-S_i)}{F_2(S)}$$

（分解定理）

$$=\lim_{s\to s_i}\frac{F_1'(S)(S-S_i)+F_1(S)}{F_2'(S)}=\frac{F_1(S_i)}{F_2'(S_i)}$$

$$k_i=\frac{F_1(S_i)}{F_2'(S_i)}$$

上例： $F(S)=\frac{4S+5}{S^2+5S+6}$

$$k_1=\frac{4S+5}{2S+5}\big|_{S=-2}=-3$$

$$k_2=\frac{4S+5}{2S+5}\big|_{S=-3}=7$$

2. Complex poles: 2. $F_2(S)$有共轭复根

一对共轭复根为 $S_{1,2}=-\alpha\pm j\omega$

$$F(S)=\frac{F_1(S)}{F_2(S)}=\frac{F_1(S)}{[S-(-\alpha+j\omega)][S-(-\alpha-j\omega)]}$$

$$=\frac{k_1}{S+\alpha-j\omega}+\frac{k_2}{S+\alpha+j\omega}$$

k_1,k_2也是一对共轭复根 设$k_1=|k|\angle\theta \quad k_2=|k|\angle-\theta$

$$f(t)=(|k|e^{j\theta}e^{-(\alpha-j\omega)t}+|k|e^{-j\theta}e^{-(\alpha+j\omega)t})\varepsilon(t)$$

$$=|k|e^{-\alpha t}[e^{j(\omega t+\theta)}+e^{-j(\omega t+\theta)}]\varepsilon(t)$$

$$=2|k|e^{-\alpha t}\cos(\omega t+\theta)\varepsilon(t)$$

例：$F(S)=\frac{S}{S^2+2S+5} \qquad S=-1\pm j2$

$$k_1=\frac{S}{S-(-1-j2)}\Big|_{S=-1+j2}=0.559\angle 26.6^\circ$$

$$k_2=\frac{S}{S-(-1+j2)}\Big|_{S=-1-j2}=0.559\angle-26.6^\circ$$

$$f(t)=2\times0.559e^{-t}\cos(2t+26.6^\circ)\varepsilon(t)$$

$$=1.118e^{-t}\cos(2t+26.6^\circ)\varepsilon(t)$$

method2: completing the square(配方） $L[\sin\omega t\varepsilon(t)]=\frac{\omega}{S^2+\omega^2}$

$$\frac{S}{S^2+2S+5}=\frac{S+1-1}{(S+1)^2+2^2}=\frac{S+1}{(S+1)^2+2^2}-\frac{1}{(S+1)^2+2^2}$$

$$f(t)=e^{-t}\cos 2t-\frac{1}{2}e^{-t}\sin 2t=1.118e^{-t}\cos(2t+26.6^\circ)\varepsilon(t)$$

3. Repeated poleas: 3.$F_2(S)$有相等的实根（重根 ）

$$F(S)=\frac{a_0S^m+a_1S^{m-1}+\cdots+a_m}{(S-S_1)^n}$$

$$F(S)=\frac{k_{11}}{S-S_1}+\frac{k_{12}}{(S-S_1)^2}+\cdots+\frac{k_{1n-1}}{(S-S_1)^{n-1}}+\frac{k_{1n}}{(S-S_1)^n}$$

$$k_{1n}=[(S-S_1)^nF(S)]\big|_{S=S_1}$$

$$k_{1n-1}=\frac{d}{ds}[(S-S_1)^nF(S)]\big|_{S=S_1}$$

$$k_{1n-2}=\frac{1}{2!}\times\frac{d^2}{ds^2}[(S-S_1)^nF(S)]\big|_{S=S_1}$$

$$\vdots$$

$$k_{11}=\frac{1}{(n-1)!}\times\frac{d^{n-1}}{ds^{n-1}}[(S-S_1)^nF(S)]\big|_{S=S_1}$$

