Friday

~~A~~ very important result is that the topology for finite dim normed space are all equivalent. They give the same open sets.

~~A key component is~~. This is related to the compactness of unit balls.

— Let X be finite dim normed space. We know that there is a basis $e_1, \dots, e_n$ so that

$$x = \sum_{j=1}^{n} \lambda_j e_j , \qquad \lambda_j \in \mathbb{C} \text{ or } \mathbb{R}$$

We call $\lambda = (\lambda_j)_{j=1}^{n}$ the coordinates of x w.r.t. this basis.

~~what we want to show:~~ Let $\|\lambda\|_{\mathbb{C}^n} = (|\lambda_1|^2 + \dots + |\lambda_n|^2)^{1/2}$ be ~~the euclidean~~ be the norm. of coordinates.

— Thm: There is $m, M > 0$ such that

$$m \|\lambda\|_{\mathbb{C}^n} \le \|x\| \le M \|\lambda\|_{\mathbb{C}^n} \quad \text{for any } x \in X$$

and m, M do not depend on ~~x~~. $\cancel{x}$ (but on e_j).

Proof: We consider $m \le \dfrac{\|x\|}{\|\lambda\|_{\mathbb{C}^n}} \le M$

By linearity, $m \le \left\| \dfrac{x}{\|\lambda\|_{\mathbb{C}}} \right\| \le M$.

def of $x = \sum_{j=1}^{n} \lambda_j e_j$

The vector $\dfrac{x}{\|\lambda\|_{\mathbb{C}}} = \sum_{j=1}^{n} \dfrac{\lambda_j}{\|\lambda\|_{\mathbb{C}}} e_j$ so ~~$\frac{x}{\|x\|_{\mathbb{C}}}$~~ it is a

vector on the set

$$S = \{ x \in X : \|\lambda\|_{\mathbb{C}^n} = 1 \}$$

(The coordinate has norm 1, we call this the unit sphere in .

We show S is compact! Let $\{ x^{(n)} \}_{n=1}^{\infty} \in S$

Then $\|\lambda^{(n)}\| = 1$ which is a vector on the unit sphere in $\mathbb{C}^n$

$$S_{\mathbb{C}^n} = \{ \lambda \in \mathbb{C}^n \mid \|\lambda\|_{\mathbb{C}^n} = 1 \}.$$

By Heine - Borel, any closed bounded set is compact

So $S_{\mathbb{C}^n}$ is compact and there is a subsequence $\lambda^{(n_k)}$ with $\lim_{n_k \to \infty} \lambda^{(n_k)} = \gamma$ on the unit sphere $S_{\mathbb{C}^n}$.

Let $y = \sum_{j=1}^{n} \gamma_j e_j \in S$ we have

$$\|x^{(n_k)} - y\| = \left\|\sum_{j=1}^{n} (\lambda_j^{(n_k)} - \gamma_j) e_j\right\|$$

$$\le \|\lambda^{(n_k)} - \gamma\|_{\mathbb{C}^n} \sqrt{\sum_{j=1}^{n} \|e_j\|^2} \to 0 \text{ as } n_k \to \infty$$

So S is compact!. ~~Now $\|\cdot\|$ is a continuous function~~

~~on X.~~ ~~Now consider $f(\lambda_1, \ldots, \lambda_n) = \|\sum \lambda_i e_j\|$~~

Now use a fact in topology that any continuous function on compact set attains its min and max. thus $m \le \|x\| \le M$!

Note: if you don't know the fact, consider

$S_{\mathbb{C}^n} = \{ \lambda \in \mathbb{C}^n \mid \|\lambda\|_{\mathbb{C}^n} = 1 \}$

and $f(\lambda_1, \dots, \lambda_n) = \|\sum_{j=1}^{n} \lambda_j e_j\|$. Show f is continuous on $S_{\mathbb{C}^n}$!

- Thm: For any two norms $\|\cdot\|_1$, $\|\cdot\|_2$ on X. there's C_1, C_2 such that $C_1\|x\|_2 \le \|x\|_1 \le C_2 \|x\|_2$

Proof: $m_1 \|\lambda\|_{\mathbb{C}^n} \le \|x\|_1 \le M_1 \|\lambda\|_{\mathbb{C}^n}$

$m_2 \|\lambda\|_{\mathbb{C}^n} \le \|x\|_2 \le M_2 \|\lambda\|_{\mathbb{C}^n}$

$\Rightarrow \frac{m_1}{M_2} \|x\|_2 \le m_1 \|\lambda\|_{\mathbb{C}^n} \le \|x\|_1 \le M_1 \|\lambda\|_{\mathbb{C}^n} \le \frac{M_1}{m_2} \|x\|_2$.

