Ch 5: Classical Fourier series

Recall: show $e_n = (2\pi)^{-\frac{1}{2}} e^{in(x)}$ $\quad -\pi < x < \pi$

is a complete orthogonal basis for $L^2([-\pi, \pi])$

WTS

Need to show: $f = \sum_{n=-\infty}^{\infty} (f, e_n) e_n$

Consider f continuous and show theorem.

WTS $F_m = \frac{1}{m+1}(S_0 + \dots + S_m) \to f$, uniformly

$$S_m = \sum_{n=-m}^{m} (f, e_n) e_n.$$

First, $F_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) K_m(x-y)\, dy$

$$K_m(t) = \frac{1}{m+1} \frac{\sin^2\left(\frac{m+1}{2}\right)t}{\sin^2\left(\frac{t}{2}\right)}$$

Let's find K_m first:

$$K_m = \frac{1}{m+1} \sum_{j=0}^{m} \left(\sum_{n=-j}^{j} e^{int} \right)$$

$$\sum_{n=-j}^{j} e^{int} = e^{i(-j)t} + \dots + e^{ijt}$$
$$= e^{-ijt}\left(1 + \dots + e^{i(2j)t}\right)$$

$$= e^{-ijt}\left(\frac{1 - e^{i(2j+1)t}}{1-e^{it}}\right)$$

$$= \frac{e^{-ijt} - e^{i(j+1)t}}{1-e^{it}}$$

bring back outer sum

$$K_m(t) = \frac{1}{m+1}\sum_{j=0}^{m} \frac{e^{-ijt} - e^{i(j+1)t}}{1-e^{it}}$$

$$\boxed{1+x+\dots+x^n = \frac{1-x^{n+1}}{1-x}}$$

$$= \frac{1}{m+1}\cdot\frac{1}{1-e^{it}}\left(\sum_{j=0}^{m} e^{-ijt} - \sum_{j=0}^{m} e^{i(j+1)t}\right)$$

$$= \frac{1}{m+1}\cdot\frac{1}{1-e^{it}}\cdot\left(\frac{1-e^{-i(m+1)t}}{1-e^{-it}} - \frac{e^{it} - e^{i(m+2)t}}{1-e^{it}}\right)$$

$$= \frac{1}{m+1}\cdot\frac{1}{1-e^{it}}\cdot\left[\frac{1-e^{-i(m+1)t}}{1-e^{-it}} + \frac{e^{it}\left(1-e^{i(m+1)t}\right)}{e^{it}\left(1-e^{-it}\right)}\right]$$

$$= \frac{1}{m+1}\cdot\left(\frac{2-e^{i(m+1)t} - e^{-i(m+1)t}}{|1-e^{it}|^2}\right)$$

$$\boxed{e^{it} = \cos t + i\sin t}$$

$$= \frac{1}{m+1}\left(\frac{2-2\cos(m+1)t}{\left|e^{-\frac{it}{2}} - e^{\frac{it}{2}}\right|^2}\right)$$

$$\left[\; e^{\frac{it}{2}} = 1 \right.$$

$\cos 2x = 1 - 2\sin^2 x$

$$= \frac{1}{m+1} \cdot \frac{\sin^2\left(\frac{m+1}{2}\right)t}{\sin^2\left(\frac{t}{2}\right)}$$

$\square$

<u>Now</u>: we want to show

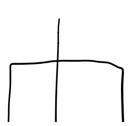
$$F_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, K_m(x-y)\, dy \longrightarrow f(x) \quad \text{as } m \to \infty$$

<u>Convolution</u>: $f(x), g(x)$, $\quad (f*g)(x) = \int f(x-y)\, g(y)\, dy$

$$= \int f(y)\, \underbrace{g(x-y)}_{\text{weight for a region around } y}\, dy$$

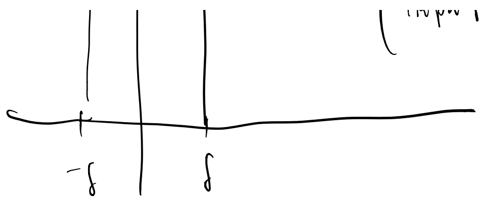
So $F_m(x) = \frac{1}{2\pi}(f * K_m)(x) \longrightarrow f(x)$ as $m \to \infty$ (we want to show)

<u>Ex</u> Suppose $K_\delta(t) = \begin{cases} 0 & \text{if } t > \delta \\ \frac{1}{2\delta} & \text{if } |t| \leq 2\delta \end{cases}$



$\frac{1}{2\delta}$

([illegible] kernel)



compute $\quad f * K_\delta(x) = \int f(y)\, K_\delta(x-y)\, dy$

$$= \int f(x-y)\, K_\delta(y)\, dy$$

$$= \int_{-\delta}^{\delta} f(x-y)\, \frac{1}{2\delta}\, dy \qquad \text{— 0 elsewhere}$$

$$= \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(y)\, dy$$

if f is continuous, then $f * K_\delta(x) \longrightarrow f(x)$ as $\delta \longrightarrow 0$.

