Last time:

<u>Thm</u> let H be separable hilbert space. let $K \in \mathcal{L}(H)$ be compact. Then $\forall \varepsilon > 0$,

$\exists$ finite rank operator K_ε s.t. $\|K - K_\varepsilon\| < \varepsilon$.

<u>Proof</u>: $K_\varepsilon(x) = \overbrace{\sum_{n=1}^{N} (Ke_n)(x, e_n)}^{\text{approximator operator}}$. Need to show

$$\mu_N = \sup\{\|Kx\| : x \in \{e_1, \dots, e_n\}^\perp, \|x\| = 1\} \to 0.$$

Prove by contradiction. μ_N bounded so $\mu_N \to \mu \geq 0$. We assume $\mu > 0$.

we found $(x_n)_{n=1}^\infty$, $\|x_n\| = 1$, $x_n \perp \operatorname{span}\{e_1, \dots, e_N\}$ and $\|Kx_n\| \geq \frac{\mu}{2}$.

We find a subsequence $Kx_n^{(1)}$ that converges to y. WTS $y = 0$.

$$(*) \quad \|Kx_n^{(1)} - y\|^2 = \|Kx_n^{(1)}\|^2 + \|y\|^2 - 2\operatorname{Re}(Kx_n^{(1)}, y).$$

We estimate $|(Kx_n^{(1)}, y)| = |(x_n^{(1)}, K^* y)|$

B/c $(x_n^{(1)})$ is a subseq of (x_n), it is also orthogonal to $\{e_1, \dots e_n\}$.

so $|(x_n^{(1)}, K^* y)| \leq \|x_n^{(1)}\| \cdot \|P_n K^* y\|$

where $P_n K^* y = \sum_{j=n+1}^{\infty} (e_n, K^* y) e_n$ (1^{st} n terms are 0 b/c)

Then $\|P_n K^* y\| = \sum_{j=n+1}^{\infty} |(e_n, K^* y)| \longrightarrow 0$ as $n \to \infty$

(Parseval's identity for Hilbert space).

Take $n \to \infty$ in $(*)$, $\|y\| = 0$. $\downarrow$

So compact operator are limits of finite rank operators in sep. Hilbert. space.

<u>Thm</u> Fredholm Alternative. let K be a compact operator on a separable Hilbert space. Then either $I - K$ is invertible or $Kf = f$ has a solution $f \neq 0$, $f \in H$.

Remarks: ① If $H = \mathbb{C}^n$. Then K (matrix) either $I-K$ invertible or $Kx = x$, $x \neq 0$.

② Fredholm is not true for any operator. Has to be compact.

Ex: Consider $Ax(t) = tx(t)$ on $L^2(0,2)$. Then $Ax = x$ has no solution but $(I-A)^{-1}$ not bounded. (exercise).

③ In terms of solving $f - Kf = g$. If for any $g \in H$, there is at most 1 solution, then there is always a solution.

(compactness + uniqueness) $\Rightarrow$ existence.

Proof: Pick a finite rank operator F so that $\|K-F\| < 1$. (for any ε we can find F by approx. thm). Then $I-(K-F)$ invertible b/c norm <1.

$$I-(K-F) = I-K+F.$$

Then $I-K = (I-T)(I-K+F)$. here $T = F(I-K+F)^{-1}$.

F is finite rank $\Rightarrow$ T is finite rank. We can find orthogonal vectors $e_1, \dots, e_N$ so that $F(x) = \sum_{n=1}^{N} \alpha_n(x) e_n$. — we can always do this on finite rank.

Then $\alpha_n(x)$ are bounded linear functionals on H. By Riesz Representation, find

$$\varphi_n \in H \quad \text{s.t.} \quad \alpha_n(x) = (\varphi_n, x).$$

So $F(x) = \sum_{n=1}^{N} (\varphi_n, x) e_n$.

$\boxed{\text{Intuition: } I-K+F \text{ is invertible. If } I-K \text{ invertible} \Leftrightarrow I-T \text{ invertible. } T \text{ is finite rank! so reduced to easier problem.}}$

Then $Tx = \sum_{n=1}^{N} (\psi_n, x) e_n$ where $\psi_n = ((I-K+F)^{-1})^* e_n$. ②

$I-K$ is invertible iff $I-T$ invertible. And $f = Kf$ has a solution iff $g = Tg$ has a solution.

If $g = Tg$, g is in the range of T, so we can write it in the form of ②.

let $g = \sum_{n=1}^{N} \beta_n e_n$ and β_n should satisfy (plug into ②).

$$\beta_n = \sum_{m=1}^{N} (\psi_n, e_m) \beta_m$$

let $A = (A_{nm})$, $A_{nm} = (\psi_n, e_m)$. So $g = Tg$ has a non zero solution if
matrix

$\det(I-A) = 0$.

On the other hand, if $\det(I-A) \neq 0$, we can solve

$$(I-T)g = h \quad \text{for any } h.$$

we set $g = h + \sum_{n=1}^{N} \beta_n \psi_n$ where $\beta_n = (\psi_n, h) + \sum_{m=1}^{N} (\psi_n, e_m)\beta_m$

(you can verify this).

So $I-T$ invertible.

□

Ex: let K be integral operator: $K: L^2(a,b) \to L^2(a,b)$ with kernel $k(s,t)$ continuous. We will see next k compact.

Then integral eqn,

$$f - Kf = g, \quad \forall g \in L^2$$

has solution if $Kf = f$ has only 0-solution.

For example $kf = \int_0^1 f(x)\,dx$

II) Hilbert - Schmidt Operators

This is a special class of compact operators.

~ Def: Let E, F Hilbert spaces. A bounded linear operator $A : E \to F$ is Hilbert-Schmidt if there is a complete orthonormal sequence $(e_n)_{n=1}^{\infty}$ in E such that

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty .$$

separable

Remark: ① we can define trace for operator A

$$\operatorname{tr} A = \sum_{n=1}^{\infty} (e_n, Ae_n)$$ if finite. Then A is Hilbert-Schmidt if $\operatorname{tr} A^*A$ finite.

$$(e_n, A^*Ae_n) = (Ae_n, Ae_n) = \|Ae_n\|^2 .$$

② The choice of $(e_n)_{n=1}^{\infty}$ does not matter.

if $(f_n)_{n=1}^{\infty}$ complete orthonormal sequence, then

$$\sum_{n=1}^{\infty} \|Af_n\|^2 < \infty \iff \sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty$$

Thm Hilbert Schmidt operators are compact.

Proof: Next time.

Coming next: Spectral Theorem.