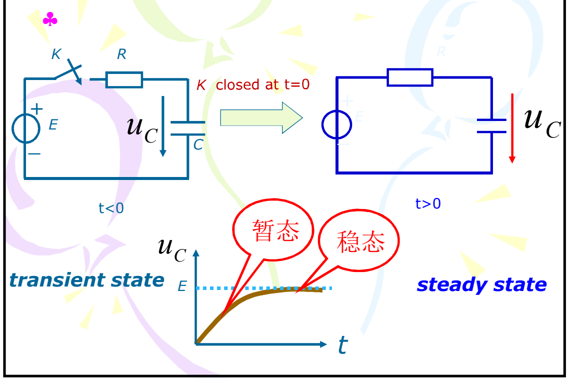
Fundamentals of Electric Circuit 2020.10

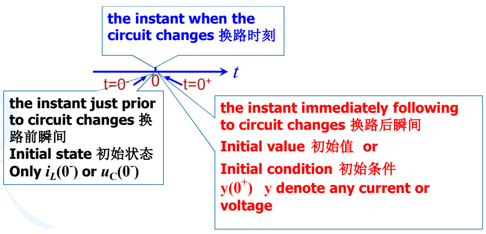
Chapter 7
First-Order Circuits

Chapter 7 First-Order Circuits



7.0 introduction

- In this chapter, we will focus on circuits that consist only of sources, resistors, and either (but not both) inductors or capacitors.
- We call them as ***RC (resistor-capacitor) circuits*** and ***RL (resistor-inductor) .***
- Networks of this form find use in electronic amplifiers, automatic control systems, operational amplifiers, communications equipment, and many other applications.



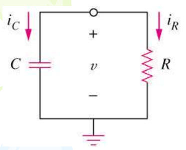
7.1 The Source-Free RC Circuit

- A ***first-order circuit*** is characterized by a first-order differential equation.

Any circuit with

- ***a single energy storage element（1个储能元件，R 或 L）***
- ***an arbitrary number of sources （任意个电源）***
- ***an arbitrary number of resistors （任意个电阻）is a circuit of order 1.***

- Apply Kirchhoff's laws to purely resistive circuit results in algebraic equations.
- Apply the laws to RC and RL circuits produces differential equations.



By KCL

$$i_R + i_C = 0 \;\rightarrow\; \frac{v}{R} + C\frac{dv}{dt} = 0$$

Ohms law   Capacitor law

Rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

Integrating both sides gives

$$\ln v = -\frac{1}{RC}t + \ln A$$

$$v(t) = Ae^{-t/RC}\,V$$

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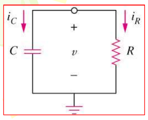

$v(t) = V_0\, e^{-t/\tau}$ where $\tau = RC$

The key to working with a source-free RC circuit is finding:

1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant τ = RC.

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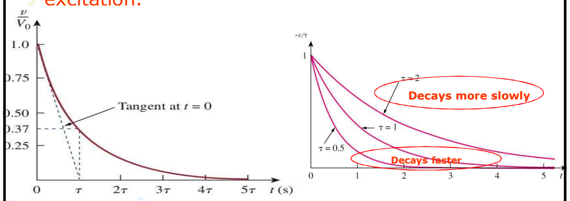
- The **natural response** of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.



- The ***time constant*** τ of a circuit is the time required for the response to decay by a factor of 1/e or 36.8% of its initial value.
- v decays faster for small τ and slower for large τ.

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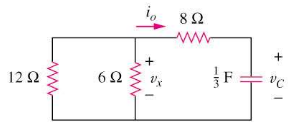
Example 1

Refer to the circuit below, determine v_C, v_x, and i_o for $t \geq 0$.

Assume that $v_C(0) = 30$ V.



- Please refer to lecture or textbook for more detail elaboration. Answer:

$v_C = 30e^{-0.25t}$ V ; $v_x = 10e^{-0.25t}$; $i_o = -2.5e^{-0.25t}$ A

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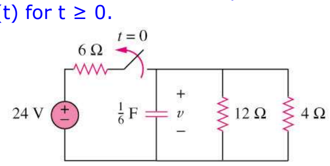
Example 2

The switch in circuit below is opened at t = 0, find v(t) for t ≥ 0.



- Please refer to lecture or textbook for more detail elaboration.

Answer: $V(t) = 8e^{-2t}$ V

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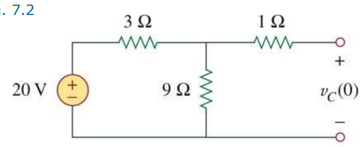
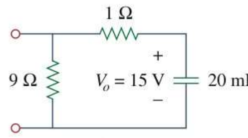
P242. E.g. 7.2



(a)



(b)

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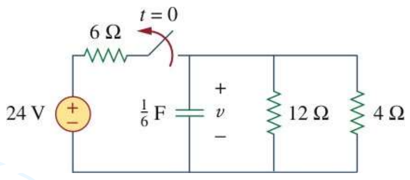
Practice problem

