Thm (Sturm Liouville): There is an infinite sequence $(\lambda_j)_{j=1}^{\infty}$ of eigenvalues, λ_j real and $|\lambda_j| \xrightarrow{j\to\infty} \infty$. The corresponding $(e_j)_{j=1}^{\infty}$ form a complete orthormal basis for $L^2[a,b]$.

we proved this under assumption that $\underline{0 \text{ is not an eigenvalue}}$. We address this now.

related to Wronskian being not 0.

To remove this assumption, we'll show:

Thm: not every real number is an eigenvalue of the Sturm-Liouville problem.

proof: let $(e_n)_{n=1}^{\infty}$ be complete orthonormal basis for $L^2(a,b)$. Assume that for any $\lambda \in \mathbb{R}$ is an eigenvalue with eigenfunction f_λ. We claim for any $\lambda \neq \mu$, $f_\lambda \perp f_\mu$.

we prove later

Then let $\|f_\lambda\| = 1$. Consider

not countable

$$E_n = \{\lambda \in \mathbb{R} : (e_n, f_\lambda) \neq 0\}$$

we'd like to consider $\bigcup_{n=1}^{\infty} E_n$. Then we write $E_n = \bigcup_{m=1}^{\infty} \{\lambda \in \mathbb{R} : |(e_n, f_\lambda)| > \frac{1}{m}\}$.

Consider $\{\lambda \in \mathbb{R} : |(g, f_\lambda)| \geq 1\}$ is finite for any $g \in L^2$.

This is a consequence of Bessel's Inequality.

So E_n is countable b/c each one is countable.

So $\bigcup_{n=1}^{\infty} E_n$ is countable.

$\Rightarrow$ This is a proper subset of $\mathbb{R}$ ($\mathbb{R}$ is not countable). Take $\lambda \in \mathbb{R} \setminus \bigcup_{n=1}^{\infty} E_n$.

Then $(e_n, f_\lambda) = 0 \ \forall n$. Then $f_\lambda = 0$. But $\|f_\lambda\| = 1$. $\downarrow$.

$\square$

we need to prove claim $f_\lambda \perp f_\mu$.

Essentially we WTS $Lu = (pu')' + qu$ is "self adjoint" but L is not bounded. So we mean "self-adjoint" by

$$(Lu, v) = (u, Lv) \quad \text{for } u, v \in C^2 \text{ solutions of Sturm Liouville problems. (subset of } L^2\text{)}.$$

To see this, $(Lu,v) - (u, Lv) = \int_a^b vLu - uLv \, dx = p(uv' - vu')\Big|_a^b$

$= 0$

B/c u, v are solutions to SL, so at say $x = a$

$$\alpha u(a) + \alpha' u'(a) = 0$$
$$\alpha v(a) + \alpha' v'(a) = 0$$

$$\Rightarrow \det \begin{pmatrix} u(a) & u'(a) \\ v(a) & v'(a) \end{pmatrix} \neq 0 \quad \text{otherwise } \alpha = \alpha' = 0$$

The same works for $x = 0$

For f_λ, f_μ eigenfunction of λ, μ, we have

$$0 = (Lf_\lambda, f_\mu) - (f_\lambda, Lf_\mu)$$
$$= (-\lambda f_\lambda, f_\mu) - (f_\lambda, -\mu f_\mu)$$
$$= (\mu - \lambda)(f_\lambda - f_\mu) \Rightarrow (f_\lambda, f_\mu) = 0 \text{ if } \lambda \neq \mu.$$

Final proof of Sturm-Liouville

Take $\mu \in \mathbb{R}$ which is not eigenvalue of SL.

Replace $(pu')' + qu = -\lambda u$ (*)

by $(pu')' + (q + \mu)u = -\lambda u$ (**)

$\Rightarrow$ f is eigen function of $(**)$ with eigen value λ

$\Leftrightarrow$ f is eigen function of $(*)$ with eigenvalue $\lambda + \mu$.

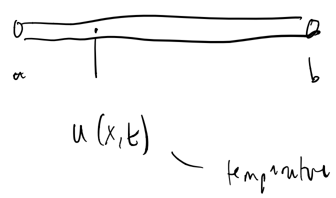
If $\lambda = 0$ is eigen value of $(*)$ $\Rightarrow$ 0 is not eigen value of $(**)$.

