Lecture 12

Recall $\mathcal{L}^2 = \{ f \in \mathcal{L}^1 : f^2 \in \mathcal{L}^1 \}$. "inner pdt" $\langle f, g \rangle = \int fg$. "norm" $\|f\|_{\mathcal{L}^2} = \left(\int |f|^2 \right)^{\frac{1}{2}}$

Thm: $\mathcal{L}^2$ is complete.

HW 4: #4

$\{f_k\} \subset \mathcal{L}^1$, $f_k \to 0$ in $\mathcal{L}^1$ but $f_k \not\to 0$ a.e.

converge in L^p space $\not\Rightarrow$ point wise convergence.

$(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 + b^2 \geq 2ab$

then $(a+b)^2 = a^2 + b^2 + 2ab \leq 2(a^2+b^2)$

Proof: for $\{f_k\}_{k=1}^{\infty} \subset \mathcal{L}^2$ cauchy, show $f_k \to f \in \mathcal{L}^2$.

First, observe the following: suppose f is the limit:

here $f_k = f_k^+ - f_k^-$, $f_k^\pm \geq 0$

$f = f^+ - f^-$, $f^\pm \geq 0$

$$\|f_k - f\|_{\mathcal{L}^2}^2 \leq \left(\|f_k^+ - f^+\|_{\mathcal{L}^2} + \|f_k^- - f^-\|_{\mathcal{L}^2} \right)^2$$

$$\leq 2\left(\int |f_k^+ - f^+|^2 + \int |f_k^- - f^-|^2 \right) = 2\left(\|f_k^+ - f^+\|^2 + \|f_k^- - f^-\|^2 \right)$$

upper bound

$$\leq 2\left(\int |f_k^+ - f^+| \cdot |f_k^+ + f^+| + \int |f_k^- - f^-| \cdot |f_k^- + f^-| \right)$$

$$= 2\left(\int |f_k^{+2} - f^{+2}| + \int |f_k^{-2} - f^{-2}| \right) \leftarrow \text{seq in } \mathcal{L}^1$$

we just need to ~~show~~ find $f^{\pm 2}$ that $f_k^{\pm 2}$ converges to in $\mathcal{L}^1$.

① First, show $\{f_k^2\}$ is cauchy in $\mathcal{L}^1$ hence $f_k^2 \to g$ in $\mathcal{L}^1$.

C-S

$$\|f_k^2 - f_\ell^2\|_{\mathcal{L}^1} = \|(f_k - f_\ell)(f_k + f_\ell)\|_{\mathcal{L}^1} = \int |f_k - f_\ell| \cdot |f_k + f_\ell| \leq \|f_k - f_\ell\|_{\mathcal{L}^2} \|f_k + f_\ell\|_{\mathcal{L}^2}$$

$$\leq \|f_k - f_\ell\|_{\mathcal{L}^2} \underbrace{\left(2 \sup_k \|f_k\|_{\mathcal{L}^2} \right)}_{\text{bounded}}$$

inner pdt is $\mathcal{L}^2$. in $\mathcal{L}^2$ space now

so $\{f_k^2\}$ cauchy in $\mathcal{L}^1$. $\exists g \in \mathcal{L}^1$ s.t. $f_k^2 \to g$ in $\mathcal{L}^1$. By a technical lemma, $\exists$ subseq $(f_{k_j}^2)$ s.t.

$f_{k_j}^2 \to g$ a.e. g is our candidate, ~~but we~~.

② Show $g = f^2$ a.e. ~~Show $\{f_{k_j}\}$~~ Consider $\|f_{k_i} - f_{k_j}\|_{\mathcal{L}^1} = \int |f_{k_i} - f_{k_j}| \cdot 1 \leq \|f_{k_i} - f_{k_j}\|_{\mathcal{L}^2} \sqrt{b-a}$

then $\{f_{k_j}\}$ cauchy in $\mathcal{L}^1$. So $\exists$ subseq $\{f_{k_{j_\ell}}\}$ s.t. $f_{k_{j_\ell}} \in \mathcal{L}^1$ a.e.

$(f_{k_{j_\ell}})$ is a subseq. of (f_{k_j}) so $(f_{k_{j_\ell}})^2 \to g$ a.e. So $g = f^2$ a.e.

Apply this ①,② to f_k^+ and f_k^-.

$\square$

Def $L^2[a,b] = \mathcal{L}^2/\sim$, $f \sim g$ means $f=g$ a.e. Then $L^2([a,b])$ Hilbert space.
This is separable (ch 5 fourier basis).