If the switch in Fig. below opens at t=0, find v(t) for t>0 and $W_c(0)$



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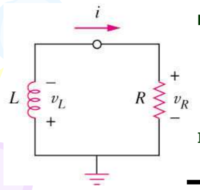
7.2 The Source-Free RL Circuit

- A **first-order RL circuit** consists of a inductor L (or its equivalent) and a resistor (or its equivalent)



By KVL $v_L + v_R = 0$

$\rightarrow$ $L\dfrac{di}{dt} + iR = 0$

Inductors law — Ohms law

$\rightarrow$ $\dfrac{di}{i} = -\dfrac{R}{L}dt$ $\rightarrow$ $i(t) = I_0\, e^{-Rt/L}$

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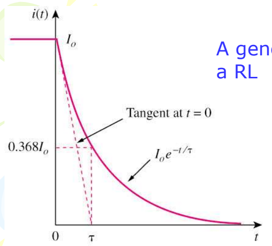

A general form representing a RL

$$i(t) = I_0\, e^{-t/\tau}$$

where $\tau = \dfrac{L}{R}$

- The **time constant** τ of a circuit is the time required for the response to decay by a factor of 1/*e* or 36.8% of its initial value.
- *i(t)* decays faster for small τ and slower for large τ.
- The general form is very similar to a RC source-free circuit.

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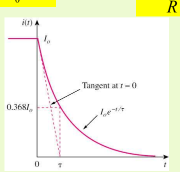
Comparison between a RL and RC circuit

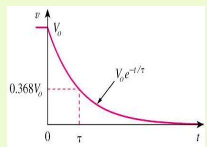
A RL source-free circuit

$i(t) = I_0\, e^{-t/\tau}$ **where** $\tau = \dfrac{L}{R}$



A RC source-free circuit

$v(t) = V_0\, e^{-t/\tau}$ **where** $\tau = RC$



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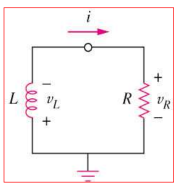
The key to working with a source-free RL circuit is finding:

$i(t) = I_0\, e^{-t/\tau}$ where $\tau = \dfrac{L}{R}$



1. The initial voltage $i(0) = I_0$ through the inductor.
2. The time constant τ = L/R.

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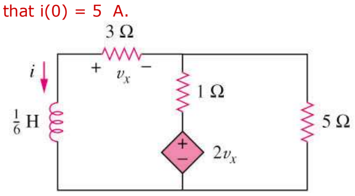
Example 3

Find i and v_x in the circuit.

Assume that i(0) = 5 A.



- Please refer to lecture or textbook for more detail elaboration.
- Answer: *i(t) = 5e^{−53t} A*

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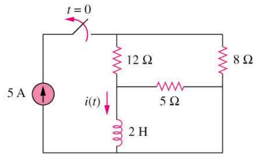
Example 4

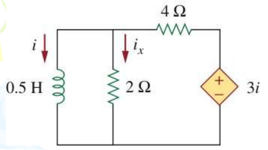
For the circuit, find i(t) for t > 0.

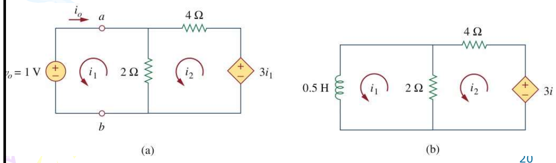


- Please refer to lecture or textbook for more detail elaboration.

Answer: $i(t) = 2e^{-2t}$ A

Assuming that i(0)=10A, calculate $i(t)$ and $i_x(t)$ in the circuit in the Fig. below





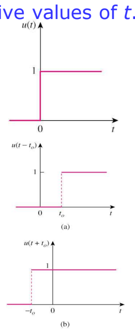
7.3 Unit-Step Function

- The ***unit step function*** $u(t)$ is 0 for negative values of t and 1 for positive values of t.

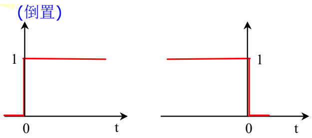
$$u(t)=\begin{cases}0, & t<0\\ 1, & t>0\end{cases}$$

$$u(t-t_o)=\begin{cases}0, & t<t_o\\ 1, & t>t_o\end{cases}$$

$$u(t+t_o)=\begin{cases}0, & t<-t_o\\ 1, & t>-t_o\end{cases}$$

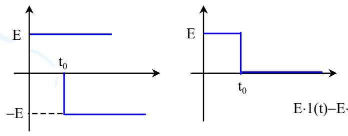


Inversion （倒置）



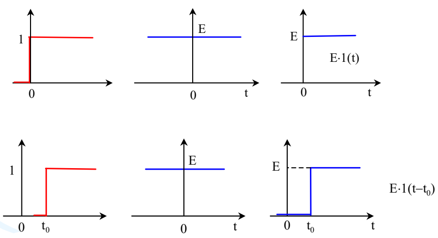
$$1(t)=\begin{cases}0 & -t<0 \qquad t>0\\ 1 & -t>0 \qquad t<0\end{cases}$$

