A biologically-plausible learning rule using reciprocal feedback connections

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Locally learning pseudoinverse feedback connections

- Linearized modification of the bio-plausible Recirculation algorithm for autoencoders is equivalent to learning a pair of pseudoinverse weight matrices. [1]
- Can be implemented with random, mean-zero noise at each layer.

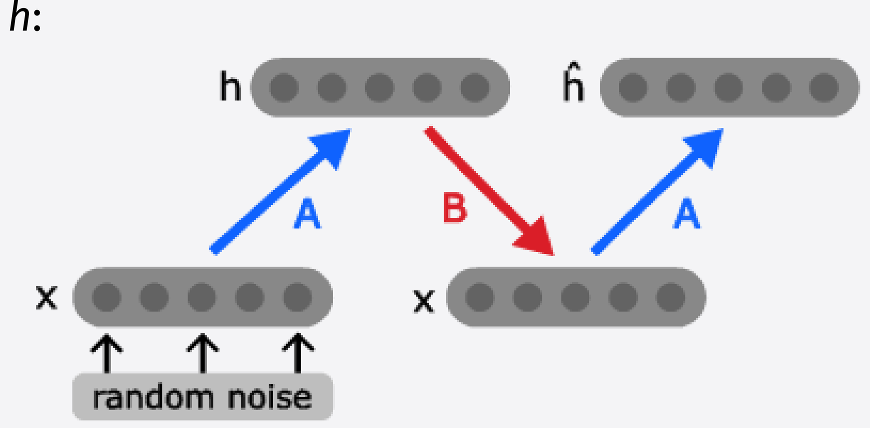
Dynamics:

$h = Ax$	initial hidden representation
$\hat{x} = Bh$	input reconstruction
$\hat{h} = A\hat{x}$	reconstructed hidden representation

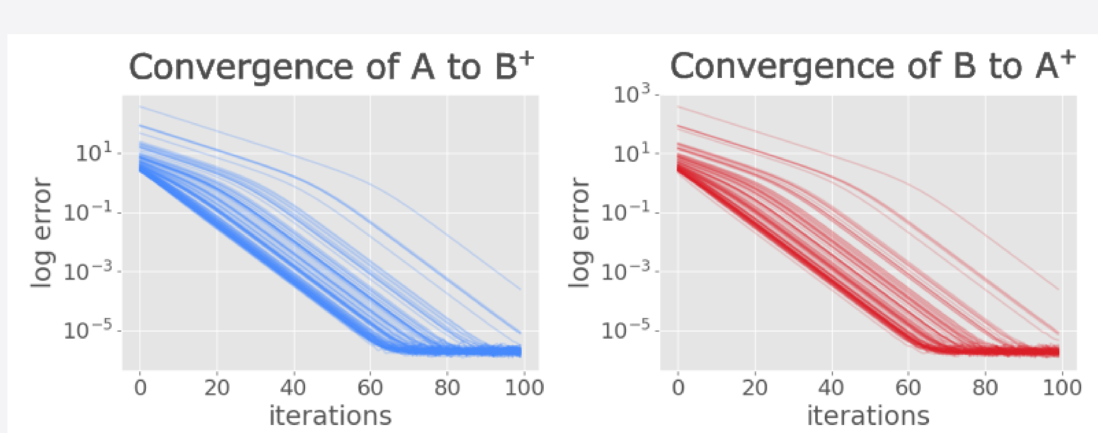
Learning rules:

$\Delta A = (h - \hat{h})x^T$	hidden reconstruction error
$\Delta B = (x - \hat{x})h^T$	input reconstruction error

Unrolling dynamics in time, with original input as x, initial hidden representation as h, reconstructed input as $\hat{x}$ and reconstructed hidden representation as $\hat{h}$:



Simulations of these learning rules (trained concurrently) show convergence to the pseudoinverse:



Def. Moore-Penrose Pseudoinverse

The unique Moore-Penrose pseudoinverse of the $n \times m$ matrix A is the $m \times n$ matrix B satisfying conditions:

1. $ABA = A$ 2. $BAB = B$ 3. $(AB)^T = AB$ 4. $(BA)^T = BA$

Acknowledgements

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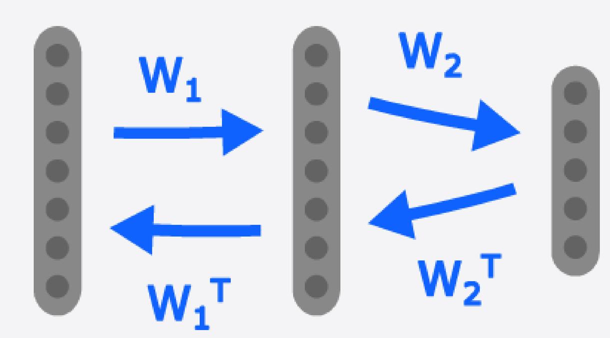
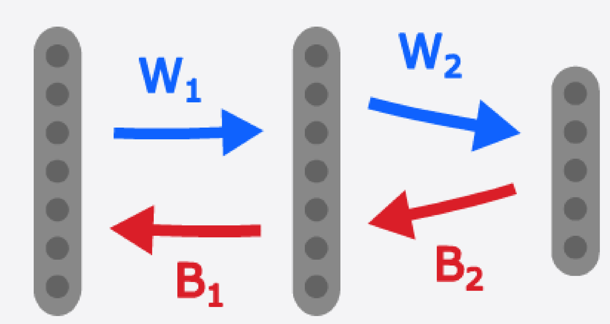
1. Hinton E., G. et al. *AIP* (1988). 2. Crick, F. *Nature* **337**, 129–132 (Jan. 1989). 3. Lillicrap, T. P. et al. *Nat. Rev. Neuros.* **21**, 335–346 (Apr. 2020). 4. Lillicrap, T. P. et al. *Nat. Comm.* **7** (Nov. 2016). 5. Levin, Y. et al. *Nonlinear Analysis: Theory, Methods and Applications* **47**, 1961–1971 (Aug. 2001).

Background

- Problem: Backpropagation is biologically implausible [2]
- Idea: Layer-wise feedback connections [3]



$$\frac{dL}{dW_2} = \frac{dL}{dh_2}\frac{dh_2}{dW_2} = eh_1^T$$



$$\frac{dL}{dW_1} = \frac{dL}{dh_1}\frac{dh_1}{dW_1} = W_2^T e h_0^T$$

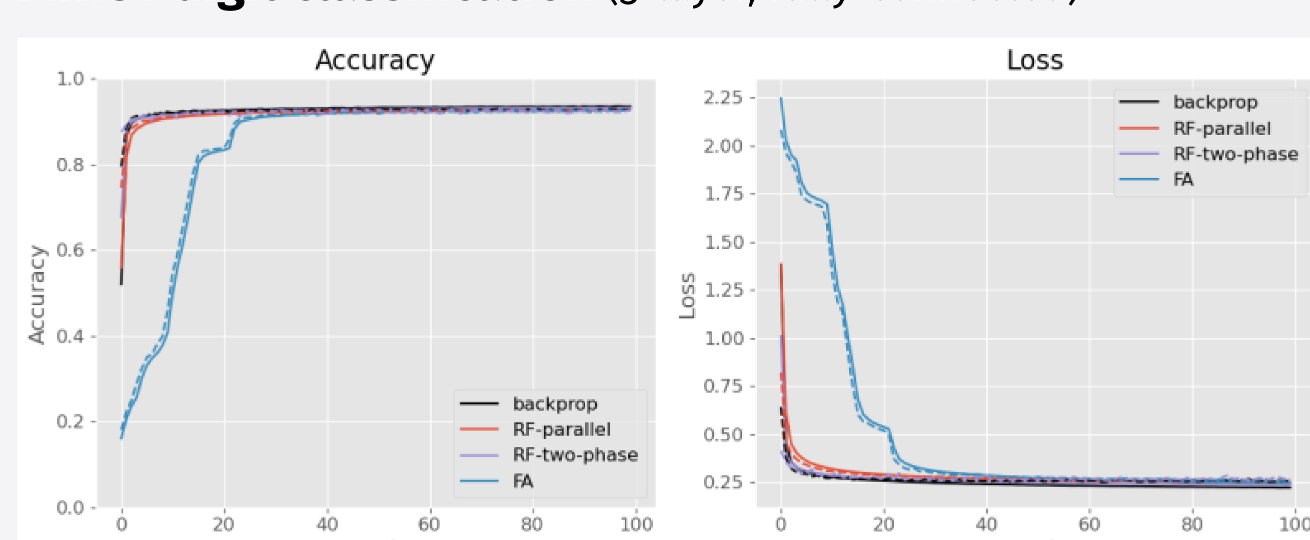
Gradient descent: Separate feedback:

$$\delta_{W_l} = -(J^L_{W_l})^T e h^T_{l-1} \qquad \delta_{W_l} = -\left(\prod_{i=l+1}^{L} B_i\right) e h_{l-1}{}^T$$

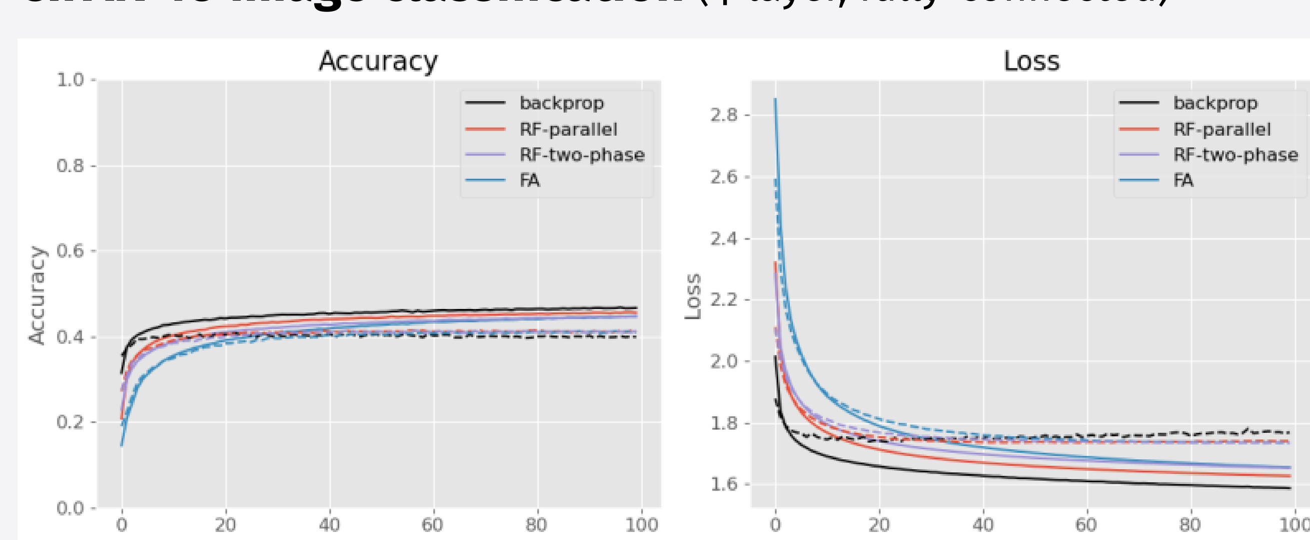
- What choice of B is biologically realistic, and capable of minimizing a global error signal?