例：$\frac{S+4}{S(S+1)^2}=\frac{K_1}{S}+\frac{K_{21}}{(S+1)}+\frac{K_{22}}{(S+1)^2}$ 频域延迟

$$K_1=\frac{S+4}{(S+1)^2}\Big|_{S=0}=4 \qquad K_{22}=\frac{S+4}{S}\Big|_{S=-1}=-3$$

$$K_{21}=\frac{d}{ds}[(S+1)^2F(S)]\big|_{S=-1}=\frac{d}{ds}[\frac{S+4}{S}]\big|_{S=-1}=-4$$

$$f(t)=4-4e^{-t}-3te^{-t}$$

小结： 由$F(S)$求$f(t)$ 的步骤

1.) 将$F(S)$化成最简真分式

例：$F(S)=\dfrac{S^2+9S+11}{S^2+5S+6}=1+\dfrac{4S+5}{S^2+5S+6}$

$=1+\dfrac{-3}{S+2}+\dfrac{7}{S+3}$

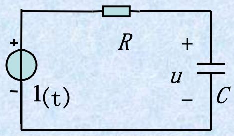
$f(t)=\delta(t)+(7e^{-3t}-3e^{-2t})\varepsilon(t)$

2.)求F（S）分母多项式等于零的根，将F（S）分解成部分分式之和

3.)求各部分分式的系数

4.)对每个部分分式和多项式逐项求拉氏反变换。

Example 2:



$u_c(0^-)=0$

$RC\dfrac{du}{dt}+u=1(t)$

$SRCU(S)+U(S)=\dfrac{1}{S}$

$U(S)=\dfrac{1}{S(1+SRC)}$

then:

$u(0^+)=\lim\limits_{s\to\infty}\cancel{S}\dfrac{1}{\cancel{S}(1+SRC)}=\lim\limits_{s\to\infty}\dfrac{1}{(1+SRC)}=0$

$u(\infty)=\lim\limits_{s\to 0}\dfrac{1}{(1+SRC)}=1$

15.4 Application to Circuits

Steps in applying the Laplace transform:

1. Transform the circuit from the *time domain* to the *s domain*.
2. *Solve the circuit* using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar.
3. *Take the inverse transform* of the solution and thus obtain the solution in the time domain.

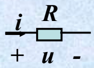
1.Kirchhoff's laws in s domain:

$KCL\quad \Sigma i=0 \qquad\qquad \Sigma I(S)=0$

$KVL\quad \Sigma u=0 \qquad\qquad \Sigma U(S)=0$

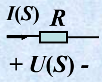
2.Circuit elements in s domain:

R:



$u=Ri \qquad U(S)=RI(S)$

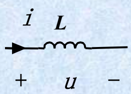
$I(S)=GU(S)$



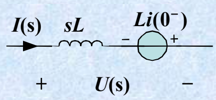
$Z(s)=R$

$Y(s)=G$

L:

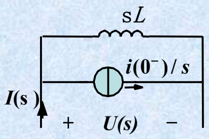


$u=L\dfrac{di}{dt}$



$U(S)=L(SI(S)-i(0^-))$

$=SLI(S)-Li(0^-)$

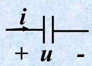


$I(S)=\dfrac{U(S)}{SL}+\dfrac{i(0^-)}{S}$

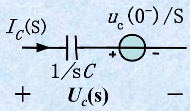
$Z(s)=sL$

$Y(s)=1/sL$

C:

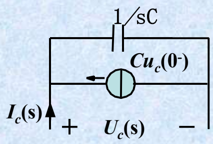


$u_c=u_c(0^-)+\dfrac{1}{C}\int_{0^-}^{t}i_c dt$



$U_c(S)=\dfrac{1}{SC}I_c(S)+\dfrac{u_c(0^-)}{S}$

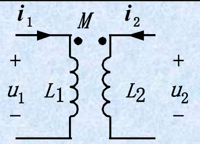
$I_C(S)=SCU_C(S)-Cu_c(0^-)$



$Z(s)=1/sC$

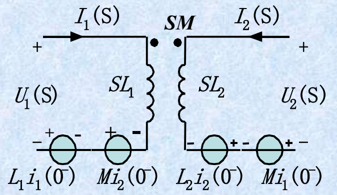
$Y(s)=sC$

M :



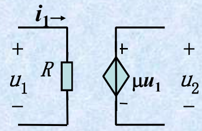
$$\begin{cases} u_1 = L_1\dfrac{di_1}{dt} + M\dfrac{di_2}{dt} \\ u_2 = L_2\dfrac{di_2}{dt} + M\dfrac{di_1}{dt} \end{cases}$$

$$\begin{cases} U_1(S) = SL_1I_1(S) - L_1i_1(0^-) + SMI_2(S) - Mi_2(0^-) \\ U_2(S) = SL_2I_2(S) - L_2i_2(0^-) + SMI_1(S) - Mi_1(0^-) \end{cases}$$



