Chapter 7: linear operators (generalization of linear functional)

I) definition and properties

Def: let E, F vector spaces. A linear operator is a mapping $T: E \to F$ such that

$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$. for any $\lambda, \mu \in \mathbb{C}$ (or $\mathbb{R}$), $x, y \in E$.

Def: A linear operator on E means $T: E \to E$.

Def: If E, F are normed spaces, then a linear operator $T: E \to F$ is bounded if $\exists M \geq 0$ s.t. $\|Tx\|_F \leq M \|x\|_E \quad \forall x \in E$.

Thm: let E, F normed space, $T: E \to F$ linear. The following are equivalent

① T is continuous at $0 \in E$

② T is continuous

③ T is bounded

Proof: Same as for linear functionals

Def: If $T: E \to F$ bounded linear operator. Define operator norm

$$\|T\| = \sup_{x \in E \setminus \{0\}} \frac{\|Tx\|_F}{\|x\|_E} = \sup_{\substack{x \in E \\ \|x\|_E \leq 1}} \left\{ \|Tx\|_F \right\}$$

In particular, $\|Tx\|_F \leq \|T\| \cdot \|x\|_E \quad \forall x \in E$.

- Notions: We define $\ker T = \{ x \in E : Tx = 0 \} \subset E$ — also called nullspace

$\operatorname{ran} T = \{ Tx : x \in E \} \subset F$

Ex 1 Integral operator: $K: L^2([a,b]) \to L^2([c,d])$. First, let $\mathcal{K}: [c,d] \times [a,b] \to \mathbb{C}$. let $\mathcal{K}$ be a continuous function. let $x(s) \in L^2([a,b])$.

Define $(Kx)(t) = \int_a^b \mathcal{K}(t,s)\, x(s)\, ds$, $c < t < d$.

Show K is linear, bounded.

① K linear, easy to check

② K is bounded. we need to compute $\|Kx\|_{L^2([c,d])} \leq M \|x\|_{L^2([a,b])}$.

note that $\|Kx\|_{L^2}^2 = \int_c^d |Kx(t)|^2 \, dt$. (*) First compute

$$|Kx(t)|^2 = \left| \int_a^b k(t,s) \, x(s) \, ds \right|^2$$

$$\leq \left(\int_a^b |k(t,s) x(s)| \, ds \right)^2 \leq \int_a^b |k(t,s)|^2 \, ds \int_a^b |x(s)|^2 \, ds$$

$$\leq \int_a^b |k(t,s)|^2 \, ds \cdot \|x\|_{L^2([a,b])} \quad \forall t \in [c,d]$$

use (*) to get

$$\|Kx\|^2_{L^2([c,d])} = \underbrace{\left(\int_c^d \int_a^b |\overset{\text{continuous}}{k(t,s)}|^2 \, ds \, dt \right)}_{\text{let this be } M} \cdot \|x\|_{L^2([a,b])}$$

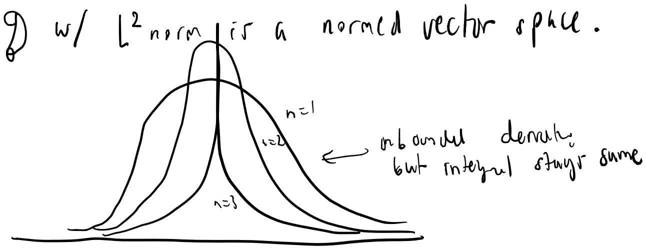
So K is bounded and $\|K\| \leq M$

Ex 2) Differential operator (unbounded)

let $\mathcal{D}$ be the space of differentiable functions $f \in L^2([-\infty, \infty])$.
Such that $f' \in L^2([-\infty,\infty])$. Namely, this implies $\|f\|_{L^2} < \infty$, $\|f'\|_{L^2} < \infty$.

Then consider $\frac{d}{dx} : \mathcal{D} \to L^2([-\infty,\infty])$. Here $\mathcal{D}$ is a subspace of L^2.

Then $\mathcal{D}$ w/ L^2 norm is a normed vector space. $\frac{d}{dx}$ is linear but not bounded.



let $f_n(x) = \sqrt{n} e^{-nx^2}$, $n = 1, 2, \dots$

$$\|f_n\|_{L^2}^2 = \int_{-\infty}^{\infty} \sqrt{n} e^{-2nx^2} dx = \frac{1}{\sqrt{2n}} \int_{-\infty}^{\infty} \sqrt{n} e^{-2nx^2} d(\sqrt{2n}x) = \frac{\sqrt{\pi}}{\sqrt{2}} \quad \text{bounded}$$

$$f_n'(x) = -2n\sqrt{n}x\, e^{-nx^2}. \quad \|f_n'\|_{L^2}^2 = \int_{-\infty}^{\infty} 2n^3 e^{-2nx^2} dx^2 = n^2\sqrt{\pi} \longrightarrow \infty$$

(unbounded! how can we fix this so we can solve diff. eqns).

<u>Note</u> if you use a different norm

$$\|f\| = \left(\int_{-\infty}^{\infty} |f|^2 + |f'|^2 \, dx\right)^{\frac{1}{2}} = \left(\|f\|_{L^2}^2 + \|f'\|_{L^2}^2\right)^{\frac{1}{2}}$$

then $\frac{d}{dx}$ is bounded. But with a new norm, there are disadvantages.

II) Banach space of operators

- <u>Def</u> let $\mathcal{L}(E,F)$ be the set of all bounded linear operators from E to F, where E, F normed vector space.

<u>Note</u> $E^* = \mathcal{L}(E, \mathbb{C})$. Had to use completeness of $\mathbb{C}$.

<u>Thm</u>: $\mathcal{L}(E,F)$ is a Banach space if F is Banach. E is any normed vector space.

The proof is the same for $E^* = \mathcal{L}(E, \mathbb{C})$.

- Define <u>composition</u> of operators, let $A: E \to F$, $B: F \to G$. Then $BA: E \to G$.

This is defined by $(BA)(x) = B(A(x)) \in G$, $\forall x \in E$.

<u>Thm</u> let E, F, G normed vector space. $A \in \mathcal{L}(E,F)$, $B \in \mathcal{L}(F,G)$, then $BA \in \mathcal{L}(E,G)$ (ie. BA is still bounded and linear). And $\|BA\| \leq \|B\| \cdot \|A\|$

operator norm

<u>Proof</u>: BA is linear. For any $x \in E$,

$$\|BAx\|_G = \|B(Ax)\|_G \leq \|B\| \cdot \|Ax\|_F \leq \|B\| \cdot \|A\| \cdot \|x\|_E$$

- Consider $\mathcal{L}(E) = \mathcal{L}(E, E)$ bounded linear operator on E.

  Then for any $T \in \mathcal{L}(E)$, then $T^n \in \mathcal{L}(E)$, $n = 1, 2, \ldots$ and $T^0 = I$ identity operator.

If $\underbrace{E \text{ banach, then } \mathcal{L}(E)}_{\text{as in proof for linear functionals}}$ banach. Next time, we show this is an algebra.