Last time: defined $L^1([a,b]) = \mathcal{L}^1/\sim$. we showed $L^1([a,b])$ is a Banach space with norm $\|f\|_{L^1} = \int |f|$.

- Some consequences:

Thm: Monotone Convergence Theorem

Suppose we have $\nearrow$ sequence $\{f_k\}_{k=1}^{\infty} \subset \mathcal{L}^1$ i.e. $f_1 \leq f_2 \leq \dots$

Assume $\{\int f_k\}_{k=1}^{\infty}$ bounded. Then there is $f \in \mathcal{L}^1$ s.t.

$f_k \to f$ a.e. and $\int f_k \to \int f$.

($\int f_k$ clearly converges ($\nearrow$ and bounded))

Pf: Show that $\{f_k\}_{k=1}^{\infty}$ is cauchy in $\mathcal{L}^1$.

$$\|f_k - f_\ell\|_{\mathcal{L}^1} = \int |f_k - f_\ell|$$

$$\text{if } k > \ell, \quad = \int f_k - \int f_\ell$$

since $\lim_{k\to\infty} \int f_k$ exists b/c $\nearrow$ bounded sequence.

duh

So cauchy b/c for k, ℓ sufficiently large, they will be ε-close.

By completeness of $\mathcal{L}^1$, there is a $f \in \mathcal{L}^1$ s.t. $\|f_k - f\| < \varepsilon$

$$\int f_k \to \int f.$$ we already know it converges but here we want to show it is to $\int f$

Need to show $f_k \to f$ a.e. Use prop 2.13 (2), $\exists$ subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ s.t. $f_{k_j} \to f$ a.e. Because $\{f_k\}$ $\nearrow$ sequence,

$f_k \to f$ a.e. (as in it can't oscillate?)

no longer guaranteed

Lemma 1: Fatou's lemma: let $\{f_k\}_{k=1}^{\infty} \in \mathcal{L}^1$ be non-negative sequence, $f_k \to f$ a.e. and $\{\int f_k\}_{k=1}^{\infty}$ bounded. Then $f \in \mathcal{L}^1$ and $\int f \leq \liminf_k \int f_k$.

Pf: set $g_k = \inf\{f_k, f_{k+1}, \dots\}$. Then $g_1 \leq g_2 \leq g_3 \dots$

Also $\lim_{k\to\infty} g_k = f$ a.e (by def). Notice that $\{\int g_k\}$ bounded

Then apply MCT. $g_k \to h \in \mathcal{L}^1$. $h = f$ a.e. (by def).

and $\underbrace{\lim_{k\to\infty} \int g_k = \int h}_{\text{by MCT}} = \int f = \int \lim_{k\to\infty} g_k$

So $\int f = \lim_{k\to\infty} \int g_k$. since $g_k \leq f_k \Rightarrow \int g_k \leq \int f_k$

we don't know $\lim_{k\to\infty} \int f_k$ exists as (f_k) is not $\uparrow$. So the best we can do is $\liminf_{k\to\infty} \int f_k$

So $\int f = \lim_{k\to\infty} \int g_k \leq \liminf_{k\to\infty} \int f_k$. $\square$

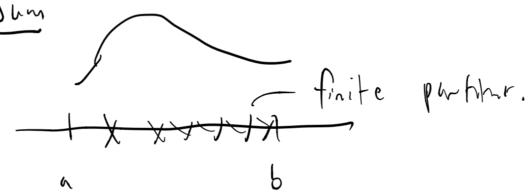
☆ Example of Lebesgue integrable but not Riemann Intgble function

Ex: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [a,b] \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{Q}$ is countable so $f(x) = 0$ a.e. so $f \in \mathcal{L}^1$ and $\int f = 0$.

f is not Riemann Int'ble.

Riemann Sum



let $P = \{x_1, x_2, \ldots, x_n\}$ be the partition of $[a,b]$.

Define upper sum $U(P,f) = \sum_{j=1}^{n} M_j (x_{j+1} - x_j)$

where $M_j = \sup_{[x_j, x_{j+1}]} f(x)$.

Lower sum $L(P,f) = \sum_{j=1}^{n} m_j (x_{j+1} - x_j)$

where $m_j = \inf_{[x_j, x_{j+1}]} f(x)$.

f is Riemann Int'ble if

$$\inf_P U(P,f) = \sup_P L(P,f) = \int_a^b f\,dx.$$

In our example $U = 1$, $L = 0$ b/c any partition has rational & irrational as rationals are dense.

$\square$

— Finish L_1 discussion. Begin L_2 discussion.

(VI) L_2 space.

<u>Def</u>: let $\mathcal{L}^2 = \{ f \in \mathcal{L}^1 : f^2 \in \mathcal{L}^1 \}$

<u>Proposition</u>: $\mathcal{L}^2$ is a vector space.

$(f+g)^2 = \underbrace{f^2 + g^2 + 2gf}_{\text{all in } \mathcal{L}^1}$

<u>Proof</u>: Need to show if $f, g \in \mathcal{L}^2$, then $f+g \in \mathcal{L}^2$.

we use definition. Since $f \in \mathcal{L}^1$, $f^2 \in \mathcal{L}^1$ There are step functions

$\varphi_k, \tilde{\varphi}_k$ s.t. $\varphi_k \to f$ a.e., $\tilde{\varphi}_k \to f^2$ a.e.

Assume $\tilde{\varphi}_k \geq 0$; otherwise take $\varphi_k = \max\{\varphi_k, 0\}$.

And also $\int |\varphi_k - f| \to 0$, $\int |\tilde{\varphi}_k - f^2| \to 0$.

Similarly we have $\psi_k, \tilde{\psi}_k$ for g, g^2. Same thing as above.

<u>Idea</u> find $\Phi_k \to f$, $\Psi_k \to g$ a.e. s.t. $(\Phi_k + \Psi_k)^2 \to (f+g)^2$ a.e.

Then Fatou's Lemma to conclude $(f+g)^2 \to \mathcal{L}^1$.

How to find Φ_k, Ψ_k?

let $\Phi_k = \operatorname{sign}(\varphi_k)\sqrt{\tilde{\varphi}_k}$. $\Psi_k = \operatorname{sign}(\psi_k)\sqrt{\tilde{\psi}_k}$.

we know $\Phi_k \to f$ a.e. & $\Phi_k^2 \to f^2$ a.e.

Also $\int |\Phi_k^2 - f^2| \to 0$ (by def).

Same thing for Ψ_k and g.

Then apply Fatou's lemma to $(\Phi_k + \psi_k)^2$. First this is not negative.

First $(\Phi_k + \psi_k)^2 \to (f+g)^2$ a.e.

Notice that $\int_k (\Phi_k + \psi_k)^2 \leq \int \Phi_k^2 + \int \psi_k^2 + 2\int \Phi_k \psi_k$

$$\leq 2\left(\int \Phi_k^2 + \int \psi_k^2\right)$$

these are both bounded

So Fatou's lemma $(f+g)^2 \in \mathcal{L}^1$

— Define inner pdt on $\mathcal{L}^2$ by $\langle f, g \rangle = \int fg$ (lesbgue)

(this is not a true inner pdt space so we consider $L^2([a,b])$

$= \mathcal{L}^2/\sim$. Then $\langle f, g \rangle = \int fg$.

It remains to show completeness of L^2.

(to make Hilbert space).

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1 2 3 4 5 6 , $\int$ midterm.