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5.a Langrangian Problem

What we have

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + \frac{1}{6}g\varphi^3 + \frac{1}{24}h\varphi^4$$

$$where$$

$$\langle 0|\varphi(0)|0\rangle = v$$
$$\langle p|\varphi(0)|0\rangle = u$$

Now we want to get a Langrangian density with a normalized φ' to statisfy

$$\langle 0|\varphi(0)'|0\rangle = 0$$
$$\langle p|\varphi(0)'|0\rangle = 1$$

What I do here is tranfering φ to φ' like this

$$\varphi' = \frac{\varphi - v}{u} \Longrightarrow \varphi = u\varphi' + v$$

substitute φ using φ' into the Lagrangian density equation, we get

$$\begin{aligned}
\mathcal{L} =& -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + \frac{1}{6}g\varphi^3 + \frac{1}{24}h\varphi^4 \\
=& -\frac{1}{2}\partial^\mu(u\varphi'+v)\partial_\mu(u\varphi'+v) - \frac{1}{2}m^2(u\varphi'+v)^2 + \frac{1}{6}g(u\varphi'+v)^3 + \frac{1}{24}h(u\varphi'+v)^4 \\
=& -\frac{1}{2}u^2\partial^\mu\varphi'\partial_\mu\varphi' - \frac{v^2}{2} \\
& - \frac{m^2u^2\varphi'^2}{2} - m^2uv\varphi' - \frac{m^2v^2}{2} \\
& + \frac{gu^3\varphi'^3}{6} + \frac{gu^2v\varphi'^2}{2} + \frac{guv^2\varphi'}{2} + \frac{gv^3}{6} \\
& + \frac{hu^4\varphi'^4}{24} + \frac{hu^3v\varphi'^3}{6} + \frac{hu^2v^2\varphi'^2}{4} + \frac{huv^3\varphi'}{6} + \frac{hv^4}{24} \\
=& -\frac{1}{2}Z_{\varphi'}\partial^\mu\varphi'\partial_\mu\varphi - \frac{1}{2}m^2Z_m\varphi'^2 + \frac{1}{6}Z_g g\varphi'^3 + \frac{1}{24}Z_h h\varphi'^4 + Y\varphi' + \Omega_0
\end{aligned}$$

Where

$$\begin{aligned}
Z_\varphi &= u^2 \\
Z_m &= u^2 - \frac{gu^2v}{m^2} - \frac{hu^2v^2}{2m^2} \\
Z_g &= u^3 + \frac{hu^3v}{g} \\
Z_h &= u^4 \\
Y &= -m^2uv + \frac{guv^2}{2} + \frac{huv^3}{6} \\
\Omega_0 &= -\frac{v^2}{2} - \frac{m^2v^2}{2} + \frac{gv^3}{6} + \frac{hv^4}{24}
\end{aligned}$$

5.b Method Of Steepest Descents[1]

Consider the asymptotic behavior (for large t, assumed real) of a function $f(t)$, where

- $f(t)$ is represented by an integral of the generic form

$$f(t) = \int_C F(z,t)dz,$$

  with $F(z,t)$ analytic in z, but also parametrically dependent on t;
- The integration path C is, or can be deformed to be, such that for large t the dominant contribution to the integral arises from a small range of z in the neighborhood of the point z_0 where $|F(z_0,t)|$ is a maximum **on the path;**
- The integration path will pass through z_0 in the orientation that causes the most rapid decrease in $|F|$ on departure from z_0 in either direction along the path (hence the name **steepest descents**); and
- In the limit of large t the contribution to the integral from the neighborhood of z_0 asymptotically approaches the exact value of $f(t)$.

We call z_0 as saddle point. For analytic functions, they do not have maxima and minima of their moduli in the range of their analyticity, they have only saddle points. Once we identify z_0 and the directions of steepest desent in $|F(z,t)|$, we complete the specification of the method of steepest descents, also called the **saddle point method** of asymptotic approximation, by assuming that the significant contributions to the integral are from a small range of $0 \leq r \leq a$ in each of the two directions along the path.

For example, take the form

$$F(z,t) = e^{w(z,t)} = e^{u(z,t)+iv(z,t)}$$

if the contour C can be deformed in such a way that it passes through the saddle point in the direction of the steepest descent, the large value of parameter t ensures that the essential contribution to the integral comes from a small vicinity of the saddle point since the exponent quickly decreaseds and the detailed shape of the curve of the steepest decent far from the saddle point is not important. In the saddle point the derivative $w' = 0$. The direction of the steepest descent is determined by the requirement that the difference $u(z,t) - u(z_0,t)$ is negative. By Taylor expanding $w(z,t)$ about z_0 and introduing polar forms, we can get

$$f(t) \approx 2e^{w_0+i\theta} \int_0^a e^{-|w_0''|r^2/2} dr$$

Now make the key assumption of the method, namely that $|w_0''|$, the measure of the rate of decrease in $|F|$ as we leave z_0, is large enough that the bulk of the value of the integral has already been attained for small a, and that the exponential decreasein the value of the intgrand enables us to replace a by infinity without making significant error. Take $a = \infty$, get the value $\sqrt{\pi/2|w_0''|}$ of the integral. Then we get

$$f(t) \approx F(z_0,t)e^{i\theta}\sqrt{\frac{2\pi}{|w''(z_0,t)|}}$$

Sometimes it is sufficient to apply the method of steepest descents only to the rapidly varying of an integral.

$$f(t) = \int_C g(z,t)F(z,t)dz \approx g(z_0,t)\int_C F(z,t)dz$$

[1] Arfken, G. "Method of Steepest Descents." §12.7 in *Mathematical Methods for Physicists, 7th ed.* Elsevier pp. 585-589, 2013