(shift of spectrum: even if $\lambda = 0$, we shift so λ cannot be 0. Apply theorem, then shift eigenvalues back).

(v) Applications to PDE:

The Sturm-Liouville theory is useful in the method called <u>separation of variables</u> for solving PDE.

Consider the heat eqn on a finite interval $[a,b] \subseteq \mathbb{R}$.



Let $u(x,t)$ be temperature of bar at position x and time t. The PDE for u is: $\underbrace{u_t = \alpha^2 u_{xx}}_{\text{derivatives}}$, $\alpha > 0$. α^2 is the diffusivity (take $\alpha^2 = 1$).

We need to prescribe boundary/initial conditions.

(BC) <u>Boundary condition</u>: $\alpha u(a,t) + \alpha' u_x(a,t) = 0$ $\quad 0 < t < \infty$

$\beta u(b,t) + \beta' u_x(b,t) = 0$

(IC) <u>Initial condition</u> $\quad u(x,0) = \varphi(x) \quad a \le x \le b$.

(Remark: can also consider wave equation $u_{tt} = c^2 u_{xx}$).

Solve using separation of variables. Look for solution

$$u(x,t) = f(x)\, g(t)$$

$\Rightarrow$ PDE becomes $\quad f(x)\, g'(t) = f''(x) \cdot g(t)$

$$\Rightarrow \quad \frac{g'(t)}{g(t)} = \frac{f''(x)}{f(x)} = \text{constant} = -\lambda$$

b/c fix t then RHS must be constant and vice versa.

$\Rightarrow$ we get 2 ODEs...

$$g'(t) + \lambda g(t) = 0 \qquad \text{and} \qquad f''(x) + \lambda f(x) = 0 \quad ①$$

BC $\Rightarrow \quad \alpha f(a)\, g(t) + \alpha' f'(a)\, g(t) = 0$

we don't want $g(t) = 0$ so

$$\alpha f(a) + \alpha' f'(a) = 0 \quad ②$$

Similarly $\quad \beta f(b) + \beta' f'(b) = 0 \quad ③$

Then ①, ②, ③ is a Sturm Liouville problem! So there is $(\lambda_j)_{j=1}^{\infty}$ and $(\varphi_j)_{j=1}^{\infty}$ eigenfunctions and $(\varphi_j)_{j=1}^{\infty}$ complete orthonormal basis for L^2.

We got λ, f

will become φ

Then solve $\quad g'(t) + \lambda_j\, g(t) = 0 \quad$ (linear 1st order ODE).

$$\Rightarrow \quad g_j(t) = c_j\, e^{-\lambda_j t}$$

Then we get $\quad u_j(x,t) = f_j(x)\, g_j(t) = c_j\, e^{-\lambda_j t}\, \varphi_j(x)$

Finally we get $u(x,t) = \sum_{j=1}^{\infty} C_j e^{-\lambda_j t} \varphi_j(x)$ (superposition principle).

To find C_j, use IC:

$$u(x,0) = \varphi(x).$$

$$\Rightarrow \quad \varphi(x) = \sum_{j=1}^{\infty} c_j \varphi_j(x), \quad c_j = (\varphi, \varphi_j)$$ (Hilbert complete for L_2). ☆

$$\Rightarrow \quad u(x,t) = \sum_{j=1}^{\infty} (\varphi, \varphi_j) e^{-\lambda_j t} \varphi_j(x).$$

Ex: Consider Heat equation on $[0, \pi]$. with $u(0)=0$, $u(\pi)=0$ . .
the eigenvalues are $\lambda_j = j^2$ and $\varphi_j(x) = \sin jx$ so

$$u(x,t) = \sum_{j=1}^{\infty} C_j e^{-j^2 t} \sin jx.$$

we need to show $\sum_{j=1}^{\infty} C_j e^{-\lambda_j t} \varphi_j(x)$ converges.

For heat equation in example, you can show that $u(x,t)$ converges in $C^2([a,b];\mathbb{R})$ using Weierstrass M-test.