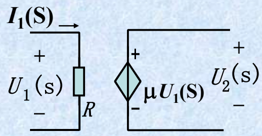
$$Z_M(s) = sM$$
$$Y_M(s) = 1/sM$$

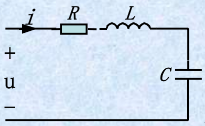
Dependent sources:



$$u_1 = i_1R$$
$$u_2 = \mu u_1$$

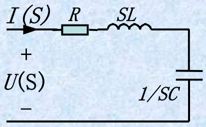
$$U_1(S) = I_1(S)R$$
$$U_2(S) = \mu U_1(S)$$





$$u_c(0^-) = 0 \quad i_L(0^-) = 0$$

$$u = iR + L\frac{di}{dt} + \frac{1}{C}\int_{0^-}^{t} i_c dt$$



$$U(S) = I(S)R + SLI(S) + \frac{1}{SC}I(S)$$
$$= I(S)\left(R + SL + \frac{1}{SC}\right)$$

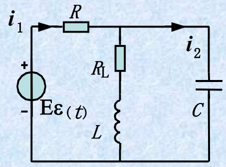
Impedance in the s domain

$$Z(S) = R + SL + \frac{1}{SC}$$

$$\begin{cases} U(S) = Z(S)I(S) \\ I(S) = Y(S)U(S) \end{cases} \quad Y(S) = \frac{1}{Z(S)}$$

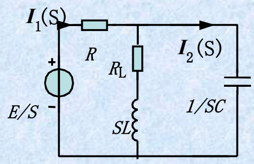
Ohm's law in s domain

3.A circuit model in s domain:



$$u_c(0^-) = 0 \quad i_L(0^-) = 0$$

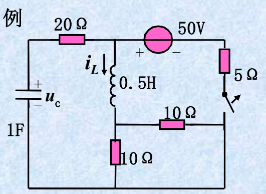
时域电路



运算电路

1. Transform the sources into s domain.
2. Transform the circuit elements into s domain.
3. Consider the initial condition of the storage elements.

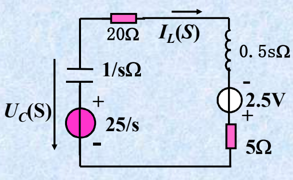
例



t=0, open the switch

The circuit in time domain

$u_c(0^-)$=25V $i_L(0^-)$=5A



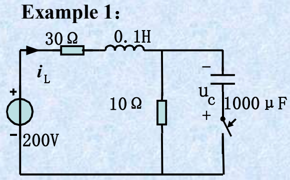
t>0 ,circuit model in s domain

4.Laplace transform application in circuits

Steps: 1.Determine $u_c(0^-)$, $i_L(0^-)$ 。

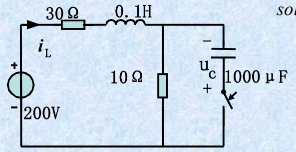
2. Set up a circuit model in s domain.
3. Solve the circuit.
4. Take the inverse transform of the solution.

Example 1:



$$u_c(0^-) = 100V$$

When $t = 0$, k closes, Calculate i_L, u_L.

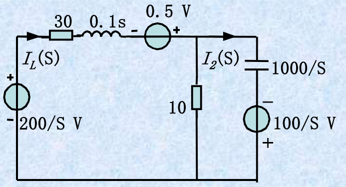


solution :(1) $i_L(0^-) = 5\,A$

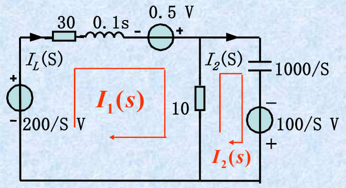
$u_c(0^-) = 100\,V$

(2)Set up the circuit model in s domain

$SL = 0.1S$



$$\frac{1}{SC} = \frac{1}{S\times1000\times10^{-6}} = \frac{1000}{S}$$



(3)*mesh equations*

$$\begin{cases} I_1(S)(40+0.1S)-10I_2(S)=\dfrac{200}{S}+0.5 \\ -10I_1(S)+(10+\dfrac{1000}{S})I_2(S)=\dfrac{100}{S} \end{cases}$$

