

Topological Hypothesis Tests for the Large-Scale Structure of the Universe

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Abstract

The large-scale structure (LSS) of the Universe is an intricate and spatially complex web. In order to understand the physics of the Universe, theoretical and computational cosmologists develop large-scale simulations that allow for visualizing and

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analyzing the LSS under varying physical assumptions. In particular, different realizations of dark matter, warm and cold, are thought to lead to contrasting velocities of cosmic structure formation. However, rigorous comparisons and inference on such complicated structures can be problematic. We present a framework for hypothesis testing of LSS using persistent homology. The randomness in the data (due to measurement error or topological noise) is transferred to randomness in the topological summaries, which provides an infrastructure for inference. These tests allow for statistical comparisons between complicated spatial data such as LSS in cosmology, but are also relevant to other areas of science. We present several test statistics using persistence diagrams, carry-out a simulation study to investigate the suitableness of the proposed test statistics, and finally apply the inference framework to study topological disparities between assumptions of warm and cold dark matter.

Keywords: persistent homology, voronoi, intensity, euler, silhouette, dark matter

1 Introduction

Rigorous comparisons of spatially complex web-like data such as the large-scale structure (LSS) of the Universe (see Figure 1) are notoriously challenging due, in part, to the difficulty in capturing the randomness of geometric and topological structures. However, these comparisons are important as it is becoming apparent that there exists information about cosmological parameters in the structure. We propose a framework for constructing topological hypothesis tests using ideas from an emerging area of topological data analysis (TDA) called persistent homology. Persistent homology offers a novel way to represent, visualize, and interpret complex data by extracting topological features, which can be used to infer properties of the underlying structures, as seen in astronomy (Sousbie 2011, Sousbie et al. 2011, Van De Weygaert et al. 2011, Cisewski et al. 2014) among other areas of science (Bendich et al. 2014, Duong et al. 2012).

The large-scale structure (LSS) of the Universe is an important example of a spatially complex structure, and is fittingly referred to as the *Cosmic Web* (Bond et al. 1996, Springel et al. 2006). The LSS of the Universe is a focus of manifold scientific research because its properties reveal information about the underlying physics and formation of our Universe (Davis et al. 1985). In order to study theoretical aspects of the formation and evolution of LSS, cosmologists develop large-scale simulations and can adjust the physical inputs and evaluate their effects on the LSS (Cooray & Sheth 2002, Centrella & Melott 1983, Doroshkevich et al. 1980, Schaye et al. 2015). One such input is related to the nature of dark matter (DM). The received position is that the Universe is made up of dark energy, DM, and baryonic matter. However, the nature of DM is still a mystery, but there are hypotheses regarding its possible particle behavior. Hot DM consists of particles that travel with ultrarelativistic speeds, while cold DM particles move much slower. For an easy introduction to DM properties, see (Hilbe et al. 2014, p. 61-63).

Though the generally accepted and best supported cosmological model assumes *cold dark matter* (known as Λ CDM), there are some elements of disagreement with observations (Schneider et al. 2012). Furthermore, it has been demonstrated through cosmological simulations that the nature of DM affects the development and formation of LSS (Schneider et al. 2012). In Figure 1, two realizations – CDM and WDM – produced from the EAGLE

cosmological simulation (Schaye et al. 2015) are displayed. Though there are similarities in shape of the densest regions (called *filaments*), there are differences in the distribution of matter about these filaments.

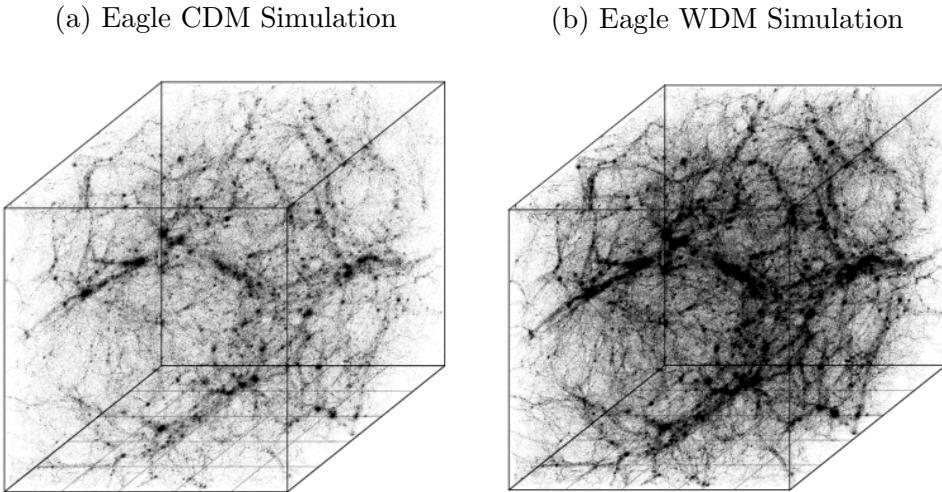


Figure 1: EAGLE simulation under (a) CDM and (b) WDM (Schaye et al. 2015). In theory, warm dark matter are hypothesized to travel at faster speeds than cold dark matter. While the larger filaments in (a) and (b) suggest similarities in skeletal structure, the faster speeds of (b) may result in disparities in density as seen by the more condensed groups in (b).

The purpose of this research is to better identify differences in the topology between structures using persistent homology. In this paper, we begin with an introduction to the algebra behind persistent homology followed by the proposed hypothesis testing framework. Then we carry out a simulation study to investigate the performance of the proposed statistics followed by more background on LSS. And finally, we apply the hypothesis tests to quantify the topological disparities between warm and cold DM. We end with concluding remarks.

2 Tools from TDA Useful for Studying LSS

Homology is the study of certain properties of topological spaces, specifically the number of different ordered holes in the space (e.g. connected components, loops, voids). Persis-

tent homology studies the spacial structure of a parameterized family of topological spaces (e.g., keeping track of the so-called births and deaths of homological features as a topological space changes with a varying parameter). The type of data we are investigating is a point cloud, where each point can represent a galaxy or, for cosmological simulation data, a certain mass of DM. With cosmological simulations, we look at cubic regions representing some part of our Universe and analyze the distribution of matter within that cube. We may define a simplicial complex directly on the point cloud, or we may compute a smoothed version of the data using kernel density estimation (KDE). The homological features mentioned above have cosmological interpretations in dimensions zero, one, and two. We discuss this briefly before going further into the algebraic topology.

Clusters A *connected component*, or zeroth-dimensional homology feature (H_0), is a maximal subspace of a topological space that cannot be covered by two disjoint open sets. In words, a connected component is a *piece* of a topological space. If our topological space is a k -nn graph, then the components are clusters of data points. In cosmology, these clusters of galaxies (or other cosmological matter) are an important structure to understand. Persistent homology tracks the appearance of new connected components and the merging of two distinct components into one.

Filaments and Loops A *loop*, or one-dimensional homology feature (H_1), provides information about the connectivity of data. As many H_0 features appear, nearby connected components can merge together. If our topological space is a k -nn graph, then loops are clusters of data points that merge into a fully connected cycle. For LSS, this would appear as filaments joining together in a loop.

Cosmological Voids A *void*, or two-dimensional homology feature (H_2), represents empty areas within the topological space. Again, if our topological space is a k -nn graph, then the voids are the unfilled spaces inside enclosed H_1 features. In cosmology, to fully appreciate the topology, it is important to understand the high-density regions (connected components and loops) but also where matter is scarce.

2.1 Persistent Homology

Various methods can be used in order to transform a discrete point set into a topological space. For example, points can be connected based on a distance (or a distance-like structure as in (Chazal et al. 2011)), or one may estimate the density from which the points were sampled. In the latter case, one can look at a KDE of a point cloud and study the topological features of super-level sets of that density. Below, we summarize some of the key components of persistent homology. See (Edelsbrunner & Harer 2010, Hatcher 2002, Munkres 1984) for a more thorough introduction to algebraic and computational topology.

Filtrations To derive the persistent homology for p , let there exist a threshold r , represented by a hyperplane that divides p into two separate segments: a super-level set, defined as $\{(x, y, z) \in p \text{ s.t. } z \geq r\}$, and a sub-level set $\{(x, y, z) \in p \text{ s.t. } z < r\}$. If r is initialized at ∞ , then the super-level set is empty and the sub-level set contains all of p . The evolving topological space is characterized by its homology as r decreases to $-\infty$. The persistent homology would then track the connected components (H_0), loops (H_1), and voids (H_2) that appear and disappear in the super-level sets $p^{-1}([r, \infty])$. More specifically, as r intersects p , the super-level set is no longer empty and is instead, composed of disjoint maxima. An example of a density with a 2-dimensional domain is presented in Figure 2a, along with the plane representing a threshold for defining super-level sets. Figures 2b and 2c display the upper-level sets for two thresholds. Figure 2d shows the persistence diagram which is discussed below.

Tracking Homology Generators Figure 2b and Figure 2c show the threshold, r decreasing from 1.1 to 0.1. In that interval, the upper-level set changed from having six connected components (H_0 's) and zero loops (H_1 's) to having one connected component and one loop. The time in the filtration when homology features appear, the *birth* of the feature, and the time when a feature joins other features, the *death* of the feature, are captured in a persistence diagram. Figure 2d displays the persistence diagram for the function in Figure 2a, where each point represents the birth time (x-axis) and death time (y-axis) of a homological feature. A point (x, x) on the diagonal represents a feature with a lifespan of 0.

