Fundamentals of Electric Circuits 2020.11

Chapter 8 Second-Order Circuits

A second-order circuit is characterized by a second-order differential equation.

It consists of resistors and the equivalence of two energy storage elements.

Response of Second-order circuits

Main points

Zero input response, zero state response

Understand:

◆overdamped response (过阻尼)

◆the critically damped response（临界阻尼）

◆underdamped responses（欠阻尼）

General solution for second-order circuit

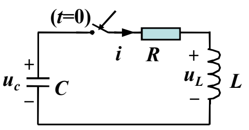
Steps:

1. Set a second-order differential equation
2. Find the natural response $y_N(t)$ from the homogeneous equation (input set to zero)
3. Find a particular solution $y_F(t)$ of the equation
4. Determine K_1 and K_2 by the initial conditions
5. Yield the required response

To solve second-order equation, there must be two initial values.

$$v_C(0) = V_0 \quad \text{and} \quad \frac{dv_C}{dt}(0) = \frac{1}{C}i(0) = \frac{I_0}{C}$$

§ 8. 2 Zero input response of the second order Circuits



known $u_C(0_-)=U_0$ $i(0_-)=0$

Find $u_C(t)$, $i(t)$, $u_L(t)$.

$$Ri + u_L - u_C = 0$$

$$i = -C\frac{du_C}{dt} \qquad u_L = L\frac{di}{dt} = -LC\frac{d^2u_C}{dt^2}$$

$$LC\frac{d^2u_C}{dt^2} + RC\frac{du_C}{dt} + u_C = 0$$

The characteristic equation $LCP^2 + RCP + 1 = 0$

$$P_{1,2}=\frac{-RC\pm\sqrt{R^2C^2-4LC}}{2LC}=-\frac{R}{2L}\pm\sqrt{(\frac{R}{2L})^2-\frac{1}{LC}}$$

$P_{1,2}$ have three distinct possibilities:

$R>2\sqrt{\frac{L}{C}}$ there are two real, unequal roots — Overdamped case

$$u_C=A_1e^{p_1t}+A_2e^{p_2t}$$

$R=2\sqrt{\frac{L}{C}}$ there are two real, equal roots — The critically damped case

$$u_C=(A_1+A_2t)e^{pt}$$

$R<2\sqrt{\frac{L}{C}}$ there are two complex conjugate roots — Underdamped case

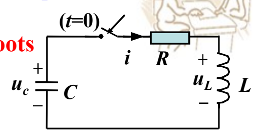
$$u_C=e^{-\alpha t}(A_1\sin\omega t+A_2\cos\omega t)=Ke^{-\alpha t}\sin(\omega t+\beta)$$

$$P_{1,2}=-\alpha\pm j\omega$$

---

(1) $R>2\sqrt{\frac{L}{C}}$ p_1, p_2 are two real, unequal roots

Overdamped : Real Unequal Roots



$$u_C=A_1e^{p_1t}+A_2e^{p_2t}$$

$$u_C(0_+)=U_0\rightarrow A_1+A_2=U_0$$

$$\frac{du_C}{dt}(0_+)=\frac{-i(0_+)}{C}=0\rightarrow P_1A_1+P_2A_2=0$$

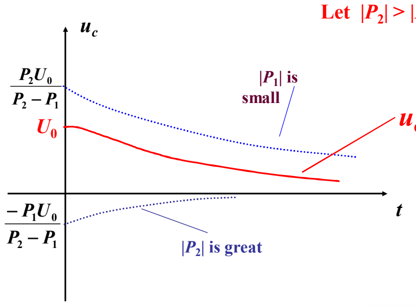
$$i=-C\frac{du_C}{dt}\qquad \frac{du_C}{dt}=-\frac{i}{C}$$

$$A_1=\frac{P_2}{P_2-P_1}U_0\qquad A_2=\frac{-P_1}{P_2-P_1}U_0$$

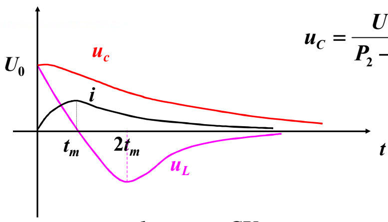
$$u_C=\frac{U_0}{P_2-P_1}(P_2e^{P_1t}-P_1e^{P_2t})$$

