Missed proof of $\mathcal{L}^1$ function. (☆)

Special Case: $S \subset \mathbb{R}^d$ Lebesgue measurable let $f = 1_S$.

Then $\Rightarrow$ $\displaystyle ⨍_{B(a,r)} 1_S \, d\lambda \xrightarrow[r\to 0]{} 1_S(x)$ a.e. x

Integral sign w/ line through it.

$$\frac{\lambda(S \cap B(a,r))}{\lambda(B(a,r))} \begin{cases} \nearrow 1 & x \in S \\ \searrow 0 & x \notin S \end{cases} \quad \text{a.e. } x$$

(what fraction of Ball is occupied by set S).

Note

$$⨍ f \, d\lambda = \frac{\int f \, d\lambda}{\lambda}$$

size of measure

(measure of average)

By Thm, $f \in L^1$ then $\displaystyle \underbrace{⨍_{B(a,r)} f \, d\lambda}_{\text{when averaging over a ball, the value goes to center.}} \longrightarrow f(a)$. (☆)

We can write $\displaystyle f(a) = ⨍_{B(a,r)} f(a) \, d\lambda$. So $\displaystyle ⨍_{B(a,r)} (f - f(a)) \, d\lambda \longrightarrow 0$ a.e.

doesn't mean f and $f(a)$ are close.
f could be + 1/2 the time &
− 1/2 the time to
average close to $f(a)$.

Lebesgue Thm: $f \in L^1(\mathbb{R}^d, \lambda)$. Then $\displaystyle ⨍_{B(a,r)} |f - f(a)| \, d\lambda \longrightarrow 0$

a.e. $a \in \mathbb{R}^d$.

(This is a much stronger Thm.)

Proof let $q \in \mathbb{R}$. Apply previous thm to $|f-q|$.

$$\fint_{B(a,r)} |f-q| \, d\lambda \longrightarrow |f(a)-q| \quad \text{a.e. } a.$$

$$\fint_{B(a,r)} |f-f(a)| \leq \fint_{B(a,r)} |f-q| + \fint_{B(a,r)} |q-f(a)|$$

$$= \left(\fint_{B(a,r)} |f-q| \right) + \underbrace{|q-f(a)|}_{\text{constant}}$$

$$= \underbrace{|f(a)-q|}_{\text{by prev. thm}} + |q-f(a)|$$

So: Except for a set $S(q)$ of measure 0.

$$\limsup_{r \to 0} \fint_{B(a,r)} |f-f(a)| \, d\lambda = 2|f(a)-q| \quad x \in S(q).$$

let $S = \bigcup_{q \in \mathbb{Q}} (S(q))$. $\lambda(S) = 0$ since each $S(q)$ is measure 0.

$x \notin S \Rightarrow x \notin S(q), \forall q \in \mathbb{Q}$

$$\Rightarrow \limsup_{r \to 0} \fint_{B(a,r)} |f-f(a)| \leq 2|f(a)-q|$$

So $\longrightarrow 0$ since it holds for all q.

$\square$

Thm (FTOC v.2): $f \in L^1(\mathbb{R})$. Then $\frac{d}{dx}\left(\int_0^x f \, d\lambda\right) = f(x)$ for a.e. x.

Proof:

$$\fint_{[a, a+h)} f \, d\lambda \longrightarrow f(a) \quad \text{a.e. point } a$$

$$\frac{\int_a^{a+h} f \, d\lambda}{\lambda[a, a+h]} = \frac{\int_{-\infty}^{a+h} f \, d\lambda - \int_{-\infty}^{a} f \, dx}{h}$$

let $g(x) = \int_{-\infty}^{x} f \, d\lambda$. Then

$$(h>0) \qquad \frac{g(x+h) - g(x)}{h} \rightarrow f(x) \quad \text{a.e. } x.$$

(def of derivative)

$\square$

Distribution Function

μ finite Borel function on $\mathbb{R}$. Define F_μ (distribution function of measure):

$$F_\mu : \mathbb{R} \rightarrow \mathbb{R}.$$

$$F_\mu(x) = \mu((-\infty, x]).$$

Properties

① F_μ is increasing and bounded (μ is finite)

