Last time: $\{e^{inx}\}$ is a complete orthonormal system of $L^2(-\pi,\pi)$.
$\qquad\qquad \{e_n\}$

Now that this is a Hilbert space, we can do a bunch of things.

- Parsevals formula.

Thm let $f \in L^2(-\pi,\pi)$ fourier series.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \Rightarrow \underbrace{\|f\|^2_{L^2}}_{\text{like } \ell^2 \text{ form}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx$$
$$= \sum_{n=-\infty}^{\infty} |c_n|^2$$

Thm let $f,g \in L^2(-\pi,\pi)$ and $f = \sum c_n e^{inx}$, $g = \sum d_n e^{inx}$, then

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\overline{g}(x)\,dx = \sum_{n=-\infty}^{\infty} c_n \overline{d_n}.$$

inner prdt

(follows from isomorphism of separable spaces w/ L^2).

How do we think about $L^2(-\pi,\pi)$? what is the lebesgue integral.

Chapter on Lebesgue Integral (not in book) on $[a,b]$.

Goal: show completeness of $L^2(-\pi,\pi)$.

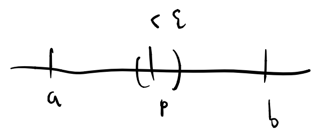
I) measure zero set.

def: a set $S \subset [a,b]$ has measure zero if

$\forall \varepsilon > 0$, $\exists$ <u>countable</u> set of open intervals $\{I_j\}_{j=1}^{\infty}$ s.t. $S \subset \bigcup_{j=1}^{\infty} I_j$ and $\sum_{j=1}^{\infty} |I_j| < \varepsilon$.

Here $|I_j| = b_j - a_j$ if $I_j = (a_j, b_j)$.

<u>Ex</u>: A point set has measure 0.



<u>Lemma</u> If $S_1, S_2, \dots$ have measure 0, then $\bigcup_{i=1}^{\infty} S_i$ has measure 0.

<u>Pf</u>: $\forall \varepsilon > 0$ and any S_j, $j = 1, 2, \dots$ we can find open intervals $\underbrace{\{I_{j,i}\}_{i=1}^{\infty}}_{\text{subset of } \{I_j\}}$ s.t. $S_j \subset \bigcup_{i=1}^{\infty} I_{j,i}$ and $\left|\sum_{i=1}^{\infty} I_{j,i}\right| < 2^{-j}\varepsilon$.

Now consider $\{I_{j,i}\}_{j,i=1}^{\infty}$, we have $\bigcup_{j=1}^{\infty} S_j \subset \bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty} |I_{j,i}|$

and $\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} |I_{j,i}|\right) < \sum_{j=1}^{\infty} 2^{-j}\varepsilon < \varepsilon$ □

<u>Example</u> The countable set of points has measure 0, i.e. <u>$\mathbb{Q} \cap [a,b]$ measure 0.</u>

Def: A property holds "Lebesgue almost everywhere" (a.e.) on $[a,b]$ if it holds on $[a,b]$ except for a measure-zero set.

Ex) $f=g$ a.e. means $\{x \in [a,b] : f \neq g\}$ has measure 0.

II) Lebesgue Integrable functions. (we build them w/o measure.)

– **Def:** A function $\varphi : [a,b] \to \mathbb{R}$ is a step function if $\exists$ a partition of $[a,b]$

$$a = x_0 < \cdots x_1 < \cdots < x_N = b$$

s.t. $\varphi\big|_{(x_i, x_{i+1})} = c_i$, some constant.

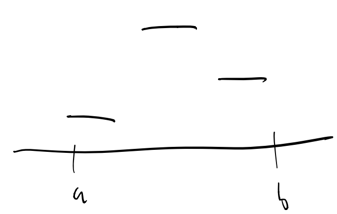
– **Def** for step function φ, define Lebesgue Integral,

$$\int \varphi = \sum_{i=1}^{N} c_i \cdot (x_i - x_{i-1})$$

no dx!

some constant

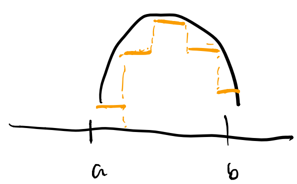
same as Riemann Integral (for step functions).



- **Def** let $\mathcal{L}_0$ be the set of functions $f:[a,b] \longrightarrow \mathbb{R}$ s.t. there is an increasing sequence of step functions $\{\varphi_i\}_{i=1}^{\infty}$,

$$\varphi_1(x) \leq \varphi_2(x) \leq \dots \quad \text{such that} \quad \lim_{i\to\infty} \varphi_i = f.$$

and $\left\{ \int \varphi_i \right\}_{i=1}^{\infty}$ is <u>bounded</u>.



for $f \in \mathcal{L}_0$, define the integral $\int f = \lim_{k\to\infty} \int \varphi_k$.

$\mathcal{L}_0$ not a vector space as negative functions are hard. To fix that...

