Thm: X normed vector space. then X^* is Banach. $\left(\|F\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |F(x)| \right)$

Pf: $\{F_n\} \subset X^*$ Cauchy $\Rightarrow$ $\{F_n(x)\}$ cauchy in $\mathbb{C}$ for any $x \in X$.

So $F(x) = \lim_{n\to\infty} F_n(x)$ is the candidate. We know $F : X \to \mathbb{C}$.

check that F is linear.

$$|F(\alpha x + \beta y) - \alpha F(x) - \beta F(y)| \leq$$

$$\left| F(\alpha x + \beta y) - F_n(\alpha x + \beta y) + \alpha F_n(x) + \beta F_n(y) - \alpha F(x) - \beta F(y) \right|$$

$$\leq |F(\alpha x + \beta y) - F_n(\alpha x + \beta y)| + \alpha |F_n(x) - F(x)| + \beta |F_n(y) - F(y)|$$

$\to 0$ as $n \to \infty$.

check continuity (we need to show $|F(x)| \leq M$ for $\|x\| \leq 1$)

For any $\varepsilon > 0$, take $N_0 > 0$ s.t. $\|F_n - F_m\| < \varepsilon$ if $n, m \geq N_0$ (we are given cauchy)

sup def of norm

$$|F_n(x) - F_m(x)| \leq \|F_n - F_m\| \|x\| \leq \varepsilon$$

For $\|x\| \leq 1$ we get $|F_n(x) - F_m(x)| \leq \varepsilon$

Then $|F_n(x)|$ bounded.

Finally WTS $F_n \to F$ in X^*

shown end of last lecture

$$\|F_n - F\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |F_n(x) - F(x)| \to 0 \text{ as } n \to \infty$$

$\square$

Def for any normed vector space, X^* Banach space is called the dual space of X.

Ex: 1) $(\mathbb{C}^n)^* = \mathbb{C}^n$

b/c for any $f \in (\mathbb{C}^n)^*$, $f(z) = \sum_{i=1}^{n} c_i z_i$

2) $(\ell^1)^*$ can be identified w/ ℓ^∞.

Recall $\ell^1(\mathbb{C}) = \{ z = (z_n)_{n=1}^\infty \mid \sum_{n=1}^\infty |z_n| < \infty \}$

$\ell^\infty(\mathbb{C}) = \{ z \in (z_n)_{n=1}^\infty \mid \sup_{n \in \mathbb{N}} |z_n| < \infty \}$

Pf: For any $c \in \ell^\infty$, define $F_c : \ell^1 \to \mathbb{C}$ s.t.

$F_c(z) = \sum_{n=1}^\infty c_n z_n$. So F_c linear and continuous.

$$|F_c(z)| \le \sum_{n=1}^\infty |c_n| |z_n| \le \sup_n |c_n| \underbrace{\sum_{n=1}^\infty |z_n|}_{\text{in } \ell^1} = \|c\|_{\ell^\infty} \|z\|_{\ell^1}$$

$$= \left| \sum_{n=1}^\infty c_n z_n \right| \le \qquad \le \sum_{n=1}^\infty \sup|c_n| |z_n|$$

So we associated a linear functional to every element in ℓ^∞.

Conversely for any $g \in (\ell^1)^*$, then $g = F_c$ for some c due to linearity. We want to show $c \in \ell^\infty$.

To find c, consider $e_n = (0, 0, \ldots, 1, 0, \ldots)$

g has the form $\sum c_n z_n$.

So $g(e_n) = F_c(e_n) = c_n$

$$|c_n| = |g(e_n)| \le \|g\| \cdot \|e_n\|_{\ell^1} = \|g\|$$

not ℓ^1 norm

so $C \in \ell^\infty$.

III Riesz-Representation Theorem (for Hilbert space).