② $\lim_{x \to \infty} F_\mu(x) = \mu(\mathbb{R})$

③ $\lim_{x \to -\infty} F_\mu(x) = 0$

Pf sketch for ③: given (x_n) sequence decreasing to $-\infty$ w/

$$x_n > x_{n+1} > \dots$$

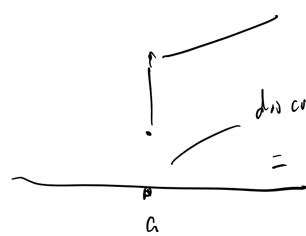
Note: $(-\infty, x_n] \supset (-\infty, x_{n+1}] \supset \dots$

$$F(x_n) = \mu((-\infty, x_n]) \longrightarrow \mu\left(\bigcap_{1}^{\infty} \mu(\infty, x_n)\right)$$

$$= \mu(\emptyset) = 0$$

④ F_μ is right continuous.

$$\lim_{\substack{x \to a \\ x > a}} F(x) = F(a)$$



discontinuity $= \mu(\{a\}) > 0$.

points w/ (+) measure

Theorem The map

$\mu \longmapsto F_\mu$ bijection from finite Borel measures on $\mathbb{R}$

to $\mathcal{F} = \{ F : \mathbb{R} \to \mathbb{R} \mid F \text{ is } \uparrow, \text{ bounded, right continuous, } F(x) \to 0 \text{ as } x \to \infty \}$

Proof: one to one? exercise

surjective: let $F \in \mathcal{F}$. Define $S \subset \mathbb{R}$.

$$\mu_F^*(S) = \inf\left\{\sum \left(F(b_i) - F(a_i)\right) : S \subset \bigcup_i^\infty (a_i, b_i)\right\}$$

Easy that μ_F^* is an outer measure. Borel sets are measurable.

$$\mu_F([a,b]) = F(b) - F(a^-) \quad \text{where } F(a^-) = \sup\{F(x) : x < a\}$$
$$= \lim_{x \uparrow a} F(a).$$

let $a \to -\infty$

$$\mu_F((-\infty, b]) = F(b).$$

So for any F, we have constructed a measure μ_F. $\Rightarrow$ surjective. $\square$

Recall: $F : \mathbb{R} \to \mathbb{R}$ increasing, then $\lim_{x \downarrow a} F(x) = \inf_{x > a} F(x) \stackrel{def}{=} F(x+)$

$$\lim_{x \uparrow a} F(x) = \sup_{x < a} F(x) \stackrel{def}{=} F(x-).$$

F is continuous iff $F(x^-) = F(x^+)$.

Also F has at most countably-many discontinuities.

Pf: **Note** if $x < y < z$ then

$$F(x) \le F(x^+) \le F(y^-) \le F(y^+) \le F(z)$$

let $D = \{x : F(x) < F(x^+)\}$. Then $(F(x^-), F(x^+))$, $x \in D$ disjoint.

"$\square$"

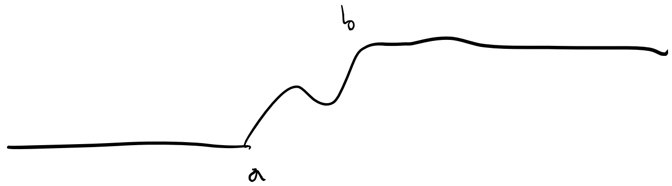
derivative

1

Thm if $F:[a,b] \to \mathbb{R}$ $\uparrow$, then F' exists a.e. and

$$\int_a^b F' \, d\lambda \le F(b) - F(a).$$

Pf: Assume F continuous. Assume $F(a) = 0$. Extend to $\mathbb{R}$ by $F(x) = 0$ for $x \le a$ and $F(b)$ for $x \ge b$.



So $F = F_\mu$ for some μ. $D\mu$ exists a.e.

$$\lim_{\substack{|t-s|\to 0 \\ x \in (s,t) \\ s<t}} \frac{\mu[s,t]}{\lambda[s,t]} \longrightarrow D\mu \text{ for } \lambda \text{ a.e. } x.$$

$$\|$$

$$\lim_{\substack{|t-s|\to 0 \\ x \in (s,t) \\ s<t}} \frac{F(t) - F(s)}{t-s}.$$ Thus F is differentiable $\Rightarrow$ F' exists.

$$\|$$

$$D\mu(x)$$

Recall $\mu[a,b] = \mu([a,b] \cap Z) + \int_a^b D\mu \, d\lambda.$

Z: $\overline{D}\mu = \infty$, $\underline{D}\mu < \overline{D}\mu$

$$= \underbrace{\mu([a,b) \cap Z)}_{\text{singular}} + \underbrace{\int_a^b F'(x) \, d\lambda x}_{\text{abs. cont. part}}$$