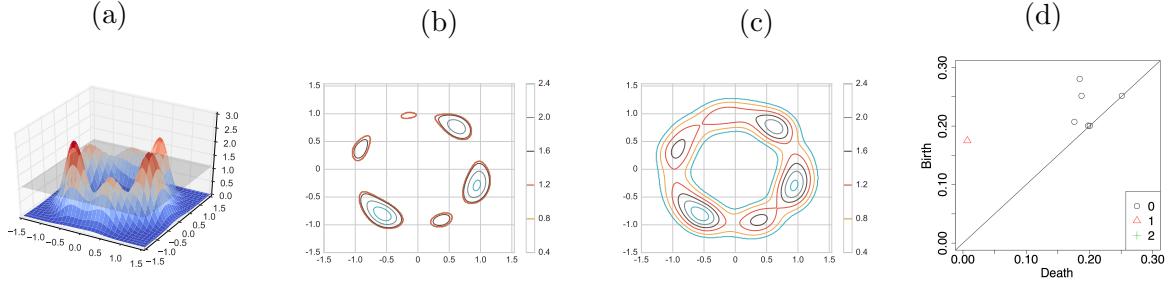


Figure 2: We illustrate persistent homology with a two-dimensional example. The density in (a) shows 3 steep peaks and 3 shallow peaks distributed around a uniform circle. In light gray, we see the hyperplane, shown (b) as $t = 1.10$. (b) and (c) plot the super-level sets $\widehat{p}^{-1}[1.10, \infty)$ and $\widehat{p}^{-1}[0.10, \infty)$ respectively. Notice that as the threshold is decreased, the contour more clearly defines a loop-like structure. In (d), we see the persistence diagram for the super-level set filtration of \widehat{p} . We highlight the upper left quadrant based at $(0.18, 0.02)$, which contains a one-dimensional persistence point. This corresponds to the loop shown in (c). The remaining 0-dimensional homologies represent the connected components from each of the 6 peaks.

The *persistence* of a point (b, d) is the length of the interval of the persistence parameter that supports that feature: $d - b$. In the persistence diagram, the distance from (b, d) to the diagonal is proportional to this value; in fact, the (Euclidean) distance to the diagonal is $\frac{(b-d)}{\sqrt{2}}$. Often, the persistence of a feature is indicative of the significance of the feature, which means that points close to the diagonal are indistinguishable from noise.

2.2 Derivatives of Persistence Diagrams

While persistence diagrams can summarize the topology of a data set, it is not straightforward to compare two different persistence diagrams. Simpler metrics like the bottleneck distance or the q -Wasserstein distance are computationally expensive. Below are several examples of methods to further summarize a persistence diagram that will be used in Section 3 to develop hypothesis tests.

Landscapes and Silhouettes Weighted silhouette functions are formed by weighting a particular functional summary of persistence diagrams called *landscape functions* (Bubenik 2015). More details and theoretical properties of landscapes and silhouettes are provided in (Chazal et al. 2014).

Landscape functions are defined as follows. Let the finite birth and death intervals of a persistence diagram with n_h points, for homology dimension $h = 0, 1, 2, \dots$, be defined as $\{(b_{hi}, d_{hi})\}_{i=1}^{n_h}$. Next, consider rotating the persistence diagram such that a given point is $p_{hi} = (\frac{b_{hi}+d_{hi}}{2}, \frac{d_{hi}-b_{hi}}{2}) \in D_h$, $i = 1, \dots, n_h$. Equilateral triangles are formed from each p_{hi} to the base as

$$\Lambda_{p_{hi}}(t) = \begin{cases} t - b_{hi} & t \in [b_{hi}, \frac{d_{hi}+b_{hi}}{2}] \\ d_{hi} - t & t \in [\frac{d_{hi}+b_{hi}}{2}, d_{hi}] \\ 0 & \text{otherwise} \end{cases}$$

where $t \in [t_{\min}, t_{\max}]$. For a given h , the persistence landscape is then defined as the following collection of functions

$$\lambda_{D_h}(k, t) = \underset{p_{hi} \in D_h}{\text{kmax}} \Lambda_{p_{hi}}(t), \quad t \in [t_{\min}, t_{\max}], k = 1, \dots, n_h$$

where kmax is the k th largest value in D_h . An example of a landscape function is displayed in Figure 3.

Rather than working with each k of $\lambda_{D_h}(k, t)$ individually, silhouettes provide a way of combining the information in the collection of landscape functions. Silhouettes are weighted averages of the individual functions for homology dimension h defined as

$$\phi_h(t) = \frac{\sum_{i=1}^m w_{hi} \Lambda_{hi}(t)}{\sum_{i=1}^m w_{hi}}$$

where the weights w_i can give more emphasis or less emphasis to features with longer lifetimes. As suggested in (Chazal et al. 2014), we use $w_{hi} = |d_{hi} - b_{hi}|^p$, where p is a tuning parameter that needs to be selected.

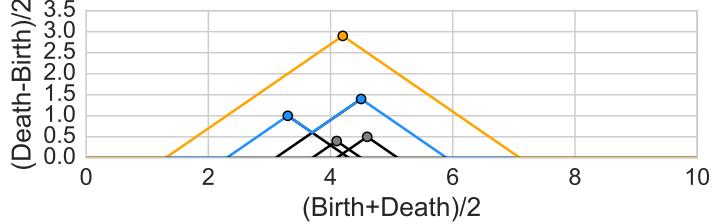


Figure 3: An example of a landscape function. The points shaded in color represent the homology coordinates in a persistence diagram D . The landscape $\lambda(k, \cdot)$ is the k -th largest of the arrangement of the graphs of $\{\Lambda_p\}$. In particular, the orange and blue segment represent landscape $\lambda(1, \cdot)$ and $\lambda(2, \cdot)$ respectively, with $\lambda(1, \cdot)$ denoted as the largest landscape function. A weighted silhouette function weighs the landscapes $\lambda(k, \cdot)$ into a single continuous function.

Euler Characteristic Function The Euler characteristic is a topological invariant and defined as: $\chi = \beta_0 - \beta_1 + \beta_2 - \beta_3 + \dots \pm \beta_N = \sum_{i=0}^N (-1)^i \beta_i$, where β_i represents the i -th Betti number (the rank of the i -th homology group) and N is the number of dimensions. When analyzing persistence diagrams of LSS, since there exist only three dimensions of data, the only non-trivial homology groups will be in dimensions 0, 1, and 2. Given the Betti numbers β_0 , β_1 , and β_2 , the Euler equation simplifies to: $\chi = \beta_0 - \beta_1 + \beta_2$. The topological space is parameterized by the threshold, t (ranging from t_{\min} to t_{\max}), defining the upper-level sets. The Euler Characteristic function, $\chi(t)$, captures the Euler characteristic for each threshold t .

3 Methods

In this section, we describe several frameworks for using persistence diagrams in two-sample hypothesis tests to compare differences between sets of data in topological structure.

Suppose we have two sets of persistence diagrams, $\{\mathcal{P}_1^{(1)}, \dots, \mathcal{P}_{n_1}^{(1)}\}$ and $\{\mathcal{P}_1^{(2)}, \dots, \mathcal{P}_{n_2}^{(2)}\}$. These samples can be used to test $H_0 : \mathcal{P}^{(1)} = \mathcal{P}^{(2)}$ vs. $H_1 : \mathcal{P}^{(1)} \neq \mathcal{P}^{(2)}$, where $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ are the true underlying distributions of persistence diagrams. Properties of probabilities on a space of persistence diagrams are addressed in (Mileyko et al. 2011). We would like

the framework to test the hypothesis that the two samples are drawn from populations with different random topologies. However, we note that without incorporating a scaling adjustment on the space of the data or the space of the diagrams, geometrical differences can also lead to a rejection of the null hypothesis. This is discussed in more detail in Section 4.2.1.

Given two samples of persistence diagrams, there are a number of possible ways to derive test statistics. We consider four general approaches. The first approach is based on functional summaries derived from the sampled persistence diagrams: Euler characteristic function (EC), Silhouette function (SIL), and a Silhouette-Euler characteristic function (SILEC). Given that each observed dataset will have a corresponding function, a *p-value* can be derived from a two-sample T-test by integrating the absolute value of the functional summaries. The next two approaches use variations on smoothed persistence diagrams called *intensity functions* (Chen et al. 2015) with p-values derived from a two-sample kernel test (Gretton et al. 2012) and an asymptotic argument through permutation (IK, WIK, PI). Additionally, we consider a test using the two-point correlation function (CORR), which directly captures the spatial behavior of the LSS without any topological summarise. Similar to the functional summaries above, p-values for the CORR test are calculated with a T-test using the integral of the absolute value of the correlation function. The proposed test statistics are discussed in more detail below.

Among the variations to test statistics considered (EC, SIL, SILEC, IK, WIK, PI, and CORR), we are seeking the summary that is most sensitive to differences in distributions of persistence diagrams produced from LSS.

