Chap 8: Compact Operator

<u>Def</u> let E, F normed vector space. let $T: E \to F$ linear. T is compact if $T\{x \in E : \|x\| \le 1\}$ is relatively compact set of F.

<u>Ex)</u> I_E, E normed vector space.

If E finite dim, E compact, I_E identity so image is itself.

If E infinite dim, I_E not compact.

Ex) finite rank operator. The rank of an operator is defined to be the dimension of range. So $T \in \mathcal{L}(E,F)$ is finite rank if $\dim \operatorname{Ran}(T) < \infty$.

All finite rank operators are compact.

Let's consider $T\{x \in E : \|x\| \le 1\} \subset V = \operatorname{Ran}(T)$, which is finite dim vector space of F. WTS $W = T\{x \in E : \|x\| \le 1\}$ bounded.

$$W \subset \{y \in V : \|y\| \le \|T\| \cdot \|x\|\} = \{y \in V : \|y\| \le \|T\|\}$$

bounded, closed $\Rightarrow$ compact set.

"Finite rank operator are like <u>matrices</u>".

<u>Thm</u> let E, F Banach space. Take $(T_n)_{n=1}^{\infty}$, $T_n \in \mathcal{L}(E,F)$, T_n compact. Assume $\|T_n - T\| \xrightarrow{n \to \infty} 0$ (operator norm) with $T \in \mathcal{L}(E,F)$. Then T is compact. (compact op. closed in operator norm).

<u>Proof</u> The proof is by standard diagonal (3ε) argument.

let $(x_n)_{n=1}^{\infty}$, $x_n \in E$ bounded i.e. $\|x_n\| < 1$.

Show $(Tx_n)_{n=1}^{\infty}$ has a convergent subsequence.

Since T_1 compact, we can take $(T_1 x_n)_{n=1}^{\infty}$ and find a convergent

subsequence: $(T_1 x_{1,k}) \subset F$. Next consider $(x_{1,k})_{n=1}^{\infty} \subset (x_n)_{n=1}^{\infty}$ bounded.
Then as T_2 compact, $\exists$ convergent subseq $(T_2 x_{2,k})_{n=1}^{\infty} \subset F$. Continue this procedure: we get $(x_{q-1,k})_{k=1}^{\infty}$ s.t. $(T_q x_{q,k})_{k=1}^{\infty}$ converges in F.

<u>Diagonal argument</u>: Consider diagonal elements, $(x_{q,q})_{q=1}^{\infty} \subset (x_{q,k})_{k=1}^{\infty}$.
For this sequence, we know $(T_p x_{q,q})_{q=1}^{\infty}$, $\forall p$ will converge (think for large $q > p$).
We show $(T x_{q,q})_{q=1}^{\infty}$ converges.

<u>3ε argument</u>: For $\varepsilon > 0$,

$$\begin{aligned}\| T x_{q,q} - T x_{r,r} \| \leq \| T x_{q,q} &- T_p x_{q,q} + \\ T_p x_{q,q} &- T_p x_{r,r} + \\ T_p x_{r,r} &- T x_{r,r} \|\end{aligned}$$

$$\leq \| T x_{q,q} - T_p x_{q,q} \| + \| T_p x_{q,q} - T_p x_{r,r} \| + \| T_p x_{r,r} - T x_{r,r} \|$$

$$\leq \| T - T_p \| + \| T_p x_{q,q} - T_p x_{r,r} \| + \| T - T_r \|$$

For $\varepsilon > 0$, choose $p >> 0$ s.t. $\|T - T_p\| < \varepsilon$ (given $T_p \xrightarrow{\|\cdot\|} T$). Fix p.
For this p, choose $N > 0$ s.t. $\| T_p x_{q,q} - T_p x_{r,r} \| < \varepsilon$ for $q, r > N$.

$$\leq 3\varepsilon.$$

So $(T x_{q,q})_{q=1}^{\infty}$ Cauchy. By completeness (Banach), it converges.

$\square$

<u>Thm</u> <u>Approximation result for Compact operator.</u> Let H be a <u>separable</u> Hilbert space.
Let $K \in \mathcal{L}(H)$ be <u>compact</u>. $\forall \varepsilon > 0$, $\exists$ <u>finite</u> rank operator K_ε s.t. $\|K - K_\varepsilon\| < \varepsilon$.
(compact operator are like the limits of finite rank operator)

<u>Proof</u>: Let $(e_n)_{n=1}^{\infty}$ be complete orthonormal system.

<u>Intuition</u>: $x = \sum_{n=1}^{\infty} (x, e_n) e_n \Rightarrow Kx = \sum_{n=1}^{\infty} (x, e_n) K e_n \Rightarrow K_\varepsilon x = \sum_{n=1}^{N} (x, e_n) K e_n$
but remains to show this approx. is good.

Define $\mu_N = \sup\{\|Kx\| : x \in \{e_1, \dots, e_N\}^{\perp}, \|x\| = 1\}$. WTS $\mu_N \to 0$ $(*)$.

(show this later). Assuming this,

For any $\varepsilon > 0$, choose $N \in \mathbb{N}$ s.t. $\mu_N < \varepsilon$. Define

$$K_\varepsilon x = \sum_{n=1}^{N} (x, e_n) K e_n = K\left(\sum_{n=1}^{N} (x, e_n) e_n\right).$$

Consider $\|(K - K_\varepsilon)x\| = \|K P_N x\|$, $\underbrace{P_N}_{\text{remainder}} x = \sum_{n=N+1}^{\infty} (x, e_n) e_n$.

we see that $\|K P_N x\| < \mu_N \|x\|$ b/c P_N lives in μ_N.

$< \varepsilon \|x\|$

$\Rightarrow \|K - K_\varepsilon\| < \varepsilon$.

Let's prove claim: $(*)$. Notice that $(\mu_N)_{N=1}^{\infty}$ bounded, decreasing. $\to$ $N\uparrow$, means smaller space

we know $\lim_{N \to \infty} \mu_N = \mu \geq 0$. Prove by contradiction that $\mu = 0$. Assume

$\mu > 0$. For any n, $\exists x_n$ where $\|x_n\| = 1$ s.t. $x_n \perp \operatorname{span}\{e_1, \dots, e_n\}$ and

$\|K x_n\| \geq \frac{\mu}{2} > 0$. $(x_n)_{n=1}^{\infty}$ bounded so $\exists$ subsequence $(x_n^{(1)})$ s.t.

$(K x_n^{(1)})$ converges to $y \in H$. We note that