$$u_C=\frac{U_0}{P_2-P_1}(P_2e^{P_1t}-P_1e^{P_2t})$$

Let $|P_2|>|P_1|$



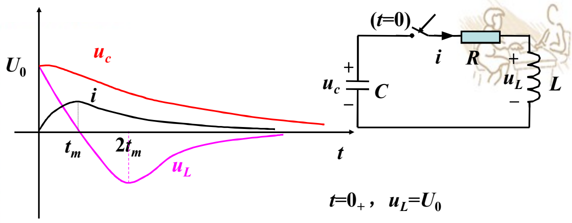
$$u_C=\frac{U_0}{P_2-P_1}(P_2e^{P_1t}-P_1e^{P_2t})$$



$$i=-C\frac{du_C}{dt}=\frac{-CU_0}{(P_2-P_1)}(p_1p_2e^{P_1t}-p_1p_2e^{P_2t})$$

$$=\frac{-U_0}{L(P_2-P_1)}(e^{P_1t}-e^{P_2t})\qquad P_1P_2=\frac{1}{LC}$$

$$u_L=L\frac{di}{dt}=\frac{-U_0}{(P_2-P_1)}(P_1e^{P_1t}-P_2e^{P_2t})$$



$t=0_+$, $i=0$

$t=\infty$, $i=0$

$t>0$ $i>0$

$t=t_m$, i is maximum

$t=0_+$, $u_L=U_0$

$t=\infty$, $u_L=0$

$0<t<t_m$, i increase, $u_L>0$

$t>t_m$, i decrease, $u_L<0$

$t=2t_m$, u_L is minimum

$t>2t_m$ u_L attenuate rapidly

---

from $u_L=0$, t_m can be calculated.

$$u_L=\frac{-U_0}{(P_2-P_1)}(P_1e^{P_1t}-P_2e^{P_2t})$$

$$p_1e^{p_1t}-p_2e^{p_2t}=0$$

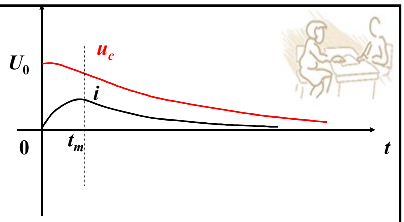
$$\frac{p_1}{p_2}=\frac{e^{p_2t}}{e^{p_1t}}=e^{(p_2-p_1)t}\qquad t_m=\frac{\ln\frac{p_1}{p_2}}{p_2-p_1}$$

The time t for minimum value of u_L can be obtained from du_L/dt .

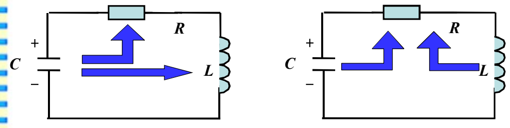
$$p_1^2e^{p_1t}-p_2^2e^{p_2t}=0\qquad (\frac{p_1}{p_2})^2=\frac{e^{p_2t}}{e^{p_1t}}=e^{(p_2-p_1)t}$$

$$t=\frac{\ln(\frac{p_1}{p_2})^2}{p_2-p_1}=2t_m$$

Where does the energy go?



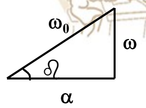
$0<t<t_m$ u_c decrease， i *increases.* $t>t_m$ u_c *decreases*， i *decreases*



Overdamped (过阻尼)

(II) $R<2\sqrt{\dfrac{L}{C}}$ there are two complex conjugate roots

$$P_{12}=-\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$



Let $\alpha=\dfrac{R}{2L}$ $\omega^2=\dfrac{1}{LC}-\left(\dfrac{R}{2L}\right)^2=\omega_0^2-\alpha^2$

$$P_{12}=-\alpha\pm j\omega$$

α neper frequency (damping factor/coefficient)
衰减因子

ω Natural radian frequency (damped natural radian frequency)
固有振荡角频率 （阻尼振荡角频率）

ω_0 resonant radian frequency
The undamped natural radian frequency (无阻尼振荡角频率)

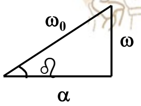
The solution of u_C $\boxed{u_C=e^{-\alpha t}[\mathrm{A}\cos\omega t+\mathrm{B}\sin\omega t]}$