$$I_1(S)=\frac{5(S^2+700S+40000)}{S(S+200)^2}$$

$$I_1(S)=\frac{5(S^2+700S+40000)}{S(S+200)^2}$$

$$i(0^+)=\lim_{s\to\infty} SF(S) = \lim_{s\to\infty}\frac{5(S^2+700S+40000)}{S^2+400S+200^2}=5$$

$$i(\infty)=\lim_{s\to 0} SF(S) = \lim_{s\to 0}\frac{5(S^2+700S+40000)}{S^2+400S+200^2}=5$$

(4)Inverse transformation:

$$I_1(S)=\frac{K_1}{S}+\frac{K_{21}}{S+200}+\frac{K_{22}}{(S+200)^2}$$

$$I_1(S)=\frac{K_1}{S}+\frac{K_{21}}{S+200}+\frac{K_{22}}{(S+200)^2}$$

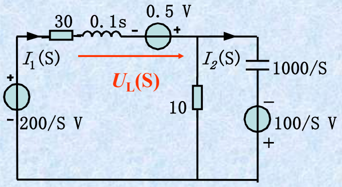
$$K_1 = F(S)S|_{S=0}=\frac{5(S^2+700S+40000)}{S^2+400S+200^2}|_{S=0}=5$$

$$K_{22}=F(S)(S+200)^2|_{S=-200}=1500$$

$$K_{21}=\frac{d}{ds}(S+200)^2F(S)|_{S=-200}=0$$

$$I_1(S)=\frac{5}{S}+\frac{0}{(S+200)}+\frac{1500}{(S+200)^2}$$

$$i_1(t)=(5+1500te^{-200t})\varepsilon(t)A$$



Find $U_L(S)$

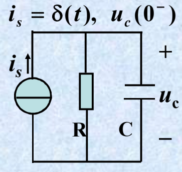
$$U_L(S) \ne I_1(S)SL$$

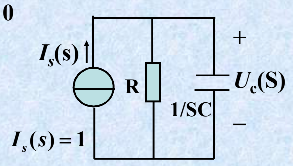
$$U_L(S)=I_1(S)SL-0.5=\frac{150}{S+200}+\frac{-30000}{(S+200)^2}$$

$$u_L(t)=150e^{-200t}-30000te^{-200t}V$$

Example 2: Find the impulse response.

$i_s=\delta(t),\ u_c(0^-)=0$

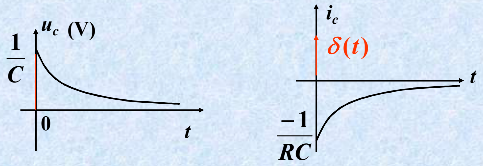




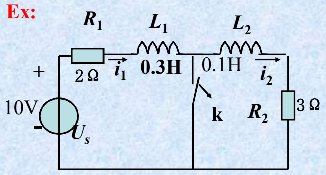
$$U_C(s)=\frac{R}{R+1/SC}I_s(s)\frac{1}{SC}=\frac{R}{RC(S+1/RC)}$$

$$I_C(S)=U_C(S)\frac{1}{SC}=\frac{RSC}{RSC+1}=\frac{RSC+1}{RSC+1}-\frac{1}{RSC+1}$$

$$u_c=\frac{1}{C}e^{-t/RC}(t\ge0) \qquad i_c=\delta(t)-\frac{1}{RC}e^{-t/RC}(t\ge0)$$



Ex:



$t = 0$, k open,
Find: i_1, i_2。

$$i_1(0^-) = 5A$$

$$i_2(0^-) = 0$$

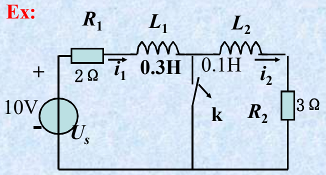
Review

Laplace transform application in circuits

steps:

1. Determine $u_c(0^-)$, $i_L(0^-)$ 。
2. Set up a circuit model in s domain.
3. Solve the circuit.
4. Take the inverse transform of the solution.