<u>def</u> let $\mathcal{L}^1$ be the set of functions

$$f = g - h \quad \text{where} \quad g, h \in \mathcal{L}_0$$

<u>def</u>: define Integral of f as

$$\int f = \underbrace{\int g}_{} - \underbrace{\int h}_{}$$

well-defined. as in $\mathcal{L}_0$

These are the Lebesgue integrable functions.

☆ does this depend on choice of $\{\varphi_i\}_{i=1}^{\infty}$? is the Integral well defined?

<u>Lemma</u> let $f \in \mathcal{L}_0$ and $\{\varphi_k\}$, $\{\psi_\ell\}$ are 2 sequences in the definition. Then $\lim_{k\to\infty} \int \varphi_k = \lim_{\ell\to\infty} \int \psi_\ell$.

Pf Fix one limit. Show $\forall \varepsilon > 0$,

$$(*) \quad \lim_{k\to\infty} \int \varphi_k \geq \int \psi_\ell - \varepsilon \quad \forall \ell \in \mathbb{N}.$$

$$\Rightarrow \lim_{k\to\infty} \int \varphi_k - \left(\psi_\ell - \frac{\varepsilon}{b-a}\right) \geq 0$$

show the set where this is negative is very small.

let $A_k = \{x \in [a,b] \mid \varphi_k \geq \psi_\ell - \frac{\varepsilon}{b-a}\}$

for any k, ℓ, φ_k, ψ_ℓ step functions. So A_k is a finite union of intervals.

As $\varphi_k \uparrow$, we know $A_k \subset A_{k+1} \subset \cdots$ Also as $\lim_{k\to\infty} \varphi_k = f$ a.e.

So $[a,b] \setminus \bigcup_{k=1}^{\infty} A_k$ has measure 0.

since $\{x \in [a,b] \mid f \geq \psi_\ell - \frac{\varepsilon}{b-a}\}$ is true by def of ψ_ℓ.

$$\text{Then} \int \varphi_k = \int_{A_k} \varphi_k + \int_{A_k^c} \varphi_k$$

$$\geq \underbrace{\int_{A} \psi_\ell - \frac{\varepsilon}{b-a}}_{\text{replace}} + \int_{A_k^c} \varphi_k$$

$$\geq \int_A \varphi_\ell - \varepsilon + \int_{A_k^c} \varphi_k$$

Take limit,

$$\lim_{k\to\infty} \int \varphi_k \geq \overbrace{\liminf_{k\to\infty} \int_{A_k} \varphi_\ell}^{\text{I}} + \overbrace{\liminf_{k\to\infty} \int_{A_k^c} \varphi_k}^{\text{IV}} - \varepsilon$$

use lim inf b/c we don't know lim exists.

I $\left| \int_{A_k} \varphi_\ell - \int \varphi_\ell \right| \leq \sup |\varphi_\ell| \cdot |A_k^c|$

small as $[a,b] \setminus \bigcup_{k=1}^{\infty} A_k$ has measure 0.

$\longrightarrow 0$ as $k \to \infty$.

II $\int_{A_k^c} \varphi_k \geq - \sup_{x \in [a,b]} |\varphi_1(x)| \cdot |A_k^c|$

first element

$\longrightarrow 0$ as $k \to \infty$.

III Properties of $\mathcal{L}_0$ $\mathcal{L}^1$ functions.

Prop: 1) $f, g \in \mathcal{L}_0$ $\alpha, \beta \geq 0$ then $\alpha f + \beta g \in \mathcal{L}_0$

and $\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$.

2) If $f, g \in \mathcal{L}_0$, then $\max\{f, g\} \in \mathcal{L}_0$ and

$\min\{f, g\} \in \mathcal{L}_0$.

3) if $f, g \in \mathcal{L}_0$, $f \leq g$ a.e. then $\int f \leq \int g$

Pf: (1) By def, $\exists \{\varphi_k\}, \{\varphi_\ell\}$ s.t.

$$\varphi_k \xrightarrow{k\to\infty} f \text{ a.e.}, \qquad \varphi_\ell \xrightarrow{\ell\to\infty} g \text{ a.e.}$$

Then $\alpha\varphi_k + \beta\varphi_\ell$ are still increasing step functions.

$\to \alpha f + \beta g$.

□

(2) Consider φ_k, φ_ℓ, $\varphi_k \to f$, $\varphi_\ell \to g$ a.e.

$\max\{\varphi_k, \varphi_\ell\}$ is still an increasing step function, ... etc...

"□"

Prop 1) $\mathcal{L}^1$ is a vector space.

if $f, g \in \mathcal{L}^1$, $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g \in \mathcal{L}^1$,

$$\int \alpha f + \beta g = \alpha \int f + \beta \int g.$$

2) $f, g \in \mathcal{L}^1$, $f \leq g$ a.e. $\int f \leq \int g$.

3) $f \in \mathcal{L}^1$, then $|f| \in \mathcal{L}^1$, $\left|\int f\right| \leq \int |f|$.