Chapter 8: III) Hilbert-Schmidt operators

Let E, F Hilbert, $A \in \mathcal{L}(E,F)$ is Hilbert Schmidt if there is a complete orthonormal sequence $(e_n)_{n=1}^{\infty}$ of E s.t.

$$\sum_{n=1}^{\infty} \|Ae_n\|_F^2 < \infty$$

<u>Thm</u> Hilbert-Schmidt are compact.

<u>Proof</u>: If A is HS, then show A is the limit of a finite-rank operator.

For $k = 1, 2, \dots$ define

$$A_k : E \to F \quad \text{by} \quad A_k x = \sum_{n=1}^{k} (x, e_n) Ae_n .$$

Here $x \in E$, $x = \sum_{n=1}^{\infty} (x, e_n) e_n$. We see A_k are finite rank operators.

$A_k = A$ on $\text{span}\{e_1, \dots, e_k\}$. WTS $\|A_k - A\| \to 0$ as $k \to \infty$.

operator norm

let's consider for any $x \in E$, $Ax - A_k x \overset{?}{=} \sum_{n=k+1}^{\infty} (x, e_n) A e_n$ but we don't know $Ax = \sum_{n=1}^{\infty} (x, e_n) Ae_n$. Need to show this.

$$Ax - A_k x = A\left(\sum_{n=k+1}^{\infty} (x, e_n) e_n \right) \quad \text{by linearity}$$

$$= \lim_{M \to \infty} \left(\sum_{n=k+1}^{M} (x, e_n) Ae_n \right) \quad \text{(can move } A \text{ in if finite).}$$

But sequence is cauchy in F:

$$\left\| \sum_{n=k+1}^{M_1} (x, e_n) Ae_n - \sum_{n=k+1}^{M_2} (x, e_n) Ae_n \right\| \leq \left\| \sum_{n=M_1+1}^{M_2} (x, e_n) Ae_n \right\|$$

$$\leq \sum_{n=M_1+1}^{M_2} |(x, e_n)| \cdot \|Ae_n\| \leq_2 \left(\sum_{n=M_1+1}^{M_2} |(x, e_n)|^2 \right)^{\frac{1}{2}} \left(\sum_{n=M_1+1}^{M_2} \|Ae_n\|^2 \right)^{\frac{1}{2}}$$

$$\leq \underset{(\text{parseval})}{\|x\|_E^2} \left(\sum_{n=M_1+1}^{M_2} \|Ae_n\|^2 \right)^{\frac{1}{2}} < \infty$$

HS

so sequence is cauchy.

Now we check $\|(A - A_k)x\| \leq \sum_{n=k+1}^{\infty} |(x, e_n)| \, \|Ae_n\|$

$$\leq \left(\sum_{n=k+1}^{\infty} |(x, e_n)|^2 \right)^{\frac{1}{2}} \left(\sum_{n=k+1}^{\infty} \|Ae_n\|^2 \right)^{\frac{1}{2}} \leq \|x\| \left(\sum_{n=k+1}^{\infty} \|Ae_n\|^2 \right)^{\frac{1}{2}} \to 0 \text{ as } k \to \infty.$$

So $\|A - A_k\| \to 0$ as $k \to \infty$.

$\square$

So HS is compact as it is the limit of a finite rank operator.

<u>Thm.</u> Integral operators are compact:

Let $k : (c,d) \times (a,b) \to \mathbb{C}$ be continuous. and

$$\int_c^d \int_a^b |k(t,s)|^2 \, ds \, dt < \infty, \text{ then}$$

$K : L^2(a,b) \longrightarrow L^2(c,d)$ with kernel $k(t,s)$ is Hilbert - Schmidt and hence compact.

<u>Proof</u>: Pick $(e_n)_{n=1}^{\infty}$ a complete orthonormal sequence of $L^2(a,b)$. Then

$K e_n(t) = \int_a^b k(t,s) \, e_n(s) \, ds$. We need to show

$$\sum_{n=1}^{\infty} \|Ke_n\|^2 < \infty.$$

Let $k_t(s) = k(t,s)$, $a < s < b$. Notice that $k_t \in L^2(a,b)$.

