Fundamentals of Electric Circuits 2020.11

Supplement:
State Equation

State Equation of Circuit

- State Variables and State Equation of Circuits
- State Equation Formulation (How to set up state equation of a circuit.)
- State Equation in complex frequency domain (*s* domain).

Motivation

Thus far in this course we have considered techniques for analyzing systems with only one output and only one input.

Many engineering systems have many inputs and many outputs.

The state variable method is a very important tool in analyzing systems and understanding such highly complex systems.

The state variable model is more general than the single-input ,single-output model, such as a transfer function.

Some conceptions

State Variables

In the state variable model, we specify ***a collection of variables*** that describe the internal behavior of the system. These variables are known as ***state variables*** of the system. They are the variables that determine the future behavior of a system when the present state of the system and the input signals are known.

In other words, they are those variables, which, if known, allow all other system parameters to be determined by using only algebraic equations.

状态变量法不仅适用于分析线性非时变电路，而且适合用来分析线性时变电路和非线性电路。

一、状态变量

状态变量：对于某个动态电路，如果已知n个独立变量在t_0时刻的初始值以及$t \geq t_0$时电路的激励，就可完全确定$t \geq t_0$时电路的响应，那么n个独立变量就称为电路的一组状态变量。

在分析网络(或系统)时，在网络内部选一组最少数量的特定变量X，$X=[X_1,X_2\ldots\ldots X_n]^T$，只要知道这组变量在某一时刻值$X(t_0)$,再知道输入$e(t)$就可以确定$t_0$及$t_0$以后任何时刻网络的性状(响应)，称这一组最少数目的特定变量为状态变量。

$$\left.\begin{array}{r} X(t_0) \\ e(t),(t \geq t_0) \end{array}\right\rangle \longrightarrow Y(t),(t \geq t_0)$$

Some conceptions

State Equation

Common examples of state variables are the pressure, volume, and temperature. In an electric circuit, the state variables are *the inductor current* and *capacitor voltage* since they collectively describe the energy state of the system.

The standard way to represent the state equations is to arrange them as ***a set of first order differential equations***.

$$\dot{X}=AX+BF$$

$$Y=CX+DF$$

特点：（1）状态方程是一组一阶微分方程，输出方程是一组代数方程，便于计算机辅助求解；
（2）容易推广到非线性和时变网络。
（3）可用于分析系统的稳定性和可控性

n阶微分方程 $\xrightarrow{\text{变量代换}}$ 含n个方程的一阶微分方程组

$$\frac{d^2y}{dt^2}+p\frac{dy}{dt}+qy=f$$

$$x_1=y \qquad x_2=\frac{dy}{dt}$$

$$\frac{dx_1}{dt}=x_2$$

$$\frac{dx_2}{dt}=-qx_1-px_2+f$$

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{pmatrix}0&1\\-q&-p\end{pmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}f$$

$$y=[1\quad 0]\begin{bmatrix}x_1\\x_2\end{bmatrix}+0\cdot f$$

$$\dot{X}=AX+BF$$

$$Y=CX+DF$$

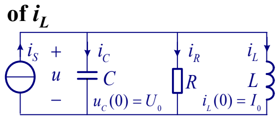
状态空间(state space):把每个状态变量作为一个坐标形成的空间。

状态轨迹(state trajectory)：—状态向量$x(t)$在任一时刻t的值称为电路在该时刻的状态。每一时刻的状态在状态空间中都对应一个“点”，所有这些“点”形成的“轨迹”，称为状态轨迹。通过状态轨迹人们就可以判断电路的基本性质

Example：*RLC* parallel circuit

（1）Differential equation of i_L

$$LC\frac{d^2i_L}{dt^2}+\frac{L}{R}\frac{di_L}{dt}+i_L=0$$



Initial values：$i_L(0_+)=I_0$、$u_C(0_+)=U_0$

（2） **Equations of i_L and u_C ：**

$$L\frac{di_L}{dt}=u_C$$

$$C\frac{du_C}{dt}=-i_L-\frac{u_C}{R}+i_S$$

Matrix form

$$\begin{bmatrix}\frac{di_L}{dt}\\\frac{du_C}{dt}\end{bmatrix}=\begin{bmatrix}0&\frac{1}{L}\\-\frac{1}{C}&-\frac{1}{RC}\end{bmatrix}\begin{bmatrix}i_L\\u_C\end{bmatrix}+\begin{bmatrix}0\\\frac{1}{C}\end{bmatrix}i_S$$

是以i_L和u_C为变量的一阶微分方程组。**(State Equation)**

初始值$i_L(0_+)=I_0$、$u_C(0_+)=U_0$也可表示成 $\begin{bmatrix}i_L(0_+)\\u_C(0_+)\end{bmatrix}=\begin{bmatrix}I_0\\U_0\end{bmatrix}$