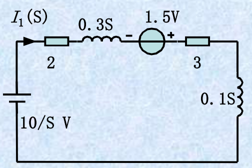
Ex:



$t = 0$, k open,
Find: i_1, i_2。

$$i_1(0^-) = 5A$$

$$i_2(0^-) = 0$$

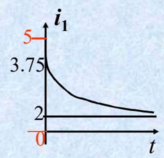


$$I_1(S) = \frac{\frac{10}{S} + 1.5}{5 + 0.4S} = \frac{10 + 1.5S}{(5 + 0.4S)S}$$

$$= \frac{25 + 3.75S}{(S + 12.5)S}$$

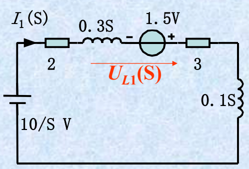
$$= \frac{2}{S} + \frac{1.75}{S + 12.5}$$

$$i_1 = 2 + 1.75e^{-12.5t} = i_2$$



$$i_1(0^+) \neq i_1(0^-)$$

$$i_2(0^+) \neq i_2(0^-)$$



$$U_{L1}(S) = 0.3SI(S) - 1.5$$

$$U_{L1}(S) = -\frac{6.56}{S + 12.5} - 0.375$$

$$u_{L1} = -0.375\delta(t) - 6.56e^{-12.5t}$$

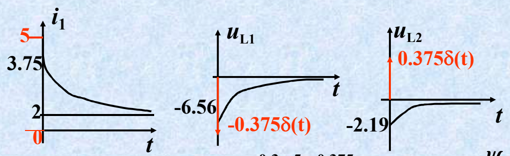
$$U_{L2}(S) = 0.1SI(S)$$

$$U_{L2}(S) = 0.375 - \frac{2.19}{S + 12.5}$$

$$u_{L2} = +0.375\delta(t) - 2.19e^{-12.5t}$$

$$u_{L1} = -0.375\delta(t) - 6.56e^{-12.5t}$$

$$u_{L2} = +0.375\delta(t) - 2.19e^{-12.5t}$$

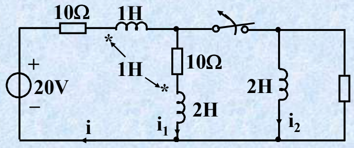


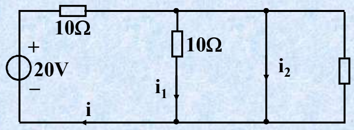
$i_2(0^+) = i_2(0^-) + \frac{0.375}{0.1} = 3.75A$ $\quad i_1(0^+) = \frac{0.3 \times 5 - 0.375}{0.3} = 3.75A$ $\quad L = \frac{\psi}{i}$

磁链守恒：$L_1 i_1(0^-) + L_2 i_2(0^-) = (L_1 + L_2) i(0^+)$

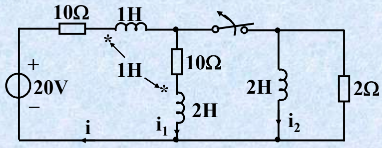
$$0.3 \times 5 + 0 = 0.4 \times 3.75$$

The circuit is shown in the Fig.,when t=0,the switch is open, calculate i.



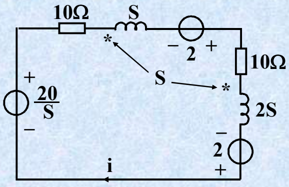


$i_1(0\_) = 0$ $i_2(0\_)= i\,(0\_)=2A$



$i_2(0\_)= i\,(0\_)=2A$

$i_1(0\_) = 0$



$$I(S)=\frac{\frac{20}{S}+4}{5S+20}$$

$$I(S)=\frac{4S+20}{5S(S+4)}$$

$$I(S)=\frac{1}{S}-\frac{0.2}{S+4}$$

$$i(t)=1-0.2e^{-4t}$$

小结： 1、运算法直接求得全响应

2、用0-初始条件，跳变情况自动包含在响应中

3、运算法分析动态电路的步骤

1).由换路前电路计算$u_c(0^-)$, $i_L(0^-)$ 。

2). 画运算电路图

3). 应用电路分析方法求象函数。

4). 反变换求原函数。

15.5 TRANSFER FUNCTIONS

$$Voltage\ gain=\frac{\dot{V}_o(s)}{\dot{V}_i(s)}$$

$$Current\ gain=\frac{\dot{I}_o(s)}{\dot{I}_i(s)}$$

$$Transfer\ impedance=\frac{\dot{V}_o(s)}{\dot{I}_i(s)}$$

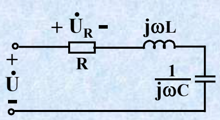
$$Transfer\ admit\tan ce=\frac{\dot{I}_o(s)}{\dot{V}_i(s)}$$

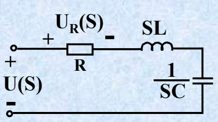
The transfer function H(s) is the ratio of the output response Y(s) to the input excitation X(s), assuming all initial conditions are zero.

