Last time: X hilbert, $\{e_n\}_{n=1}^{\infty}$ orthonormal sequence

$\forall x \in X$, fourier series $\sum_{j=1}^{\infty} (x, e_j) e_j$ — we showed this converges

WTS $x = \sum_{j=1}^{\infty} (x, e_j) e_j$ — converges to the right value

we need

III) complete orthonomal sequence

- Def $\{e_n\}_{n=1}^{\infty}$ complete orthonormal sequence.

  if $x \perp e_n$, $n = 1, \dots, \infty$ $\Rightarrow$ $x = \vec{0}$

(Note: also works for finite orthonormal system $\{e_n\}_{n=1}^{N}$)

(every hilbert space has an orthonormal seq by Gram-Schmidt but not every hilbert space has complete orthonormal seq)

Thm $\{e_n\}_{n=1}^{\infty}$ is complete orthonormal sequence in Hilbert space X

① $\forall x \in X$, $x = \sum_{j=1}^{\infty} (x, e_j) e_j$ # any vector can be represented w/ e_j ($\{e_j\}$ is the basis)

② $\|x\|^2 = \sum_{j=1}^{\infty} |(x, e_j)|^2$ (parseval's identity) equality for Bessel's

Pf: ① let $y = x - \sum_{j=1}^{\infty} (x, e_j) e_j$. Consider (y, e_n). (we know it converges already)

$$(y, e_n) = (x, e_n) - \left(\sum_{j=1}^{\infty} (x, e_j) e_j, e_n\right)$$

$$= (x, e_n) - (x, e_n) = 0 \quad \forall n, = 1, 2, \dots$$

By completeness of sequence, $y = 0$

② Consider partial sum:

$$\left\| \sum_{j=1}^{N} (x, e_j) e_j \right\|^2 \overset{\text{Pythag}}{=} \sum_{j=1}^{N} \left| (x, e_j) \right|^2 \|e_j\|^2$$

$$= \sum_{j=1}^{N} |(x, e_j)|^2$$

take limit (as norm is continuous)

$$\Rightarrow \underset{\text{by def}}{\|x\|^2} = \sum_{j=1}^{\infty} |(x, e_j)|^2$$

$\square$

Equivalent definition of complete orthonormal sequence:

Prop: $\{e_n\}_{n=1}^{\infty}$ <u>complete</u> orthonormal seq iff Parseval's identity holds for all $x \in X$.

we assume orthonormality
we just want to show completeness

<u>Proof</u> "$\Rightarrow$" done. (from above)

"$\Leftarrow$" Suppose $\{e_n\}_{n=1}^{\infty}$ is not complete.

$\Rightarrow \exists\, x \in X,\ x \neq 0$ s.t. $x \perp e_n,\ n = 1, 2, \ldots$

for this x,

$$\|x\|^2 = \sum_{j=1}^{\infty} |(x, e_j)|^2 \quad \text{by assumption}$$

$$0 \neq \|x\|^2 = 0$$

def of norm
$x=0 \iff \|x\|=0$

def of orthonormal

$\square$

A 3rd characterization:

<u>Prop</u> let $\{e_n\}_{n=1}^{\infty}$ orthonormal sequence, then $\{e_n\}_{n=1}^{\infty}$ is complete is equivalent to $\overline{\operatorname{span}\{\{e_n\}_{n=1}^{\infty}\}} = X$
(closure)

Pf: "$\Rightarrow$" done b/c we showed we can build any vector w/ fourier series.

"$\Leftarrow$" let $x \in X$, $x \perp e_n$, $n=1,\dots$

Let's consider vectors perpendicular to x.

Let $E = \{y \in X, (x,y)=0\}$. This is closed in X b/c inner pdt is a continuous function. Precisely

$$f(y) = (x,y) \text{ is continuous } X \to \mathbb{C}.$$

$$E = f^{-1}(0) \text{ is closed.} \quad (\{0\} \text{ is closed})$$

Note that $\operatorname{span}\{\{e_n\}_{n=1}^{\infty}\} \in E$, then b/c all $e_n \perp x$ ie. $(e_n, x)=0$

$$\overline{\operatorname{span}\{\{e_n\}_{n=1}^{\infty}\}} \in E \text{ as } E \text{ is closed.}$$

By assumption $X \subset E \Rightarrow E = X$

$$\Rightarrow (x,x)=0 \Rightarrow x=0.$$

$\square$

Use these characterizations to describe separability.

IV) Separable Hilbert space

- Def A Hilbert space X is separable if it has a complete orthonormal sequence (can be finite).

- Def: A mapping $U: X \to Y$ (inner pdt spaces). is called unitary if U is linear

i.e. $U(\alpha x + \beta y) = \alpha U(x) + \beta U(y)$.

and bijective and preserves the inner product.

i.e. $(U(x), U(y))_Y = (x, y)_X$

Exercise Show if U linear, surjective then U unitary is equivalent to $\underbrace{\|Ux\| = \|x\|}_{\text{injectivity is trivially implied.}}$.

$(U(x), U(x))_Y = (x, x)_X$
$\Rightarrow (U(x), U(x))_Y^{\frac{1}{2}} = (x, x)_X^{\frac{1}{2}}$

Thm let X separable Hilbert space, then X is isomorphic to either $\mathbb{C}^n$ or $\ell^2(\mathbb{C})$.

Isomorphic: $\exists$ Unitary map U from X to $\mathbb{C}^n$ or $\ell^2(\mathbb{C})$.

Pf: define U? Suppose X has a complete orthonormal sequence (by separability), either $\{e_n\}_1^N$, assume finite case (could be infinite for ℓ^2).

Then any $x = \sum_{j=1}^{N} (x, e_j) e_j$. Then define $U: X \to \mathbb{C}^N$

then $x \to (\xi_j)_{j=1}^{N}$, $\xi_j = (x, e_j)$.

(map x to its coordinates)

literally

It is easy to show U is linear, bijective and $\|Ux\| = \|x\|$ (parsevals).

If $N = \infty$, define $U: X \to \ell^2(\mathbb{C})$, $x \to (\xi_j)_{j=1}^{\infty}$, $\xi_j = (x, e_j)$

Rest is straight forward.

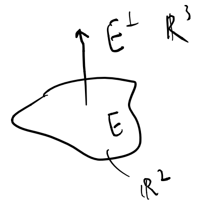
$\square$

V <u>orthogonal complements</u> where do we require ~~the~~ completeness?

Can we decompose <u>Hilbert space</u> to different spaces?

(assume it is <u>not</u> separable).

- <u>Def</u>: for any subset E of a Hilbert space X, the <u>orthogonal complement</u> of E is $E^{\perp} = \{x \in X : (x, y) = 0, \forall y \in E\}$



- <u>Thm</u> $E^{\perp}$ is a closed linear subspace of X. (E is subset)

look up in book

Pf: Easy to check that $E^{\perp}$ is a subspace.

Closed? let (x_k) be sequence in $E^{\perp}$ s.t. $(x_k) \longrightarrow x \in X$.

Then wts $x \in E^{\perp}$. Then $\forall y \in E$, $(x, y) = \lim_{k \to \infty} (x_k, y) = 0$

b/c all $x_k \in E^{\perp}$ $\forall k$.

$\square$

High level Goal — WTS $X = E \oplus E^{\perp}$

Def let M, N be the subspaces of vector space X. X is the direct sum of M and N, denoted $X = M \oplus N$ if for any $x \in X$, there is $m \in M$, $n \in N$ s.t. $x = m + n$.

and $\underbrace{M \cap N = \{0\}}$.

makes it a "direct" sum.