称这一阶微分方程组为*RLC*并联电路动态过程的状态方程，并可简写成

$$\dot{x}=Ax+Bw$$

$$\dot{x}=Ax+Bw$$

其中$x=[\,i_L\ u_C]^T$称为电路的状态

x中的元素i_L和u_C称为状态变量

A、B —为系数矩阵，取决于电路拓扑结构和元件参数

W—为输入向量

$x(0_+)=[\,I_0\ U_0]^T$ —为电路的初始状态

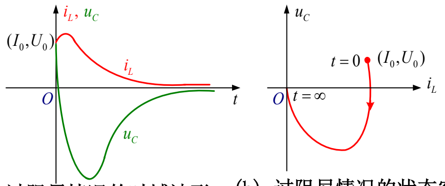
$x(0_-)$ —电路的原始状态

根据换路定律有$x(0_+)=x(0_-)=x(0)=x_0$

（1）当$w=0$，$x_0 \neq 0$时，状态方程描述零输入响应；

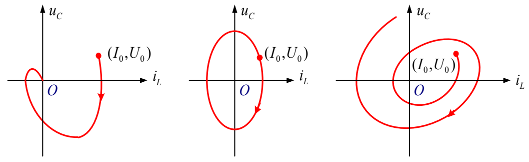
（2）当$w \neq 0$，$x_0 = 0$时，状态方程描述零状态响应；

（3）当$w \neq 0$，$x_0 \neq 0$时，状态方程描述完全响应。



(a) 过阻尼情况的时域波形 (b) 过阻尼情况的状态空间轨迹

***RLC*并联电路的零输入响应**

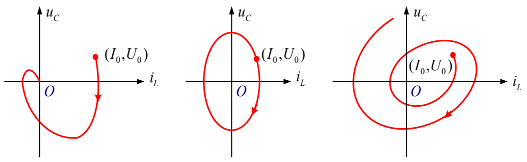


(a) 欠阻尼情况 **(b)** 无阻尼情况 **(c)** 发散情况

电路的状态空间轨迹能够反映电路的特性

1. 过阻尼情况：

状态轨迹从t=0_+的初始状态x_0=$[I_0\ \ U_0]^T$开始，在t=∞时终止于坐标原点



(a) 欠阻尼情况 **(b)** 无阻尼情况 **(c)** 发散情况

（2）欠阻尼情况：状态轨迹是从t=0_+到t=∞ 时的螺旋线

（3）无阻尼情况：状态轨迹是以原点为对称的椭圆

（4）响应为增幅振荡情况：在t趋于∞ 时，零输入响应成为无界，状态轨迹是向外发散的。

Steps to Apply the State Variable Method to Circuit Analysis:

1.Select the **inductor current *i*** and **capacitor voltage *v*** as the state variables, making sure they are consistent with the passive sign convention.

2.Apply KCL and KVL to the circuit and obtain circuit variables (voltages and currents) in terms of the state variables. This should lead to a set of first-order differential equations (**State Equation**) necessary and sufficient to determine all state variables.

3.Obtain the output equation and put the final result in state-space representation.

电路状态方程的列写

列写方法 { 直接观察、置换方法：不太复杂的电路；系统法：复杂的电路 }

这里介绍直接观察或置换方法列写电路的状态方程。

一、直接观察法 步骤

(1) 选一个树，使它包含全部电容（和无伴电压源支路）而不含电感（和无伴电流源支路）。

(2) 对每个电容树支确定的基本割集列写KCL方程；对每个电感连支确定的基本回路列写KVL方程。

(3) 消去以上两组方程中的非状态变量（就是将非状态变量用状态变量和激励来表示），并整理成标准形式的状态方程。

二、输出方程的列写

（1）用置换定理将每个电容C用电压源u_C置换 将每个电感L用电流源i_L置换

（2）将非状态变量用状态变量和输入激励表示

（3）整理成标准形式的输出方程

电路不含下列情况之一

(1)仅由电容元件构成的回路(全电容电路)；

(2)仅由电感元件构成的割集(全电感割集)；

(3)仅由电压源与电容构成的回路；

(4)仅由电流源和电感元件构成的割集；

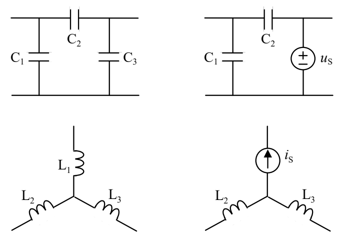
状态变量的数目 ＝ 动态元件的数目

常态树(proper tree)的概念

树包含所有电容支路，而不含任何电感支路。

常态网络：既无纯电容回路，又无纯电感割集的网络

常态网络中所有的电容电压和电感电流，均应被选作状态变量。

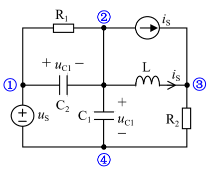


纯电容回路

纯电感割集

(常态网络)

(非常态网络)：存在纯电容回路和纯电感割集的网络



纯电容回路数目=1

纯电感割集数目=0

电路阶数=3-1-0=2

对LTI网络，选择电容电压和电感电流作为状态变量

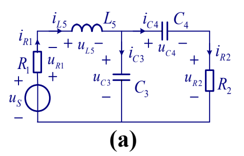
$$\frac{du_c}{dt} \longrightarrow C\frac{du_c}{dt}=i_c \longrightarrow \text{建立包含电容支路的KCL方程}$$

$$\frac{di_L}{dt} \longrightarrow L\frac{di_L}{dt}=u_L \longrightarrow \text{建立包含电感支路的KVL方程}$$

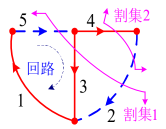
- 只有将KCL应用于割集才能最大限度得到满足，只要使所选取的每个割集仅含一个电容支路（单电容割集）；从方程的右边考虑，所选取的每个割集应尽可能多地包含电感元件的支路。
- 所选取的每个回路只含一条电感元件的支路（单电感回路）；另外回路应尽可能多地包含电容支路。

电容为树支，电感为连支，电压源为树支，电流源为连支；即树包含所有电容和电压源，不含电感和电流源；这样的树称为常态树(proper tree)

例1　试列出图(a)所示电路的状态方程。



(a)