For $\mathbb{C}^n$, $(\mathbb{C}^n)^* = \mathbb{C}^n$. we know that $f \in (\mathbb{C}^n)^* \Rightarrow f = \sum_{n=1}^{\infty} c_n z_n$.

change to $f = \underbrace{\sum_{n=1}^{\infty} \bar{c}_n z_n}_{\text{adding bar should be okay}} \Rightarrow f = \langle z, c \rangle$.

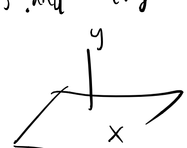
we want to generalize this to ∞-dim Hilbert space. Such that any linear functional is the inner prod of z w/ some element in the Hilbert space.

Thm let X be a Hilbert space. F is a continuous linear functional. Then exists a unique $y \in X$ s.t. $F(x) = \langle x, y \rangle \ \forall x \in X$.

Moreover $\underbrace{\|F\|}_{\text{norm for dual space}} = \underbrace{\|y\|}_{\text{norm for Hilbert space.}}$

Proof: If $F = 0$ then $y = 0$. Assume $F \neq 0$.

idea: Suppose y exists and $F(x) = \langle x, y \rangle = 0$, then $y \perp \operatorname{Ker}(F)$.



If y continuous, $\operatorname{Ker}(F)$ should be closed. Then it has an orthogonal complement $\Rightarrow$ $y \in$ orthogonal complement.

$\Leftarrow$

Consider kernel of F : $\ker F = \{ x \in X : F(x) = 0 \}$

since F is continuous, $\ker F$ is closed in X (pre-image of closed set).

Note that $\ker F \neq X$. Let $M = \ker F$.

Then we have an orthogonal decomposition of X.) structure of hilbert space.

$$X = M \oplus M^{\perp}$$

For any $x \in X$, $x = \underset{\in M}{w} + \underset{\in M^{\perp}}{z}$ in general.

$\Rightarrow F(x) = F(w) + F(z)$ (linear)

$\Rightarrow F(x) = F(z)$.

Fix $z_0 \in M^{\perp}$ s.t. $F(z_0) = 1$. So we write $x = w + \alpha z_0$

$\Rightarrow \alpha = F(x)$. Then $\langle x, z_0 \rangle = \underset{0}{\langle w, z_0 \rangle} + \langle \alpha, z_0 \rangle = F(x) \|z_0\|$
(in $M^{\perp}$)

let $y = \dfrac{z_0}{\|z_0\|}$, then $\langle x, y \rangle = F(x)$.

Next, prove the norm.

(1) Take $\|x\| \leq 1$, then $|f(x)| = |\langle x, y\rangle| \leq \|x\| \cdot \|y\| \leq \|y\|$

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)| \leq \|y\|$$

(a) Find vector $x = \frac{y}{\|y\|}$. So $\|x\| \le 1$. Then $\|F\| \geq |F(x)| = |\langle x, y \rangle| = \left| \frac{\langle y, y \rangle}{\|y\|} \right|$
$= \|y\|$

Thus $\|F\| = \|y\|$. $\square$

$\Rightarrow$ Main pt: we can identify X^* with X for Hilbert space.

We can find map $T: X \to X^*$ s.t. for $y \in X$.

$T_y = \langle \cdot, y \rangle$ is linear bijection & preserves norm. (isometry).

<u>Example</u> 1) $(\ell^2)^* \simeq \ell^2$

2) $(L^2([a,b]))^* \simeq L^2([a,b])$

<u>Note</u> ℓ^1 is Banach but not Hilbert.

Midterm up here

IV) Hahn-Banach theorem. <u>Goal</u>: For normed vector space X, show that X^* has non-trivial element. Discuss applications.

— <u>Def</u>: Let X vector space over $\mathbb{R}$. A functional $q: X \to \mathbb{R}$ is called sublinear if

① $q(x+y) \le q(x) + q(y)$ for any $x, y \in X$ ($\triangle$ ineq)

② $q(\lambda x) = \lambda q(x)$, $\lambda \geq 0$

Ex) X normed space. $q(x) = \|x\|$ sublinear.