Recall Adjoints

X, Y Hilbert spaces. $A \in \mathcal{L}(X,Y)$. The adjoint $A^* \in \mathcal{L}(Y,X)$ is the unique op. s.t.

$(Ax, y)_Y = (x, A^*y)_X, \ \forall x \in X, y \in Y.$

In the case $X = Y$,

def X hilbert space, we say $A \in \mathcal{L}(X)$ bounded linear operator is self-adjoint / hermitian if $A = A^*$ ie. $(Ax,y) = (x, Ay)$.

[You should think Hermitian matrices]

Thm X Hilbert space, A Hermitian. Then $\|A\| = \sup\limits_{\|x\|=1} \{(Ax, x)\}$.

[decode this a bit.

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \underbrace{\sup_{\|x\|=1}}_{\text{riecz repr.}} \Big\{ \underbrace{\sup_{\|y\|=1} |(Ax,y)|}_{\text{dual norm}} \Big\}$$

If A hermitian, we can take $x = y$.]

Remark: Hermitian-ness is necessary. Otherwise $X = \mathbb{C}^2$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, this thm does not hold.

Proof of thm: "$\geq$" obvious b/c $\|A\| = \underbrace{\sup_{\|x\|=1} \sup_{\|y\|=1} |(Ax,y)|}_{\text{smaller set}} \geq \sup_{\|x\|=1} |(Ax,x)|$

"$\leq$". Let $m := \sup\limits_{\|x\|=1} |(Ax,x)|$. We compute $\text{Re}\,(Ax,y) = \frac{1}{2}\left((Ax,y) + \overline{(Ax,y)} \right)$

$= \frac{1}{2}\left((Ax,y) + (y, Ax) \right) = \frac{1}{2}\left((Ax,y) + (Ay, x) \right)$

$= \frac{1}{4}\Big((Ax,x) + (Ax,y) + (Ay,x) + (Ay,y) - (Ax,x) + (Ax,y) + (Ay,x) - (Ay,y) \Big)$

$$= \frac{1}{4}\Big((A(x+y), x+y) - (A(x-y), x-y) \Big)$$

$$\leq \frac{1}{4}\left(m\,\|x+y\|^2 + m\,\|x-y\|^2 \right) = \frac{m}{4}\left(\|x+y\|^2 + \|x-y\|^2 \right)$$

$$= \frac{m}{4}\cdot 2\left(\|x\|^2 + \|y\|^2 \right) = \frac{m}{2}\left(\|x\|^2 + \|y\|^2 \right)$$

$$\sup_{\substack{\|x\|=1 \\ \|y\|=1}} \left\{ |(Ax,y)| \right\} = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} \left\{ \mathrm{Re}(Ax,y) \right\} \leq \sup_{\substack{\|x\|=1 \\ \|y\|=1}} \left\{ \frac{m}{2}\left(\|x\|^2 + \|y\|^2 \right) \right\} = m$$

$|(Ax,y)| \neq \mathrm{Re}(Ax,y)$
but sup are same

$$\therefore \sup_{\substack{\|x\|=1 \\ \|y\|=1}} \left\{ |(Ax,y)| \right\} \leq \sup_{\|x\|=1} |(Ax,x)|$$

$\square$

How do we generalize eigenvalues to ∞-dim space? ~~Spectrum~~.

<u>Spectrum</u> X Hilbert space, $A \in \mathcal{L}(X)$. Define the spectrum of A, denoted $\sigma(A)$, as:

$$\sigma(A) = \{ \underbrace{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is not invertible}}_{\text{set of numbers}} \}$$

Recall B invertible means $\exists C \in \mathcal{L}(X)$ s.t. $BC = CB = I$.
(must also be bounded operator)

<u>Inverse Mapping Theorem</u> B is invertible iff B is injective & surjective.
Not trivial that B must be bounded.

In other words, if $\lambda \in \sigma(A)$ iff $\lambda I - A$ is either <u>not</u> injective or not surjective.

Def If $\lambda I - A$ is not injective, (equivalently $\ker(\lambda I - A) \neq \{0\}$, equivalently $Ax = \lambda x$ for some $x \neq 0$). We call λ an eigenvalue.

Remark In finite dimensions,

$$\left.\begin{aligned} \lambda \in \sigma(A) &\iff \lambda I - A \text{ not invertible} \\ &\iff \det(\lambda I - A) = 0 \\ &\iff \lambda I - A \text{ is not injective} \\ &\iff \lambda I - A \text{ is not surjective} \end{aligned}\right) \text{ does not hold in } \infty \text{ dim}$$

Ex In ∞-dim, this is diff. Consider $X = \ell^2$. $S : \ell^2 \to \ell^2$ is the right shift. defined by

$$S(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$$

S is injective, not surjective.

In fact, $\sigma(S) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$ but S has no eigenvalues, meaning $\ker\{\lambda I - A\} = \{0\}$. Verify this is homework.

Thm X Hilbert, $A \in \mathcal{L}(X)$. Then $\sigma(A)$ is a closed subset of $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|A\|\}$.

Recall If $\|B\| < 1$, then $I - B$ is invertible. $(*)$

Proof of Thm:

1. $\sigma(A)$ is closed. We'll show $\mathbb{C} \setminus \sigma(A)$ open. In other words, for $\lambda \notin \sigma(A)$, we will show $\exists \delta > 0$ s.t. if $|\mu - \lambda| < \delta$ then $\mu \notin \sigma(A)$.

Consider
$$\begin{aligned} \mu I - A &= (\mu - \lambda) I + (\lambda I - A) \\ &= (\lambda I - A)\left[(\mu - \lambda)(\lambda I - A)^{-1} + I \right] \end{aligned}$$

Since $\lambda I - A$ invertible, by $(*)$, it suffices to ensure that $\|\underbrace{(\mu - \lambda)}_{\#}(\lambda I - A)^{-1}\| \leq 1$ but this is true if $|(\mu - \lambda)| < \dfrac{1}{\|(\lambda I - A)^{-1}\|} = \delta$.

So we've proven what we've wanted. ($\mu I - A$ invertible).

2. $\sigma(A) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|A\|\}$

Equivalent to show that if $|\lambda| > \|A\|$ then $\lambda I - A$ is invertible

(complement of $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|A\|\}$ is in complement of $\sigma(A)$)

Now take $|\lambda| > \|A\|$,

$\lambda I - A = \lambda(I - \lambda^{-1}A)$. Since $\|\lambda^{-1}A\| = |\lambda|^{-1}\|A\| < 1$, by $(*)$

$\lambda I - A$ is invertible.

$\square$

Compact operators

Def: X, Y Hilbert. We say a linear map $A: X \to Y$ compact if

$$A(\{x \in X \mid \|x\| \leq 1\})$$

is precompact in Y, meaning its closure is compact.

Remark: This is equivalent to saying for any bounded sequence $(x_n)_{n=1}^{\infty}$ in X, $(Ax_n)_{n=1}^{\infty}$ has a convergent sub-sequence.

Remark Compact operators are bounded.

[Pf otherwise if A is compact and not bounded

$\exists (x_n)_{n=1}^{\infty}$ bounded but $(Ax_n)_{n=1}^{\infty}$ unbounded s.t. $\|Ax_n\| \to \infty$.

But then not possible to have a convergent subsequence. $\square$]

Remark Not every bounded operator is compact.

(Consider $I: X \to X$. I is compact iff X is finite-dim)