解：1. 直接观察法写状态方程

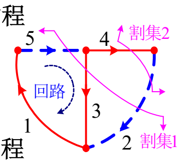
(1) 选1、3、4作为树支，则2、5为连支。

(2) 对电容C_3确定的基本割集1列写KCL方程

$$C_3\frac{du_{C3}}{dt}=i_{L5}-i_{R2}$$

对电容C_4确定的基本割集2列写KCL方程

$$C_4\frac{du_{C4}}{dt}=i_{R2}$$



对电感L_5确定的基本回路列写KVL方程

$$L_5\frac{di_{L5}}{dt}=-u_{C3}-R_1 i_{R1}+u_s$$

(3) 用u_{C3}、u_{C4}、i_{L5}和u_s表示非状态变量i_{R1}和i_{R2}，得到

$$i_{R1}=i_{L5},\quad i_{R2}=\frac{u_{C3}-u_{C4}}{R_2}$$

代入基本割集和基本回路方程，有

$$C_3\frac{du_{C3}}{dt}=-\frac{u_{C3}}{R_2}+\frac{u_{C4}}{R_2}+i_{L5}$$
$$C_4\frac{du_{C4}}{dt}=\frac{u_{C3}}{R_2}-\frac{u_{C4}}{R_2}$$
$$L_5\frac{di_{L5}}{dt}=-u_{C3}-R_1 i_{L5}+u_s$$

整理成标准形式的状态方程为

$$\begin{bmatrix}\frac{du_{C3}}{dt}\\ \frac{du_{C4}}{dt}\\ \frac{di_{L5}}{dt}\end{bmatrix}=\begin{bmatrix}-\frac{1}{C_3R_2} & \frac{1}{C_3R_2} & \frac{1}{C_3}\\ \frac{1}{C_4R_2} & -\frac{1}{C_4R_2} & 0\\ -\frac{1}{L_5} & 0 & -\frac{R_1}{L_5}\end{bmatrix}\begin{bmatrix}u_{C3}\\ u_{C4}\\ i_{L5}\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \frac{1}{L_5}\end{bmatrix}u_s$$

2. 写输出方程

若以i_{C3}和u_{L5}作为输出变量，则有

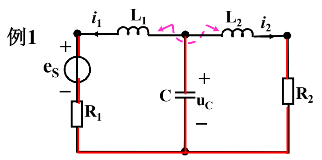
$$i_{C3}=i_{L5}-i_{R2}$$
$$u_{L5}=-u_{R1}-u_{C3}+u_S$$

整理后可得标准形式的输出方程

$$\begin{bmatrix}i_{C3}\\ u_{L5}\end{bmatrix}=\begin{bmatrix}-\frac{1}{R_2} & \frac{1}{R_2} & 1\\ -1 & 0 & -R_1\end{bmatrix}\begin{bmatrix}u_{C3}\\ u_{C4}\\ i_{L5}\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u_S$$

Example:

例1



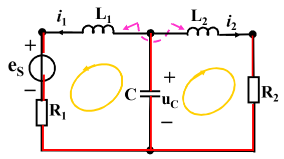
(1) Select a tree and state variables（u_C，i_1，i_2）

(2) To every fundamental cut-set with capacitor，write KCL equations.

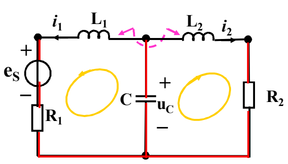
$$C\frac{du_c}{dt}=-i_1-i_2 \qquad \frac{du_c}{dt}=-\frac{1}{C}i_1-\frac{1}{C}i_2$$

(3) To every fundamental loops with inductor, write KVL equations

$$\frac{du_c}{dt}=-\frac{1}{C}i_1-\frac{1}{C}i_2$$



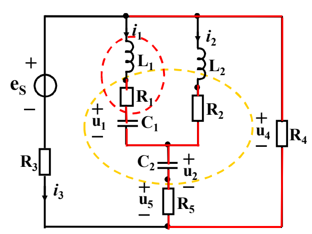
$$L_1\frac{di_1}{dt}=u_c-R_1 i_1-e_S \qquad \frac{di_1}{dt}=\frac{u_c}{L_1}-\frac{R_1}{L_1}i_1-\frac{1}{L_1}e_S$$
$$L_2\frac{di_2}{dt}=u_c-R_2 i_2 \qquad \frac{di_2}{dt}=\frac{u_c}{L_2}-\frac{R_1}{L_2}i_2$$



$$\frac{du_c}{dt}=-\frac{1}{C}i_1-\frac{1}{C}i_2$$
$$\frac{di_1}{dt}=\frac{u_c}{L_1}-\frac{R_1}{L_1}i_1-\frac{1}{L_1}e_S$$
$$\frac{di_2}{dt}=\frac{u_c}{L_2}-\frac{R_1}{L_2}i_2$$

$$\frac{d}{dt}\begin{pmatrix}u_C\\ i_1\\ i_2\end{pmatrix}=\begin{pmatrix}0 & -\frac{1}{C} & -\frac{1}{C}\\ \frac{1}{L_1} & -\frac{R_1}{L_1} & 0\\ \frac{1}{L_2} & 0 & -\frac{R_2}{L_2}\end{pmatrix}\begin{pmatrix}u_C\\ i_1\\ i_2\end{pmatrix}+\begin{pmatrix}0\\ -\frac{1}{L_1}\\ 0\end{pmatrix}e_S$$