Results

MNIST digit classification (5-layer, fully-connected)

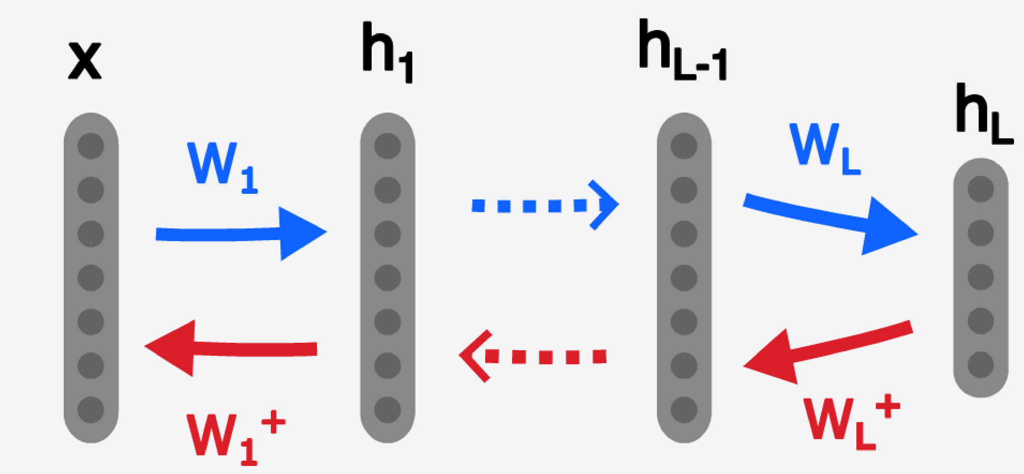


CIFAR-10 image classification (4-layer, fully-connected)



In both CIFAR-10 and MNIST image classification tasks, our method (RF) reaches a similar asymptotic error to backpropagation, with faster rate of convergence than the Feedback Alignment algorithm [4]

Global error minimization using pseudoinverse feedback



Global loss (for one input): $\mathcal{L} = \frac{1}{2}||h_L - h_L^*||^2$

Residual vector: $e = h_L - h_L^*$

Layer-wise Jacobians:

$$J^{\mathcal{L}}_{W_l} = J^e_{W_l} e$$

$$J^e_{W_l} = (h_{l-1} \otimes J^e_{h_l}) \quad \text{w.r.t layer weight matrix}$$

$$J^e_{h_l} = J^e_{h_{l+1}} W_{l+1} = W_L W_{L-1} \ldots W_{l+1} \quad \text{w.r.t layer activation vector}$$

A generalized left inverse can also be defined recursively, corresponding, physically, to the "backwards" application of each layer-wise pseudoinverse to a top-level vector.

Layer-wise left reciprocals:

$$B_{W_l} = (h^+_{l-1} \otimes B_{h_l}) \quad \text{w.r.t layer weight matrix}$$

$$B_{h_l} = W^+_{l+1} B_{h_{l+1}} = W^+_{l+1} \ldots W^+_{L-1} W^+_L \quad \text{w.r.t layer activation vector}$$

Note: $(W_L W_{L-1} \ldots W_{l+1})^+ \neq W^+_{l+1} \ldots W^+_{L-1} W^+_L$, in general! So, $(J^e_{W_l})^+ \neq B_{W_l}$.

However, the recursion on B preserves Moore-Penrose properties 1,2 and 3 for a full-rank, contracting architecture.

Learning rule:

$$\delta_{W_l} = \left(\prod_{i=l+1}^{L} W_i^+\right) e h_{l-1}{}^T$$

Theorem (Levin and Ben-Israel, 2001)

(Informal) Let $f \rightarrow \mathbb{R}^n \rightarrow R^m$ be a vector function we want to minimize, and let T_x be a $\{2\}$-inverse of it's Jacobian matrix, J^f_x. Then, under certain conditions, the iteration

$$x_{k+1} - x_k = -T_{x_k} f(x_k)$$

converges to a fixed point x^* which satisfies

$$T_{x^*} f(x^*) = 0$$

[5]. If T_x is also a $\{1,2,3\}$ inverse, and is full-rank, this solution is in the same nullspace as that reached by gradient descent on $\frac{1}{2}||f||^2$.