Inductively, we constructed $e_1, \dots, e_n$ of K with eigenvalue $\lambda_1, \dots, \lambda_n$ such that $|\lambda_j| = \|K_j\|$, $1 \le j \le n$. This can be continued if $K_n \ne 0$

If $K_n = 0$, the construction stop and we get

$$y = x - \sum_{j=1}^{n-1} (x, e_j) e_j \in H_n$$

$$\Rightarrow 0 = K_n y = Ky = Kx - \sum_{j=1}^{n-1} (x, e_j) K e_j$$

$$\Rightarrow Kx = \sum_{j=1}^{n-1} \lambda_j (x, e_j) e_j.$$ We are done with finite sum

If $K_n \ne 0$ $\forall n$, Consider $x \in X$ and $y_n = x - \sum_{j=1}^{n-1} (x, e_j) e_j$.

Then $y_n \in X_n$, write $x = y_n + \sum_{j=1}^{n-1} (x, e_j) e_j$

$$\Rightarrow \|Kx - \sum_{j=1}^{n-1} \lambda_j (x, e_j) e_j\| = \|K y_n\| = \|K_n y_n\| \le \|K_n\| \|y_n\|$$

$\le |\lambda_n| \|y_n\|$. But $\|x\|^2 = \|y_n\|^2 + \sum_{j=1}^{n-1} |(x, e_j)|^2 \Rightarrow \|y_n\| \le \|x\|$

$\le |\lambda_n| \|x\|$ $\Rightarrow$ limit $\to 0$. as $\lambda_n \to 0$. ! (lemma)

So $Kx = \sum_{j=1}^{\infty} \lambda_j (x, e_j) e_j$.

(ii) We need to show the completeness when X is separable!

We know that $\forall x \in X$, $Kx = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n$.

We can assume that $\lambda_n \ne 0$. Here e_n is a finite or infinite orthonormal sequence. Each e_n is eigen function

So we observe that $x - \sum_{n=1}^{\infty} (x, e_n) e_n \in \ker K$.

Then let ~~ϕ~~ f_m be a complete orthonormal basis for $\ker K$. which is a Hilbert space separable. Here we have $X = \underbrace{\ker K}_{f_m} \oplus \underbrace{(\ker K)^{\perp}}_{e_n}$

~~let e_n and~~ Because f_m are eigen functions of 0, so $f_m \perp e_n$

Thus $\{f_m\} \cup \{e_n\}$ is a countable orthonormal set in X.

For any $x \in X$, $x - \sum_n (x, e_n) e_n = \sum (x, f_m) f_m$

~~so $x =$~~ $\{f_m\} \cup \{e_n\}$ is a complete orthonormal set in X. □

Remark: ① + ⑦

Ex: let K be a cpt Hermitian operator on a Hilbert space X. let K be ~~positive~~ positive operator namely $(Kx, x) \geq 0$ $\forall x \in X$. Then we can define a square root $A = K^{1/2}$ such that $A^2 = K$.

?

Proof: $Kx = \sum \lambda_n (x, e_n) e_n$, $\lambda_n \geq 0$

Define $Ax = \sum \sqrt{\lambda_n} (x, e_n) e_n$.

Ex: (Canonical form of cpt operators) let A be cpt operator on X. Then there is orthonormal set $\{e_n\}_{n=1}^{\infty}$, $\{f_n\}_{n=1}$ and positive real numbers λ_n with $\lambda_n \to 0$ so that

$$Ax = \sum_{n=1}^{\infty} \lambda_n (x, e_n) f_n.$$

Then λ_n are called singular values of A

?

Proof: Since A is cpt, so is A^*A. and A^*A is cpt, Hermitian

so there is orthonormal set $\{e_n\}_{n=1}^N$ such that $A^*A e_n = \mu_n e_n$, $\mu_n \neq 0$

and A^*A is zero on the sub space orthogonal to $\{e_n\}_1^N$. (N could be ∞)

Since A^*A is positive, each $\mu_n > 0$. Let $\lambda_n = \sqrt{\mu_n}$, let $f_n = A e_n / \lambda_n$

We can check f_n is orthonormal and

$$\cancel{Ax} \quad Ax = \sum \lambda_n (e_n, x) f_n$$

Chap ~~9~~ 9-11: Sturm-Liouville systems.

We apply the Hilbert space theories to ODEs. then we study PDE later.

1) Sturm-Liouville eigenvalue problem.

For $u \in C^2([a,b])$ (twice ~~continuous~~ continuous differentiable functions on $[a,b]$).