例2



(1) Select proper tree and state variables

(2) $\frac{du_1}{dt}=\frac{i_1}{C_1} \qquad \frac{du_2}{dt}=\frac{i_1}{C_2}+\frac{i_2}{C_2}$

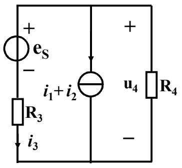
$$\frac{di_1}{dt}=\frac{1}{L_1}(u_4-u_5-u_2-u_1-R_1 i_1)$$
$$\frac{di_2}{dt}=\frac{1}{L_2}(u_4-u_5-u_2-R_2 i_2)$$

(3) 消去除输入外的非状态变量

消去除输入外的非状态变量，就是用状态变量和输入去表示那些非状态变量。

$u_5=R_5(i_1+i_2)$

将电容元件用电压源代替，电感元件用电流源代替



$$u_4=\frac{R_4}{R_3+R_4}e_s-\frac{R_3R_4}{R_3+R_4}(i_1+i_2)$$

$$\frac{d}{dt}\begin{pmatrix}u_1\\u_2\\i_1\\i_2\end{pmatrix}=\begin{pmatrix}0&0&\frac{1}{C_1}&0\\0&0&\frac{1}{C_2}&\frac{1}{C_2}\\-\frac{1}{L_1}&-\frac{1}{L_1}&a_{33}&a_{34}\\0&-\frac{1}{L_2}&a_{43}&a_{44}\end{pmatrix}\begin{pmatrix}u_1\\u_2\\i_1\\i_2\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{R_4}{L_1(R_3+R_4)}\\\frac{R_4}{L_2(R_3+R_4)}\end{pmatrix}e_S$$

$$\frac{d}{dt}\begin{pmatrix}u_1\\u_2\\i_1\\i_2\end{pmatrix}=\begin{pmatrix}0&0&\frac{1}{C_1}&0\\0&0&\frac{1}{C_2}&\frac{1}{C_2}\\-\frac{1}{L_1}&-\frac{1}{L_1}&a_{33}&a_{34}\\0&-\frac{1}{L_2}&a_{43}&a_{44}\end{pmatrix}\begin{pmatrix}u_1\\u_2\\i_1\\i_2\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{R_4}{L_1(R_3+R_4)}\\\frac{R_4}{L_2(R_3+R_4)}\end{pmatrix}e_S$$

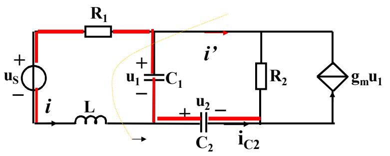
$$a_{33}=-\frac{1}{L_1}\left(R_1+R_5+\frac{R_3R_4}{R_3+R_4}\right)$$

$$a_{34}=-\frac{1}{L_1}\left(R_5+\frac{R_3R_4}{R_3+R_4}\right)$$

$$a_{43}=-\frac{1}{L_2}\left(R_5+\frac{R_3R_4}{R_3+R_4}\right)$$

$$a_{44}=-\frac{1}{L_2}\left(R_2+R_5+\frac{R_3R_4}{R_3+R_4}\right)$$

例3



(1) Select proper tree and state variables（u_1， u_2， i）

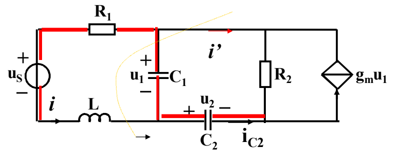
(2) KVL, fundamental loop, inductor

$$L\frac{di}{dt}=u_1-iR_1-u_s$$

(3) KCL, fundamental cut-set, capacitor

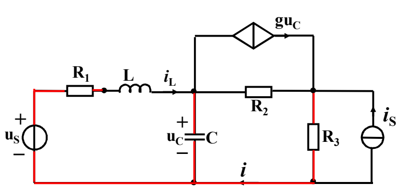
$$C_1\frac{du_1}{dt}=-i-i',C_2\frac{du_2}{dt}=-i',i'=\frac{u_1+u_2}{R_2}-g_mu_1$$

例3



$$\frac{d}{dt}\begin{pmatrix}i\\u_1\\u_2\end{pmatrix}=\begin{pmatrix}-\frac{R_1}{L}&\frac{1}{L_1}&0\\-\frac{1}{C_1}&\frac{g_m}{C_1}-\frac{1}{R_2C_1}&-\frac{1}{R_2C_1}\\0&-\frac{1}{R_2C_2}+\frac{g_m}{C_2}&-\frac{1}{R_2C_2}\end{pmatrix}\begin{pmatrix}i\\u_1\\u_2\end{pmatrix}+\begin{pmatrix}-\frac{1}{L_1}\\0\\0\end{pmatrix}e_S$$

例4

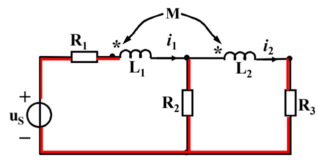


$$\frac{du_c}{dt}=\frac{1}{C}(i_L-i)\qquad\frac{di_L}{dt}=\frac{1}{L}(-u_C-R_1i_L+u_S)$$

$$i=\frac{1}{R_2+R_3}(u_C-R_3i_S+R_2gu_C)$$

$$\frac{d}{dt}\begin{pmatrix}u_C\\i_L\end{pmatrix}=\begin{pmatrix}-\frac{1+gR_2}{(R_2+R_3)C}&\frac{1}{C}\\-\frac{1}{L}&-\frac{R_1}{L}\end{pmatrix}\begin{pmatrix}u_C\\i_L\end{pmatrix}+\begin{pmatrix}0&\frac{R_3}{(R_2+R_3)C}\\\frac{1}{L}&0\end{pmatrix}\begin{pmatrix}u_S\\i_S\end{pmatrix}$$

