Ch 4 (book): Orthogonal Expansions

I) orthogonality

- Def let X be an inner pdt space. for $x, y \in X$, x, y are orthogonal (denoted $x \perp y$) if $(x,y) = 0$.

- Def A sequence $\{e_n\}_{n=1}^{\infty}$ is called an orthogonal sequence if $e_n \perp e_m$ when $n \neq m$.

goal: represent hilbert space any thru basis.

If in addition, $\|e_n\| = 1$ (or $(e_n, e_n) = 1$ same thing) then sequence is called orthonormal sequence.

Ex In $\mathbb{R}^n$

$$\left.\begin{aligned} e_1 &= (1, 0, \dots) \\ e_2 &= (0, 1, \dots) \\ &\vdots \\ e_n &= (0, \dots, 1) \end{aligned}\right\} \text{orthonormal system.}$$

There is a method called Gram-Schmidt to construct thus orthonormal system from linearly independent vectors (HW).

- Some consequences.

① Thm (pythagoras's thm).

Let X inner pdt space. If $e_1, \dots e_n$ orthogonal system. Then

$$\left\| \sum_{j=1}^{n} e_j \right\|^2 = \sum_{j=1}^{n} \| e_j \|^2$$ (only finite-dim)

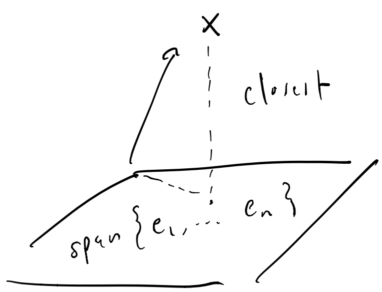
pf: $\text{LHS} = \left(\sum_{j=1}^{n} e_j , \sum_{\ell=1}^{n} e_\ell \right)$ — inner pdt

b/c $(u+v, u+v) = (u,u)+(v,v)+(u,v)+(v,u)$

$$= \sum_{j,\ell=1}^{n} (e_j, e_\ell)$$

$$= \sum_{j=1}^{n} (e_j, e_j) = \sum_{j=1}^{n} \|e_j\|^2 .$$

- For span$\{e_1, \dots, e_n\}$, we can find the closest point to $x \in X$ explicitly



(see last lecture), for Hilbert spaces

here we don't need Hilbert space, we can do it for orthogonal finite-dim spaces.

<u>Lemma</u> let X inner pdt space. let $\{e_1, \dots, e_n\}$ orthonormal system.

e_j is a vector

let $\lambda_1, \dots \lambda_n$ scalars. For $x \in X$ let $c_j = (x, e_j)$, then

$$\|x - \underbrace{\sum_{j=1}^{n} \lambda_j e_j}_{\text{any vector in linear span}} \|^2 = \|x\|^2 + \sum_{j=1}^{n} |\lambda_j - c_j|^2 - \sum_{j=1}^{n} |c_j|^2 \qquad (1)$$

Pf: $\left(x - \sum_{j=1}^{n} \lambda_j e_j \, , \, x - \sum_{j=1}^{n} \lambda_j e_j \right)$

$$= (x, x) - \sum_{j=1}^{n} \lambda_j (e_j, x) - \sum_{j=1}^{n} \bar{\lambda}_j (x, e_j) + \sum_{j,l=1}^{n} \lambda_j \bar{\lambda}_l (e_j, e_l)$$

$$= \|x\|^2 - \sum_{j=1}^{n} \lambda_j \bar{c}_j - \sum_{j=1}^{n} \bar{\lambda}_j c_j + \sum_{j=1}^{n} |\lambda_j|^2$$

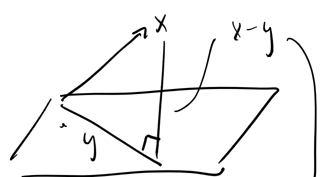
$\cdots$

$= RHS$ $\square$

to minimize ①, $\lambda_j = c_j$

Thm: under assumptions of lemma. for any $x \in X$, the closest point to x in $\operatorname{span}\{e_1, \dots, e_n\}$ is $y = \sum_{i=1}^{n} c_j e_j = \sum_{j=1}^{n} (x, e_j) e_j$

This closest vector is the projection of x onto the span



- Lemma The vector $x - \sum_{j=1}^{n} (x, e_j) e_j$ is orthogonal to $\operatorname{span}\{e_1, \dots, e_n\}$.

$= (x, e_l) - \left(\sum_{j=1}^{n} (x, e_j) e_j , e_l \right)$

Pf: $\left(x - \sum_{j=1}^{n} (x, e_j) e_j \, , \, e_l \right)$ for any l $- \sum_{j=1}^{n} (x, e_j) (e_j, e_l)$

$$= (x, e_l) - \sum_{j=1}^{n} (x, e_j)(e_j, e_l)$$

$$= (x, e_l) - (x, e_l) = 0$$

$\square$

how to extend this to ∞ dimensional spans?