- Thm: All norms on finite dim space give the same topo.

Proof: Consider $(X, \|\cdot\|_1)$ the open ball is $B_r(x)$

show this is open in $(X, \|y\|_2$.

$\|y - x\|_1 < r \Rightarrow \|y - x\|_2 \le cr$..

In the proof we see that closed unit ball is compact for finite dim normed space. This is not true for infinite dim space. so we get an characterization

Ex: show $B = \{ x \in X : \|x\| \le 1 \}$ is ~~closed~~ cpt!

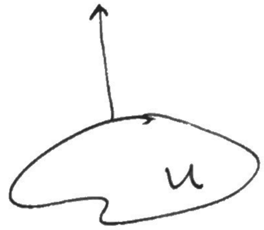
Pf: If $\|x\| \le 1 \Rightarrow \|\lambda\|_{\mathbb{C}^n} \le \frac{1}{m}$. But $\{\lambda \in \mathbb{C}^n \mid \|\lambda\| \le c\}$ is cpt Use this to show B is cpt.

— Thm: If X is infinite dimensional normed space, the closed unit ball is not compact!

Proof: A key is the following lemma due to Riesz.

<u>Riesz lemma:</u> Let $U \subset X$ be a closed subspace and X is normed ~~sp~~ vector space. $U \neq X$. For any $0<\delta<1$, there is a unit vector $x = x_\delta \in X$, $\|x_\delta\|=1$ such that

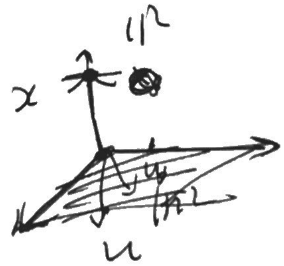
$$\|x_\delta - u\| \geq 1-\delta, \quad \forall u \in U.$$



~~Note: equal~~

Proof: Choose $x \notin U$ and let

$$d = \inf\{\|x-u\| : u \in U\} \geq 0$$



So there is $u_\delta \in U$ such that

$$\|x - u_\delta\| \leq \frac{d}{1-\delta} \qquad \left(\frac{d}{1-\delta} > d\right)$$

Let $x_\delta = \dfrac{x-u_\delta}{\|x-u_\delta\|}$. Then $\|x_\delta - u\| = \left\| \dfrac{x}{\|x-u_\delta\|} - \dfrac{u_\delta}{\|x-u_\delta\|} - u \right\|$

$$= \frac{1}{\|x-u_\delta\|} \|x - (u_\delta + \|x-u_\delta\| u)\| \geq \frac{d}{\|x-u_\delta\|} > 1-d$$

~~because $\frac{d}{1-\delta} > d$.~~

Now Take $e_1 \in X$, with $\|e_1\| = 1$

Take $e_2 \in X \setminus \operatorname{span}\{e_1\}$ with $\|e_2\| = 1$ and $\|e_2 - e_1\| \geq 1 - \delta$.

\- - - - - $\{e_n\}$ has no convergent subsequence

Final remark: $\operatorname{span} A = \{c_1 v_1 + \dots + c_n v_n\}$ $n \in \mathbb{N}$ finite, $v_i \in A$, c_i scalars

Then $\operatorname{span} A = \operatorname{lin} A$ (linear span of A)

= intersection of all subspaces that contain A.

define $\operatorname{clin} A$ = intersection of all closed subspace that contain A

and $\operatorname{clin} A = \overline{\operatorname{span} A}$ (see 2.11 – 2.12)

Thm: closure of a subspace of a normed space is subspace

Proof: let $F \subset X$ be a subspace, $\overline{F}$ the closure

Then $x, y \in \overline{F} \Rightarrow x + y \in \overline{F}$

let $x_n \to x$, $y_n \to y$. use continuity.

Thm: for any set A, $\overline{\operatorname{span} A} = \operatorname{clin} A$.

$\boxed{\text{Read } 2.7 - 2.12}$