例5



$$L_1\frac{di_1}{dt}+M\frac{di_2}{dt}=-R_1i_1+u_S-R_2(i_1-i_2)$$

$$M\frac{di_1}{dt}+L_2\frac{di_2}{dt}=R_2(i_1-i_2)-R_3i_2$$

$$\frac{d}{dt}\begin{pmatrix}i_1\\i_2\end{pmatrix}=\frac{1}{L_1L_2-M^2}\begin{pmatrix}-(R_1+R_2)L_2-MR_2&R_2L_2+(R_2+R_3)M\\(R_1+R_2)M+L_1R_2&-(R_2+R_3)L_1-MR_2\end{pmatrix}\begin{pmatrix}i_1\\i_2\end{pmatrix}+\begin{pmatrix}L_2\\-M\end{pmatrix}u_S$$

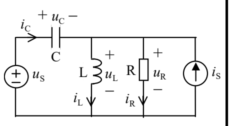
讨论： 若 $L_1L_2-M^2=0$ （全耦合）

则 $\det L=0$，L^{-1}不存在，i_1、i_2线性相关。

Example:

Given $i_L(0^-)$, $u_C(0^-)$, find i_R, i_C , u_L = ?

Solution: State variables: u_C i_L



$$L\frac{di_L}{dt}=u_L=-u_C+u_S\ ,$$

$$C\frac{du_C}{dt}=i_C=i_L+\frac{u_R}{R}-i_S=i_L+\frac{u_S-u_C}{R}-i_S$$

$$\frac{di_L}{dt}=-\frac{1}{L}u_C+\frac{1}{L}u_S\ ,\quad \frac{du_C}{dt}=\frac{1}{C}i_L-\frac{1}{RC}u_C+\frac{1}{RC}u_S-\frac{1}{C}i_S$$

In matrix form, the equation becomes

$$\frac{d}{dt}\begin{bmatrix} i_L \\ u_C \end{bmatrix}=\begin{bmatrix} \frac{di_L}{dt} \\ \frac{du_C}{dt} \end{bmatrix}=\begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}\begin{bmatrix} i_L \\ u_C \end{bmatrix}+\begin{bmatrix} \frac{1}{L} & 0 \\ \frac{1}{RC} & -\frac{1}{C} \end{bmatrix}\begin{bmatrix} u_S \\ i_S \end{bmatrix}$$

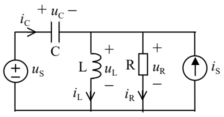
Output Equations

$$\frac{d}{dt}\begin{bmatrix} i_L \\ u_C \end{bmatrix}=\begin{bmatrix} \frac{di_L}{dt} \\ \frac{du_C}{dt} \end{bmatrix}=\begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}\begin{bmatrix} i_L \\ u_C \end{bmatrix}+\begin{bmatrix} \frac{1}{L} & 0 \\ \frac{1}{RC} & -\frac{1}{C} \end{bmatrix}\begin{bmatrix} u_S \\ i_S \end{bmatrix}$$

$$i_C=i_L+\frac{u_R}{R}-i_S=i_L+\frac{u_S-u_C}{R}-i_S\ ,$$

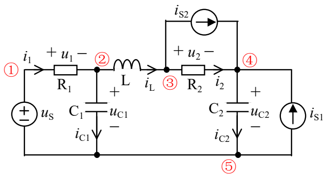
$$i_R=\frac{u_R}{R}=\frac{u_S-u_C}{R}=-\frac{1}{R}u_C+\frac{1}{R}u_S\ ,$$

$$u_L=-u_C+u_S$$

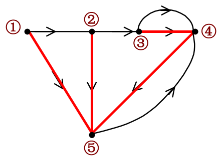


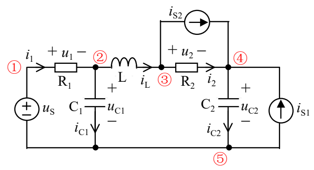
In matrix form, the equation becomes

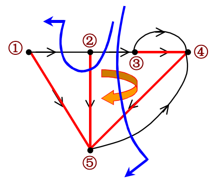
$$\begin{bmatrix} i_C \\ i_R \\ u_L \end{bmatrix}=\begin{bmatrix} 1 & -\frac{1}{R} \\ 0 & -\frac{1}{R} \\ 0 & -1 \end{bmatrix}\begin{bmatrix} i_L \\ u_C \end{bmatrix}+\begin{bmatrix} \frac{1}{R} & -1 \\ \frac{1}{R} & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_S \\ i_S \end{bmatrix}$$



State Equation:







$$KVL:\quad L\frac{di_L}{dt}=u_{C1}-u_{C2}-u_2$$

$$KCL:\quad C_1\frac{du_{C1}}{dt}=-i_L+i_1 \qquad C_2\frac{du_{C2}}{dt}=i_L+i_{S1}$$

$$i_2=i_L-i_{S2}\ ,\quad u_2=R_2i_2=R_2i_L-R_2i_{S2}$$

$$u_1=-u_{C1}+u_S\ ,\quad i_1=\frac{u_1}{R_1}=-\frac{1}{R_1}u_{C1}+\frac{1}{R_1}u_S$$

$$C_1\frac{du_{C1}}{dt}=-\frac{1}{R_1}u_{C1}-i_L+\frac{1}{R_1}u_S\ ,$$

$$C_2\frac{du_{C2}}{dt}=i_L+i_{S1}\ ,$$

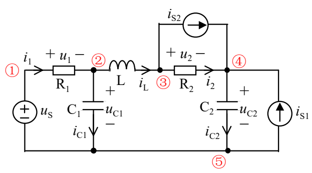
$$L\frac{di_L}{dt}=u_{C1}-u_{C2}-R_2i_L+R_2i_{S2}$$