II) Fourier Series

- Def: X hilbert space. let $\{e_n\}_{n=1}^{\infty}$ be orthonormal in X.

For any $x \in X$, define j^{th} fourier coefficient to be (x, e_j)

and fourier series $\sum_{j=1}^{\infty} (x, e_j) e_j$.

(this is clearly true if x is in the span ie. $x = \overbrace{\sum_{j=1}^{n} (x, e_j) e_j}^{\text{finite dim}}$

but is this true for any $x \in X$?

☆ Our goal is to show $x = \sum_{j=1}^{\infty} (x, e_j) e_j$ under some conditions!

- **Bessel's Inequality:**

Thm: If $\{e_n\}_{n=1}^{\infty}$ is a orthonormal sequence in inner pdt space X, then for any $x \in X$,

$$\sum_{j=1}^{\infty} |(x, e_j)|^2 \leq \|x\|^2.$$

Proof: for any $N \in \mathbb{N}$, let $y_N = \sum_{j=1}^{N} (x, e_j) e_j$. From previous lemma, we know $\|x - y_N\|^2 = \|x\|^2 - \sum_{j=1}^{N} |(x, e_j)|^2$ (see ① but w/ $\lambda_j = c_j$.

$$\Rightarrow \sum_{j=1}^{N} |(x, e_j)|^2 = \|x\|^2 - \|x - y_N\|^2 \leq \|x\|^2.$$

take $N \to \infty$

- **Def** we say $\sum_{j=1}^{\infty} x_j = x$ (converges to x) in inner pdt space X.

This means $\left(\sum_{j=1}^{k} x_j \right) \longrightarrow x$ as $k \to \infty$

in this norm: $\left\| \sum_{j=1}^{k} x_j - x \right\| \to 0$ as $k \to \infty$.

- ☆ **Thm** $\sum_{j=1}^{\infty} \lambda_j e_j$ converges if and only if $\sum_{j=1}^{\infty} |\lambda_j|^2 < \infty$ (converges).

($\{e_n\}_{n=1}^{\infty}$ is an orthonormal sequence in Hilbert space X).

we actually need this

Pf: "$\Rightarrow$" let $\sum_{j=1}^{\infty} \lambda_j e_j$ converge to x. For any k, w/ $m \geq k$

$$\left(\sum_{j=1}^{m} \lambda_j e_j , e_k \right) = \lambda_k \quad \text{(orthogonality)}.$$

$$= \sum_{j=1}^{m} \lambda_j (e_j, e_k)$$

Take $m \to \infty$. $\lim_{m \to \infty} \left(\sum_{j=1}^{m} \lambda_j e_j, e_k \right) = \underbrace{(x, e_k)}_{\lambda_k \text{ is the Fourier coefficient}} = \lambda_k$.

Use Bessel's to show $\sum_{j=1}^{\infty} |\lambda_j|^2 < \infty$.

$$\sum_{j=1}^{\infty} \left| (x, e_j)^2 \right| \leq \|x\|^2 < \infty$$

$$\sum_{j=1}^{\infty} |\lambda|^2$$

"$\Leftarrow$" . Assume $\sum_{j=1}^{\infty} |\lambda_j|^2 < \infty$. Consider partial sum $x_m = \sum_{j=1}^{m} \lambda_j e_j$.

Show that this is cauchy.

$$\|x_{m+p} - x_m\|^2 = \left\| \sum_{j=m+1}^{m+p} \lambda_j e_j \right\|^2 = \sum_{j=m+1}^{m+p} \|\lambda_j e_j\|^2$$

$$= \sum_{j=m+1}^{m+p} |\lambda_j|^2 \|e_j\|^2 = \underbrace{\sum_{j=m+1}^{m+p} |\lambda_j|^2}_{\text{converges since } \sum_{j=1}^{\infty} |\lambda_j|^2 < \infty}$$

(pythagoras)
(b/c e_j's are orthogonal)

By completeness of Hilbert space $(x_m) \longrightarrow x \in X$. $\square$

We know fourier series converges but we don't know what it converges to.

III) Complete orthonormal sequence. WTS $x = \sum_{j=1}^{\infty} (x, e_j) e_j$.

(we want to also show $\overline{\operatorname{span} \{e_n\}_{n=1}^{\infty}} = X$)

- Def: An orthogonal sequence $\{e_n\}$ in Hilbert space X is called complete if the only vector in X that is orthogonal to every e_n is the $\vec{0}$.