$u_C=e^{-\alpha t}[\mathrm{A}\cos\omega t+\mathrm{B}\sin\omega t]$

By the initial values:

$$u_C(0^+)=U_0\rightarrow \mathrm{A}=U_0$$

$$\frac{du_C}{dt}(0^+)=0\rightarrow-\alpha\mathrm{A}+\omega\mathrm{B}=0$$



$\Rightarrow$ $\mathrm{A}=U_0$ $\mathrm{B}=\dfrac{\alpha}{\omega}U_0$

$$u_C=e^{-\alpha t}U_0(\cos\omega t+\frac{\alpha}{\omega}\sin\omega t)$$
$$=\frac{\omega_0}{\omega}U_0e^{-\alpha t}\sin(\omega t+\beta)$$
$$=Ke^{-\alpha t}\sin(\omega t+\beta)$$

$\dfrac{\alpha}{\omega_0}=\cos\beta$

$\dfrac{\omega}{\omega_0}=\sin\beta$

$\beta=\arctan\dfrac{\omega}{\alpha}$

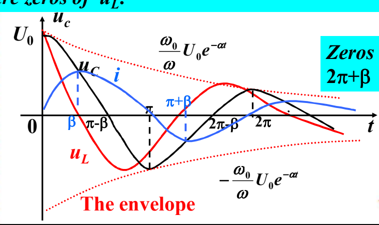
$u_C=\dfrac{\omega_0}{\omega}U_0e^{-\alpha t}\sin(\omega t+\beta)$ $i=-C\dfrac{du_C}{dt}=\dfrac{U_0}{\omega L}e^{-\alpha t}\sin\omega t$

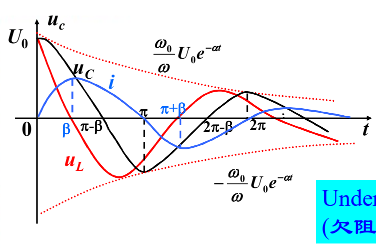
$$u_L=L\frac{di}{dt}=-\frac{\omega_0}{\omega}U_0e^{-\alpha t}\sin(\omega t-\beta)$$

Zeros of u_C: $\omega t=\pi-\beta,\ 2\pi-\beta\ ...\ n\pi-\beta$， the zeros of i is the extreme values of u_C .

Zeros of i ： $\omega t=0,\ \pi,\ 2\pi\ ...\ n\pi$ ， the extreme values of i *are zeros of* u_L.

Zeros of u_L: $\omega t=\beta,\ \pi+\beta,\ 2\pi+\beta\ ...\ n\pi+\beta$





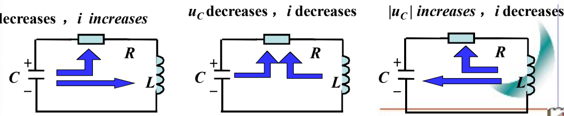
Underdamped case
(欠阻尼，衰减振荡)

Energy

$0<\omega t<\beta$ — u_C decreases， i *increases*

$\beta<\omega t<\pi-\beta$ — u_C decreases， i decreases

$\pi-\beta<\omega t<\pi$ — $|u_C|$ *increases*， i decreases



---

specially $R=0\Rightarrow\alpha=0$

$$P_{1,2}=\pm j\omega_0$$

$$\omega=\omega_0=\frac{1}{\sqrt{LC}},\ \beta=\frac{\pi}{2}$$

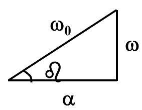
$$u_C=U_0\sin(\omega_0 t+\frac{\pi}{2})=u_L$$

$$i=\frac{U_0}{\omega_0 L}\sin\omega_0 t$$

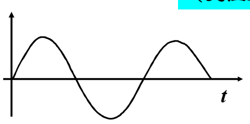
$u_C=\dfrac{\omega_0}{\omega}U_0e^{-\alpha t}\sin(\omega t+\beta)$

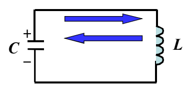
$i=-C\dfrac{du_C}{dt}=\dfrac{U_0}{\omega L}e^{-\alpha t}\sin\omega t$

$u_L=L\dfrac{di}{dt}=-\dfrac{\omega_0}{\omega}U_0e^{-\alpha t}\sin(\omega t-\beta)$