Consider a differential operator $Lu = (pu')' + qu = pu'' + p'u' + qu$

where $p \in C^1([a,b], \mathbb{R})$; $p > 0$ and $q \in C^0([a,b]; \mathbb{R})$.

Def: Sturm-Liouville eigenvalue problem:

$$\begin{cases} -Lu = \lambda u & \text{on } [a,b] \\ \alpha u(a) + \alpha' u'(a) = 0 \\ \beta u(b) + \beta' u'(b) = 0 \end{cases}$$

where, $\alpha, \alpha', \beta, \beta'$ are real constants and $\alpha = \alpha' = 0$, $\beta = \beta' = 0$ is excluded.

An eigenfunction to a scalar λ is a non-zero C^2 function u.

λ is called eigenvalue if such function exist.

Ex: ~~For~~ let $[a,b] = [0,\pi]$, ~~$q \equiv 0$, $p = 1$ on $[0,\pi]$~~. Consider

$$\begin{cases} -u'' = \lambda u & \text{on } [a,b] \\ u(0) = 0 \\ u(\pi) = 0 \end{cases}$$

Here $q \equiv 0$, $p \equiv 1$, $\alpha = 1$, $\alpha' = 0$, $\beta = 1$, $\beta' = 0$.

(so $p' = 0$)

Find all eigenvalue and eigen functions. ②

Sol: We search for all positive values

- $\lambda > 0 \Rightarrow$ general solution is $u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$.

(how do we know this?)

Use ~~initial~~ boundary condition, $u(0) = 0 \Rightarrow A = 0$, $u(\pi) = 0 \Rightarrow$ ~~$B\cos\sqrt{\lambda}$~~

$\sqrt{\lambda}B\cos\sqrt{\lambda}\pi = 0$. If $B = 0$, $u = 0$ trivial so $B \neq 0 \Rightarrow$

$\cos\sqrt{\lambda}\pi = 0 \Rightarrow \lambda = j^2, j = 1, 2, \ldots$ So corresponding eigenfunction is

(from prop of cosine)

$u_j = B\sin jx$.. (not unique!)

(why do we in this form?) ?

- $\lambda = 0 \Rightarrow u'' = 0 \Rightarrow u = Ax + B$. Use B.C. $\Rightarrow A = B = 0$.
- $\lambda < 0 \Rightarrow u'' - \lambda u = 0 \Rightarrow u = Ae^{+\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x} \Rightarrow A = B = 0$.

We find that eigenvalues $\underset{\text{index}}{\lambda_j} = j^2$, $j = 1, 2, \ldots$. eigenfunction

$u_j = \sin jx$.

Remark: ① Notice that u_j form a complete orthonormal basis for $L^2((0,\pi]; \mathbb{R})$.

② the eigenvalue $\lambda_j \to \infty$ as $j \to \infty$.

③ ~~why~~ Is λ_j the "eigenvalue" of the differential operator?

(Unbounded operator).

Q: How to find these for general Sturm-Liouville problem?

The idea is to look at the "inverse" L^{-1} of L. We show that L^{-1} is an integral operator (Green's function) It is cpt, self adjoint hence we can apply spectral theory to get eigenvalues of L^{-1}.

Finally, we connect them to L.

I) Green's function

walking through this might be useful

Consider the solution of the ODE problem

$$\begin{cases} Lw = g \\ \alpha w(a) + \alpha' w'(a) = 0 \\ \beta w(b) + \beta' w'(b) = 0 \end{cases} \qquad (*)$$

The inverse of L is that given g we find w. There is a standard way to ~~first~~ solve this ODE called variation of parameters.

Let u, v be two linearly independent solutions of the homogeneous equation (means $g(x)=0$)

$Lu = 0, \quad Lv = 0.$ The general solution is $w = au + bv$.

(is this something we know? or take this for granted)

Then we look for a solution of $(*)$ of the form

$$w = \varphi(x) u(x) + \psi(x) v(x).$$

where φ, ψ to be determined (tbd)

Take derivative $\Rightarrow$ $w' = \varphi u' + \psi v' + \varphi' u + \psi' v$

We choose $\varphi' u + \psi' v = 0$ and get $w' = \varphi u' + \psi v'$.

(Lw) (how do we get this?) ④

Then $(pw')' + qw = \cancel{\varphi' p w} \; \varphi' p u' + \gamma' p v' = g$

So $\begin{cases} p(\varphi' u' + \gamma' v') = g \\ \varphi' u + \gamma' v = 0 \end{cases}$
and

$A = \begin{pmatrix} pu' & pv' \\ u & v \end{pmatrix} \Rightarrow \det A = \underbrace{p(u'v - uv')}$. (This is called Wronskian). what this say?