Euler Characteristic Test (EC) Each persistence diagram in the two samples result in an individual Euler characteristic (EC) function, $\chi^{(J)}(t)$, $J = 1, 2$. The two-sample T-test is based on the following sample means:

$$\frac{1}{n_J} \sum_{i=1}^{n_J} \int_{t_{\min}}^{t_{\max}} |\chi_i^{(J)}(t)| dt.$$

Silhouette Test (SIL) As with the Euler Characteristic (EC) function, a weighted silhouette summarizes a persistence diagram through a continuous function. And as with EC,

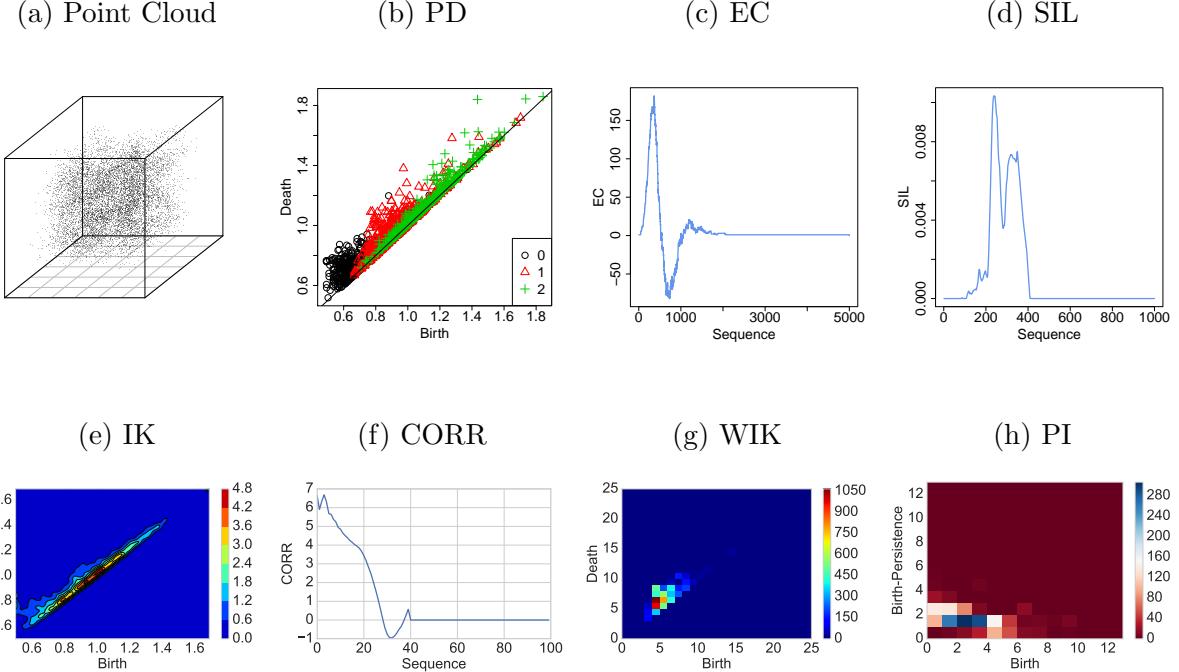


Figure 4: Examples of eight test statistics used in the hypothesis tests. A Voronoi foam with a percent filament of 0.5 generates the point cloud dataset shown in (a). Visually, (a) shows dense and sparse regions of sampled points that produce the topological signatures shown in (b). (b) contains three homology features, where 0, 1, and 2 represent connected components, loops, and voids respectively. (c) shows the euler characteristic function derived from (b) as an alternating sum of Betti numbers. (d) shows the weighted silhouette of the landscape functions. See Figure 3 for more details. (e), the intensity kernel, smooths a persistence diagram using a 1-D Gaussian kernel. Because points further from the diagonal are more topologically interesting, (g) builds on (e) by weighting points closer to the diagram less in order to prevent over-smoothing. (h) is similar to (g) but performs a linear transformation on both axis and uses a 2-D Gaussian on all 3 homologies simultaneously. (f) shows the two-point correlation function, which summarizes (a) directly without (b). See Section 3 for more details.

a two-sample T-test is carried-out by integrating the weighted silhouette functions for each sample. However, the SIL tests are computed individually for each homology dimension, $h = 0, 1, 2$, resulting in SIL_0 , SIL_1 , and SIL_2 . In an effort to combine information across homologies, we also considered Hotelling's T^2 test using the all three dimensions at once, denoted $SIL_{0:2}$.

Silhouette-Euler Characteristic (SILEC). Another method for simultaneously considering individual silhouettes, $S_h(t)$, across dimensions $h = 0, 1$, and 2 , is a modified Euler characteristic function. Instead of calculating the alternating sum of Betti numbers, the Silhouette-Euler characteristic function (SILEC) computes the alternating sum of individual silhouette functions across the threshold parameter, t , $SILEC(t) = S_1(t) - S_2(t) + S_3(t)$. A p-value is calculated using a T-test in the same fashion as done for EC.

Intensity Kernel Test (IK). Rather than working with raw persistence diagrams, the Intensity Kernel Test (IK) pre-smooths the diagrams with a Gaussian kernel, similar to the intensity function seen in (Chen et al. 2015). The IK statistic used in this paper is computed for two sets of persistence diagrams, and the discrepancy between the diagrams is calculated as the integrated squared difference between two *unweighted* intensity functions instead of points directly from the persistence diagrams. (Unweighted intensity functions assigns uniform weights to points on the diagram and therefore does not account for the diagonal boundary on persistence diagrams where birth = death; we considered weighted intensity functions next.) The two-sample test statistic for $X = X_1, \dots, X_{n_1}$ and $Y = Y_1, \dots, Y_{n_2}$ is defined as

$$\widehat{T}_{IK}(X, Y) = \frac{1}{n_1^2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} K_\sigma(X_i, X_j) - \frac{2}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} K_\sigma(X_i, Y_j) + \frac{1}{n_2^2} \sum_{i=1}^{n_2} \sum_{j=1}^{n_2} K_\sigma(Y_i, Y_j), \quad (1)$$

where n_1 and n_2 are the sizes of the two samples, and $\{X_1, \dots, X_{n_1}\}$ and $\{Y_1, \dots, Y_{n_2}\}$ are the two sets of intensity functions. $K_\sigma(X, Y)$ can be thought of as a similarity measure between intensity functions X and Y , and in this case is a Gaussian kernel $K_\sigma(X, Y) = \exp(-\frac{\|X-Y\|^2}{\sigma^2})$ with $\|X-Y\| = \int (X(t_1, t_2) - Y(t_1, t_2))^2 dt_1 dt_2$. The σ is a hyperparameter that sets the standard deviation of the Gaussian distribution used in the kernel K_σ : a larger σ will reduce sensitivity to small differences between X and Y , while a smaller σ will

heighten sensitivity. The optimal σ value was found to be 0.1 ± 0.04 using grid search from 0 to 5. A permutation test is used to calculate a p-value for each homology dimension, IK_0 , IK_1 , and IK_2 .

Weighted Intensity Kernel Test (WIK) The intensity function in WIK, unlike IK, uses weighted kernel density estimates (Chen et al. 2015) where the weights are a function of a point's persistence. Let (b_i, d_i) , $i = 1, \dots, m$ be the features on a persistence diagram corresponding to some dimension h . Then the weighted intensity function for the persistence diagram is

$$X(t_1, t_2) = \sum_{j=1}^m (d_j - b_j) \frac{1}{\tau^2} K\left(\frac{t_1 - d_j}{\tau}\right) K\left(\frac{t_2 - b_j}{\tau}\right)$$

such that the sum over all persistent features, K is a symmetric kernel function, and τ is a smoothing parameter. Using weighted intensity functions, a p-value is calculated by considering the same kernel test statistic as Equation (1). A permutation test for each homology dimension produces three analogous statistics: WIK_0 , WIK_1 , and WIK_2 .

Persistent Image Test (PI) The Persistent Image test (Adams et al. 2015) is a variation of the WIK test with an alternative kernel function and sub-sampling. Again, let $\mathcal{D} = (b_j, d_j) : i = 1, \dots, m$ be a persistence diagram where m is the number of persistent features. Unlike WIK, the PI test first transposes the persistent diagram using the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $T(x, y) = (x, y - x)$. Therefore, let $T(\mathcal{D})$ represent the transformed diagram with the new birth-persistence coordinates. Let $P_u : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a differentiable probability distribution with mean $u = (u_x, u_y)$. In our application, we choose a normalized symmetric two-dimensional Gaussian with mean (u_x, u_y) and variance σ^2 .

$$P_u(x, y) = \frac{1}{2\pi\sigma^2} e^{-[(x-u_x)^2 + (y-u_y)^2]/2\sigma^2}$$

Finally, let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a non-negative weighting function that is zero along the horizontal axis, continuous and differentiable. f is chosen to only depend on the rotated vertical persistence coordinate y , and like WIK, weight points of higher persistence more

heavily. In our application, we use a piecewise linear weighting function that was found to be stable. Given $b > 0$, define f_b as

$$f_b(t) \rightarrow \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{t}{b} & \text{if } 0 < t < b, \text{ and} \\ 1 & \text{if } t \geq b \end{cases}$$

Given f , P , and $T(\mathcal{D})$, we can define a persistence surface (Adams et al. 2015), $\rho_{\mathcal{D}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$\rho_{\mathcal{D}}(x, y) = \sum_{u \in T(\mathcal{D})} f(u) P_u(x, y)$$

Like the WIK test, ρ is calculated over the two-dimensional grid

$$\{(x, y) \text{ s.t. } b_{min} \leq x \leq b_{max} \wedge d_{min} \leq y \leq d_{max}\}$$

. The step sizes h_x and h_y were chosen to make the grid size 30 by 30. Given $\rho_{\mathcal{D}}$, a persistence image (Adams et al. 2015) $I(\rho_{\mathcal{D}})_p$ is defined as a grid of pixels calculated from convolving the persistent surface with a uniform matrix of size n by n of 1's. In other words, $I(\rho_{\mathcal{D}})_p = \int \int_p \rho_{\mathcal{D}} dy dx$. This matrix is then flattened into a 1D vector. The accuracy of the p-value using this PI framework is found to be robust to the choice of the convolution filter size. In our applications, a filter size of 3 is used, resulting in a 10 by 10 matrix, or 100 length vector when flattened.

