Midterm medium 35/50

P1 : 15, P2: 15, P3: 10, P4: 10

Last time: if E Banach, $\underbrace{\mathcal{L}(E)}_{\text{set of bounded linear operators from } E\to E} = \mathcal{L}(E,E)$ Banach and $\underbrace{\|AB\|}_{\text{composition}} \le \|A\|\|B\|$, $A,B \in \mathcal{L}(E)$.

$\mathcal{L}(E)$ is a Banach Algebra. (Not really going into this).

Ex def $e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$ b/c $A^n \in \mathcal{L}(E)$ if $A \in \mathcal{L}(E)$.

III Inverse of operator

- Def: let E, F be normed vector space. An operator $A \in \mathcal{L}(E,F)$ is invertible if $\exists B \in \mathcal{L}(F,E)$ s.t. $A B = \underbrace{I_F}_{\text{identity "on" } F}$ and $BA = \underbrace{I_E}_{\text{identity "on" } E}$.

Note: B must be bounded / continuous, and must be left & right inverse to A.

- If such B exists, then B is unique. It is the inverse of A, denoted A^{-1}.

Remark If $A: E\to F$ is one-to-one and onto i.e. bijective. Then there exists an inverse in the set-theoretic sense.

The question is boundedness.

Thm (Inverse Mapping Theorem).

If E,F Banach space, $A \in \mathcal{L}(E,F)$ is linear, one-to-one and onto, then A^{-1} inverse and is bounded linear operator.

Will not prove in this course.

Ex: Consider shift operator: S on $\ell^2(\mathbb{R})$.

$$S(x) = S(x_1, x_2, \dots), \quad x \in \ell^2(\mathbb{R}).$$

$$= (0, x_1, x_2, \dots) \longleftarrow \text{shifted over } 1$$

This is linear, $S \in \mathcal{L}(\ell^2)$. In fact, $\|Sx\|_{\ell^2} = \|x\|_{\ell^2} < \infty$ by membership in ℓ^2.

This is not "onto". So S^{-1} does not exist.

Question what is the set of invertible operators from $\mathcal{L}(E,F)$?
(Since not every operator is invertible).

Thm let E Banach space. Take $A \in \mathcal{L}(E)$. If $\|A\| < 1$, then $I-A$ is invertible.
And $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$ in $\mathcal{L}(E)$.

when $n=0$, $A^0 = I_E$. Convergence is in $\mathcal{L}(E)$. $(I-A)^{-1} = \lim_{N \to \infty} \sum_{n=0}^{N} A^n$.

Intuition think $1-z$. Then $(1-z)^{-1} = \frac{1}{1-z}$ is well defined if $|z|<1$.
Also $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ so that's where the series comes out.
(replace z w/ A).

Proof: WTS $\left(\sum_{n=0}^{\infty} A^n\right)(I-A)^{-1} = I$. First need to show $\sum_{n=0}^{\infty} A^n$ well defined.

Consider partial sum, $S_N = \sum_{n=1}^{N} A^n$. show $(S_N)_{N=1}^{\infty}$ cauchy

$$\|S_N - S_M\|_E = \underbrace{\left\|\sum_{n=N+1}^{M} A^n\right\| \leq \sum_{n=N+1}^{M} \|A\|^n}_{\text{prop of } \mathcal{L}(E)} = \frac{\|A\|^{N+1}}{1-\|A\|} \quad (\text{assume } M>N)$$

$\uparrow < \varepsilon$ for N large as $\|A\|<1$.

So S_N cauchy. $\exists$ limit $\sum_{n=0}^{\infty} A^n \in \mathcal{L}(E)$ by completeness.

To show this is the inverse.

$$(I-A)\circ\sum_{n=1}^{\infty}A^n = \sum_{n=1}^{\infty}A^n - \sum_{n=1}^{\infty}A^{n+1} = I$$

Similarly, $\sum_{n=0}^{\infty}A^n\circ(I-A) = I$.

$\square$

<u>Corollary</u>: If E Banach, then the set of invertible operators on E is open in $\mathcal{L}(E)$.

(i.e. If E invertible, nearby operators are invertible).

<u>Proof</u>: Suppose $A\in\mathcal{L}(E)$ invertible. Consider $B\in\mathcal{L}(E)$, and $\|B-A\|<\frac{1}{\|A^{-1}\|}$.

We claim B invertible.

First, consider $\|(B-A)(A^{-1})\|\le\|B-A\|\,\|A^{-1}\|<1$.

Then, $I+(B-A)(A^{-1}) = I+BA^{-1}-I = BA^{-1}$ is invertible (by previous result).

Then $B = \underbrace{(BA^{-1})}_{\text{invertible}}\underbrace{A}_{\text{invertible}}$ is invertible.

<u>Ex</u>: let K be an integral operator on $L^2(0,1)$.

$$(Kf)(t) = \int_0^1 k(t,s)\,f(s)\,ds, \quad f\in L^2(0,1).$$

and $k(t,s)$ is continuous on $[0,1]\times[0,1]$.

We showed (last lecture) that

$$\left.\|K\|\le\int_0^1\int_0^1|k(t,s)|^2\,ds\,dt<1.\right\}\text{ assumption}$$

Then we want to solve

$$f-kf=g \qquad \Big(\Rightarrow (I-k)f=y \Rightarrow \overbrace{Af=y}^{\text{classic problem we want to solve}}\Big)$$

Then $\underbrace{f(t)-\int_0^1 k(t,s)f(s)\,ds = g(t)}_{\text{not easy to solve}}$

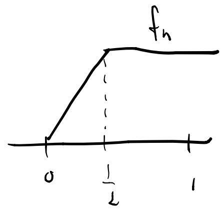
But we can solve this as:

$f(t) = (I - K)^{-1} g = \sum_{n=0}^{\infty} K^n g$ and convergence is guaranteed.

MIDTERM

P1: need to show its not complete. want to find limit that is not continuous.

Find (f_n) cauchy in $\underbrace{L^2(0,1)}_{\text{that's the norm we used}}$ and $f_n \to f \in L^2(0,1)$ but $f \notin C([0,1])$.



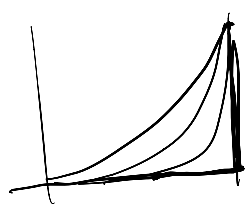
f_n continuous.

$\|f_n - f_m\|_{L^2}^2 = \int_0^1 |f_n - f_m| \, dx$ is small.

Show $f = \begin{cases} 0 & x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}$

$\|f_n - f\|_{L^2}^2 \to 0$ as $n \to \infty$. limit should be $f \in L^2([0,1])$

But no continuous function $h = f$ a.e.



doesn't work b/c only just 1 point.

in measure 0.

P2: Use fact that unit ball in ∞-dim will not be compact.

P3 M is closed in H. $H = M \oplus M^\perp$.

$x_0 \in M^\perp$, $x = y_0 + \alpha x_0$.) need to prove what

for any $z \in M^\perp$, $z = \alpha x_0$

P4: H separable.

① $H = \overline{\operatorname{span}\{e_n\}}$

② $(v, e_n) = 0 \Rightarrow v = 0$

③ $\|v^2\| = \sum |(v, e_n)|^2$

Take $(v, e_n) = 0$, show $v = 0$

$\|v\|^2 = \sum |(v, en)|^2 = \sum |(v, e_n - f_n)|^2$

$\leq \sum \|v\|^2 \|e_n - f_n\|^2 < \|v\|^2$