Let: $x_1=u_{C1}$, $x_2=u_{C2}$, $x_3=i_L$, $f_1=u_S$, $f_2=i_{S1}$, $f_3=i_{S2}$

State Equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}=\begin{bmatrix} \frac{1}{R_1C_1} & 0 & -\frac{1}{C_1} \\ 0 & 0 & \frac{1}{C_2} \\ \frac{1}{L} & -\frac{1}{L} & -\frac{R_2}{L} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}+\begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 \\ 0 & \frac{1}{C_2} & 0 \\ 0 & 0 & \frac{R_2}{L} \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Output: i_{C1}, i_{C2}, u_2, u_L



$$i_{C1}=-\frac{1}{R_1}u_{C1}-i_L+\frac{1}{R_1}u_S$$

$$i_{C2}=i_L+i_{S1}$$

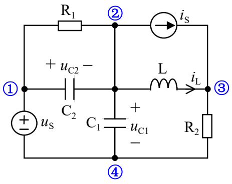
$$u_2=R_2i_L-R_2i_{S2}$$

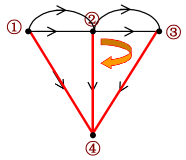
$$u_L=u_{C1}-u_{C2}-R_2i_L+R_2i_{S2}$$

令：$y_1=i_{C1}$, $y_2=i_{C2}$, $y_3=u_2$, $y_4=u_L$

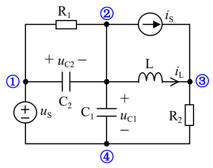
Output Equation:

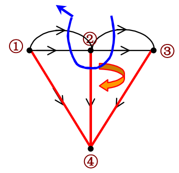
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}=\begin{bmatrix} \frac{1}{R_1} & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & R_2 \\ 1 & -1 & -R_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}+\begin{bmatrix} \frac{1}{R_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -R_2 \\ 0 & 0 & R_2 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$





$$KVL:\ L\frac{di_L}{dt} = u_{C1} - (i_L + i_S)R_2 \ \rightarrow\ \frac{di_L}{dt} = \frac{-R_2}{L}i_L + \frac{1}{L}u_{C1} - \frac{R_2}{L}i_S$$





$$KCL:\ C_1\frac{du_{C1}}{dt} = C_2\frac{d}{dt}(u_S - u_{C1}) + \frac{1}{R_1}(u_S - u_{C1}) - i_L - i_S$$

$$(C_1 + C_2)\frac{du_{C1}}{dt} = -i_L - \frac{1}{R_1}u_{C1} + \frac{1}{R_1}u_S + C_2\frac{du_S}{dt} - i_S$$

$$\frac{du_{C1}}{dt} = -\frac{1}{C_1 + C_2}i_L - \frac{1}{R_1(C_1 + C_2)}u_{C1} + \frac{1}{R_1(C_1 + C_2)}u_S + \frac{C_2}{C_1 + C_2}\frac{du_S}{dt} - \frac{1}{C_1 + C_2}i_S$$

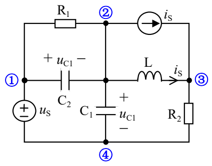
$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{du_{C1}}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_2}{L} & \frac{1}{L} \\ -\frac{1}{C_1 + C_2} & \frac{-1}{R_1(C_1 + C_2)} \end{bmatrix} \begin{bmatrix} i_L \\ u_{C1} \end{bmatrix} + \begin{bmatrix} 0 & \frac{-R_2}{L} \\ \frac{1}{R_1(C_1 + C_2)} & -\frac{1}{C_1 + C_2} \end{bmatrix} \begin{bmatrix} u_S \\ i_S \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ \frac{C_2}{C_1 + C_2} & 0 \end{bmatrix} \begin{bmatrix} \frac{du_S}{dt} \\ \frac{di_S}{dt} \end{bmatrix}$$

Output: i_{C1}，i_{C2}

$$i_{C1} = C_1\frac{du_{C1}}{dt} = -\frac{C_1}{C_1 + C_2}i_L - \frac{C_1}{R_1(C_1 + C_2)}u_{C1} + \frac{C_1}{R_1(C_1 + C_2)}u_S$$

$$-\frac{C_1}{C_1 + C_2}i_S + \frac{C_1C_2}{C_1 + C_2}\frac{du_S}{dt}$$



Node 2：

$$i_{C2} = i_{C1} + i_L + \frac{u_{C1} - u_S}{R_1} + i_S$$

$$i_{C2} = \frac{C_2}{C_1 + C_2}i_L + \frac{C_2}{R_1(C_1 + C_2)}u_{C1} - \frac{C_2}{R_1(C_1 + C_2)}u_S + \frac{C_2}{C_1 + C_2}i_S + \frac{C_1C_2}{C_1 + C_2}\frac{du_S}{dt}$$