Unlike the WIK test, the PI test combines all homologies into a single statistic. Suppose the homologies H_0, \dots, H_k are computed. One can concatenate the PI vectors for H_0, \dots, H_k into a single vector representing all dimensions simultaneously. Like the WIK test, the p-value is calculated using the permutation method.

Two-point Correlation Function Test (CORR) The Two Point Correlation Test (Landy & Szalay 1993), unlike previous methods, quantitatively measures LSS through tracing the amplitude of clustering as a function of scale, directly summarizing the point cloud instead of a persistence diagram. Such a trace is determined by the correlation function, $\xi(r)$. $\xi(r)$ is defined as the measure of the excess probability dP , above the

expectation for an unclustered random Poisson distribution of finding a cluster in a given volume element dV at a separation radius r from another cluster such that

$$dP = n[1 + \xi(r)] dV$$

where n is the mean number density of the sample dataset.

$\xi(r)$ is measured by counting pairs of clusters as a function of the separation radius compared to the count for an unclustered distribution. To do this, one must construct a *random catalog* that has similar three dimensional coverage as the data but is populated with randomly distributed points. Define DD , DR , RR as counts of pairs of clusters (in bins of varying separation radii) in the data only, between the data and the random catalog, and in the random catalog only; let n_D , n_R respectively define the mean number of densities of clusters in the data and random catalogs. The correlation function, defined by (Landy & Szalay 1993), is:

$$\xi = \frac{1}{RR} \left[DD \left(\frac{n_R}{n_D} \right)^2 - 2DR \left(\frac{n_R}{n_D} \right) + RR \right]$$

Provided a point cloud, clusters are derived and compared to a random catalog that is generated within the same dimensional space. Unlike other statistics, because \widehat{CORR} is not dependent on persistence diagrams, only a parallel test is applicable. The test statistic, \widehat{CORR} is defined by the area under the absolute value of ξ .

$$\widehat{CORR} = \int_r |\xi(r)| dr$$

To generate the random catalog, $|\mathcal{P}|$ points were drawn from a Uniform distribution $U(\min(\mathcal{P}), \max(\mathcal{P}))$ where \mathcal{P} is a point cloud. If \mathcal{P} was standardized, points were sampled from $U(0, 1)$.

4 Simulation Study

To evaluate the performance of the proposed test statistics, we carried out a simulation study by generating realizations of web-like spatial structures. The simulation model is discussed in detail below.

4.1 Simulation model

Motivated by LSS, we developed our simulation model to approximate the Cosmic Web. In particular, we drew from ideas that use Voronoi tessellations to model the filament structure of the Universe, known as *Voronoi Foam* (Icke & van de Weygaert 1987, 1991, Van De Weygaert 2007). The Voronoi Foam model offers an approximation to the distribution of matter in the Universe at large scales (e.g. galactic clusters, filaments, walls) (Icke & van de Weygaert 1991).

The cells of the Voronoi tessellation become the cosmological voids, the outline of the cells are the filaments and walls, and the points of intersection are the superclusters (large clusters of galaxies). Once the tessellation is defined, points are added according to several parameters - the points can represent individual galaxies, clusters of galaxies, or dark matter halos (which would host gravitationally-bound galaxies or galactic clusters). The elements of our approximate Voronoi Foam model include (i) the number of voids (the number of cells in the Voronoi tessellation), (ii) the number of galaxies/clusters/halos (the number of points to generate), and (iii) the percentage of the points that should fall on the cluster, filaments, and walls; see Table 1. In this simulation study, we varied the filament percentage ($percFil$) from 0.1 to 0.3 by a 0.05 step size. See the Appendix for additional tests for filament percentages ranging from 0.1 to 0.9.

Abbrev	Definition	Value
percWall	Percentage of particles on the walls	$0.98 - p_f$
percFil	Percentage of particles on the filaments	p_f
percClust	Percentage of particles in the clusters	0.02

Table 1: Parameters of LSS model. For the simulation study, p_f will vary from 0.1 to 0.3 by 0.05 increments. See Figure 5 for a visual representation of walls, filaments, and clusters.

Figure 5 displays the construction procedure of one realization of our simulation model: (i) First a grid is defined at a specified resolution within a specified volume; (ii) then a specified number of points are randomly selected within the volume - these will be used to define the Voronoi tessellation and will act as voids (these will be called *void points*; (iii)

the nearest void point to each grid point is found and stored, call this the *void label* of a grid point; (iv) the void labels of the eight nearest neighbors of each grid point is noted - if there are more than three unique void labels among the eight then that grid point is assigned to be a cluster point, if there are exactly three unique void labels among the eight nearest neighbors then that grid point is assigned to be a filament point, and if there are exactly two unique void labels among the eight nearest neighbors then that grid point is assigned to be a wall point. The black, empty circles in Figures 5b, 5c, 5d display the grid points that were selected to be cluster points, filament points, and wall points, respectively. Depending on the parameter assignments in Table 1 and the total desired sample size of dataset, the number of points are randomly selected among the cluster, filament, and wall points. Specified Gaussian noise is also added to the selected points so they do not fall exactly on the defined grid.

Examples of three Voronoi Foam models with percFil 0.1, 0.5, 0.9 are shown in Figure 6 along with their persistence diagrams. One can see that as the percFil increases, the web-like structure becomes more pronounced, changing the distribution of topological features.

4.2 Simulation Study Results

The Voronoi foams used in this paper were generated under 1.25×10^5 box volume, 0.1 resolution, 1×10^4 points, 64 cells (voids), 0.02 percClust, and [0.1, 0.3] percFil and [0.68, 0.88] percWall. Persistent diagrams are generated using distance-to-measure (DTM) with a 0.01 tuning parameter. The diagrams are preprocessed to remove the known 0-dimensional artifact, a vestigial H_0 element with birth time of 0 and a death time of ∞ (with exception to the EC test in which the artifact is preserved). The hypothesis tests were performed on 100 independent iterations of 15 independent realizations from each of the two populations. The 15 independent datasets are each generated using a percFil setting from 0.1 to 0.3 (0.05 step size); we also include a control per iteration with a percFil of 0.1. (Hence 5 sets of 15 datasets, repeated 100 times.) Using the proposed hypothesis tests, each of the variable models will be compared to the control. Given the data drawn from a model

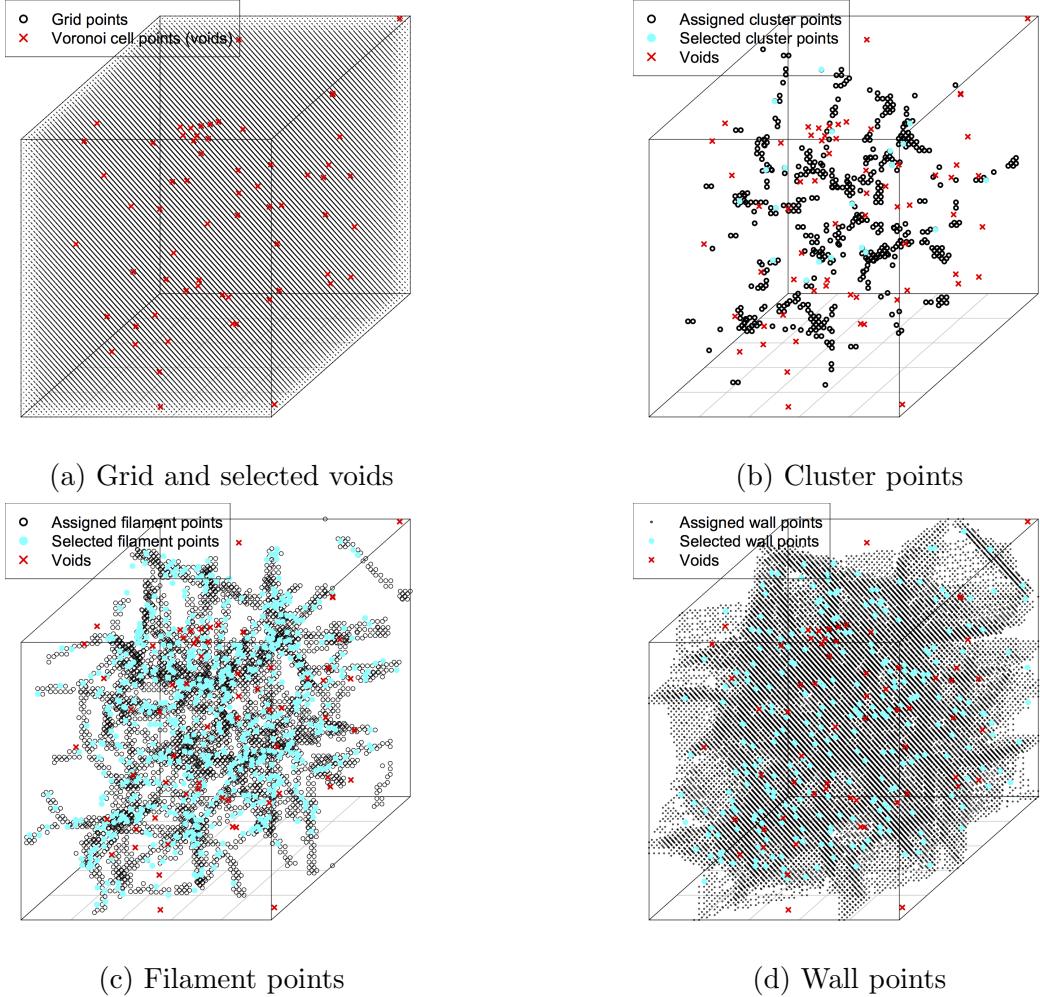


Figure 5: Simulation model construction. (a) A grid is defined and points are randomly selected to define the Voronoi tessellation - the Voronoi cells are the voids. (b) - (d) Based on the location of the voids and the grid, points are defined to be cluster points, filament points or wall points - these are the empty black circles. Based on the values assigned from Table 1, a number of points are randomly selected from the assigned points to be in the dataset.