Prop: Suppose that $u, v \in C^2([a,b])$ are not identically zero.

Suppose that $Lu = 0$, $Lv = 0$. Then either $u \equiv cv$ for some constant c or $p(u'v - uv') \neq 0$ for all $t \in [a,b]$.

Moreover $p(u'v - uv') = $ constant

~~Proof~~ (For this we need a existence result for ODEs).

Proof: We consider

$$\frac{d}{dt}(p(uv' - vu')) = \frac{d}{dt}((pv')u - (pu')v) = (pv')'u + pv'u' - pu'v' - (pu')'v = + quv - quv = 0$$

So the quantity is a constant.

Now we consider the problem at $t_0 \in [a,b]$. If $p(u'v - uv')(t_0) \neq 0$ $\Rightarrow$ nonvanishing for all t. If $p(u'v - uv')(t_0) = 0 \Rightarrow \vec{u} = \begin{pmatrix} u(t_0) \\ u'(t_0) \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v(t_0) \\ v'(t_0) \end{pmatrix}$ are linearly dependent $\Rightarrow \vec{u} = c\vec{v}$ for some constant c. Then

$Lu = 0$ satisfy $u(t_0) = cv(t_0)$, $u'(t_0) = cv'(t_0)$

$Lv = 0$ - - - - $v(t_0) = v(t_0)$ $\quad v'(t_0) = v'(t_0)$

$\Rightarrow w = u - cv$ satisfy $Lw = 0$, $w(t_0) = 0$, $w'(t_0) = 0 \Rightarrow w = 0$ by uniqueness

$\square$

Then we can solve for φ', ψ':

$$\varphi' = -\frac{vg}{c}, \quad \psi' = \frac{ug}{c}$$

So $\varphi(t) = \frac{1}{c}\left(\int_t^b v(z)g(z)\,dz + A\right)$

$\psi(t) = \frac{1}{c}\left(\int_a^t u(z)\,g(z)\,dz + B\right)$

Then the solution $w(t) = \varphi(t)\,u(t) + \psi(t)\,v(t)$

Finally, we use Boundary condition to find A and B. (Actually, they are zeros). If $A = B = 0$, we get

$$w(t) = \frac{1}{c}\int_t^b v(z)g(z)\,dz\; u(t) + \frac{1}{c}\int_a^t u(z)\,g(z)\,dz\; v(t)$$

We check BC:

• at $t=a$: $\quad w(a) = \frac{u(a)}{c}\int_a^b v(z)g(z)\,dz, \quad w'(a) = \frac{u'(a)}{c}\int_a^b g(z)\,v(z)\,dz$

Then $\alpha w(a) + \alpha' w'(a) = (\alpha u(a) + \alpha' u'(a))\frac{1}{c}\int_a^b g(z)v(z)\,dz$

$= 0$

if u satisfies the BC. at a.

• at $t=b$, $\quad w(b) = \frac{v(b)}{c}\int_a^b g(z)\,u(z)\,dz, \quad w'(b) = \frac{v'(b)}{c}\int_a^b g(z)u(z)\,dz$

$\Rightarrow \beta w(b) + \beta' w'(b) = (\beta v(b) + \beta' v'(b))\int_a^b g(z)u(z)\,dz$

$= 0$

if v satisfies the BC at b.

If these are satisfied, we get

$$w(t) = \frac{1}{C}\int_a^b g(s)\left(H(s-t)u(t)v(s) - H(t-s)u(s)v(t)\right)ds$$

$$= \int_a^b k(s,t)\, g(s)\, ds \qquad k(s,t) = \begin{cases} - \\ - \end{cases}$$

where $H(s)$ is the ~~Heaviside~~ Heaviside function $H(s) = \begin{cases} 0 & s<0 \\ 1 & s>0 \end{cases}$

Here $k(s,t)$ is the integral kernel. It's a symmetric function, real valued, continuous on $[a,b] \times [a,b]$.

Now we need to check that the desired u, v can be found.

Thm: (Existence and uniqueness of ODE)

Let $g: [a,b] \to \mathbb{R}$ be a given C^0 function and suppose that $u(t_0) = c_1$, $u'(t_0) = c_2$ at some $t_0 \in [a,b]$. Then there is a unique solution $C^2([a,b])$ ~~with~~ of $Lu = g$ with $u(t_0) = c_1$, $u'(t_0) = c_2$.

Using this result, we can find u with

$$\begin{cases} Lu = 0 \\ u(a) = c_1 \\ u'(a) = c_2 \end{cases}$$

such that $\alpha c_1 + \beta \alpha' c_2 = 0$! Similarly for v. This shows that the inverse operator is found.