$$\begin{bmatrix} i_{C1} \\ i_{C2} \end{bmatrix} = \begin{bmatrix} \frac{-C_1}{C_1 + C_2} & \frac{-C_1}{R_1(C_1 + C_2)} \\ \frac{C_2}{C_1 + C_2} & \frac{C_2}{R_1(C_1 + C_2)} \end{bmatrix} \begin{bmatrix} i_L \\ u_{C1} \end{bmatrix} + \begin{bmatrix} \frac{C_1}{R_1(C_1 + C_2)} & \frac{-C_1}{C_1 + C_2} \\ \frac{-C_2}{R_1(C_1 + C_2)} & \frac{C_2}{C_1 + C_2} \end{bmatrix} \begin{bmatrix} u_S \\ i_S \end{bmatrix} + \begin{bmatrix} \frac{C_1C_2}{C_1 + C_2} & 0 \\ \frac{C_1C_2}{C_1 + C_2} & 0 \end{bmatrix} \begin{bmatrix} \frac{du_S}{dt} \\ \frac{di_S}{dt} \end{bmatrix}$$

状态方程的解法：

1 时域解法

2 Laplace 解法

Solution of the state equations

The process of finding the solution to the state equation is known as state-variable analysis.

$$\dot{X} = AX + BF$$

$$X(S) = \mathcal{L}[X]$$

$$F(S) = \mathcal{L}[F]$$

$$\mathcal{L}[\dot{X}] = SX(S) - X(0^-)$$

$SX(S)–X(0)=AX(S)+BF(S)$

$(SI–A)X(S)=X(0)+BF(S)$ (I is the identity matrix)

$X(S)= (SI–A)^{-1}X(0)+(SI–A)^{-1}BF(S)$

$\lambda(S)=(S\mathbf{1}-\boldsymbol{A})^{-1}$ 预解矩阵 ($\mathbf{1}$ is the identity matrix)

$\boldsymbol{X}(S)=\lambda(S)\boldsymbol{X}(0)+\lambda(S)\boldsymbol{BF}(S)$

$\boldsymbol{X}(t)=\mathcal{L}^{-1}[\lambda(S)\boldsymbol{X}(0)]+\mathcal{L}^{-1}[\lambda(S)\boldsymbol{BF}(S)]$

Complete response = zero-input response + zero-state response

Output equations $\boldsymbol{Y}=\boldsymbol{A}'\boldsymbol{X}+\boldsymbol{B}'\boldsymbol{F}$

$\boldsymbol{Y}(S)=\boldsymbol{A}'\boldsymbol{X}(S)+\boldsymbol{B}'\boldsymbol{F}(S)$

$\boldsymbol{Y}(S)=\boldsymbol{A}'\lambda(S)\boldsymbol{X}(0)+\boldsymbol{A}'\lambda(S)\boldsymbol{BF}(S)+\boldsymbol{B}'\boldsymbol{F}(S)$

$\boldsymbol{Y}(t)=\mathcal{L}^{-1}[\boldsymbol{A}'\lambda(S)\boldsymbol{X}(0)]+\mathcal{L}^{-1}[\boldsymbol{A}'\lambda(S)\boldsymbol{BF}(S)+\boldsymbol{B}'\boldsymbol{F}(S)]$

17-3 状态方程的求解

17. 3. 1 解析解法

解析解法
- 时域的解法
  - 幂级数法
  - 矩阵函数的有限项表示法
  - 对角线化变换矩阵法
- 复频域解法

$\dot{X}=AX+BF \xrightarrow{\pounds} (S\mathbf{1}-A)X(s)=X(0)+BF(s)$

令 $\Phi(S)=(S\mathbf{1}-A)^{-1}$ 状态方程的预解矩阵

则 $X(S)=\Phi(s)X(0)+\Phi(s)BF(s)$

$\det(S\mathbf{1}-A)$ 矩阵A的特征多项式

方程 $\det(S\mathbf{1}-A)=0$ 的根称为矩阵A的特征值,

17.3.1 解析法解法 复频域解法

$X(S)=\Phi(s)X(0)+\Phi(s)BF(s)$

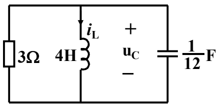
由输出方程 $Y=CX+DF$

$Y(S)=\underbrace{C\Phi(s)X(0)}_{\text{零输入响应}}+\underbrace{[C\Phi(s)B+D]F(s)}_{\text{零状态响应}}$

$C\Phi(s)B+D=H(S)$——转移函数矩阵

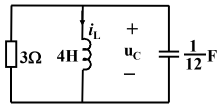
$Y(t)=\pounds^{-1}[C\Phi(s)X(0)]+\pounds^{-1}[H(s)F(s)]$

例1 对图示电路,列出状态方程,并求解。



$\begin{bmatrix}u_c(0)\\ i_L(0)\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix}u_C\\ i_L\end{bmatrix}=\begin{bmatrix}-4 & -12\\ \frac{1}{4} & 0\end{bmatrix}\begin{bmatrix}u_C\\ i_L\end{bmatrix}$$



$$(S\mathbf{1}-A)=\begin{bmatrix}(S+4) & 12\\ -\frac{1}{4} & S\end{bmatrix}\qquad \begin{bmatrix}u_c(0)\\ i_L(0)\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