with percFil p , each of the proposed test statistics are used to compute a p-value for the test $H_0 : \mathcal{P}^{(1)} = \mathcal{P}^{(2)}$ vs. $H_1 : \mathcal{P}^{(1)} \neq \mathcal{P}^{(2)}$, based on two samples of persistence diagrams: $\{\mathcal{P}_1^{(1)}, \dots, \mathcal{P}_{15}^{(1)}\}$ drawn from the model with $\text{percFil} = 0.1$, and $\{\mathcal{P}_1^{(2)}, \dots, \mathcal{P}_{15}^{(2)}\}$ drawn from the model with $\text{percFil} = p$, $p = 0.1, 0.15, \dots, 0.3$. (Recall that $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ are the true underlying distributions of persistence diagrams) Similar tests were also completed against

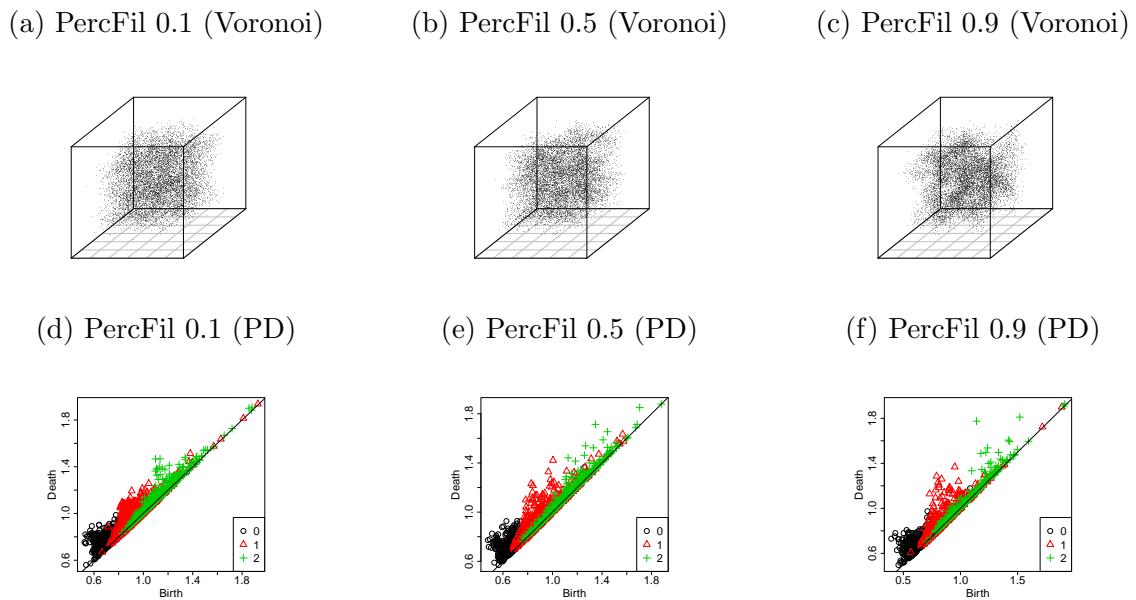


Figure 6: Three examples of Voronoi Foam simulations and their corresponding persistence diagrams using 3 different filament densities. As the percent filament increases, the point clouds tend to co-aggregate into dense and sparse regions; the persistence diagrams also form more features with longer lifespans. All other parameters used to generate the Voronoi foams are as defined in Table 1.

a control model with 0.3 percFil, and agreeable results were found. The simulation study results are displayed in Figure 16, which shows the median p-values along with a (25%, 75%) confidence interval across the 100 iterations of the following test statistics: (i) Euler-based tests, (ii) Silhouette-based tests, (iii) Kernel-based tests, (iv) Weighted Kernel-based tests, and (v) Correlation-based tests.

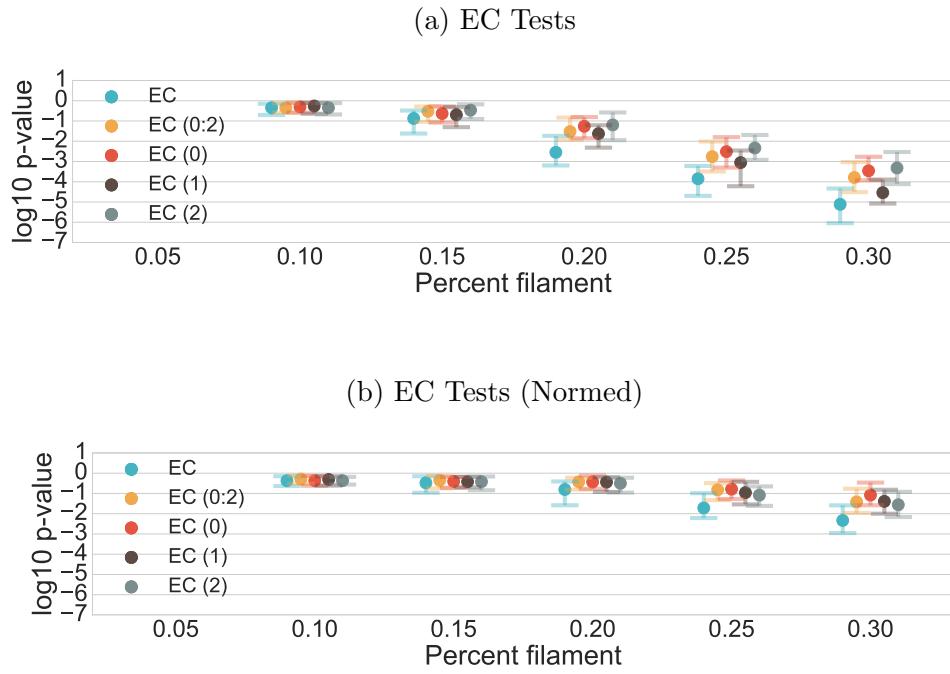
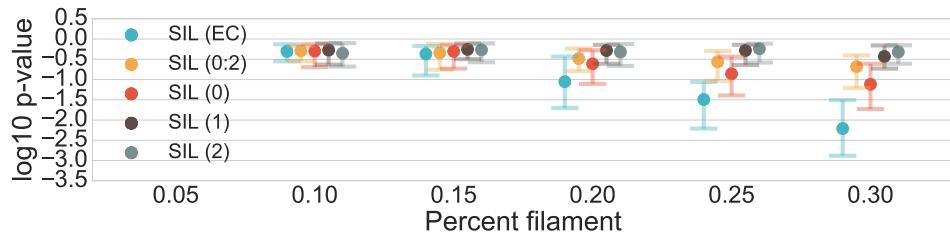


Figure 7: P-values from a suite of EC tests. X-axis represents a percent filament (0.1 to 0.3) being compared to a baseline of 0.1; Y-axis shows the p-values in \log_{10} space. The lines plot the median \log_{10} p-value and error bars show the 25th and 75th percentile p-values of the 100 iterations. Figure 7b is identical to Figure 7a with the exception of standardizing the persistence diagrams prior to testing.

From Figure 7 to Figure 11, we see that the EC test, the CORR test, and the SILEC test are the most effective in distinguishing differences between the Voronoi simulations. All other tests are not as sensitive, eventually finding differences between the test and control once percFil approaches 0.7 (see Appendix). Additionally, all tests derived from Euler characteristics perform relatively well compared to the other tests, suggesting that the Betti numbers, by being topologically invariant, are much better functional summaries

(a) SIL Tests



(b) SIL Tests (Normed)

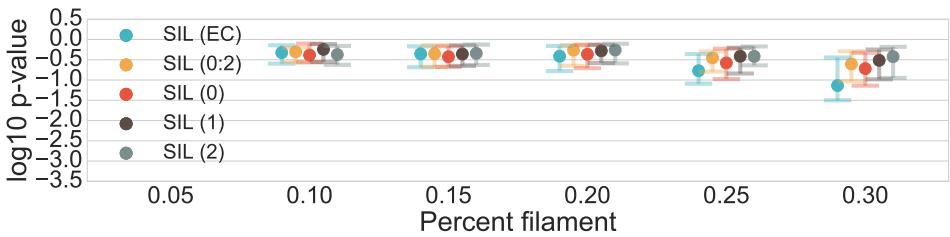
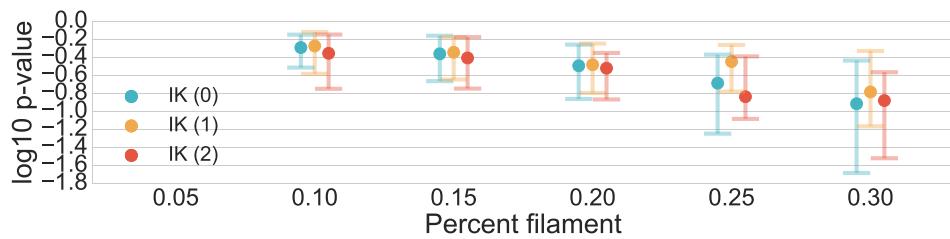


Figure 8: P-values from a suite of SIL tests.

