Direct Finite Difference Simulations of 2D Incompressible Navier Stokes

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The incompressible navier stokes equation is the following:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \tag{1}$$

We ignore the continuity equation for now. For now, we will consider it in two dimensions as in $\mathbf{u} = (u, v)$. In general, bold will represent vectors. We want to derive an explicit representation for pressure (so that we can apply numerical methods). To do so, we can take the divergence of the NS equations (assuming smoothness).

$$\nabla \cdot (\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}) = \nabla \cdot (-\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}) \tag{2}$$

We do the derivation piece by piece, starting with the LHS.

$$\begin{aligned}\nabla \cdot (\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}) &= \frac{\partial}{\partial t}(\nabla \cdot \mathbf{u}) + \nabla \cdot (\mathbf{u} \cdot \nabla)\mathbf{u} \\ &= 0 + \nabla \cdot (\mathbf{u} \cdot \nabla)\mathbf{u}\end{aligned}$$

Now the RHS.

$$\begin{aligned}\nabla \cdot (-\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}) &= \nu \nabla(\nabla^2 \mathbf{u}) - \frac{1}{\rho}\nabla^2 p + \nabla \cdot \mathbf{F} \\ &= 0 - \frac{1}{\rho}\nabla^2 p + \nabla \cdot \mathbf{F}\end{aligned}$$

Putting it together:

$$\begin{aligned}\nabla \cdot (\mathbf{u} \cdot \nabla)\mathbf{u} &= -\frac{1}{\rho}\nabla^2 p + \nabla \cdot \mathbf{F} \\ \nabla^2 p &= \rho(\nabla \cdot (\mathbf{F} - (\mathbf{u} \cdot \nabla)\mathbf{u}))\end{aligned}$$

For now, we assume no source flow ($\mathbf{F} = 0$). We rewrite the equations in non-vector format as the following set of differential equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{3}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \tag{4}$$

$$\frac{\partial^2 p}{dx^2} + \frac{\partial^2 p}{\partial y^2} = -\rho(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{]\partial y}) \tag{5}$$

We use finite difference methods to discretize each formula, using a combination of forward-difference, backward-difference and central-difference estimators.

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + u_{i,j}^{n}\frac{u_{i,j}^{n} - u_{i-1,j}^{n}}{\Delta x} + v_{i,j}^{n}\frac{u_{i,j}^{n} - u_{i,j-1}^{n}}{\Delta y} = \tag{6}$$

$$-\frac{1}{\rho}(\frac{p_{i+1,j}^{n} - p_{i-1,j}^{n}}{2\Delta x}) + \nu(\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j}^{n}}{\Delta y^2}) \tag{7}$$

A similar one can be done for the v momentum equation. We also write out the estimator for the Poisson pressure equation:

$$\frac{p_{i+1,j}^{n} - 2p_{i,j}^{n} + p_{i-1,j}^{n}}{\Delta x^2} + \frac{p_{i,j+1}^{n} - 2p_{i,j}^{n} + p_{i,j-1}^{n}}{\Delta y^2} = \tag{8}$$

$$-\rho((\frac{u_{i+1,j}^{n} - u_{i-1,j}^{n}}{2\Delta x})^2 + 2(\frac{u_{i,j+1}^{n} - u_{i,j-1}^{n}}{2\Delta y})(\frac{v_{i+1,j}^{n} - v_{i-1,j}^{n}}{2\Delta x}) + (\frac{v_{i,j+1}^{n} - v_{i,j-1}^{n}}{2\Delta y})^2) \tag{9}$$

Notice in the first finite difference equation, there is only one unknown: $u_{i,j}^{n+1}$, everything else is known. For pressure, it is an iterative algorithm so we need to run it many times.