(use Fubini's thm or assume $k(t,s)$ is continuous on $[c,d] \times [a,b]$)
bounded implies continuity

$$Ke_n(t) = \int_a^b k(t,s)\, e_n(s)\, ds = (k(t), \bar{e}_n).$$

Then $\|Ke_n\|^2 = \int_c^d |(k_t, \bar{e}_n)|^2 \, dt$

norm includes integral b/c L^2.

$$\Rightarrow \sum_{n=1}^{\infty} \|Ke_n\|^2 = \sum_{n=1}^{\infty} \int_c^d |(k_t, \bar{e}_n)|^2 \, dt$$

$$\Rightarrow \text{MCT} \Rightarrow \int_c^d \sum_{n=1}^{\infty} |(k_t, \bar{e}_n)|^2 \, dt$$

if (e_n) is complete orthonormal, so is $(\bar{e}_n)$.

$$= \int_c^d \|k_t\|^2 \, dt \qquad \text{(parsevals)}$$

$$= \int_c^d \int_a^b |k(t,s)|^2 \, ds \, dt < \infty \quad \text{by assumption}$$

Therefore, it is Hilbert-Schmidt.

$\square$

doesn't have to be integral operator

Remark: Consider $L^2(a,b)$. If K is HS on $L^2(a,b)$, then there is a function $k \in L^2((a,b) \times (a,b))$ such that

$$Kx(t) = \int_a^b k(t,s)\, x(s)\, ds$$

but we can write it as an integral operator.

Ref: Reed-Simon Vol 1.

IV) Spectral Theorem for compact Hermitian operator.

- Recall that for $A \in \mathcal{L}(H)$, H hilbert. We defined the spectrum $\sigma(A) \subset \mathbb{C}$ consisting of $\lambda \in \mathbb{C}$ s.t. $\lambda I - A$ is not invertible. We say that λ is an eigenvalue of A if $\lambda I - A$ is not injective $\Rightarrow$ there is $x \in H$, $x \neq 0$ s.t. $Ax = \lambda x$. x is called eigenvector.

Ex let $H = \mathbb{C}^n$. $A \in \mathcal{L}(H)$ can be identified with a matrix $(n \times n)$. Then A has n eigenvalues λ_n. If $A = A^*$ hermitian, then we have a diagonalization: we can find a Unitary matrix P s.t.

$$P^{-1} A P = \operatorname{diag}(\lambda_1, \dots \lambda_n).$$

$(P^* P = I)$.

If $e_i = (0, 0, 0 \dots \overset{i\text{th}}{1}, \dots 0)$. let $\tilde{e}_i = P e_i$ i.e. another basis. For $x \in \mathbb{C}^n$, $x = \sum_{i=1}^{n} (x, \tilde{e}_i) \tilde{e}_i$. Then $Ax = \sum_{i=1}^{n} \lambda_i (x, \tilde{e}_i) \tilde{e}_i$.

Another way to look at this problem is that we let

$$U : \mathbb{C}^{n'} \longrightarrow \mathbb{C}^n.$$

add dash to differentiate

maps $(\beta_i)_{i=1}^{n} \longrightarrow x = \sum_{i=1}^{n} \beta_i \tilde{e}_i$. Then $U^{-1} A U : \mathbb{C}^{n'} \longrightarrow \mathbb{C}^{n'}$ is multiplication of λ_i to each component.

How do we generalize this to compact Hermitian operators?

Thm (Spectral Theorem):

we do not know that X is separable

① Let K be compact Hermitian operator on Hilbert space X. Then there is a finite or infinite orthonormal sequence (e_n) of <u>eigenvectors</u> of K, with a corresponding eigenvalue λ_n s.t.

$$\forall x \in X, \quad Kx = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n$$

Here (λ_n) if infinite tends to 0 as $n \to \infty$.

② In addition if X separable, infinite-dim Hilbert space, then there is a <u>complete</u> orthogonal sequence as above s.t.

$$\forall x \in X, \quad \underbrace{x = \sum_{n=1}^{\infty} (x, e_n) e_n}_{\text{we could not say this in non-separable}}, \quad Kx = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n$$

Thus we can find $U : \ell^2 \to \mathbb{C}^n$ s.t. $U^{-1} K U$ is multiplication operator.

<u>Proof on Monday</u>.