(a) IK Tests



(b) IK Tests (Normed)

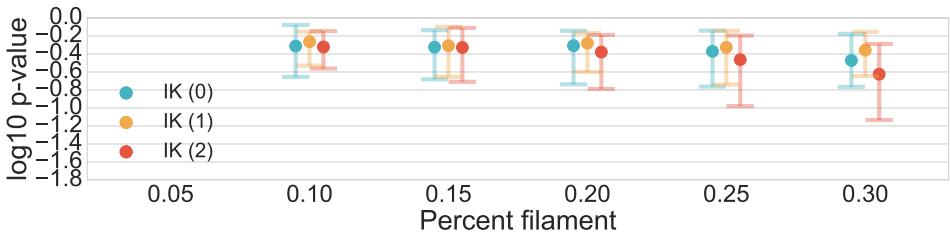
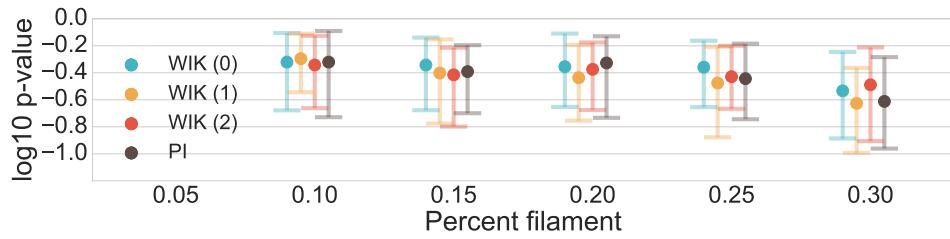


Figure 9: P-values from a suite of IK tests.

(a) WIK/PI Tests



(b) WIK/PI Tests (Normed)

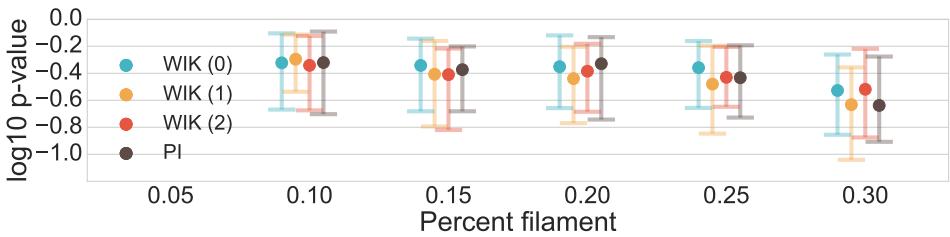
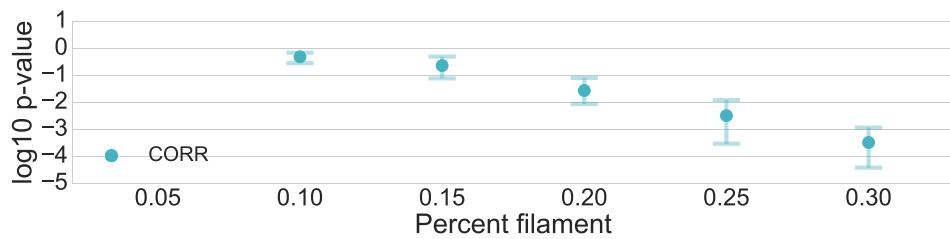


Figure 10: P-values from a suite of WIK tests.

(a) CORR Tests



(b) CORR Tests (Normed)

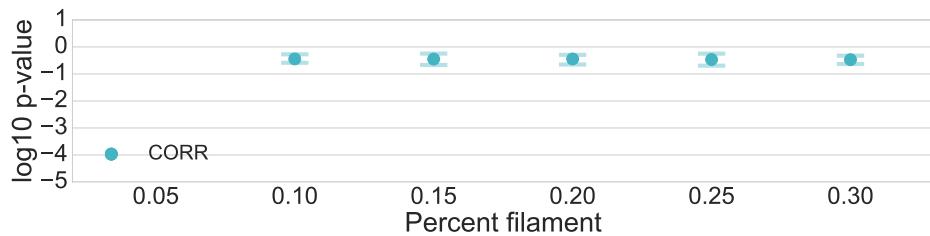


Figure 11: P-values from a suite of CORR tests.

of the persistence diagrams than intensity functions, silhouettes, and landscapes. More interestingly, it is possible that the alternating linear function by which the Betti numbers are combined may better preserve topological information given that the SILEC test, which combines silhouettes through a similar linear fashion, performed better than any individual silhouette counterpart and the naive cross-dimension statistic (SIL (0:2)). Finally, the CORR test, being independent of persistence diagrams, acts as a powerful alternative to the EC test, being nearly as sensitive as percFil increases.

4.2.1 Standardization of persistence diagrams

In the methods proposed above, difference in scale can also result in rejection when, in fact there are not statistically significant topological differences. For example, suppose we are considering two datasets - each has points randomly sampled from the perimeter of a circle with a radius of 1 and 10 respectively. It may or may not be desirable to conclude that the two datasets come from different persistence diagram generators (i.e. conclude $\mathcal{P}^{(1)} \neq \mathcal{P}^{(2)}$) since inference would be based on geometrical (scaling) differences rather than topological differences. If we wish to focus only on topological differences, we propose a possible preprocessing step to normalize scaling. Specifically, we standardize the persistence diagrams so that all the homological features are re-scaled to $[0, 1] \times [0, 1]$. This simple standardization takes the persistence diagram window and shrinks it or expands it to fill the $[0, 1] \times [0, 1]$ window, maintaining the same relationship among all the homological features. If there is concern about outliers, then other quantiles could be used for the standardization. The exception is with the CORR, in which the point cloud itself is standardized to $[0, 1] \times [0, 1] \times [0, 1]$.

We repeated the simulation study from Section 4.2 except including our standardization; the results are displayed in Figure 16. Standardization had an appreciable impact on hypothesis test results, decreasing the effectiveness and sensitivity of all test statistics. Notice, however, that the best-performing test statistics remain the same: EC, CORR, SILEC tests, confirming that the properties underlying those three tests are better able to capture purely topological differences than any other test presented in this paper. Notably after standardization, the KC, WIK, and other SIL tests have essentially constant p-values

across the percFil variation, losing effectiveness in distinguishing differences. One interpretation may be that these tests captured only geometric dissimilarities in the unstandardized setting.

5 Application: Cosmological Simulation Data

Similar to the simulation models from Section 4.1, Megaparsec cosmic mass distributions are likewise characterized by intricate multiscale configuration of web-like filaments and voids. We apply the proposed methodology to cosmological simulation data.

- Discuss LSS
- Discuss Cosmological simulations
- Discuss Warm vs. Cold DM

Real observations of cosmic web: Great Wall (Geller & HUCHRA 1989), Sloan Great Wall (Gott III et al. 2005), gas (Cantalupo et al. 2014).

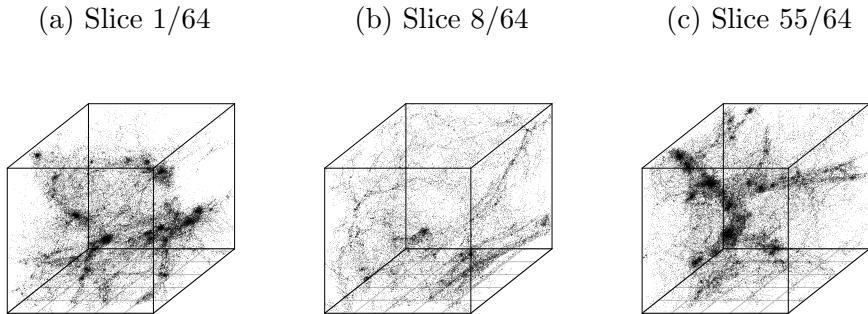


Figure 12: Examples of three quadruple-split samples from CDM simulation. Each of the three figures represents $\frac{1}{64}$ th of a piece of the entire EAGLE simulation.

With regard to the cosmic mass, warm and cold dark matter are traditionally believed to produce very different realizations of the observable Universe, with the latter involving more slowly moving particles prone to produce a more lumpy distribution of galaxies and

clusters. The former, containing more kinetic energy, has higher resistance to formation of global structure, and is theorized to render a less topologically-interesting cosmic mass. Applying the hypothesis testing framework would offer a method to *quantify* the topological differences between simulations with underlying WDM and CDM assumptions.