Moving to Ch. 6: Dual Spaces

I) Linear functionals.

- Def let X be vector space. A linear functional on X is a mapping ~~from~~ $f: X \to \mathbb{R}$ (or $\mathbb{C}$). which satisfies $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$. $\forall \lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), $x, y \in X$.

Ex 1 On $\underbrace{X = \mathbb{C}^n}_{\text{finite-dim.}}$. All linear functionals are $F(z_1, \dots z_n) = \alpha_1 z_1 + \dots \alpha_n z_n$, $\alpha_i \in \mathbb{C}$.

Ex 2 on $X = \ell^2(\mathbb{C})$. we define $\alpha \in \ell^2(\mathbb{C})$, $F(z) = \sum_{i=1}^{\infty} \alpha_i z_i < \infty$ well-defined by CS

- let X be a normed vector space. Consider continuous linear functionals $f: X \to \mathbb{R}$ (or $\mathbb{C}$) continuous.
$\exists$ a topology so we can talk about continuity.

Thm: let F linear functional on normed vector space X. The following are equivalent:

(i) F is continuous on X
(ii) F is continuous at $0 \in X$
(iii) F is bounded meaning $\exists M \geq 0$ s.t. $|F(x)| \leq M\|x\|$ for any $x \in X$.

(bounded linear functional = continuous linear func.).

Pf: (i) $\to$ (ii) trivial.

(ii) $\to$ (iii) use continuity of F at 0. $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $|F(x) - F(0)| < \varepsilon$ if $\|x - 0\| < \delta$.
($\mathbb{R}$ abs val.; norm in X)

since $F(x)$ linear, $F(0) = 0$. for $\varepsilon = 1$, we have

$$|F(x)| < 1 \text{ if } \|x\| < \delta.$$

For any $y \in X$, consider $\frac{\delta y}{2\|y\|}$, $y \neq 0$. let $\tilde{y} = \frac{\delta y}{2\|y\|} < \delta$. Then $\|\tilde{y}\| < \delta$. Then $|F(\tilde{y})| < 1$

$$\Rightarrow \left|F\left(\frac{\delta y}{2\|y\|}\right)\right| < 1 \Rightarrow F(y) < \frac{2\|y\|}{\delta} = M\|y\|, \ M = \frac{2}{\delta}. \quad \forall y.$$

□

(iii) $\to$ (i) for any $x, y \in X$, $|F(x) - F(y)| = |F(x-y)| \leq M\|x-y\|$. Use def of continuity. □

Note linearity is very important.

II) <u>Dual Space</u> let X be normed vector space. let X^* be set of all ~~linear~~ continuous linear functionals on X.

- <u>Lemma</u> X^* vector space. (exercise).

- define a norm on X^*.

$$\|F\| = \sup_{x} \left\{ \frac{|F(x)|}{\|x\|} ; \; \underbrace{x \in X \setminus \{0\}}_{\text{non-zero}} \right\}$$

from before $|F(x)| \leq M\|x\| \Rightarrow \frac{|F(x)|}{\|x\|} \leq M$
find smallest M.
(linear functional is bounded by some M)

$$= \sup_{\substack{x \in X \\ \|x\| \leq 1}} \{ |F(x)| \}$$

we can always rescale as this is a linear space.

Need to check it is a norm. (≤ 1)

<u>Note</u> we have $F(x) \leq \|F\| \cdot \|x\|$ so $\|F\|$ is the smallest M in the bound of F.

- <u>Lemma</u> This is a norm on X^*. (exercise).
  check 3 axioms.

- <u>Thm</u>: ~~X^* is a~~ $(X^*, \|\cdot\|)$ is a banach space. Note X does not need to banach (complete).

For now, assume X^* is not empty (this is for later, proven by Hahn-Banach).

<u>Proof</u>: we only need to check the completeness. let $\{F_n\}$ cauchy seq. in X^*.
so $\|F_n - F_m\| \to 0$ as $n, m \to \infty$

For a fixed pt $x \in X$, we have $|F_n(x) - F_m(x)| \underset{\text{linearity}}{=} |(F_n - F_m)(x)| \leq \underbrace{\|F_n - F_m\|}_{\text{small by assumption}} \|x\|$.

so $\{F_n(x)\}$ is cauchy in $\mathbb{C}$ <u>for fixed x</u>. $\mathbb{C}$ is complete so $\exists$ $F_n(x) \longrightarrow \underbrace{F(x)}_{\text{some } F}$ for every x.

This is a function on X. let F be the candidate.