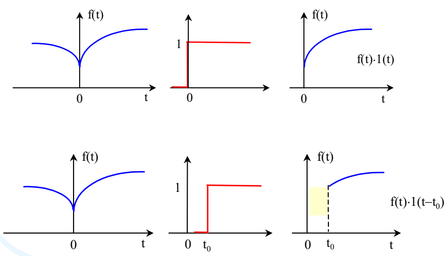
The sum of two functions



E·1(t)−E·1(t−t_0)

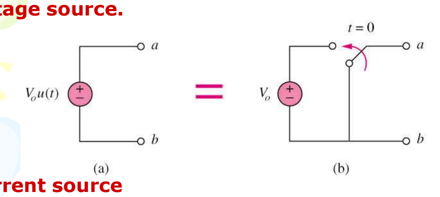
The multiplication of two functions



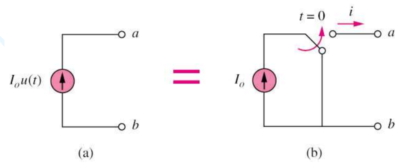


Represent an abrupt change for:

Voltage source.



Current source



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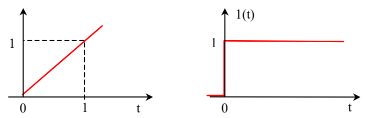
Other Singularity Functions（奇异函数）

The relationship between the unit ramp function and the unit step function:

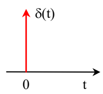


$$r(t)=\begin{cases}0 & t<0\\ t & t>0\end{cases}\qquad 1(t)=\frac{dr(t)}{dt}\ ,\quad r(t)=\int_0^t 1(\tau)d\tau$$

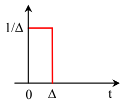
2．The unit impulse function （单位冲激函数）

$$\delta(t)=\begin{cases}0 & t\neq 0\\ \infty & t=0\end{cases}$$

$$\int_{-\infty}^{\infty}\delta(t)dt=1$$



$$f(t)=\begin{cases}0 & t<0\\ \frac{1}{\Delta} & 0<t<\Delta\ ,\quad \frac{1}{\Delta}\cdot\Delta=1\\ 0 & t>\Delta\end{cases}$$



The relationship between the unit impulse function and the unit step function:

$$\int_{-\infty}^{t}\delta(t)dt=\begin{cases}0 & t<0\\ 1 & t>0\end{cases}=1(t)\ ,\quad \frac{d1(t)}{dt}=\delta(t)$$

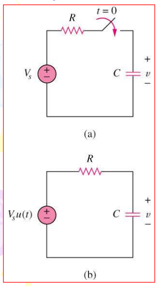
Sifting (sampling) property:（筛分）

$f(t)\delta(t)=f(0)\delta(t)$; $f(t)\delta(t-t_0)=f(t_0)\delta(t-t_0)$

$$\int_{-\infty}^{\infty}f(t)\delta(t)dt=f(0)\ ,\quad \int_{-\infty}^{\infty}f(t)\delta(t-t_0)dt=f(t_0)$$

7.4 The Step-Response of a RC Circuit

- The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.



- **Initial condition**:

$v(0-) = v(0+) = V_0$

Applying KCL,

$$C\frac{dv}{dt}+\frac{v-V_s u(t)}{R}=0$$

or

$$\frac{dv}{dt}=-\frac{v-V_s}{RC}u(t)$$

- Where u(t) is the unit-step function

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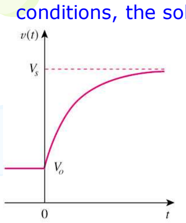
- Integrating both sides and considering the initial conditions, the solution of the equation is:



$$v(t)=\begin{cases}V_0 & t<0\\ V_s+(V_0-V_s)e^{-t/\tau} & t>0\end{cases}$$

Final value at t -> ∞ **Initial value at t = 0** **Source-free Response**

Complete Response = Natural response + Forced Response

(stored energy) (independent source)

= $V_0e^{-t/\tau}$ + $V_s(1-e^{-t/\tau})$

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Three steps to find out the step response of an **RC circuit**:

1. The initial capacitor voltage v(0).
2. The final capacitor voltage v(∞) — DC voltage across C.
3. The time constant τ.

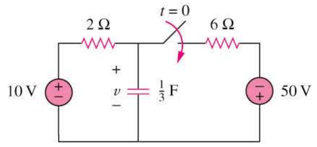
$$v(t) = v(\infty) + [v(0+) - v(\infty)]e^{-t/\tau}$$

Note: The above method is a **short-cut method**.
You may also determine the solution by setting up the circuit formula directly using KCL, KVL , ohms law, capacitor and inductor VI laws.

Example 5

Find v(t) for t > 0 in the circuit in below. Assume the switch has been open for a long time and is closed at t = 0.

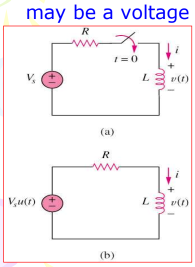
Calculate v(t) at t = 0.5.