Starting with two cosmological simulation cubes (one representing CDM and the other WDM), one might consider cutting up the cubes into sub-cubes to bootstrap similar *sets* of samples as used in the Voronoi experiments. Let a *double-split* be defined as splitting the simulation cube along both the x and y axis, creating 4 equally sized cubes in total. Similarly, a *quadruple-split* produces 64 equally sized cubes, further splitting each of the cubes in a *double-split* set into four sub-cubes. Using this splitting technique on both the CDM and WDM simulations, we end up with two usable data sets similar to the Voronoi Foam. However, by artificially expanding our data set, the WDM and CDM data sets are then correlated due to their identical initial conditions. Because of this, the skeleton structure of the warm and cold DM simulations are nearly identical, allowing for paired T-tests. Note that there is also correlation between the samples due to the large-scale structure crossing the boundaries of the sub-cubes. We investigate this by varying sub-cube sizes, but consequently by doing so, change our sample size.

5.1 The EAGLE project

We analyze N-body simulations of structure formation. The simulation box is 100 co-moving Mpc on a side, and the numerical integration of the gravitational forces is run from redshift 127, when the age of the Universe is assumed to be less than 10Myr, to the present day (13.8 Gyr). The cosmological parameters are consistent with the 7-year results from the WMAP satellites: matter density $\Omega_0 = 0.272$, dark energy density $\Omega_\Lambda = 0.728$, Hubble parameter $h_0 = 0.704$, spectral index $n_s = 0.967$, and power spectrum normalization $\sigma_8 = 0.81$. The mass of the simulation particle is $8.8 \times 10^6 M_{\text{sun}}$. Haloes and subhaloes were identified using the SUBFIND algorithm (Springel et al. 2001), and the smallest halo that can be resolved has 20 particles. These runs were performed to be dark matter-only counterparts to the hydrodynamical runs of the Eagle project (Schaye et al. 2015); we stress that the runs used in this paper use gravity alone.

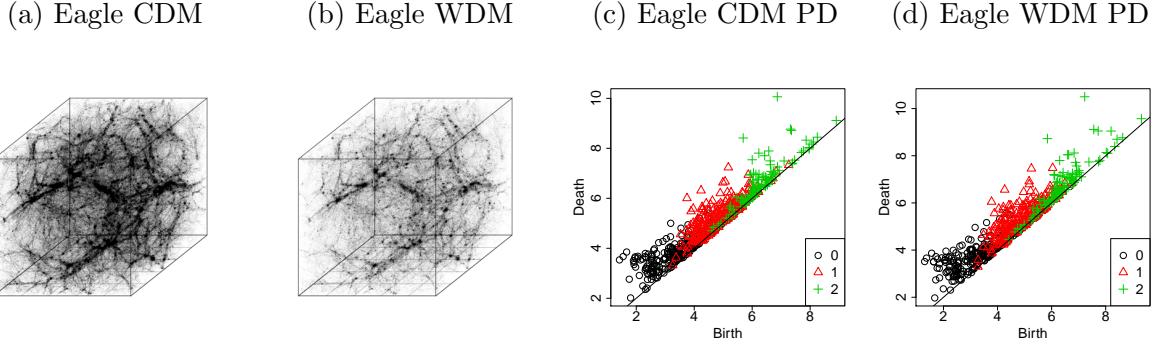


Figure 13: (a, b) Point clouds of the complete EAGLE CDM and WDM simulations. (c, d) Their corresponding persistence diagrams. Although the CDM structure is denser, the persistence diagrams appear comparable. See the results of the hypothesis testing for more details.

Both the CDM and WDM simulations make use of the same initial phases, and differ in that the latter has wave amplitudes rescaled using the transfer function of a 3.3keV thermal relic, with the relic mass chosen to be in agreement with the Lyman-alpha constraints of (Viel et al. 2013). This results in the suppression of structure on the scale of dwarf galaxies. Spurious subhaloes have been removed using the algorithm of (Lovell et al. 2014). Figure 13 shows a scatter plot of both the CDM and WDM simulations along with their respective persistent diagrams. Visually, we can confirm that the CDM is far more dense than WDM but seem to share similar internal structure; the persistence diagrams also share a general structure but we can identify smaller differences in homology groups that we hope to quantify using the hypothesis testing framework. These diagrams were generated under a volume of 1×10^6 , resolution of 2 and a DTM distance function with hyper-parameter 0.001.

5.2 Results

Table 2 shows both the standardized (right) and unstandardized (left) results of the five categories of hypothesis tests. The unstandardized results suggest that with a higher number of splits, we are able to focus on smaller-scale topology, finding more significant differences.

Through standardizing the persistence diagrams, we see that all the frameworks produce higher p-values on average and are less confident in the difference between double-split and quadruple-split datasets. This seems to explain that similar to the Voronoi foams, a large degree of the difference between WDM and CDM assumptions in the LSS are due to geometrical properties such as size. Judging from our most sensitive test statistics, Table 2 suggest that statistically significant differences in topology exist between WDM and CDM realizations.

5.3 Localizing Differences

After discovering differences in both geometry and topology, a good question would be how are these differences distributed? One hypothesis might be that differences are grouped among clustered sections of the observable Universe while other galaxies are practically identical. It is also possible that the differences are uniformly distributed across our cosmic data. To explore these questions more, we looked at the Bottleneck distances between different cubic splits as the number of splits increased. Because a higher number of splits focuses on smaller parts of the CDM/WDM simulation, we expect higher resolution into the topological features specific to that region. Figure 14 shows 3 sets of heatmaps for each of the 3 persistent homologies. Each set includes another 3 heatmaps, detailing the Bottleneck distances between congruent CDM and WDM for unsplit, double-split, and quadruple-split partitions respectively.

From Figure 14, we might infer that the topological and geometric differences are certainly not uniform. For each of the three homologies, introducing a greater number of splits uncovers greater magnitudes of Bottleneck distances between CDM and WDM cubes. This suggests that it is possible for topological differences to be masked by local similarities from a global perspective. At the very least, we can be confident that there exist partitions of topologically similar areas and partitions of topologically dissimilar areas within the EAGLE simulation. Further work should explore what areas in the Universe occupy these two partitions and if there exist physical evidence of differences.

Provided that EC and CORR were the two most effective hypothesis tests, to confirm the validity of these heatmaps, we compared these two functions for the unstandardized

Unstandardized			Standardized		
Test	Double Split	Quadruple Split	Test	Double Split	Quadruple Split
EC	1.156e-06	7.379e-26	EC	1.102e-04	7.345e-12
EC _{0:2}	2.104e-05	7.379e-28	EC _{0:2}	1.327e-05	3.972e-14
EC ₀	3.273e-08	3.811e-30	EC ₀	1.387e-06	1.614e-12
EC ₁	1.849e-05	0.935	EC ₁	0.0235	0.00143
EC ₂	0.340	0.0877	EC ₂	0.00385	1.977e-07
Sil _{EC}	7.709e-08	2.455e-20	Sil _{EC}	0.0122	0.00500
Sil _{0:2}	1.892e-06	1.114e-33	Sil _{0:2}	4.295e-04	8.072e-12
Sil ₀	2.958e-08	1.489e-34	Sil ₀	6.471e-06	1.762e-11
Sil ₁	1.169e-05	2.884e-23	Sil ₁	0.0282	3.899e-07
Sil ₂	0.925	0.0345	Sil ₂	0.00294	1.545e-04
KC ₀	0.442	0.000	GC ₀	0.986	0.856
KC ₁	0.192	0.000	GC ₁	0.962	0.956
KC ₂	0.248	0.00199	GC ₂	0.962	0.260
WIK ₀	0.084	0.000	WIK ₀	0.092	0.000
WIK ₁	0.051	0.000	WIK ₁	0.066	0.000
WIK ₂	0.496	0.000	WIK ₂	0.459	0.001
PI	0.923	0.281	PI	0.999	0.306
CORR	6.656e-04	7.355e-16	CORR	0.0289	0.918

Table 2: P-values from hypothesis tests on the unstandardized (left) and standardized (right) WDM & CDM simulations by double, and quadruple splits. A p-value of 0.000 comes from a permutation test with no positive examples. As the number of splits increases, the p-values dramatically decrease for the unstandardized setting, suggesting small-scale geometric and topological differences. A similar pattern is present, though less pronounced in the standardized setting.

