Chorin's Projection Method with Spectral Collocation for 2D Incompressible Navier Stokes

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The setup and time discretization with Crank Nicholson remains the same as the finite difference version of this (see other directory in repo). Here, we focus on deriving a Chebyshev pseudo-spectral estimator.

Recall the NSE (with no force function)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \Delta \mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0$$
$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega$$

Chorin: Step 1 Also recall that the Chorin's projection method first ignores pressure to comute an intermediate velocity field

$$\frac{\partial \mathbf{u}^*}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \Delta \mathbf{u}^*$$
$$\mathbf{u}^* = 0 \quad \text{on} \quad \partial\Omega$$

We discretize time first with Adams-Bashford and implicit Crank-Nicholson

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\triangle t} + (\mathbf{u}^{\frac{n+1}{2}} \cdot \nabla)\mathbf{u}^{\frac{n+1}{2}} = \Delta \mathbf{u}^{\frac{n+1}{2}}$$

So far things are identical to the finite difference version of Chorin's. But to solve this, we do not discretize spatial dimensions anymore: we use (pseudo)spectral methods:

We want to approximate the solution $\mathbf{u}^*$ as truncated series of Chebyshev polynomials: $\{T_k(x)\}_{k=1}^{\infty}$ where $T_k(x) = \cos(k \cos^{-1} x)$; each polynomial is restricted to $[-1, 1]$.

$$u_N^*(x) = \sum_{k=0}^{\infty} \hat{u}_k^* T_k(x) \approx \sum_{k=0}^{N} \hat{u}_k^* T_k(x)$$

where $\mathbf{u}^* = (u^*, v^*)$. We just write the derivation for the first equation for simplicity. To get the Chebyshev coefficients, we have to compute the (normalized) inner product:

$$\hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^{1} u T_k w dx$$

where $c_k = \begin{Bmatrix} 2 & \text{if} & k = 0 \\ 1 & \text{if} & k \geq 1 \end{Bmatrix}$. This is hard to compute, so we need to estimate the integral. Normally, this would require a lot of points but we can get away with carefully selected ones and a Gaussian quadrature. The best points turn out to be the roots of another Chebyshev polynomial with degree one higher; these are called the Gauss-Lobatto points – they end up with denser spread near the boundaries (-1 and 1).

$$x_i = \cos \frac{\pi i}{N}, i = 0, ..., N \tag{1}$$

resulting in,

$$\hat{u}_k = \frac{2}{\bar{c}_k N} \sum_{i=0}^{N} \frac{1}{\bar{c}_i} u_i T_k(x_i), k = 0, ..., N \tag{2}$$

where $\bar{c}_k = \begin{Bmatrix} 2 & \text{if} & k = 0 \\ 1 & \text{if} & 1 \leq k \leq N - 1 \\ 2 & \text{if} & k = N \end{Bmatrix}$. Note that to convert back and forth between spectral coefficients ($\hat{u}_k$) and the values at the collocation points ($u_N(x_i)$) is a matrix multiplication. To be explicit, let $\mathcal{T} = [cosk\pi i/N], k, i = 0, ..., N$ and $\mathcal{T}^{-1} = [2(\cos \pi i/N)/(\bar{c}_k \bar{c}_i N)]$. Then,

$$\mathcal{U}^* = \mathcal{T} \hat{\mathcal{U}}^*$$
$$\hat{\mathcal{U}}^* = \mathcal{T} \mathcal{U}^*$$

where $\mathcal{U}^* = [u^*(x_0), ..., u^*(x_N)]$ and $\hat{\mathcal{U}}^* = [\hat{u}_0, ..., \hat{u}_N]$.