Answer:
$v(t) = 15e^{-2t} - 5$ *and v(0.5) = 0.5182V*

7.5 The Step-Response of a RL Circuit

- The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.



- **Initial current**
  i(0-) = i(0+) = I_o
- **Final inductor current**
  i(∞) = Vs/R
  Time constant τ = L/R

$$i(t) = \frac{V_s}{R} + (I_o - \frac{V_s}{R})e^{-\frac{t}{\tau}}u(t)$$

Three steps to find out the step response of an **RL circuit**:

1. The initial inductor current i(0) at t = 0+.
2. The final inductor current i(∞).
3. The time constant τ.

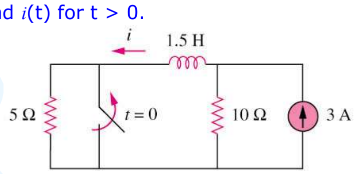
$$i(t) = i(\infty) + [i(0+) - i(\infty)]e^{-t/\tau}$$

Note: The above method is a ***short-cut method***.
You may also determine the solution by setting up the circuit formula directly using KCL, KVL , ohms law, capacitor and inductor VI laws.

Example 6

The switch in the circuit shown below has been closed for a long time. It opens at t = 0.

Find *i*(t) for t > 0.

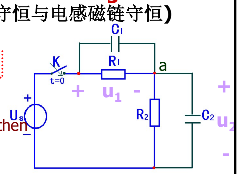


Answer:

$$i(t) = 2 + e^{-10t}$$

Capacitive Charge Conservation and Magnetism Chain Conversation（电容电荷守恒与电感磁链守恒）

Charge Conservation电荷守恒定律



Example

t<0， K has been opened for a long time, then

$$u_1(0^-) = 0 \quad u_2(0^-) = 0$$

t=0， K is closed. Apply KCL at node a

$$\frac{u_1}{R_1} + C_1\frac{du_1}{dt} = \frac{u_2}{R_2} + C_2\frac{du_2}{dt} \qquad \int_{0^-}^{0^+}[\frac{u_1}{R_1} + C_1\frac{du_1}{dt}]dt = \int_{0^-}^{0^+}[\frac{u_2}{R_2} + C_2\frac{du_2}{dt}]dt$$

If both $u_1$and u_2 are limited, then

$$C_1u_1(o^+) - C_1u_1(o^-) = C_2u_2(o^+) - C_2u_2(o^-)$$

$$-C_1u_1(o^+) + C_2u_2(o^+) = -C_1u_1(o^-) + C_2u_2(o^-)$$

$$u_1(o^+) + u_2(o^+) = U_s$$

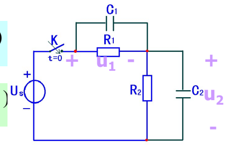
so

$$u_1(o^+) = \frac{C_2}{C_1 + C_2}U_s \qquad u_2(o^+) = \frac{C_1}{C_1 + C_2}U_s$$

Charge Conservation

$$\sum_{k=1} C_k u_k(o^+) = \sum_{k=1} C_k u_k(o^-)$$

or $$\sum_{k=1} q_k(o^+) = \sum_{k=1} q_k(o^-)$$



To determine $u_c(0^-)$:

i_c is limited?
- no: charge conservation
  - a. Capacitor-only loop
  - b. Loop consists only capacitors and voltage sources
  - c. Impulse source
- yes: switching law

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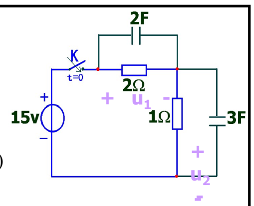
Example: Find $u_2(t)$ for $t>0$ 。



Solution:

t<0: $u_1(0^-) = u_2(0^-) = 0$

t=0⁺:

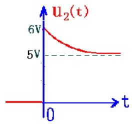
$$-2u_1(o^+) + 3u_2(o^+) = -2u_1(o^-) + 3u_2(o^-)$$

$$u_1(o^+) + u_2(o^+) = 15$$

$$u_2(o^+) = \frac{2}{2+1} \times 15 = 6V$$

t>0 $u_2(\infty) = \frac{1}{2+1} \times 15 = 5V \qquad C = 3+2 = 5F$

$$R = \frac{2\times 1}{2+1} = \frac{2}{3}\Omega \qquad \tau = RC = \frac{10}{3}s$$



$$u_2(t) = u_2(\infty) + [u_2(0^+) - u_2(\infty)]e^{-\frac{t}{\tau}} = 5 + e^{-\frac{3t}{10}} V \qquad t>0$$

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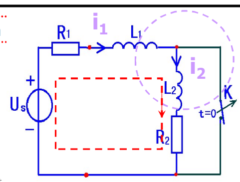
Magnetism Chain Conversation

Example



t<0 $i_1(0^-) = \frac{U_s}{R_1} \qquad i_2(0^-) = 0$

t=0

$$i_1 R_1 + L_1 \frac{di_1}{dt} + i_2 R_2 + L_2 \frac{di_2}{dt} = U_s$$

$$\int_{0^-}^{0^+} [i_1 R_1 + L_1 \frac{di_1}{dt} + i_2 R_2 + L_2 \frac{di_2}{dt}]dt = \int_{0^-}^{0^+} U_s dt$$

