<u>Goal</u> show completeness of L^1 function.

<u>Review</u> $\mathcal{L}^1$, $\mathcal{L}_0$

$\mathcal{L}^1$: diff. of a step fn.

$\mathcal{L}_0$: limit of step fn

<u>Lemma</u> Let $F \in \mathcal{L}^1$, then there's a step function φ s.t.

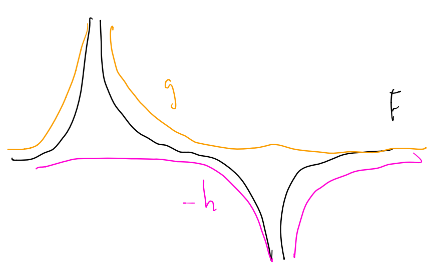
$$\|F-\varphi\|_{\mathcal{L}^1} \left(= \int |F-\varphi| \right) < \varepsilon.$$

Pf: Let $F = g-h$, $g,h \in \mathcal{L}_0$. By def of $\mathcal{L}_0$ functions, we can find ψ, η step functions s.t. $\|g-\psi\|_{\mathcal{L}^1} < \frac{\varepsilon}{2}$ and $\|h-\eta\|_{\mathcal{L}^1} < \frac{\varepsilon}{2}$.

$$\psi - \eta = \varphi$$

diff of 2 step fns is a step fn.

<u>Proposition</u>: Let $F \in \mathcal{L}^1$. Then there is a decreasing sequence $\{g_n\}_{n\geq 1}^{\infty}$, $g_n \in \mathcal{L}_0$ s.t. $g_n \to F$ a.e. $\int g_n \to \int F$.



Pf: Let $F = g-h$, $g,h \in \mathcal{L}_0$. This is an increasing seq of step functions $\{\varphi_n\}_{n\geq 1}^{\infty}$

$\varphi_n \to h$ a.e. and $\int \varphi_n \to \int h$.

Let $g_n = g - \varphi_n \to g - h$ a.e. $= F$

$$\int g_n = \int g - \int \varphi_n = \int g - h = \int F$$

$(g_n)_{n=1}^{\infty}$ is $\searrow$ as $(\varphi_n)_{n=1}^{\infty}$ $\nearrow$.

Now we can approximate $\mathcal{L}^1$ functions w/ ① step fnt and ② $\mathcal{L}_0$ functions.

Q: If $\int |F_k - F| \to 0$ then do you have $F_k \to F$ a.e.

Lemma: Let $\{\varphi_n\}_{n=1}^{\infty}$ $\nearrow$ seq of step functions s.t. $\{\int \varphi_n\}_{n=1}^{\infty}$ is bounded.
Then $\{\varphi_n\}_{n=1}^{\infty}$ is bounded a.e. (and thus convergent).

(i.e. does this arbitrary $\{\varphi_n\}$ approx. some $\mathcal{L}_0$ function?).

Prop: If $\{F_n\}_{n=1}^{\infty} \subseteq \mathcal{L}^1$ and F_n non-neg, and $\int F_n \xrightarrow{n\to\infty} 0$, then exists a sub-sequence $\{F_{n_j}\}_{j=1}^{\infty} \to 0$ a.e.

w/ neg, you can have weird cancelling.

Corollary: If $F \in \mathcal{L}^1$ non-negative s.t. $\int F = 0$, then F is 0 a.e.
Pf: $\{F, F, F, \dots\} \subseteq \mathcal{L}^1$. Apply Prop.

For $F \in \mathcal{L}^1$, we want to define $\|F\|_{\mathcal{L}^1} = \int |F|$.
Not a norm b/c if $F = 0$ a.e. then $\int |F| = 0$. There are lots of such F (a norm dictates that only one function $F = 0$ can have $\|F\|_{\mathcal{L}^1}$).

The corollary tells us if $\|F\|_{\mathcal{L}^1} \Rightarrow F = 0$ a.e.

L^1 is $\mathcal{L}^1/\sim$ where $F \sim g$ if $F = g$ a.e.

Then $\|\cdot\|_{\mathcal{L}^1}$ is a norm on L^1. Because if $F = g$ a.e. then $\|F\|_{\mathcal{L}^1} = \|g\|_{\mathcal{L}^1}$.

We want to show L^1 is Banach. i.e. completeness.

i.e. need to show that if $\{F_n\}_{n=1}^{\infty}$ cauchy, $F_n \in L^1$.. then there is F in L^1 s.t. $\|F_n - F\| \xrightarrow{n\to\infty} 0$.

Proof: ① propose a candidate for F.

Take a subsequence $\{F_{k_j}\}$ such that for $l > k_j$, $\|F_l - F_{k_j}\| < 2^{-j}$.

(use def. of cauchy sequence).

Approx. each F_{k_j}: for each j, pick φ_j s.t. $\|F_{k_j} - \varphi_j\| < 2^{-j}$.

Let $\varphi_0 = 0$.

Let $\varphi_j = \varphi_0 + (\varphi_1 - \varphi_0) + \dots + (\varphi_j - \varphi_{j-1})$ telescoping trick

$$= \sum_{i=1}^{j} (\varphi_i - \varphi_{i-1})$$

$$= \underbrace{\sum_{i=1}^{j} (\varphi_i - \varphi_{i-1})_+}_{\psi_j} - \underbrace{\sum_{i=1}^{j} (\varphi_i - \varphi_{i-1})_-}_{\eta_j}$$

$\{\psi_j\}$, $\{\eta_j\}$ are increasing seq of non-neg. step functions.

b/c sum?

To define ψ s.t. $\psi_j \to \psi$ a.e. and η s.t. $\eta_j \to \eta$ a.e.

we need $\{\int \psi_j\}$ and $\{\int \eta_j\}$ be bounded.

$$\int \psi_j \leq \underbrace{\sum_{i=1}^{j} \int}_{\text{can swap b/c finite sum}} (\varphi_i - \varphi_{i-1})_+ \leq \sum_{i=1}^{j} \int |\varphi_i - \varphi_{i-1}|$$

since we removed the neg. part

$$= \sum_{i=1}^{j} \|\varphi_i - \varphi_{i-1}\|$$

Cauchy in $\mathcal{L}^1$

$$\leq \sum_{i=1}^{j} \left(\underbrace{\|\varphi_i - F_{k_i}\|}_{\text{def}} + \|F_{k_i} - F_{k_{i-1}}\| + \|F_{k_{i-1}} - \varphi_{i-1}\| \right)$$

$$= \sum_{i=1}^{j} \left(2^{-i} + 2^{-(i-1)} + 2^{-(i-1)}\right) \leq 5$$

bounded

So we can find ψ and η. Take $F = \psi - \eta$.

our candidate!

We now need to show $\|F_j - F\| \to 0$ and $F \in L^1$.