$H(S) \rightleftarrows H(j\omega)$

$$H(s)=\frac{Y(s)}{X(s)}$$

与对应正弦稳态响应的关系 $H(S) \rightleftarrows H(j\omega)$



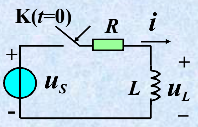


$$H(j\omega)=\frac{\dot{U}_R}{\dot{U}} \qquad H(S)=\frac{U_R(S)}{U(S)}$$

$$=\frac{R}{R+j\omega L+\frac{1}{j\omega C}} \qquad =\frac{R}{R+SL+\frac{1}{SC}}$$

A DISCUSSION: PHASOR DOMAIN AND S DOMAIN

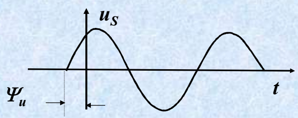
1. Sinusoidal response:



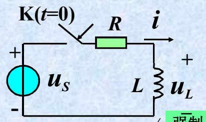
$i_L(0^-)=0$

$$u_s=U_m\sin(\omega t+\psi_u)$$

$i\ (0^-)=0$



calculate：$i(t)$



$i_L(0^-)=0$

$$u_s = U_m \sin(\omega t + \psi_u)$$

$$Ri + L\frac{di}{dt} = U_m \sin(\omega t + \psi_u)$$

强制分量(稳态分量)

$$i = i' + i''$$

自由分量(暂态分量)

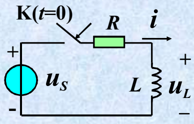
$$i'' = Ae^{-\frac{t}{\tau}}$$

Particular solution: i'

$$I_m = \frac{U_m}{\sqrt{R^2 + (\omega L)^2}}$$

$$\varphi = \text{arctg}\frac{\omega L}{R}$$

$$i' = I_m \sin(\omega t + \psi_u - \varphi)$$



$i_L(0^-)=0$

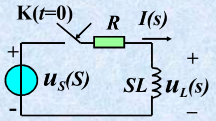
$$u_s = U_m \sin(\omega t + \psi_u)$$

$$i = i' + i'' = I_m \sin(\omega t + \psi_u - \varphi) + Ae^{-\frac{t}{\tau}}$$

由 $i(0^+) = 0 = I_m \sin(\psi_u - \varphi) + A$ 定积分常数A

$$A = -I_m \sin(\psi_u - \varphi)$$

$$i = I_m \sin(\omega t + \psi_u - \varphi) - I_m \sin(\psi_u - \varphi)e^{-\frac{t}{\tau}}$$



$i_L(0^-)=0$

$$RI(S) + SLI(S) = \frac{S\sin\psi_u + \omega\cos\psi_u}{S^2+\omega^2}U_m$$

$$I(S) = \frac{S\sin\psi_u + \omega\cos\psi_u}{(S^2+\omega^2)(R+SL)}U_m$$

$$= \left(\frac{K_1}{S+R/L} + \frac{K_2}{S+j\omega} + \frac{K_3}{S-j\omega}\right)U_m$$

$$K_1 = \frac{S\sin\psi_u + \omega\cos\psi_u}{(S^2+\omega^2)(R+SL)}(S+R/L)|_{S=-R/L}$$

$$= \frac{\omega L\cos\psi_u - R\sin\psi_u}{R^2+(\omega L)^2}$$

$$= \frac{1}{\sqrt{R^2+(\omega L)^2}}\left[\frac{\omega L}{\sqrt{R^2+(\omega L)^2}}\cos\psi_u - \frac{R}{\sqrt{R^2+(\omega L)^2}}\sin\psi_u\right]$$