If and i_1 and i_2 are limited

$$L_1 i_1(o^+) - L_1 i_1(o^-) + L_2 i_2(o^+) - L_2 i_2(o^-) = 0$$

$$L_1 i_1(o^+) + L_2 i_2(o^+) = L_1 i_1(o^-) + L_2 i_2(o^-) \qquad i_1(o^+) = i_2(o^+)$$

so $$i_1(o^+) = i_2(o^+) = \frac{L_1}{L_1 + L_2} \frac{U_s}{R_1}$$

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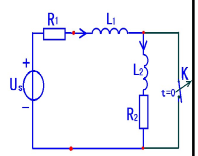
Magnetism Chain Conversation

$$\sum_{k=1} L_k i_k(o^+) = \sum_{k=1} L_k i_k(o^-)$$

or $$\sum_{k=1} \psi_k(o^+) = \sum_{k=1} \psi_k(o^-)$$



To determine $i_L(0^-)$:

If u_L is limited?
- no: Magnetism Chain Conversation
  - a. inductor-only cutset
  - b. Cutset with only inductors and current sources.
  - c. Impulse sources
- yes: switching law

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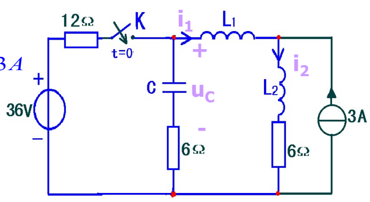
Example. Find $u_c(o^+)$、$i_1(o^+)$ and $i_2(o^+)$ 。

Solution:



$i_1(0^-) = 0 \quad i_2(0^-) = 3A$

$u_c(0^-) = 18V$

so

$u_c(0^+) = u_c(0^-) = 18V$

Apply Magnetism Chain Conversation

$$L_1 i_1(o^+) + L_2 i_2(o^+) = L_1 i_1(o^-) + L_2 i_2(o^-)$$

and $i_1(o^+) - i_2(o^+) + 3 = 0 \qquad i_1(o^+) = 0, i_2(o^+) = 3A$

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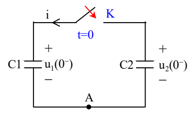
4. Singularity Circuit

C_1=1F, C_2=2F, $u_1(0^-)$=0V, $u_2(0^-)$=2V

$q_1(0^-)$=$C_1u_1(0^-)$=0, $q_2(0^-)$=$C_2u_2(0^-)$=4C



$$w_{C1}(0^-) = \frac{1}{2}C_1 u_1^2(0^-) = 0,$$

$$w_{C2}(0^-) = \frac{1}{2}C_2 u_2^2(0^-) = 4J,$$

$$w_{C1}(0^-) + w_{C2}(0^-) = 4J$$

$$\int_{0^-}^{0^+} idt = \int_{0^-}^{0^+} C_1 \frac{du_{C1}}{dt}dt = -\int_{0^-}^{0^+} C_2 \frac{du_{C2}}{dt}dt$$

$$C_1[u_1(0^+) - u_1(0^-)] = -C_2[u_2(0^+) - u_2(0^-)]$$

$$C_1 u_1(0^+) + C_2 u_2(0^+) = C_1 u_1(0^-) + C_2 u_2(0^-)$$

$$q_1(0^+) + q_2(0^+) = q_1(0^-) + q_2(0^-)$$

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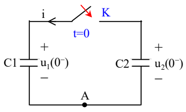
$u_1(0^+)=u_2(0^+)=u(0^+)$，$C_1u_1(0^+)+C_2u_2(0^+)=(C_1+C_2)u(0^+)=4C$



$$\Rightarrow \quad u(0^+)=\frac{4}{C_1+C_2}=\frac{4}{3}V$$

$$q_1(0^+)=C_1u(0^+)=\frac{4}{3}C\,,$$

$$q_2(0^+)=C_2u(0^+)=\frac{8}{3}C$$

$$w_{C1}(0^+)=\frac{1}{2}C_1u^2(0^+)=\frac{8}{9}J\,,\ w_{C2}(0^+)=\frac{1}{2}C_2u^2(0^+)=\frac{16}{9}J\,,$$

$$w_{C1}(0^+)+w_{C2}(0^+)=\frac{8}{3}J$$

$$\Delta w=(4-\frac{8}{3})J=\frac{4}{3}J$$

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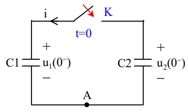
$$u_1=\frac{4}{3}\cdot 1(t)\,V\,,\quad u_2=2\cdot 1(-t)+\frac{4}{3}\cdot 1(t)\,V$$



$$i_1=C_1\frac{du_1}{dt}=\frac{4}{3}\frac{d}{dt}[1(t)]=\frac{4}{3}\delta(t)\ A$$

$$i_2=C_2\frac{du_2}{dt}=-4\delta(t)+\frac{8}{3}\delta(t)=-\frac{4}{3}\delta(t)\ A$$