$\det(S\mathbf{1}-A)=S(S+4)+3=(S+1)(S+3)$ 特征多项式的零点

$$\Phi(s)=(S\mathbf{1}-A)^{-1}=\frac{\begin{bmatrix}S & -12\\ \frac{1}{4} & (S+4)\end{bmatrix}}{(S+1)(S+3)}$$

$$\begin{bmatrix}U_C(s)\\ I_L(s)\end{bmatrix}=\frac{1}{(S+1)(S+3)}\begin{bmatrix}S & -12\\ \frac{1}{4} & (S+4)\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}\frac{S-12}{(S+1)(S+3)}\\ \frac{S+17/4}{(S+1)(S+3)}\end{bmatrix}$$

$$\begin{bmatrix}U_C(s)\\ I_L(s)\end{bmatrix}=\frac{1}{(S+1)(S+3)}\begin{bmatrix}S & -12\\ \frac{1}{4} & (S+4)\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}\frac{S-12}{(S+1)(S+3)}\\ \frac{S+17/4}{(S+1)(S+3)}\end{bmatrix}$$

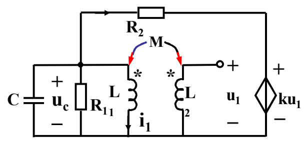
$$\begin{bmatrix}u_C(t)\\ i_L(t)\end{bmatrix}=\pounds^{-1}\begin{bmatrix}U_C(s)\\ I_L(s)\end{bmatrix}=\begin{bmatrix}-\frac{13}{2}e^{-t}+\frac{15}{2}e^{-3t}\\ \frac{13}{8}e^{-t}-\frac{5}{8}e^{-3t}\end{bmatrix}\qquad t\geq 0$$

讨论:关于网络的固有频率和网络函数的极点

$$\Phi(S)=(S\mathbf{1}-A)^{-1}=\frac{(S\mathbf{1}-A)\text{的伴随矩阵}}{\det(S\mathbf{1}-A)}$$

$|SI-A|=0$ 的根就是网络的固有频率,决定了网络零输入的形式

例2 图示电路为一LC振荡电路,耦合电感的付线圈开路,
(1)试写出该电路的状态方程(矩阵形式);
(2)欲使此电路产生正弦振荡,K应满足什么条件,振荡角频率为何值?



$$\frac{du_c}{dt}=\frac{1}{C}\left(-\frac{u_c}{R_1}-i_1-\frac{u_c-ku_1}{R_2}\right)$$

$$\frac{di_1}{dt}=\frac{1}{L_1}u_C\qquad u_1=\frac{M}{L_1}u_C$$

$$\frac{d}{dt}\begin{bmatrix}u_C\\ i_1\end{bmatrix}=\begin{bmatrix}-\frac{1}{R_1C}-\frac{1}{R_2C}+\frac{kM}{L_1R_2C} & -\frac{1}{C}\\ \frac{1}{L_1} & 0\end{bmatrix}\begin{bmatrix}u_C\\ i_1\end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix}u_C\\ i_1\end{bmatrix}=\begin{bmatrix}-\frac{1}{R_1C}-\frac{1}{R_2C}+\frac{kM}{L_1R_2C} & -\frac{1}{C}\\ \frac{1}{L_1} & 0\end{bmatrix}\begin{bmatrix}u_C\\ i_1\end{bmatrix}$$

$$\begin{vmatrix}S+\frac{1}{R_1C}+\frac{1}{R_2C}-\frac{kM}{L_1R_2C} & \frac{1}{C}\\ -\frac{1}{L_1} & S\end{vmatrix}=0$$

$$S^2+\left(\frac{1}{R_1C}+\frac{1}{R_2C}-\frac{kM}{L_1R_2C}\right)S+\frac{1}{L_1C}=0$$

产生正弦振荡的条件： $\frac{1}{R_1C}+\frac{1}{R_2C}-\frac{kM}{L_1R_2C}=0$

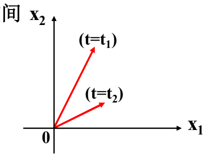
$$k=\frac{L_1(R_1+R_2)}{R_1M}$$

振荡角频率： $\omega=\frac{1}{\sqrt{L_1C}}$

17.4 状态空间与状态轨迹

$$X(t)=\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}$$

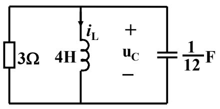
状态空间:由状态变量为基底构成的空间



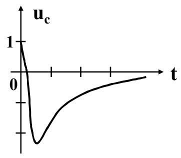
状态矢量

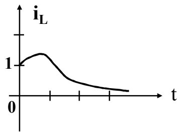
状态轨迹　　状态矢量末端的轨迹

例　如图，$u_C(0)$=1V, $i_L(0)$=1A求零输入响应的状态轨迹。

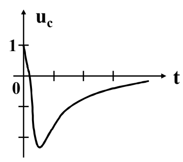


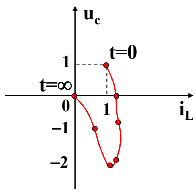
$$\begin{pmatrix}u_C(t)\\ i_L(t)\end{pmatrix}=\begin{pmatrix}\frac{-13}{2}e^{-t}+\frac{15}{2}e^{-3t}\\ \frac{13}{8}e^{-t}-\frac{5}{8}e^{-3t}\end{pmatrix}$$

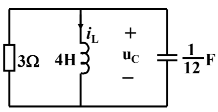


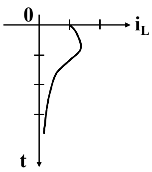


例　如图，$u_C(0)$=1V, $i_L(0)$=1A求零输入响应的状态轨迹。









$$\begin{pmatrix}u_C(t)\\ i_L(t)\end{pmatrix}=\begin{pmatrix}\frac{-13}{2}e^{-t}+\frac{15}{2}e^{-3t}\\ \frac{13}{8}e^{-t}-\frac{5}{8}e^{-3t}\end{pmatrix}$$

The end