$$= \frac{1}{\sqrt{R^2+(\omega L)^2}}\sin(\psi_u - \varphi)$$

$$\varphi = arctg\ \frac{\omega L}{R}$$

$$K_2 = \frac{S\sin\psi_u + \omega\cos\psi_u}{(S^2+\omega^2)(R+SL)}U_m(S+j\omega)|_{S=-j\omega}$$

$$= \frac{-\cos\psi_u + j\sin\psi_u}{2Rj + 2\omega L}U_m$$

$$K_3 = \frac{S\sin\psi_u + \omega\cos\psi_u}{(S^2+\omega^2)(R+SL)}U_m(S-j\omega)|_{S=j\omega}$$

$$= \frac{\cos\psi_u + j\sin\psi_u}{2Rj - 2\omega L}U_m$$

$$\frac{K_2}{S+j\omega} + \frac{K_3}{S-j\omega} = \frac{(R\sin\psi_u - \omega L\cos\psi_u)S - (\omega L\sin\psi_u + R\cos\psi_u)\omega}{[R^2+(\omega L)^2](S^2+\omega^2)}$$

$$= \frac{1}{\sqrt{R^2+(\omega L)^2}}\frac{\left(\frac{R}{\sqrt{R^2+(\omega L)^2}}\sin\psi_u - \frac{\omega L}{\sqrt{R^2+(\omega L)^2}}\cos\psi_u\right)S + \left(\frac{\omega L}{\sqrt{R^2+(\omega L)^2}}\sin\psi_u + \frac{R}{\sqrt{R^2+(\omega L)^2}}\cos\psi_u\right)\omega}{S^2+\omega^2}$$

$$\varphi = arctg\ \frac{\omega L}{R}$$

$$\frac{K_2}{S+j\omega} + \frac{K_3}{S-j\omega} = \frac{S\sin(\psi_u - \varphi) + \omega\cos(\psi_u - \varphi)}{S^2+\omega^2}\frac{1}{\sqrt{R^2+(\omega L)^2}}$$

$$I(S) = \frac{S\sin\psi_u + \omega\cos\psi_u}{(S^2+\omega^2)(R+SL)}U_m$$

$$= \left(\frac{K_1}{S+R/L} + \frac{K_2}{S+j\omega} + \frac{K_3}{S-j\omega}\right)U_m$$

$$K_1 = \frac{1}{\sqrt{R^2+(\omega L)^2}}\sin(\psi_u - \varphi)$$

$$\varphi = arctg\ \frac{\omega L}{R}$$

$$I(S) = \frac{U_m}{\sqrt{R^2+(\omega L)^2}}\left[\sin(\psi_u - \varphi)\frac{1}{S+R/L} + \frac{\sin(\psi_u-\varphi)S + \cos(\psi_u - \varphi)\omega}{S^2+\omega^2}\right]$$

$$\varphi = arctg\ \frac{\omega L}{R} \qquad I_m = \frac{U_m}{\sqrt{R^2+(\omega L)^2}}$$

$$i = I_m \sin(\omega t + \psi_u - \varphi) - I_m \sin(\psi_u - \varphi)e^{-\frac{t}{\tau}}$$

Phasor method:

It's a convenient way to get the sinusoidal steady_state response.

Laplace transform:

A powerful tool to analyze dynamics of the higher_order circuits.

It can provide us the complete response of an ***arbitrary input***.

a complex variable s=σ+jω

If σ=0, natural response is beyond consideration

H(S) ⇄ H(jω)

We know the circuit equations:

$$\begin{cases} \dfrac{d^2y}{dt^2}+3\dfrac{dy}{dt}+2y=2\dfrac{dx}{dt}+x \\ y(0_-)=0, \left.\dfrac{dy}{dt}\right|_{t=0_-}=0 \end{cases}$$

Where, y is the output response, x is the excitation.
Calculate H(s) and the unit impulse response h(t).

