Computer-Assisted Proofs of Solitons in the Gross-Pitaevskii Equation

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We are interested in solutions to the Gross-Pitaevskii equation

$$i\partial_t \psi = -\partial_x^2 \psi + \cos(2x)\psi + c|\psi|^2\psi$$

of the form ψ(t,x) = exp(-iat) u(x),where u : ℝ → ℝ satisfies:

$$u'' + au - \cos(2x)u - cu^3 = 0, \qquad \lim_{x\to\pm\infty}(u(x), u'(x)) = (0,0)$$

These are known as standing wave solutions, localized solutions, or soliton solutions. *Given a numerically approximated soliton, we ask: under what conditions can we prove the existence of a true, rigorous solution close to it?* To explore this question, we use tools from the field of computer-assisted proofs (CAPs).

Problem Setup

We split the proof of a soliton into three parts:

Since the equation is even in x, we solve the equation on half of the domain and extend the solution symmetrically.

The non-homogeneous potential cos(2x) in the equation satisfies the ODE

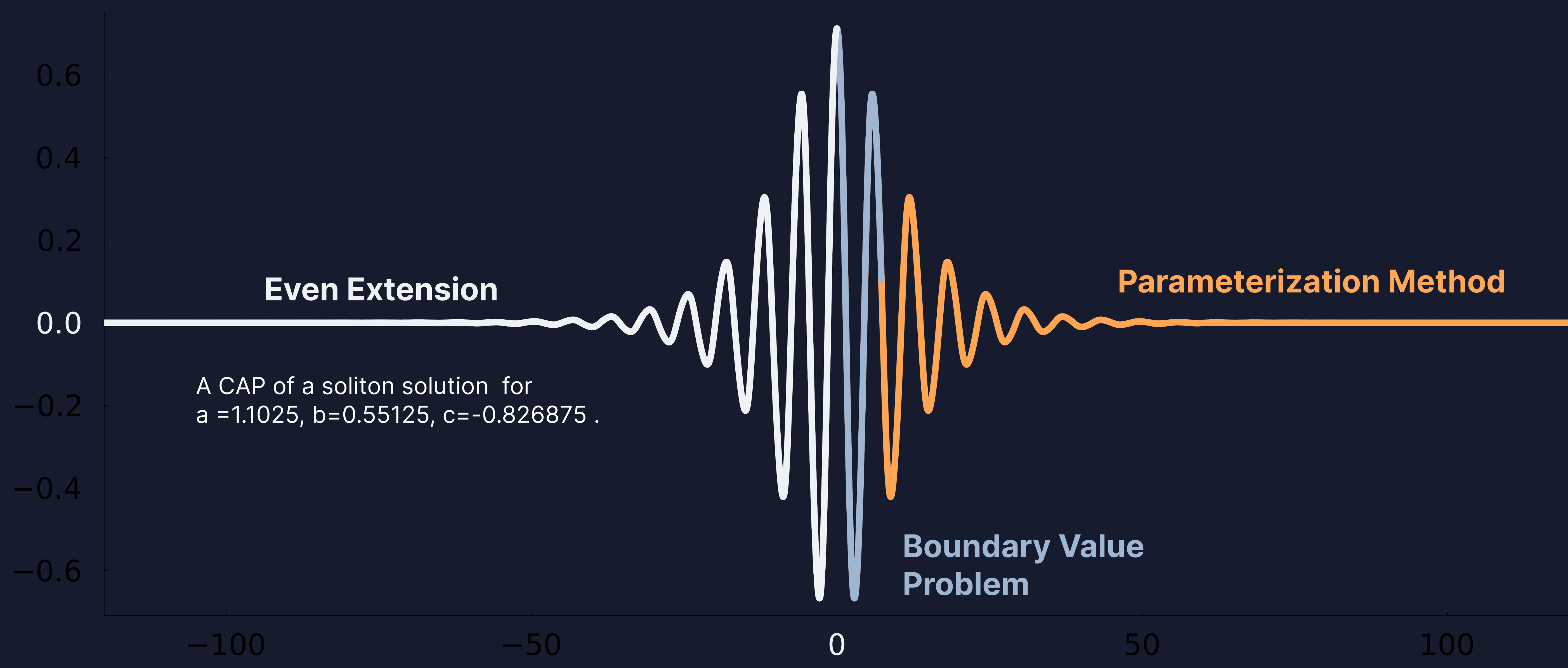
$$y'' = -4y, \quad y(0) = 1, \quad y'(0) = 0.$$

This structure allows us to write the problem as a boundary value problem:

$$\mathbf{u} = (u_1, u_2, u_3, u_4), \qquad \gamma(x) = (0, 0, \cos(2x), -2\sin(2x)),$$

$$\mathbf{u}' = \begin{bmatrix} u_2 \\ -au_1 + bu_3u_1 + cu_1^3 \\ u_4 \\ -4u_3 \end{bmatrix}, \quad \mathbf{u}(0) = \begin{bmatrix} u_0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}(L) \in W(\sigma, \theta)$$

for some $u_0, L, \sigma, \theta \in \mathbb{R}.$



Here W is a parameterization of the stable manifold attached to the periodic orbit γ , as constructed by the parameterization method:

$$\begin{aligned}\frac{\partial}{\partial\theta}W(\sigma,\theta) + \lambda\sigma\frac{\partial}{\partial\sigma}W(\sigma,\theta) &= f(W(\sigma,\theta)),\\ W(0,\theta) &= \gamma(\theta),\\ \frac{\partial}{\partial\sigma}W(0,\theta) &= v(\theta),\\ v' + \lambda v &= Df(\gamma(\theta))v.\end{aligned}$$

Zero-Finding Approach to CAPs

To solve the differential equations involved in the BVP:

Each solution is expanded in a basis adapted to the problem: Taylor and Fourier series for the parameterization of the stable manifold, and Chebyshev series for the boundary value problem.

We then define a zero-finding problem F(x) = 0 for the corresponding coefficients in an infinite-dimensional sequence space. We use a Newton-type theorem (also called the radii polynomial theorem) to show the existence of true zeros of F.

The theorem requires bounding quantities involving both the evaluation of F and its derivative DF. Since these operators act on infinite-dimensional spaces,the main challenge is to derive explicit, bounds that can be implemented as a computer program. To rigorously evaluate these bounds we use interval arithmetic.

We apply this approach to validate several numerically observed solitons appearing in the Gross-Pitaevskii literature, including the one illustrated in the figure.

Full details of the bounds, references, and source code links are available in our preprint:

https://arxiv.org/abs/2503.04701

The preprint can also be accessed via the QR code.