7.4 Adjoints

Generalizes A^T of real-valued matrix
A^* of complex valued matrix

def/prop Let X, Y be Hilbert spaces and $A \in \mathcal{L}(X,Y)$ is a bounded linear operator.
Then there is a unique operator $A^* \in \mathcal{L}(Y,X)$ with

$$(Ax, y)_Y = (x, A^* y)_X, \quad x \in X, \quad y \in Y$$

$$\begin{tikzcd} X \arrow[r, shift left, "A"] & Y \arrow[l, shift left, "A^*"] \end{tikzcd}$$

Example (matrices)

$$\mathbb{C}^2 \longrightarrow \mathbb{C}^3$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \qquad A^* = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

$$\left(A\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right) = \left(\begin{pmatrix}a_{11}\\a_{21}\\a_{31}\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right) = a_{21}$$

This should be equal to

$$\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} \right) = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}b_{12}\\b_{22}\end{pmatrix} \right) = \bar{b}_{12} = a_{21}$$

Proof: Use Riesz Representation theorem (Goal: define $A^* y$)
Consider for fixed $y \in Y$, the mapping $x \overset{\Phi}{\longmapsto} (Ax, y)_Y$, $\Phi \in \mathcal{L}(X, \mathbb{F}) \in X^*$ field ($\mathbb{R}$ or $\mathbb{C}$).
So by Riesz representation, $\exists$ unique $z \in X$ s.t.

$$(Ax, y) = (x, z) \quad \forall x$$

Define $A^* y = z$. We need to show $A^* : Y \longrightarrow X$, linear, continuous and unique.

① linearity $\forall y, w \in Y$, $\lambda, \mu \in \mathbb{C}$ we have $\forall x$

$$(x, A^*(\lambda y + \mu w)) = (Ax, \lambda y + \mu w) = \bar{\lambda}(Ax, y) + \bar{\mu}(Ax, w)$$
$$= \bar{\lambda}(x, A^* y) + \bar{\mu}(x, A^* w)$$
$$= (x, \lambda A^* y) + (x, \mu A^* w)$$
$$= (x, \lambda A^* y + \mu A^* w)$$

$\Rightarrow A^*(\lambda y + \mu w) = \lambda A^* y + \mu A^* w$

② Continuity: For any $y \in Y$, $\|A^* y\| = (A^* y, A^* y) = (AA^* y, y) \leq \|AA^* y\| \|y\|$

$\leq \|A\| \|A^* y\| \|y\|$

if $\|A^* y\| > 0 \Rightarrow \|A^* y\| \leq \|A\| \|y\|$ so that $\|A^*\| \leq \|A\|$.

③ Uniqueness: If there is a B s.t. $(Ax, y) = (x, A^* y) = (x, By)$ $\forall x, y$.

Then $0 = (x, By - A^* y)$ $\forall x \Rightarrow By = A^* y$

$\square$

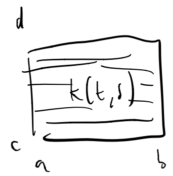
ex) $X = L^2(a,b)$, $Y = L^2(c,d)$. take $x(t) \in X$.

Take Integral operator $K: X \to Y$ s.t.

$$K(x(t)) = \int_a^b k(t,s)\, x(t)\, dt$$

for $k(t,s)$ some function $t \in (a,b)$, $s \in (c,d)$

$\in (a,b)$ $\in (c,d)$



example $(K(x(t)))(c) = \int_a^b k(t,c)\, x(t)\, dt$

As you might expect, $K^*(y(s)) = \int_c^d \ell(t,s)\, y(s)\, dt$, $\ell(t,s) = \overline{k(t,s)}$

Consider

$$(Kx, y) = \int_c^d K(x(t))\, \overline{y(t)}\, dt = \int_c^d \int_a^b k(s,t)\, x(s)\, \overline{y(t)}\, dt \quad \text{or}$$

$$= \int_a^b \left(\int_c^d \overline{k(s,t)}\ \overline{y(t)}\, dt \right) x(s) \Big)\, ds$$

$$(x, K^* y) = \int_a^b x(t) \left(\int_c^d \overline{\ell(t,s)\ \overline{y(s)}}\, ds \right) dt$$

$$= \int_a^b x(t) \int_c^d k(t,s)\ \overline{y(s)}\, ds\ dt = \int_a^b \int_c^d k(t,s)\, x(t)\, \overline{y(s)}\, ds\, dt$$

We discuss properties of A^*.

<u>Thm</u> For X, Y hilbert spaces, $A \in \mathcal{L}(X,Y)$, $A^{**} = A$ and $\|A^*\| = \|A\|$.
(already proved $\|A^*\| \leq \|A\|$).

<u>Proof</u>: $(y, (A^*)^* x)_Y = (A^* y, x)_X = \overline{(x, A^* y)_X} = \overline{(Ax, y)}_X = (y, Ax)_Y$

so $(A^*)^* = A$.

$$\|A\| = \|(A^*)^*\| \leq \|A^*\| \leq \|A\| \Rightarrow \|A^*\| = \|A\|$$

<u>Thm</u> (i) if $A: X \longrightarrow Y$, $B: Y \longrightarrow Z$, $B \circ A: X \longrightarrow Z$.
Then $(B \circ A)^* = A^* \circ B^*$

(ii) $(\lambda A + \mu B)^* = \bar{\lambda} A^* + \bar{\mu} B^*$.

<u>Proof</u> Exercise

<u>def</u>: let X Hilbert, $A \in \mathcal{L}(X)$ is <u>self-adjoint</u> or Hermitian if $A^* = A$.

<u>Thm</u> If A Hermitian on Hilbert space, then $\|A\| = \sup_{\|x\| \leq 1} |(Ax, x)|$

(For a matrix, A Hermitian $\Rightarrow$ diagonalizable, $\|A\|$ = largest eigenvalue)
x here where sup is achieved = eigenvector for largest eigenvalue.

Pf: By def $\|A\|^2 = \sup_{\|x\|\le 1} (Ax, Ax)$

less than ?

To show $\|A\| \geq \sup_{\|x\|\le 1} |(Ax,x)|$, note $\|(Ax,x)\| \le \|Ax\|\|x\| \le \|A\|\|x\|^2 \le \|A\|$

For any $x, y \in X$, $(A(x \pm y), x \pm y) = (Ax,x) \underbrace{\pm 2\operatorname{Re}(Ax,y)} + (Ay,y)$

$= (Ax,y, Ax,y)$

$- (Axy,$

$= 4 \operatorname{Re}(Axy) = |A(x+y), x+y| - |A(x-y), x-y|$

If $m = \sup_{\|x\|\le 1} \|(Ax, x)\|$, then $\le m(\|x+y\|^2 + \|x-y\|^2) = 2m(\|x\| + \|y\|^2)$

parallelogram

To get $|(Ax,y)|$, replace x by λx, λ rescale. Choose λ s.t. $\lambda(Ax,y) \geq 0$

Then it follows $|(Ax,y)| = \operatorname{Re}(Ax,y) \le \frac{1}{2}m(\|x\|^2 + \|y\|^2)$

If $\lambda x \neq 0$, choose $y = \frac{\|x\|}{\|Ax\|} Ax$. Note $\|y\| \|x\|$

Then, $\|x\| \cdot \|Ax\| \le m\|x\|^2 \Rightarrow \|Ax\| \le m\|x\| = \sup_{\|x\|\le 1} |(Ax,x)|$.