Hahn-Banach

- Def: $q : X \to \mathbb{R}$ is sublinear if (i) $q(x+y) \leq q(x) + q(y)$
  (vectorspace) (ii) $q(\lambda x) = \lambda q(x)$, $\lambda \geq 0$

Theorem: let X be real vector space. let $q : X \to \mathbb{R}$ be sublinear functional. let $Y \subset X$ subspace. let $f : Y \to \mathbb{R}$ be a linear functional and $f(y) \leq q(y)$ $\forall y \in Y$.

Then $\exists$ $F : X \to \mathbb{R}$ linear functional s.t.

$$F\big|_Y = f \quad \text{and} \quad F(x) \leq q(x) \quad \forall x \in X.$$

Example $q(x) = \|x\|$.

Proof: ① Banach Lemma, ② Zorn's Lemma.

First: Banach Lemma. HB is true if in addition, we assume that Y has co-dimension 1 in X namely,

$$\exists\, x_0 \in X \setminus Y \text{ s.t. }$$

$$X = \{ \lambda x_0 + y \mid y \in Y, \lambda \in \mathbb{R} \}$$

(basically like adding a dimension to Y)

Proof: define $f_0 : X \to \mathbb{R}$ by

$$f_0(\lambda x_0 + y) = f_0(y) + \lambda f_0(x_0)$$ ideally but we can't write this b/c $x_0 \notin Y$.

$$= f(y) + \lambda \alpha$$ ← some #

This is linear and $f_0|_Y = f(y)$. Need to find α.

(Suppose the lemma is true.)

$f_0(\lambda x_0 + y)$. let $\lambda = 1$. $f(x_0 + y) = f(y) + f(x_0)$

let $\lambda = -1$. $f_0(z - x_0) = f(z) - f_0(x_0)$

Also $f_0(x_0 + y) \leq g(x_0 + y)$ (assume true).

$$f_0(x_0) = f_0(x_0 + y) - f(y) \leq g(x_0 + y) - f(y) \qquad (**)$$

$$-f_0(x_0) = f_0(z - x_0) - f(z) \leq g(z - x_0) - f(z)$$

$\Rightarrow$

$$f(z) - g(z - x_0) \leq \underbrace{f_0(x_0)}_{\alpha} \leq g(x_0 + y) - f(y). \qquad (*)$$

for any $y, z \in Y$. Does α exist for all y, z?

Need to show $\underbrace{f(z) - g(z - x_0) \leq g(x_0 + y) - f(y)}_{\text{note, no } f_0 \text{ in here.}}$. otherwise $f_0(x_0)$ DNE.

$$f(y) + f(z) = f(y + z) \leq g(y + z)$$
$$\leq g(z - x_0) + g(y + x_0)$$
$$=$$

we've shown $f(z) - g(z - x_0) \leq g(y + x_0) - f(y)$ for any $y, z \in Y$

$$\Rightarrow \sup_{z \in Y} \left(f(z) - g(z - x_0) \right) \leq \inf_{y \in Y} \left(g(y + x_0) - f(y) \right)$$

So there is some α in between. Now f_0 is well defined. But we need to check that it is bounded by g.

So,
$$f_0(y+x_0) = f(y) + \alpha \le q(x_0+y) \qquad \text{from } (**)$$
$$f_0(z-x_0) = f(z) - \alpha \le q(z-x_0)$$

Now we extend this to any λ (not just $\lambda=-1$, or $\lambda=1$)

for $\lambda > 0$,
$$\begin{aligned} f_0(y+\lambda x_0) &= f(y) + \lambda\alpha \\ &= \lambda\left(f\left(\frac{y}{\lambda}\right) + \alpha\right) \\ &= \lambda\left(f\left(\frac{y}{\lambda} + x_0\right)\right) \\ &\le \lambda\left(q\left(\frac{y}{\lambda} + x_0\right)\right) \\ &= q(y+\lambda x_0) \end{aligned}$$

$\lambda < 0$:
$$\begin{aligned} f_0(y+\lambda x_0) &= f(y) + \lambda\alpha \\ &= |\lambda|\left(f\left(\frac{y}{\lambda}\right) - \alpha\right) \\ &= \ \dots \end{aligned}$$

$\square$

(ii) Zorn's Lemma.

- **Def**: A partial ordering on a set Q means $\exists$ binary relation "$\precsim$"
  s.t. (1) $x \precsim x$, $\forall x \in Q$
  (2) $x \precsim y$ and $y \precsim x \Rightarrow x = y$
  (3) $x \precsim y$ and $y \precsim z \Rightarrow x \precsim z$

we don't require that every 2 elements can be compared.

Q is totally ordered if for any $x, y \in Q$, either $x \precsim y$ or $y \precsim x$.

Ex: $(\mathbb{R}, \text{"}\leq\text{"})$ is totally ordered.

Ex: let $\mathcal{A}$ be set of all subsets of X w/ binary relation, "inclusion" $\subset$. Not every 2 subsets are included $\Rightarrow$ partial ordering.

Zorn's Lemma says if S is any partially ordered set s.t. any totally ordered set $J \subset S$ has an upper bound in S. Then S has at 1 maximal element.

= Finish HB proof:
we need to introduce a relation on the linear space.

let S be set of ordered pairs (W, f_W) s.t. W is a subspace of X. $f_W : W \to \mathbb{R}$ linear functional bounded by sublinear functional: $f_W \leq q$ on W. We define a partial ordering on S:

$$(W, f_W) \precsim (T, f_T) \quad \text{if} \quad W \subset T \text{ and } f_T\big|_W = f_W.$$

(f_T is an extension on f_W i.e. added dims).

If $\mathcal{Y}$ is totally ordered subset of S. let $\overline{T} = \underbrace{\overline{\bigcup_{(T, f_T) \in \mathcal{Y}} T}}$ subspace. (not closure!)

This is candidate for maximal element.

Define $\overline{f} : \overline{T} \to \mathbb{R}$, s.t. for any $x \in \overline{T}$,

$$\overline{f}(x) = f_T(x) \text{ for some } T \text{ s.t. } x \in T.$$

So $\bar{f}$ linear and $\bar{f} \le q$ on $\bar{T}$. So $(\bar{T}, \bar{f})$ is upper bound of $\mathcal{I}$.

By Zorn's, $\exists$ at least 1 maximal element in S. Call it (U, f_U).

we want to show $U = X$ and $f_U = f_X$.

If $U \neq X$, we can find a W bigger than U and extend f_U more to fill X. So $U = X$.

<u>Corollary</u> let X be a normed space. let $q(x) = \|x\|$ (sublinear functional).

For any $y \in X \setminus \{0\}$ and let $\underbrace{Y = \text{span}\{y\}}_{\text{has 1 vector}}$. Then

$f(\lambda y) = \lambda\|y\|$ is linear on y. HB says $\exists f_0 : X \to \mathbb{R}$

(fixed treat as constant.

s.t. $f_0|_Y = f$ and $f_0 \le \|x\|$ for $x \in X$.

(bounded linear functional.

The pt again is so we know the dual space X^* of a hilbert space X is not empty.