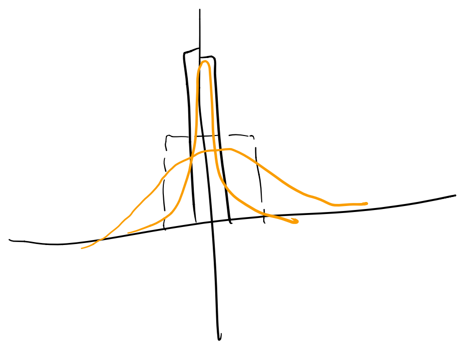
$$\left| f(x) - f * K_\delta(x) \right| = \left| \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(x)\, dy - \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(y)\, dy \right|$$

$$\leq \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} \left| f(x) - f(y) \right| dy$$

$$\leq \frac{1}{2\delta} (2\delta)\, \varepsilon \quad \text{if } |f(x) - f(y)| < \varepsilon$$

If kernel is impulse, $F(x) \xrightarrow{\text{clearly}} f(x)$.

Intuition then make kernel K_m get closer to K_δ.



Finish the proof of theorem. (Fejér theorem).

(ie. $F_m(x) \longrightarrow f(x)$ uniformly as $m \longrightarrow \infty$)

Consider $x \in [-\pi, \pi]$. Use property (ii) of K_m.

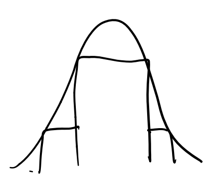
So $$\int_{x-\pi}^{x+\pi} K_m(x-y)\,dy = 2\pi \qquad \left(\int_{-\pi}^{\pi} K_m(t)\,dt = 2\pi \right)$$

$$\Rightarrow f(x) = \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(x)\, K_m(x-y)\,dy$$

Notice that $$F_m(x) = \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(y)\, K_m(x-y)\,dy$$

not precise but can we bound the error

Then $$\left| f(x) - F_m(x) \right| = \left| \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} (f(x) - f(y))\, K_m(x-y)\,dy \right|$$



Split into 3 parts.

$$\leq \frac{1}{2\pi}\Bigg(\underbrace{\int_{x-m}^{x-\delta}}_{(b)} + \underbrace{\int_{x-\delta}^{x+\delta}}_{(a)} + \underbrace{\int_{x+\delta}^{x+m}}_{(b)}\Bigg) |f(x)-f(y)|\, K_m(x-y)\, dy$$

(b) Use property 3 of K_m: Also f is continuous on $[-\pi, \pi]$.
f is bounded by some M. $\big(|f(x)| \leq M\big)$.

Thus (b) becomes $\leq \left(\int_{-\delta}^{-\pi} + \int_{\delta}^{\pi}\right) 2M\, K_m(t)\, dt$

$$\leq \varepsilon \text{ if } m \geq m_0.$$

(a) Need uniform continuity of f on $[-\pi, \pi]$.
(cont. function on compact interval $\Rightarrow$ uniformly cont.).

This means $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.
$$|f(x) - f(y)| < \varepsilon \text{ if } |x-y| < \delta$$

$$(a) \leq \frac{1}{2\pi}\int_{x-\delta}^{x+\delta} \varepsilon\, K_m(x-y)\, dy \leq \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} K_m(t)\, dt$$
$$\leq \varepsilon.$$

Then $|f(x) - F_m(x)| \leq 2\varepsilon$ if $m \geq m_0$.

$\square$

We showed $F_m(x) \to f(x)$ uniformly. (Arithmetic mean converges to f).

we've only showed pointwise convergence. but need to show $F_m(x) \longrightarrow f(x)$ in $L^2(-\pi, \pi)$ to show completeness. f continuous.

Then
$$\|F_m - f\|_{L^2} = \left(\int_{-\pi}^{\pi} |F_m(x) - f(x)|^2 \, dx \right)^{\frac{1}{2}}$$

$$\leq \left(\int_{-\pi}^{\pi} (2\varepsilon)^2 \, dx \right)^{\frac{1}{2}} \text{ if } m \geq m_0$$

$$\leq 2\varepsilon (2\pi)^{\frac{1}{2}} \quad \text{if } m \geq m_0$$

This shows $\{e_n\}$ is complete orthonormal basis.