- 纯电容回路：当开关K闭合后形成一个由电压源、电容组成的回路。
- 电荷守恒

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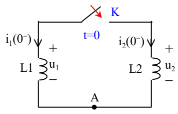
$L_1=1H$，$L_2=2H$，$i_1(0^-)=0A$，$i_2(0^-)=2A$



$\psi_1(0^-)=L_1i_1(0^-)=0$，$\psi_2(0^-)=L_2i_2(0^-)=4Wb$

$$w_{L1}(0^-)=\frac{1}{2}L_1i_1^2=0\,,\ w_{L2}(0^-)=\frac{1}{2}L_2i_2^2(0^-)=4\,J$$

$$w_{L1}(0^-)+w_{L2}(0^-)=4J$$

$$L_1\frac{di_1}{dt}=L_2\frac{di_2}{dt}=(-L_2\frac{di_1}{dt}) \qquad \int_{0^-}^{0^+}L_1\frac{di_1}{dt}=\int_{0^-}^{0^+}L_2\frac{di_2}{dt}$$

$\Rightarrow L_1i_1(0^+)-L_1i_1(0^-)=L_2i_2(0^+)-L_2i_2(0^-)$

$\psi_1(0^+)-\psi_2(0^+)=\psi_1(0^-)-\psi_2(0^-)$

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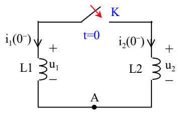
$L_1 i_1(0^+)-L_2i_2(0^+)=L_1i_1(0^-)-L_2i_2(0^-)=-4$



$$\Rightarrow \quad i_1(0^+)=\frac{-4}{L_1+L_2}=-\frac{4}{3}A$$

$$i_2(0^+)=\frac{4}{3}A$$

$$\psi_1(0^+)=-\frac{4}{3}Wb\,,\ \psi_2(0^+)=\frac{8}{3}Wb$$

$$w_{L1}(0^+)=\frac{1}{2}L_1i_1^2(0^+)=\frac{8}{9}J\,,\ w_{L2}(0^+)=\frac{1}{2}L_2i_2^2(0^+)=\frac{16}{9}J$$

$$\Delta w=\frac{4}{3}J$$

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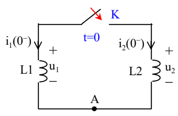
$$i_1=-\frac{4}{3}\cdot 1(t)\ A\ ,\quad i_2=2\cdot 1(-t)+\frac{4}{3}\cdot 1(t)\ A$$



$$u_1=L_1\frac{di_1}{dt}=-\frac{4}{3}\frac{d}{dt}[1(t)]=-\frac{4}{3}\delta(t)\,V$$

$$u_2=L_2\frac{di_2}{dt}=-4\delta(t)+\frac{8}{3}\delta(t)=-\frac{4}{3}\delta(t)\,V$$

- 当K闭合后形成一个由电流源、电感支路组成的节点
- 磁链守恒

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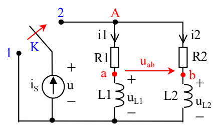
例：L_1=1H，L_2=2H，R_1=R_2=1Ω，i_S=1A，t=0 开关 K 从 1→2，计算$t\geq 0$后的零状态响应 i_1，i_2；并分析换路前后的能量变化。

解：



K从1→2：
$$\tau=\frac{L_1+L_2}{R_1+R_2}=\frac{3}{2}s$$

$$i_1(\infty)=\frac{R_2}{R_1+R_2}i_s=0.5A\,,\quad i_2(\infty)=0.5A$$

$$\psi_1(0^+)-\psi_2(0^+)=\psi(0^-)=0$$

$$L_1i_1(0^+)-L_2i_2(0^+)=0\ ,\quad \Rightarrow \quad i_1(0^+)-2i_2(0^+)=0$$

$$i_1(0^+)+i_2(0^+)=i_S=1 \qquad \Rightarrow \quad i_1(0^+)=\frac{2}{3}A,\quad i_2(0^+)=\frac{1}{3}A$$

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$$i_1=\left\{i_{1p}+\left[i_1(0^+)-i_{1p}(0^+)e^{-\frac{t}{\tau}}\right]\right\}\cdot 1(t)=(0.5+\frac{1}{6}e^{-\frac{2}{3}t})\cdot 1(t)\ A \qquad t\ge 0$$

$$i_2=i_S-i_1=0.5-\frac{1}{6}e^{-\frac{2}{3}t}\ A \qquad t\ge 0$$

$$w_{L1}(0^+)=\frac{1}{2}L_1 i_1^2(0^+)=\frac{2}{9}J\,,\ w_{L2}(0^+)=\frac{1}{2}L_2 i_2^2(0^+)=\frac{1}{9}J$$

$$w_{R1}(0^-,0^+)=\int_{0^-}^{0^+}R_1 i_1^2 dt=0\,,\ w_{R2}(0^-,0^+)=\int_{0^-}^{0^+}R_2 i_2^2 dt=0$$

