Last Lecture: Review

Wed 3:30 pm – 6:00 pm Final

Chap 1-3: Inner pdt space, norm space, Hilbert + Banach

- How to show completeness of a space
- Riesz lemma: $U \subset X$ closed subspace, X normed vector space. If $U \neq X$ for any $0 < \delta < 1$, $\exists\ x_\delta \in X$ s.t. $\|x_\delta\| = 1$ and $\|x_\delta - u\| \geq 1-\delta$, $u \in U$

Consequences: Closed unit ball is not compact for ∞-dim space. This is however, compact for finite dim.

- All the norms (or topology) in a finite dim normed space are equivalent.
- Examples of Banach / Hilbert space:

$\ell^1(\mathbb{R})$, $\ell^2(\mathbb{R})$

$L^1(\mathbb{R})$, $L^2(\mathbb{R})$

How to show completeness?

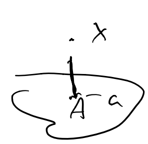
Chap 4: Orthogonality in Hilbert space

- Parallelogram Law $\|x-y\|^2 + \|x+y\|^2 = 2(\|x\|^2 + \|y\|)^2$

  Bessel's Inequality

  Pythagorean Thm

  Nearest pt property: A closed convex subset of H, H hilbert, then $\exists x \in H$ unique $a \in A$ s.t. $\|x - a\| \leq \|x - y\|$ $\forall y \in A$.



- Orthogonal Complement $E^{\perp}$.

  Thm M closed subspace, then $H = M \oplus M^{\perp}$, if H hilbert

- Complete orthonormal sequence : 3 equivalent definitions
  - Parseval's identity

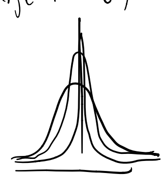
Chap 5 : Classical Fourier Series

- $L^2(a,b)$ is a separable Hilbert space.

  $(e_n)_{n=1}^{\infty}$, $e_n = \frac{1}{\sqrt{2\pi}} e^{inx}$ is a complete orthonormal basis

- Fejer's kernel: $f_m = \sum_{n=-m}^{m} (f, e_n) e_n$, $F_m = \frac{1}{m+1}(f_0 + \dots + f_m) = \underbrace{K_m}_{\text{kernel}} * f$

  Then we showed $F_m \longrightarrow f$ uniformly if K_m has certain properties. (converge to δ)



- Talk about Lebesgue Integral function $C_0(a,b)$
  - How to approximate them using continuous function & step functions
  - MCT

Midterm ——— 30-40%

Chap 6 : Linear Functionals

- For linear functionals, continuity = boundedness (how to prove?)
- Define $\|F\| = \sup_{\substack{x \in X \\ \|x\|=1}} |F(x)|$ on set of linear functionals, X^*.

  Showed X^* is Banach if X is normed vector space (proof?)

- For Hilbert space, Riesz Representation Thm.

  For $F \in X^*$, $\exists y \in X$ s.t. $F(x) = (x, y)$ $\forall x \in X$.
  (y unique)

  and $\|F\| = \|y\|_X$.

- For normed vector space, Hahn-Banach Theorem

  Thm: Let X be real vector space. Let $q : X \to \mathbb{R}$ be sublinear functional.

  Let $Y \subset X$ linear subspace, $f : Y \to \mathbb{R}$ linear and $f(y) \leq q(y)$.

  Then exists $\exists F : X \to \mathbb{R}$ s.t. $F|_Y = f$ and $F(x) \leq q(x)$ $\forall x \in X$.

Example: (Banach Space Projection Theorem)

Let X be normed vector space and let $N \subset X$ be finite dim linear subspace. Then $\exists Y \subset X$ subspace that $X = N + Y$ and $N \cap Y = 0$

Proof: N is finite dim. Say $N = \text{span}\{x_1, \dots, x_n\}$. Find $f, f_i \in N^*$

s.t. $f_i f_j = \delta_{ij} = \begin{cases} 0 \text{ if } & \text{if } i \neq j \\ 1 \text{ if } & \text{if } i = j \end{cases}$.

Extend f to X^* by Hahn Banach.

If x in N, $\left(x = \sum_{i=1}^{n} f(x)\, x_i - y\right)$

Let $Y = \ker f \cdots \cap \cdots \ker(f_n)$. Then $y \in Y$ for any $y \in X$.

Then show $N \cap Y = \{0\}$

(exercise).

Chap 7 : Linear Operators

- Equivalence of boundedness and continuity.
- $\mathcal{L}(E,F)$ Banach if F Banach.
- Inverse: Inverse mapping Theorem (no proof).

  If $A \in \mathcal{L}(X,Y)$. If X, Y Banach, if A injective & surjective then $A^{-1} \in \mathcal{L}(Y,X)$.

  (why does it need completeness)

- If $\|A\| < 1$, $I-A$ invertible. Implies set of invertible operators in $\mathcal{L}(E)$ is open.

- Adjoints definition relies on Riesz Representation. (look at proof)

  - If A self adjoint / hermitian, then $\|A\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |(Ax, x)|$

HW 7.29. ($\operatorname{Ran} A^{\perp} = \ker A^*$)

- Spectrum: Know definition.

  $\sigma(A)$ is compact in $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|A\|\}$.

Chap 8 : Compact Operator

- Definition (sequence vs set def.)
- Examples: finite rank operator
- Thm If $T_n \to T$, T_n cpt $\Rightarrow$ T cpt (3-ε proof - review)

- H separable, cpt operator are limits of finite rank operator (review proof).

  Suppose K cpt, $K_\varepsilon x = \sum_{n=1}^{N} (x, e_n)(Ke_n)$

  $(e_n)_{n=1}^{\infty}$ complete orthonormal sequence. Amounts to showing (Ke_n) small.

  If K cpt & Hermitian, then

  $$Ke_n = \lambda e_n$$

  $\Rightarrow \lambda_n \longrightarrow 0$.

  But if K not Hermitian, still show this. (see proof)

- <u>Fredholm Alternative</u>. If K cpt, either $I-K$ invertible or $Kf = f$ has non-zero solution.

  (we assumed separability in proof but we don't need it since we can get a finite approx. w/ spectral thm)

- <u>Hilbert Schmidt operator</u>

  Main Example: Integral operator on $L^2(a,b)$.

- <u>Spectral Thm</u>: Cpt, Hermitian

- <u>Sturm Liouville</u>. Find Green's Function and convert to functional Analysis problem.