Undamped oscillation
（无阻尼振荡）





(III) $R=2\sqrt{\frac{L}{C}}$ p_1 , p_2 **are two real, equal roots**

$$P=P_1=P_2=-\frac{R}{2L}=-\alpha \qquad u_C=e^{-\alpha t}(A_1+A_2 t)$$

By using initial conditions $u_C(0_+)=U_0 \rightarrow A_1=U_0$

$$\frac{du_C}{dt}(0_+)=0 \rightarrow A_1(-\alpha)+A_2=0$$

$$\begin{cases} A_1=U_0 \\ A_2=U_0\alpha \end{cases}$$

$$u_C=U_0(1+\alpha t)e^{-\alpha t}$$

$$i=-C\frac{du_C}{dt}=\frac{U_0}{L}te^{-\alpha t}$$

$$u_L=L\frac{di}{dt}=U_0(1-\alpha t)e^{-\alpha t}$$

Critically damped situation

$R=2\sqrt{\frac{L}{C}}$ critically resistance

非振荡放电 临界阻尼

Summary for series RLC circuit:

$R>2\sqrt{\frac{L}{C}}$ 过阻尼，非振荡放电 $u_c=A_1e^{p_1t}+A_2e^{p_2t}$

$R=2\sqrt{\frac{L}{C}}$ 临界阻尼，非振荡放电 $u_c=e^{-\alpha t}(A_1+A_2t)$

$R<2\sqrt{\frac{L}{C}}$ 欠阻尼，振荡放电 $u_c=Ae^{-\alpha t}\sin(\omega t+\beta)$

由 $\begin{cases} u_C(0^+) \\ \frac{du_C}{dt}(0^+) \end{cases}$ 定积分常数

可推广应用于一般二阶电路

小结

1. 一阶电路是单调的响应，可用时间常数τ表示过渡过程的时间；二阶电路用三个参数α，ω和ω_0来表示动态响应。

$$P=-\alpha\pm j\omega \qquad \omega^2=\omega_0^2-\alpha^2$$

	特征根	响应性质	自由分量形式
$R=0$	共轭虚根	等幅振荡(无阻尼)	$K\sin(\omega_0 t+\beta)$
$R<2\sqrt{\frac{L}{C}}$	共轭复根	衰减振荡(欠阻尼)	$Ke^{-\alpha t}\sin(\omega t+\beta)$ 或$e^{-\alpha t}(A\sin\omega t+B\cos\omega t)$
$R=2\sqrt{\frac{L}{C}}$	相等的实根	非振荡放电 (临界阻尼)	$e^{-\alpha t}(A_1+A_2t)$
$R>2\sqrt{\frac{L}{C}}$	不等的实根	非振荡放电 (过阻尼)	$A_1e^{p_1t}+A_2e^{p_2t}$

2. 电路是否振荡取决于特征根，特征根仅仅取决于电路的结构和参数，而与初始条件和激励的大小没有关系。

General steps for second-order circuit analysis :

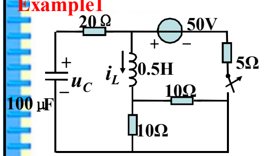
1. Write the second-order differential equation in terms of u_C or i_L according to KVL, KCL and element VCR.

2. Use switching law { $u_C(0_-)=u_C(0_+)$ or $i_L(0_-)=i_L(0_+)$} , along with circuit analysis techniques , to determine the initial conditions of the circuit. $\begin{cases} u_C(0^+) \\ \frac{du_C}{dt}(0^+) \end{cases}$ or $\begin{cases} i_L(0^+) \\ \frac{di_L}{dt}(0^+) \end{cases}$

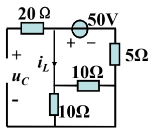
3. Determine the solution form of the natural response according to the roots of the characteristic equation: p_1、p_2。

4. Determine the constants of the homogenous solution by the initial values.

Example1



The circuit is as shown in the Fig. at $t=0$, the switch is opened. Find capacitance voltage u_C , And draw the waveform.

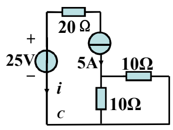


Equivalent circuit for t=0_

Solution:

