- <u>Spectral Theorem</u> Let K be compact-Hermitian on Hilbert space X.

$$Kx = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n \qquad \forall x \in X$$

(e_n) are eigenfunctions.

<u>Intuition</u>: find (e_1, λ_1). Consider $X \setminus \operatorname{Span}\{e_1\}$, which is also Hilbert. Recurse.

Start with some results about K:

<u>Thm</u> Let A be Hermitian on Hilbert space X. Then eigenvalues are real and eigenfunctions corresponding to distinct eigenvalues are orthogonal.

<u>Proof</u>: We know $A = A^*$. If $Ax = \lambda x$,

$$(Ax, x) = (x, Ax) \Rightarrow (\lambda x, x) = (x, \lambda x) \Rightarrow \lambda(x,x) = \bar{\lambda}(x,x)$$

$\Rightarrow \lambda = \bar{\lambda}$. λ is real.

Let $\lambda \neq \mu$ be eigenvalues:

$$Ax = \lambda x, \quad Ay = \mu y.$$

Then $\lambda(x,y) = (\lambda x, y) = (Ax, y) = (x, Ay) = (x, \mu y) = \mu (x,y)$

$\Rightarrow (\lambda - \mu)(x,y) = 0$ but $\lambda \neq \mu$. Then $(x,y) = 0$ (orthogonal).

- <u>Thm</u>: Let A be a <u>compact</u> Hermitian operator on Hilbert space X. Then either $\|A\|$ or $-\|A\|$ is an eigenvalue of A.

<u>Proof</u>: WLOG, assume $A \neq 0$. Recall that

$$\|A\| = \sup_{\|x\|=1} |(Ax, x)|$$

took out abs value

Then there is $(x_n)_{n=1}^{\infty} \subset X$, $\|x_n\| = 1$ such that $(Ax_n, x_n) \longrightarrow \lambda$. Then λ is either $\|A\|$ or $-\|A\|$.

Show $Ax_n - \lambda x_n \to 0$. Notice that $\|Ax_n - \lambda x_n\|^2 = \|Ax_n\|^2 - 2\lambda(Ax_n, x_n) + \lambda^2 \|x_n\|^2$.

$$\leq 2\lambda^2 - 2\lambda(Ax_n, x_n).$$

As $n \to \infty$, $Ax_n - \lambda x_n \to 0$.

Since A is compact, find $(y_n) \subset (x_n)$ subseq. such that $Ay_n \xrightarrow{n\to\infty} y \in X$. (we can't use (x_n) b/c we don't know it converges in X)

Use $Ay_n - \lambda y_n \to 0$. So $\lambda y_n \to y$ as $n \to \infty$.

So we have $Ay = \lambda y$. If only compact, not hermitian, then eigenvalue may not exist.

WTS $y \neq 0$. This is b/c, $\|y\| = \lim_{n\to\infty} \|\lambda y_n\| = |\lambda|$ b/c $\|x_n\| = 1$ and $\lambda \neq 0$ since $|\lambda| = \|A\| \neq 0$ by assumption.

$\square$

- Thm: Let M be closed linear subspace of of Hilbert space X, let $A \in \mathcal{L}(X)$ be Hermitian. If $AM \subseteq M$, then $A(M^\perp) \subseteq M^\perp$. ($M$ is invariant under A).

Proof: Take $y \in A(M^\perp)$. There is a $z \in M^\perp$ s.t. $y = Az$. For any $x \in M$, consider $(x, y) = (x, Az) = (Ax, z)$. $Ax \in M$ but $z \in M^\perp$. $(Ax, z) = 0$.

Proof of the spectral Theorem

Proof by Induction:

First, K is compact Hermitian on Hilbert space X. We know that $\lambda_1 = \|K\|$ or $-\|K\|$. λ_1 is an eigenvalue. Let e_1 be an eigen vector and $\|e_1\| = 1$.

Then consider $M_1 = \operatorname{span}\{e_1\}$. It is obvious that $KM_1 \subseteq M_1$. Then by Prev. Thm, $K(M_1^\perp) \subseteq M_1^\perp$. Then, define K_2 to be the restriction of K to $M_1^\perp$. Call $X_2 = M_1^\perp$. X_2 is hilbert space. WTS K_2 is compact Hermitian on X_2.

The compactness of K_2 follows from K. The self-adjointness follows from:

$$\text{if } x, y \in X_2, \quad (K_2 x, y) = \underbrace{(Kx, y) = (x, Ky)}_{K \text{ is self-adj}} = (x, K_2 y) \quad \text{since } y \in X_2.$$

Then $K_2 = K_2^*$. We can find $\lambda_2 = \pm\|K_2\|$ and eigen function $e_2 \in X_2$ with $\|e_2\| = 1$. Notice that $e_1 \perp e_2$ b/c $e_1 \in M_1$, $e_2 \in M_2$.

Note e_2 is an eigen funct for K (since K_2 is a restriction). Continue this argument.

we get $e_1, \dots e_n$ eigenvectors of K corresponding to $\lambda_1, \dots, \lambda_n$, and $\lambda_j = \pm \|K_j\|$.

This will stop if for some n, $K_n = 0$. Then consider,

$$y = x - \sum_{i=1}^{n} (x, e_i) e_i \in X_n$$

make y is not 0 by $K_n y$ is 0.

Then $Ky = K_n y = 0$ $\left(b/c \; K_n = 0 \right)$. So we get

$$Kx = \sum_{i=1}^{n} (x, e_i) K e_i = \sum_{i=1}^{n} \lambda_i (x, e_i) e_i$$

we've shown the spectral thm for finite cases. The infinite case is to follow.