Solution:

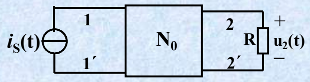
$$s^2Y(s)+3sY(s)+2Y(s)=2sX(s)+3X(s)$$

$$(s^2+3s+2)Y(s)=(2s+3)X(s)$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{2s+3}{s^2+3s+2}=\frac{1}{s+1}+\frac{1}{s+2}$$

$$h(t)=\ell^{-1}[H(s)]=\ell^{-1}[\frac{1}{s+1}+\frac{1}{s+2}]=(e^{-t}+e^{-2t})1(t)$$

Example: N_0 is a linear passive network, if $i_S(t)=1(t)$A, zero_state response $u_2(t)=2\,1(t)\,(1-e^{-2t})$V, when $i_S(t)=10\sin 2t$ A, calculate the sinusoidal steady_state response $u_2(t)$.



$$H(S)=\frac{\pounds[u_2(t)]}{\pounds[i_S(t)]}=\frac{2(\frac{1}{S}-\frac{1}{S+2})}{\frac{1}{S}}=\frac{4}{S+2}$$

$$\dot{U}_2=H(j2)\dot{I}_S=\frac{4}{2+j2}\times 10=14.1\angle -45^o$$

$$u_2(t)=14.1\sin(2t-45^o)$$

The transfer function H(s) is the radio of the output response Y(s) to the input excitation X(s), assuming all initial conditions are zero.

$$H(s)=\frac{Y(s)}{X(s)}$$

A special case is when the input is the unit impulse function,

$$x(t)=\delta(t)$$

so that X(s)=1.

For this case, Y(s)=H(s) or y(t)=h(t)

where

$$h(t)=L^{-1}[H(s)]$$

The term h(t) represents the unit impulse response——it is the time-domain response of the network to a unit impulse.

Example1: The transfer function

H(jω)=(2jω+3)/(-ω²+3jω+2),when the input excitation is e^{-t}, find the zero-state response.

Solution :

We use *s* instead of *jω*, and we obtain the network function of complex frequency domain.

$$H(s)=\frac{2s+3}{s^2+3s+2}$$

$$Y(s)=H(s)X(s)$$
$$=\frac{2s+3}{s^2+3s+2}\cdot\frac{1}{s+1}$$
$$=\frac{-1}{s+2}+\frac{1}{(s+1)^2}+\frac{1}{s+1}$$

$$\therefore y(t)=-e^{-2t}+te^{-t}+e^{-t} \qquad t\geq 0$$

Example2: The transfer function

H(s)=(2s+3)/(s²+3s+2),when the input excitation is *Sin t*, find the sinusoidal steady-state response.

Solution : We use *jω* instead of *s*, we obtain the network function of frequency domain:

$$H(j\omega)=\frac{3+2j\omega}{-\omega^2+3j\omega+2} \qquad \because \omega=1rad/s$$

If we assume Y_m is the phasor of the input sinusoidal steady-state response,

$$Y_m=1\angle 0^0\cdot\frac{3+2j\omega}{-\omega^2+3j\omega+2}\bigg|_{\omega=1}$$

$$Y_m = 1\angle 0^0 \cdot \frac{3+2j\omega}{-\omega^2+3j\omega+2}\Big|_{\omega=1}$$

$$= \frac{3+j2}{1+j3} = 0.9 - j0.7 = 1.14\angle -37.87^0$$

∴ The steady-state response is

$$y(t) = 1.14Sin(t - 37.87^0)$$

Example3: The network's zero-state response of unit step function is (1-e^{-t}), then find the network function H(s).

Solution : $\because s(t) = (1-e^{-t})\varepsilon(t)$

The unit impulse function

$$h(t) = \frac{ds(t)}{dt} = e^{-t}\varepsilon(t) + (1-e^{-t})\delta(t) = e^{-t}\varepsilon(t)$$

$$\therefore H(s) = L[h(t)] = L[e^{-t}\varepsilon(t)] = \frac{1}{s+1}$$

Example4: The input impedance

$$Z(s) = \frac{6s^3+3s^2+3s+1}{6s^3+3s}$$

Try to construct the network.

Solution: (method1:)

$$Z(s) = 1 + \frac{3s^2+1}{6s^3+3s} = 1 + \cfrac{1}{2s + \cfrac{s}{3s^2+1}}$$

$$= 1 + \cfrac{1}{2s + \cfrac{1}{3s + \cfrac{1}{s}}}$$

As in Fig.

(method2:)

$$Z(s) = 1 + \frac{3s^2+1}{6s^3+3s} = 1 + \frac{1}{3s} + \frac{s}{6s^2+3}$$

$$= 1 + \frac{1}{3s} + \cfrac{1}{6s + \cfrac{3}{s}}$$

As in Fig.