(1) $u_c(0_-)$=25V

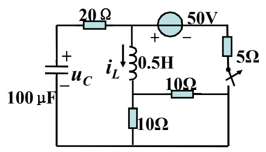
$i_L(0_-)$=5A



Equivalent circuit for t=0+

(2) $u_c(0_+)$=25V

$i_C(0_+)$= -5A



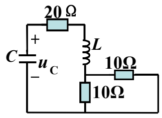
(1) $u_c(0_-)$=25V

$i_L(0_-)$=5A

(2) $u_c(0_+)$=25V

$i_C(0_+)$= -5A

(3) $L\frac{d}{dt}[-C\frac{du_C}{dt}]-25C\frac{du_C}{dt}-u_C=0$



Equivalent circuit for t >0

Characteristic equation

$50P^2+2500P+10^6=0$

$$P=-25\pm j139$$

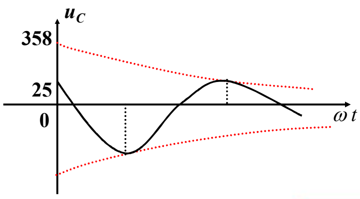
$$u_C=Ke^{-25t}\sin(139t+\beta)$$

$$u_C = Ke^{-25t}\sin(139t+\beta)$$

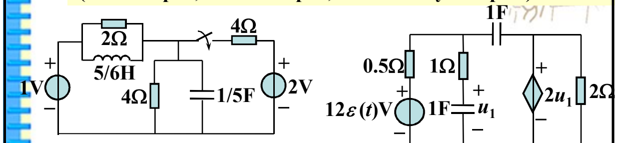
(4) from $\begin{cases} u_C(0^+) = 25 \\ C\dfrac{du_C}{dt} = -5 \end{cases} \Rightarrow \begin{cases} K\sin\beta = 25 \\ 139K\cos\beta - 25K\sin\beta = \dfrac{-5}{10^{-4}} \end{cases}$

$K = 358$, $\beta = 176^\circ$

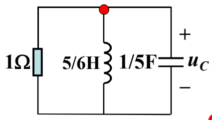
$u_C = 358e^{-25t}\sin(139t+176^\circ)\text{V} \quad t \ge 0$



Example2 Determine the character of the transient response (overdamped, underdamped, the critically damped)



solution: (a) Simplify the circuit



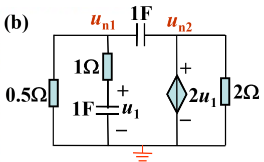
$$\frac{u_C}{R} + \frac{1}{5}\frac{du_C}{dt} + \frac{6}{5}\int u_C dt = 0$$

$$\frac{d^2u_C}{dt^2} + 5\frac{du_C}{dt} + 6u_C = 0$$

Characteristic roots: $p_1 = -2$, $p_2 = -3$

非振荡衰减(over damped)

(b)



$$\begin{cases} u_{n2} = 2u_1 \\ u_{n1} = RC_1\dfrac{du_1}{dt} + u_1 \\ \dfrac{u_{n1}}{0.5} + C_1\dfrac{du_1}{dt} = -C_2\dfrac{d}{dt}(u_{n1} - u_{n2}) \end{cases}$$

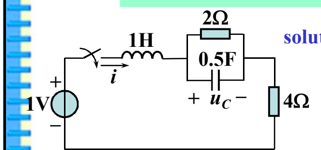
$$\frac{d^2u_1}{dt^2} + 2\frac{du_1}{dt} + 2u_1 = 0 \qquad p^2 + 2p + 2 = 0$$

Characteristic roots: $p = -1 \pm j1$

$\therefore \alpha = -1,\ \omega = 1,\ \omega_0 = \sqrt{2}$

衰减振荡 (underdamped)

Example3 Given that: $u_C(0^-)=1\text{V}$, $i(0^-)=2\text{A}$, the switch is closed at $t=0$. Find u_C.



solution: $i = \dfrac{u_C}{2} + C\dfrac{du_C}{dt}$

$$L\frac{di}{dt} + u_C + 4i = 1$$

Rearrange the equation:

$$0.5\frac{d^2u_C}{dt^2} + 2.5\frac{du_C}{dt} + 3u_C = 1$$

$p_1 = -2,\ p_2 = -3$

$$u_C = \frac{1}{3} + A_1e^{-2t} + A_2e^{-3t} \qquad C\frac{du_C}{dt}(0^+) = i(0^+) - \frac{u_C(0^+)}{2}$$

Determine the constant: $\begin{cases} \dfrac{1}{3} + A_1 + A_2 = 1 \\ -2A_1 - 3A_2 = 3 \end{cases} \Rightarrow \begin{cases} A_1 = 5 \\ A_2 = -\dfrac{13}{3} \end{cases}$

$$u_C = \frac{1}{3} + 5e^{-2t} - \frac{13}{3}e^{-3t}\text{V} \quad (t \ge 0)$$

Summary

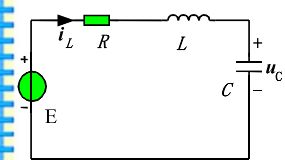
•Circuits containing linear resistors and the equivalent of two energy storage elements are described by second-order differential equations in which the dependent variable is one of the state variables. The initial conditions are the values of the two state variables at t=0.