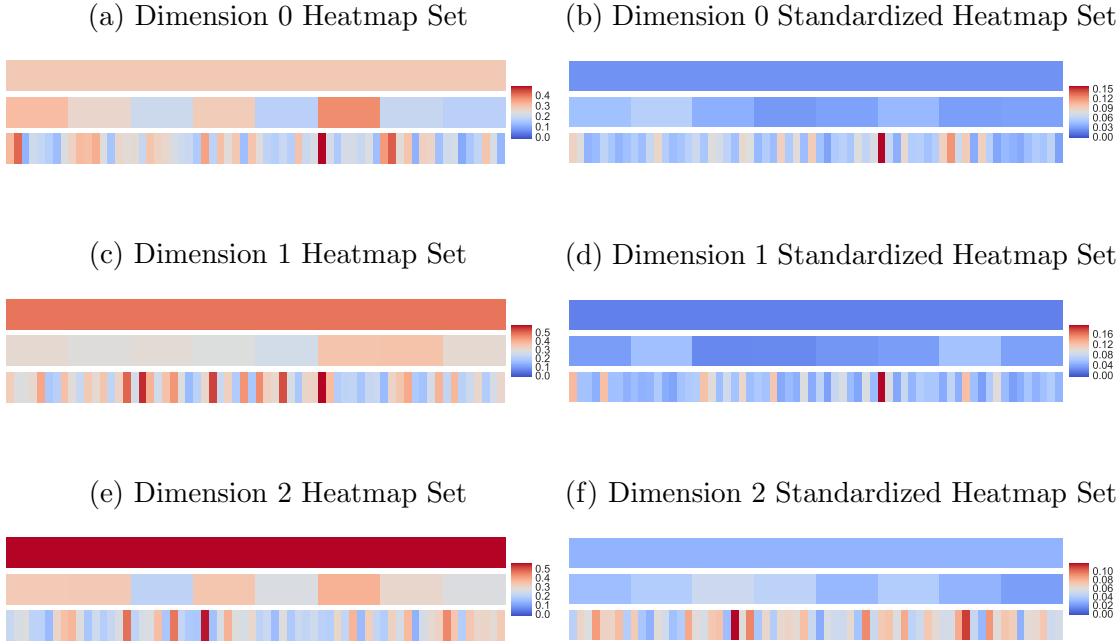


Figure 14: Each subfigure (a-f) contains three horizontal heatmaps representing (from top to bottom) the Bottleneck distances between each slice of unsplit, double split, and quadruple split pairs from respective WDM and CDM data sets by dimension. Each individual heatmap is a vectorization of the cubic slices produced from the WDM and CDM Eagle simulations. For example, Figure 14a's top row contains only 1 slice, since the whole EA-GLE simulation itself is one part; the second row contains 8 slices, since the dataset was split along two axis; similarly, the third row contains 64 slices. Subfigures (a, c, e) show the bottleneck distances derived from unstandardized persistence diagrams while the subfigures (b, d, f) use standardized diagrams.

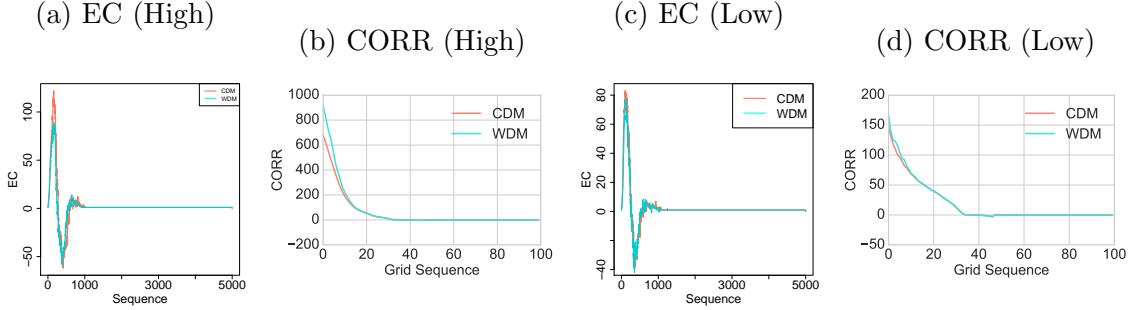


Figure 15: A comparison of EC and CORR for dimension 1 between the cube with the highest bottleneck distance and the cube with the lowest. (a) and (b) clearly depict larger distinctions between the CDM (red) and WDM (blue) slices than (c) and (d).

cubes with the highest Bottleneck distance and the unstandardized cubes with the lowest Bottleneck distance. As expected, Figure 15 confirms that the (High) EC and (High) CORR functions have far more discrepancies than their (Low) counterparts, also suggesting that certain partitions are more similar topologically than others. The same pattern arises for dimension 0 and dimension 2.

We can do the same heatmap experiment with the standardized persistence diagrams. As confirmed in the hypothesis testing of both the Voronoi and the EAGLE simulations, standardizing removes geometric differences, greatly reducing the Bottleneck distances, as seen in Figure 14 (b,d,f). Notably, for dimensions 0 and 1, the slice with the largest bottleneck distance remained the same as in the unstandardized heatmaps, suggesting that those differences are largely not due to size, but true topological distinctions. Dimension 2, however, was less consistent, possibly indicating that voids in CDM and WDM are more aberrant in terms of size, not topological shape. Despite the differences standardizing induced, we are still confident that differences in topology alone are not uniform across the EAGLE Universe: certain splits are consistently indicating differences in dark matter make-up between warm and cold assumptions.

6 Conclusion

In this paper, we presented a hypothesis testing framework built on persistent homology to compare topological summaries of two sets of point cloud data. We showed empirically that such a framework is able to infer differences in the true distribution of topology by comparing Voronoi tessellations with controlled hyperparameters. Additionally, we presented the application of this framework on the EAGLE data set to analyze the topology of the cosmic mass given assumptions of warm and cold dark matter, resulting in the discovering of locally significant spatial differences in geometry and topology. We believe this framework may provide a standard method for evaluating topological hypotheses in a diverse array of fields that greatly improve over currently existing methods.

SUPPLEMENTARY MATERIAL

Additional voronoi simulations: Hypothesis testing p-values for voronoi simulations of percent filament varying from 0.1 to 0.9. See Figure 16.

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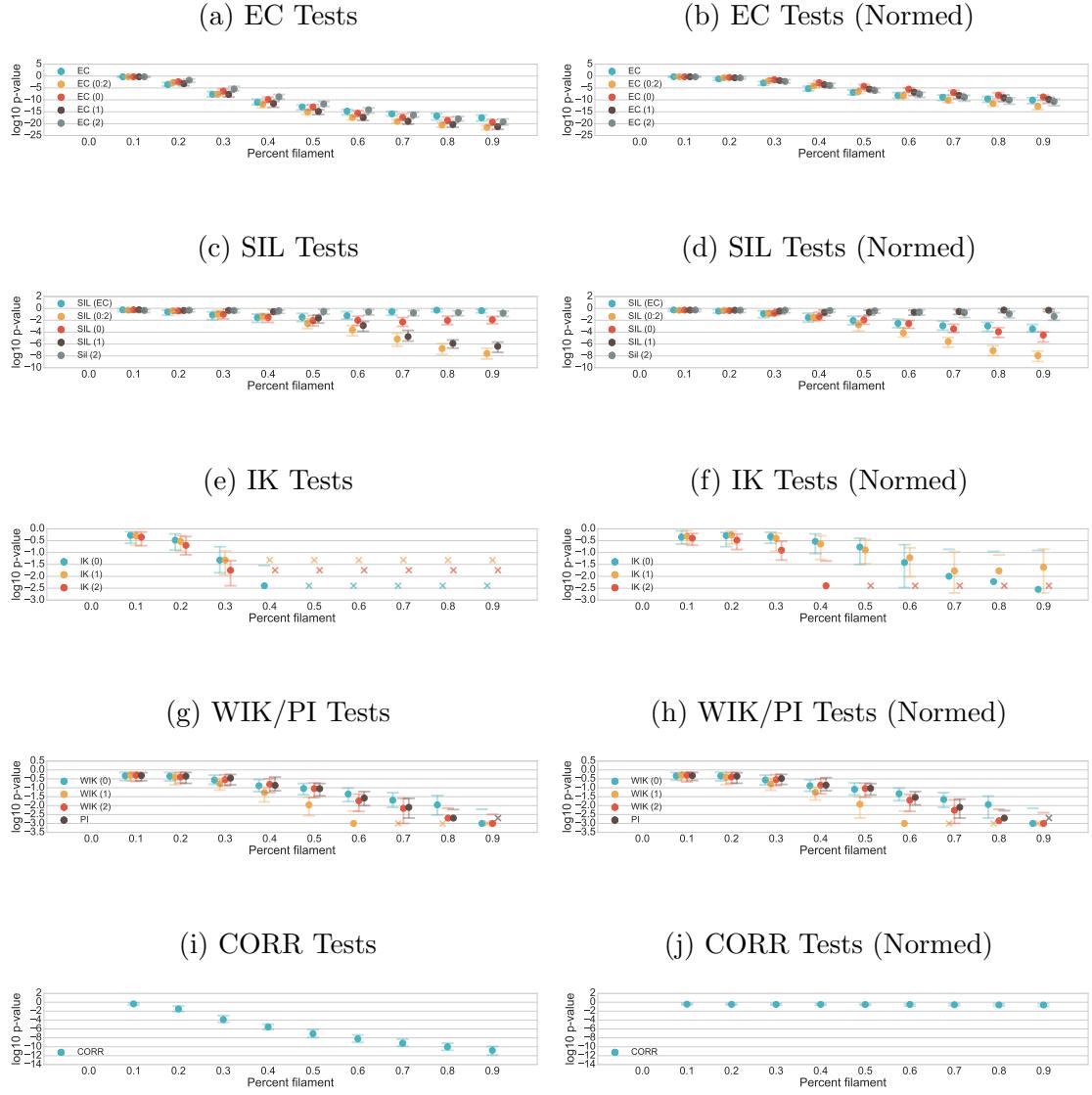


Figure 16: P-values for EC, SIL, IK, WIK, PI, and CORR. X-axis represents a varying percent filament (0.1 to 0.9 with 0.1 step increments) being compared to a baseline of 0.1; Y-axis shows the p-values in \log_{10} space. The lines plot the median \log_{10} p-value and peripheral points show the 25th and 75th percentile p-values of the 100 iterations. The right column of plots replicates the left with the exception of normalizing the persistence diagrams prior to testing. Points containing X marks indicate a p-value of 0.

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