$$u=R_1 i_1+L_1\frac{di_1}{dt}=(0.5+\frac{1}{6}e^{-\frac{2}{3}t})\cdot 1(t)+\frac{d}{dt}[(0.5+\frac{1}{6}e^{-\frac{2}{3}t})\cdot 1(t)]$$

$$=\frac{2}{3}\delta(t)+\left[0.5+\frac{1}{18}e^{-\frac{2}{3}t}\right]\cdot 1(t)\ V$$

$$w_{i_s}(0^-,0^+)=\int_{0^-}^{0^+}u i_S dt=\int_{0^-}^{0^+}[\frac{2}{3}\delta(t)+0.5\cdot 1(t)\cdot(1+\frac{1}{9}e^{-\frac{2}{3}t})]\cdot 1\cdot dt=\frac{2}{3}J$$

$$w_{i_s}(0^-,0^+)-\left[w_{L1}(0^+)+w_{L2}(0^+)+w_{R1}(0^-,0^+)+w_{R2}(0^-,0^+)\right]=\frac{2}{3}-(\frac{2}{9}+\frac{1}{9})=\frac{1}{3}J$$

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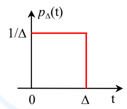
Impulse Response

Unit impulse response h(t)： the zero-state response with the unit impulse function excitation.

Method a：



1(t)→s(t)； δ(t)→h(t)

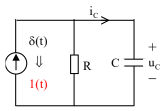
$$\delta(t)=\lim_{\Delta\to 0}p_\Delta(t)\ ,\quad p_\Delta(t)=\frac{1}{\Delta}[1(t)-1(t-\Delta)]$$

$$\delta(t)=\lim_{\Delta\to 0}\frac{1}{\Delta}[1(t)-1(t-\Delta)]$$

$$\Rightarrow\ h(t)=\lim_{\Delta\to 0}\frac{1}{\Delta}[s(t)-s(t-\Delta)]$$

$$h(t)=\frac{ds(t)}{dt}$$

例：求单位冲激响应 u_C 与 i_C 。



解：

$$u_C=s(t)=R(1-e^{\frac{-t}{RC}})\cdot 1(t)$$

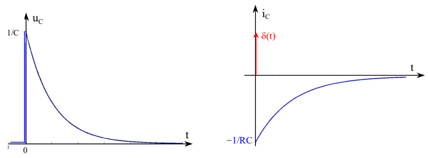
$$u_{Cp}=u_C(\infty)=R\cdot 1(t)\ ,\quad \tau=RC$$

$$u_C=h(t)=\frac{ds(t)}{dt}=\frac{d}{dt}[R(1-e^{\frac{-t}{RC}})\cdot 1(t)]$$

$$=\frac{1}{C}e^{\frac{-t}{RC}}\cdot 1(t)+(1-e^{\frac{-t}{RC}})R\,\delta(t)\big|_{t=0}=\frac{1}{C}e^{\frac{-t}{RC}}\cdot 1(t)$$

$$i_C=h(t)=C\frac{du_C(t)}{dt}=C\frac{d}{dt}[\frac{1}{C}e^{\frac{-t}{RC}}\cdot 1(t)]=-\frac{1}{RC}e^{\frac{-t}{RC}}+e^{\frac{-t}{RC}}\,\delta(t)\big|_{t=0}$$

$$=\delta(t)-\frac{1}{RC}e^{\frac{-t}{RC}}\cdot 1(t)$$



R=1Ω，C=1F，i_S=δ(t)A

$$u_C(0^+)=1V\,,\ q(0^+)=Cu_C(0^+)=1C\,,\ w_C(0^+)=\frac{1}{2}Cu_C^2(0^+)=\frac{1}{2}\times 1\times 1^2=0.5J$$

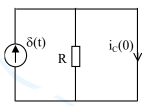
- δ(t) ⇒ $u_C(0^+)$，$i_L(0^+)$

Method b：

u_C and i_L： $y=y(0^+)e^{\frac{-t}{\tau}}1(t)$

L：open circuit；C：short circuit

$$i_L(0^+)=i_L(0^-)+\frac{1}{L}\int_{0^-}^{0^+}u_L(0)dt\,,\quad u_C(0^+)=u_C(0^-)+\frac{1}{C}\int_{0^-}^{0^+}i_C(0)dt$$

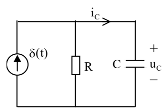


$$u_C(0^+)=u_C(0^-)+\frac{1}{C}\int_{0^-}^{0^+}i_C(0^-)dt$$

$$=\frac{1}{C}\int_{0^-}^{0^+}\delta(t)dt=\frac{1}{C}$$

$$u_C=\frac{1}{C}e^{-\frac{t}{RC}}\cdot 1(t)$$

Method c：



$$u_C=h(t)=u_{Cp}+Ae^{-\frac{t}{\tau}}$$

$u_{Cp}=a\delta(t)$，τ=RC

$$u_C=a\delta(t)+Ae^{-\frac{t}{\tau}}\cdot 1(t)$$

$$C\frac{du_C}{dt}+\frac{1}{R}u_C=\delta(t)$$

$$C\frac{d}{dt}[a\delta(t)+Ae^{-\frac{t}{\tau}}\cdot 1(t)]+\frac{1}{R}[a\delta(t)+Ae^{-\frac{t}{\tau}}\cdot 1(t)]=\delta(t)$$

$$C[a\delta'(t)+A\delta(t)-\frac{A}{\tau}e^{-\frac{t}{\tau}}\cdot 1(t)]+\frac{1}{R}[a\delta(t)+Ae^{-\frac{t}{\tau}}\cdot 1(t)]=\delta(t)$$

