Final next wednesday. Ch 1 → 11 (Review on Friday).

III) Property of operator K ($\approx L^{-1}$)

Recall we found Integral operator

$$Kf(t) = \int_a^b k(t,s)\, f(s)\, ds$$

where $k(t,s)$ is continuous and symmetric. And if $f \in C^0[a,b]$, then $Kf(t)$ is $C^2[a,b]$ solution of $Lu = f$ with boundary conditions.

We know $K: L^2(a,b) \to L^2(a,b)$ is compact Hermitian. By spectral theorem, there is a complete orthonormal sequence $(e_j)_{j=1}^{\infty}$ in $L^2(a,b)$ which are eigenfunctions of K with eigenvalue λ_j (we know $\lambda_j \xrightarrow{j\to\infty} 0$). In particular, $K e_j = \lambda_j e_j$.

<u>How do translate back to L?</u>

Intuition: if $e_j \in C^0([a,b]) \Rightarrow Ke_j \in C^2([a,b]) \Rightarrow e_j \in C^2([a,b])$

$$\Rightarrow L(Ke_j) = e_j \Rightarrow \lambda_j L(e_j) = e_j \Rightarrow L(e_j) = \frac{1}{\lambda_j} e_j$$

First, we show:

<u>Lemma</u>: $K: L^2(a,b) \to C^0(a,b)$

<u>Proof</u>: For $\varepsilon > 0$. There is $\delta > 0$ s.t.

$$|k(t_1,s) - k(t_2,s)| < \varepsilon \quad \text{if} \quad |t_1 - t_2| < \delta \text{ and } s \in [a,b].$$

Now, for $f \in L^2(a,b)$, $\left| Kf(t_1) - Kf(t_2) \right| = \left| \int_a^b \left(k(t_1,s) - k(t_2,s)\right) f(s)\,ds \right|$

$$\leq \int_a^b \varepsilon |f(s)|\,ds \leq \varepsilon \left(\int_a^b 1^2\,ds \right)^{\frac{1}{2}} \left(\int_a^b |f(s)|^2\,ds \right)^{\frac{1}{2}} \quad (CS)$$

$$= \varepsilon \sqrt{b-a}\, \|f\|_{L^2}$$

So $Kf(t)$ is continuous.

$\square$

<u>Remark</u>: If (e_j) are eigen functions in $L^2([a,b])$, then $e_j \in C^0([a,b])$.

(assuming $Ke_i = \lambda_j e_j$; $\lambda_j \neq 0$)

Now we found $e_j \in C^2([a,b])$ s.t.

$$\begin{cases} L(e_j) = \frac{1}{\lambda_j} e_j \\ \alpha e_j(a) + \alpha' e_j'(a) = 0 \\ \beta e_j(b) + \beta' e_j'(b) = 0 \end{cases}$$

To complete the proof, WTS

ⓐ 0 is not an eigen value for K (or $\operatorname{Ker} K = 0$)

ⓑ We derived green's function under assumption that 0 is not eigenvalue of Sturm-Liouville problem. We need to prove this.

ⓐ <u>Thm</u> if 0 is not eigenvalue of Sturm-Liouville $\Rightarrow \operatorname{Ker} K = 0$.

<u>Proof</u>: Recall that $\forall g \in L^2(a,b)$.

$$Kg(t) = \int_a^b k(t,s)\, g(s)\,ds = u(t)\,\varphi(t) + v(t)\,\psi(t)$$

where $u(t), v(t)$ are solutions of $Lu = 0$ and

$$\varphi(t) = \frac{1}{W}\int_t^b v(s)\, g(s)\, ds$$

$$W = p(u'v - v'u) = \text{Wronskian}$$

$$\psi(t) = \frac{1}{W}\int_a^t u(s)\, g(s)\, ds$$

Need to show if $Kg = 0$, then $g = 0$. If $g \in C^0([a,b])$, then $Kg \in C^2([a,b])$. Then

$$(\bullet) \qquad Kg = u\varphi + v\psi$$

$$(\star) \qquad (Kg)' = u'\varphi + v'\psi \quad \text{(other 2 are assumed to be 0 by construction)}$$

If $Kg = 0$, then $\begin{pmatrix} u & v \\ u' & v' \end{pmatrix}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

But Wronskian not 0 so $\begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \neq 0 \Rightarrow \varphi, \psi = 0$.

$$\Rightarrow \int_t^b v(s)\, g(s)\, ds = 0, \quad \int_a^t u(s)\, g(s)\, ds = 0$$

$$\Rightarrow v(t)\, g(t) = 0, \quad u(t)\, g(t) = 0$$

$\Rightarrow g(t) = 0$ b/c $v(t)$ and $u(t)$ cannot simultaneously be 0 as W is not 0.