•The zero-input response of a second-order circuit takes different forms depending on the roots of the characteristic equation. Unequal real roots produce the overdamped response, equal real roots produce the critically damped response, and complex conjugate roots produce underdamped responses.

§ 8. 3 Zero state response and complete response

(I) Zero state response

$u_c(0_-)=0$, $i_L(0_-)=0$



The differential equation

$$LC\frac{d^2u_c}{dt^2} + RC\frac{du_c}{dt} + u_c = E$$

$$u_c = u_c' + u_c''$$

Solution for homogeneous

Particular solution

The characteristic equation

$$LCP^2 + RCP + 1 = 0$$

particular: $u_c'' = E$

Solution of u_c:

$$u_c = E + A_1 e^{p_1 t} + A_2 e^{p_2 t} \quad (p_1 \neq p_2)$$

$$u_c = E + A_1 e^{-\alpha t} + A_2 t e^{-\alpha t} \quad (P_1 = P_2 = -\alpha)$$

$$u_c = E + A e^{-\alpha t}\sin(\omega t + \beta) \quad (P_{1,2} = -\alpha \pm j\omega)$$

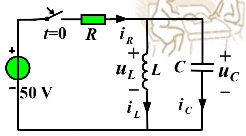
from the initial values $\begin{cases} u_c(0_+) \\ \dfrac{du_c}{dt}(0_+) \end{cases}$ the constants can be determined

(II) Complete response

known $i_L(0_-)$=2A, $u_C(0_-)$=0

R=50Ω , L=0.5H , C=100μF

Find $i_L(t)$.



solution **(1) write the differential equation**

$$\frac{50 - u_C}{R} = i_L + C\frac{du_C}{dt} \qquad u_C = u_L = L\frac{di_L}{dt}$$

$$RLC\frac{d^2 i_L}{dt^2} + L\frac{di_L}{dt} + Ri_L = 50$$

$$\frac{d^2 i_L}{dt^2} + 200\frac{di_L}{dt} + 2\times10^4 i_L = 2\times10^4$$

$$\frac{d^2 i_L}{dt^2} + 200\frac{di_L}{dt} + 2\times10^4 i_L = 2\times10^4$$

(2) Find the solution of homogeneous equation

Characteristic equation $P^2 + 200P + 20000 = 0$

roots P= -100 ± j100

$$i_L(t) = Ke^{-100t}\sin(100t + \beta)$$

(3) Find the particular solution（forced response）

$$i_L'' = 1A$$

(4) Complete solution

$$i_L(t) = 1 + Ke^{-100t}\sin(100t + \beta)$$

$$i_L(t) = 1 + Ke^{-100t}\sin(100t + \beta)$$

(5) Determine the constants by using the initial conditions

$i_L(0_+)$=2A , $u_C(0_+)$=0 （given）

$$\frac{di_L}{dt}\Big|_{0+} = \frac{1}{L}u_L(0^+) = \frac{1}{L}u_C(0^+) = 0$$

$$\frac{di_L}{dt} = -100Ke^{-100t}\sin(100t + \beta) + 100Ke^{-100t}\cos(100t + \beta)$$

$$\begin{cases} i_L(0^+) = 2 \;\rightarrow\; 1 + K\sin\beta = 2 \\ \dfrac{di_L}{dt}\Big|_{0+} = 0 \;\rightarrow\; -100K\sin\beta + 100K\cos\beta = 0 \end{cases}$$

yields $K = \sqrt{2} \quad \beta = 45^\circ$

$$\therefore i_L(t) = 1 + \sqrt{2}e^{-100t}\sin(100t + 45^\circ)A \quad t \geq 0$$

本章小结

重点掌握

二阶电路的零输入响应、零状态响应和完全响应。

学习要求

1. 掌握求解二阶电路的方法、步骤。
2. 会建立二阶电路的微分方程并写出初始条件。
3. 理解二阶电路在不同参数条件下，电路的不同状态：过阻尼、欠阻尼、临界阻尼；振荡与非振荡。