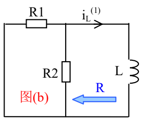
$$Ca=0\,,\quad CA+\frac{a}{R}=1 \qquad \Rightarrow a=0\,,\quad A=\frac{1}{C} \qquad \therefore\ u_C=\frac{1}{C}e^{-\frac{t}{RC}}\cdot 1(t)$$

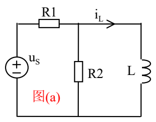
$\delta(t)\rightarrow h(t)$， $k\delta(t)\rightarrow kh(t)$， $k\delta(t-t_0)\rightarrow kh(t-t_0)$

The complete response=zero-input response $y(0^+)e^{-\frac{t}{\tau}}$ +zero-state response kh(t)

例：$R_1=1\Omega$，$R_2=2\Omega$，L=3H，$i_L(0^-)=1A$，$u_S=2\delta(t\text{-}2)V$，求冲激响应 i_L

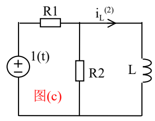
解：





$$\tau=\frac{L}{R}=\frac{L}{R_1 \parallel R_2}=4.5s \qquad i_L^{(1)}(0^+)=i_L(0^-)=1A \qquad \therefore\ i_L^{(1)}=e^{-\frac{t}{4.5}}\ A$$

设$u_S=1(t)$V，图(c)



$$i_{Lp}^{(s)}=\frac{u_S}{R_1}=1\ A$$

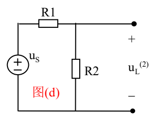
$$s(t)=i_L^{(s)}=(1-e^{-\frac{t}{4.5}})\cdot 1(t)\ A$$

$$\therefore\ i_L^{(h)}=h(t)=\frac{ds(t)}{dt}=\frac{d}{dt}[(1-e^{-\frac{t}{4.5}})\cdot 1(t)]=\frac{1}{4.5}e^{-\frac{t}{4.5}}\cdot 1(t)\ \ A$$

$$u_S=2\delta(t-2)\ \rightarrow\ i_L^{(2)}=2h(t-2)=\frac{2}{4.5}e^{-\frac{t-2}{4.5}}\cdot 1(t-2)\ \ A$$

$$i_L=i_L^{(1)}+i_L^{(2)}=e^{-\frac{t}{4.5}}+\frac{2}{4.5}e^{-\frac{t-2}{4.5}}\cdot 1(t-2)\ \ A \qquad t\geq 0$$

$$i_L(0^-)=1A\ \rightarrow\ i_L^{(1)}=e^{-\frac{t}{4.5}}\ A \quad t\geq 0$$

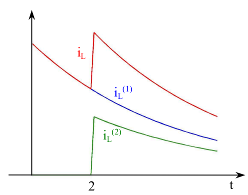


$u_S=2\delta(t-2)$ V：图(d)

$$u_L(2)=\frac{R_2}{R_1+R_2}u_S=\frac{4}{3}\delta(t-2)\,V$$

$$i_L(2^+)=i_L(2^-)+\frac{1}{L}\int_{2^-}^{2^+}u_L(2)dt=0+\frac{1}{3}\int_{2^-}^{2^+}\frac{4}{3}\delta(t-2)dt=\frac{2}{4.5}A$$

$$i_L^{(2)}=\frac{2}{4.5}e^{-\frac{t-2}{4.5}}\cdot 1(t-2)\ \ A$$



Summary and Review

- The response of a circuit having sources suddenly switched in or out of a circuit containing capacitors and inductors will always be composed of two parts: **a natural response** and **a forced response.**
- The form of the natural response (also referred to as the transient response) depends only on the component values and the way they are wired together.

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- The form of forced response mirrors the form of the forcing function. Therefore, a dc forcing function always leads to a constant forced response.
- A circuit reduced to a single equivalent inductance L and a single equivalent resistance R will have a natural response given by $i(t)=I_0e^{-t/\tau}$, where $\tau=L/R$ is the circuit time constant.

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- A circuit reduced to a single equivalent capacitance C and a single equivalent resistance R will have a natural response given by $v(t)=V_0e^{-t/\tau}$, where $\tau=RC$ is the circuit time constant.
- The unit-step function is a useful way to model the closing or opening of a switch, provided we are careful to keep an eye on the initial conditions.
- The complete response of an RL or RC circuit excited by a dc source will have the form $\boldsymbol{y(0^+)=y(\infty)+A}$ and $\boldsymbol{y(t)=y(\infty)+[y(0^+)-y(\infty)]e^{-t/\tau}}$, or ***total response = final value + (initial value − final value) e^{-t/τ}***

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