What if $g \in C^0([a,b])$ (all we know $g \in L^2([a,b])$), then we know $Kg \in C^0([a,b])$. We need $Kg \in C^1$.

Choose $g_k \in C^0([a,b])$ and $\|g_k - g\|_{L^2} \to 0$ as $k \to \infty$.

Then WTS $Kg_k \to Kg$. (nothing we can do except approx.)

and $(Kg_k)' \to (Kg)'$. b/c (★)

Then we consider

$$|\varphi_k(t) - \varphi(t)| = \frac{1}{|W|} \int_a^b v(s)\left(g_k(s) - g(s)\right) ds$$

relax bounds to a to b

$$\leq \frac{1}{|W|} \int_a^b |v(s)| \cdot |g_k(s) - g(s)|\, ds$$

$v(s) \in C^2$ so must have a max M

$$\leq \frac{1}{|W|} M \sqrt{b-a}\, \|g_k - g\|_{L^2}$$

we are given $g_k \to g$ so $\varphi_k \to \varphi$ uniformly.

Same argument for ψ_k and ψ. Also

$Kg_k(t) \to Kg(t)$ uniformly. (use (•)).

Next, consider $(Kg_k)' = u'(t)\varphi_k(t) + v'(t)\psi_k(t)$.

$\to u'\varphi + v'\psi$ uniformly.

(but we don't know $(Kg)'$ exists).

We want to claim $u'\varphi + v'\psi = (Kg)'$. To see this, note

$$Kg_k(t) - Kg_k(a) = \int_a^t \left(u'(s)\,\varphi_k(s) + v'(s)\,\psi_k(s) \right) ds$$

As $k \to \infty$

$$Kg(t) - Kg(a) = \int_a^t u'(s)\,\varphi(s) + v'(s)\,\psi(s)\,ds$$

$$\Rightarrow \quad (Kg)' = u'\varphi + v'\psi.$$

By previous argument for $C^0([a,b])$ functions, we get

$$\varphi(t) = \int_a^t v(s)\,g(s)\,ds = 0, \quad \psi(t) = \int_t^b u(s)\,g(s)\,ds = 0.$$

but $g \in L^2$ not C_0.

Approx $v(s)\,g(s)$ by a function $f \in L^2(a,b)$.

$$\int_a^t f(s)\,ds = 0 \Rightarrow f = 0 \text{ a.e.}$$

For $f \in L^2$, we approx f by step functions

say $$f_k = \sum_{j=1}^{N_k} a_j^{(k)} \chi_{[x_{j-1}^{(k)},\, x_j^{(k)}]} \quad \text{so } \|f - f_k\|_{L^2} \to 0$$

Then $\|f - f_k\|^2 = \|f\|^2 + \|f_k\|^2 - 2(f, f_k)$. Now

$$(f, f_k) = \int_a^b f_k(s)\,f(s)\,ds = \sum_{j=1}^{N_k} a_j^{(k)} \int_{x_{j-1}^{(k)}}^{x_j^{(k)}} f(s)\,ds = 0$$

$$\Rightarrow \|f\|^2 + \|f_k\|^2 \Rightarrow 0 \text{ as } k \to \infty \Rightarrow \|f\| = 0. \Rightarrow f = 0 \text{ a.e.}$$

B/c f approx. $v(s)\,g(s)$, this shows that

$$vg = 0 \text{ a.e.}, \quad ug = 0 \text{ a.e.}$$

v & u are linearly $\perp$ so both can't be 0 $\Rightarrow$

$g = 0$ a.e $\Rightarrow$ $g = 0$ in L^2. Then $\ker K \neq 0$.

$\square$