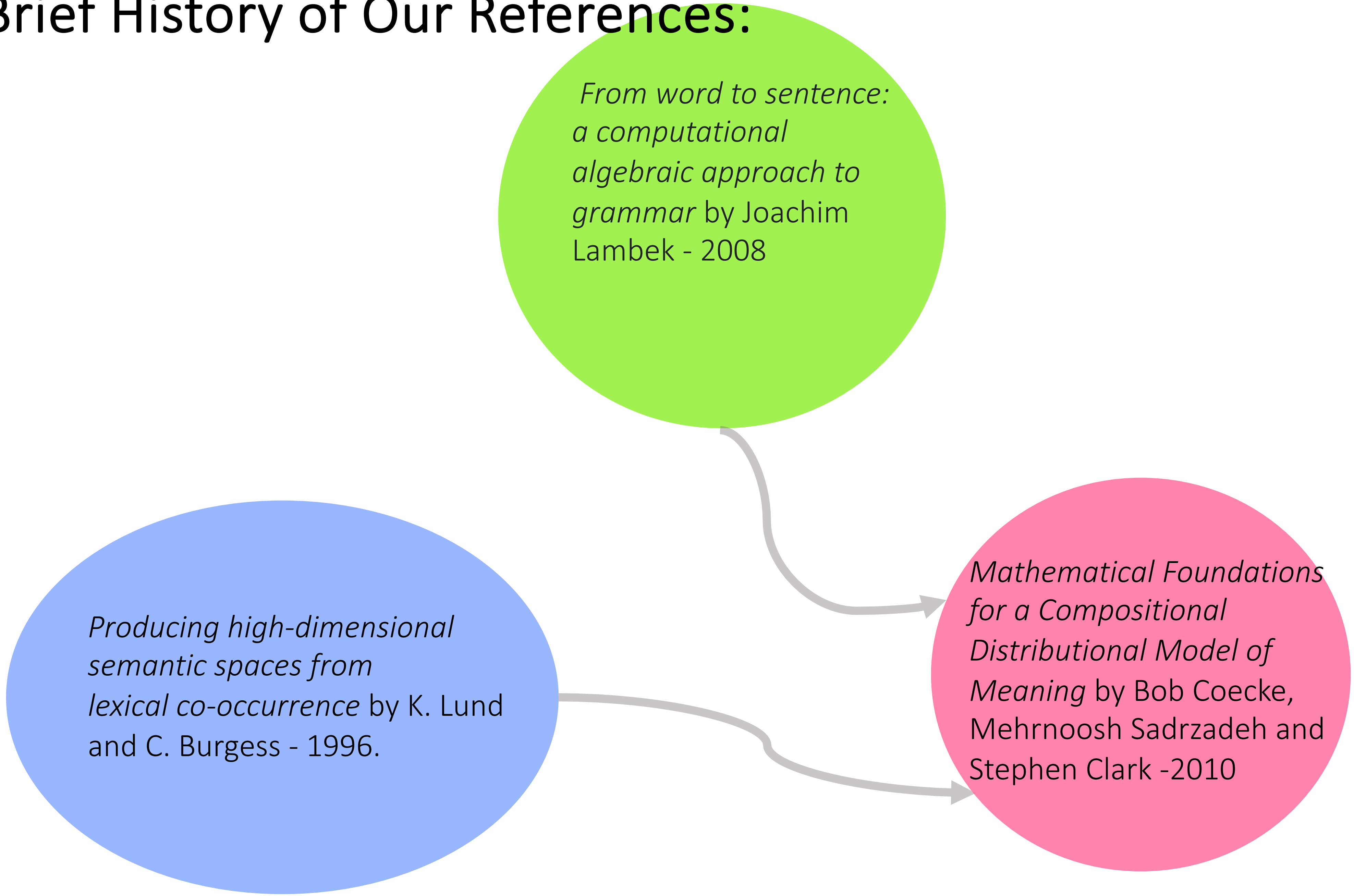
Explicit Categorical Constructions Used in Modeling Sentences

Monoidal Product Category

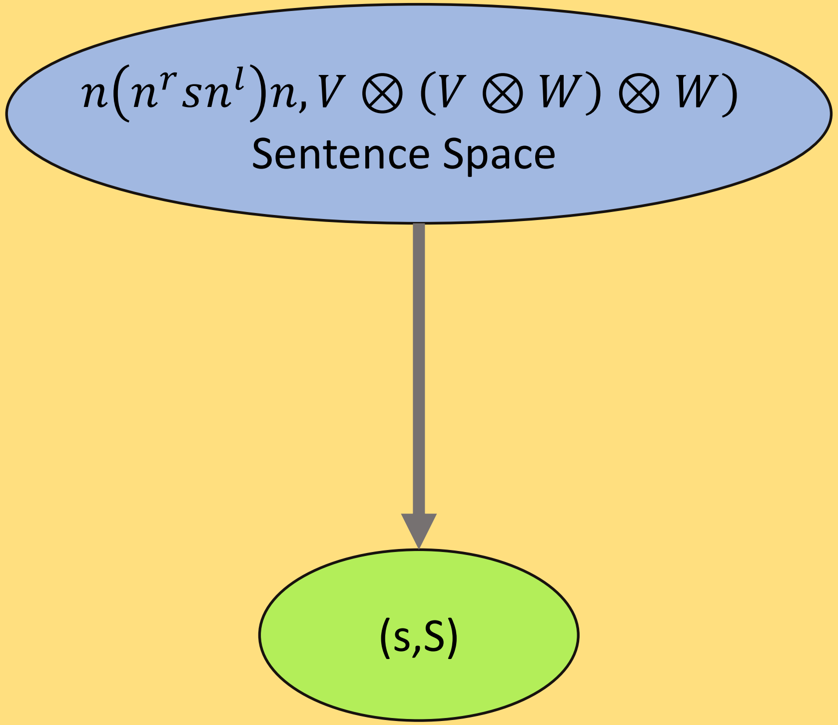
Mia Goldstein and Emily Herbert
SUNY New Paltz

Thank you to RSCA and NCUWM for getting us here!

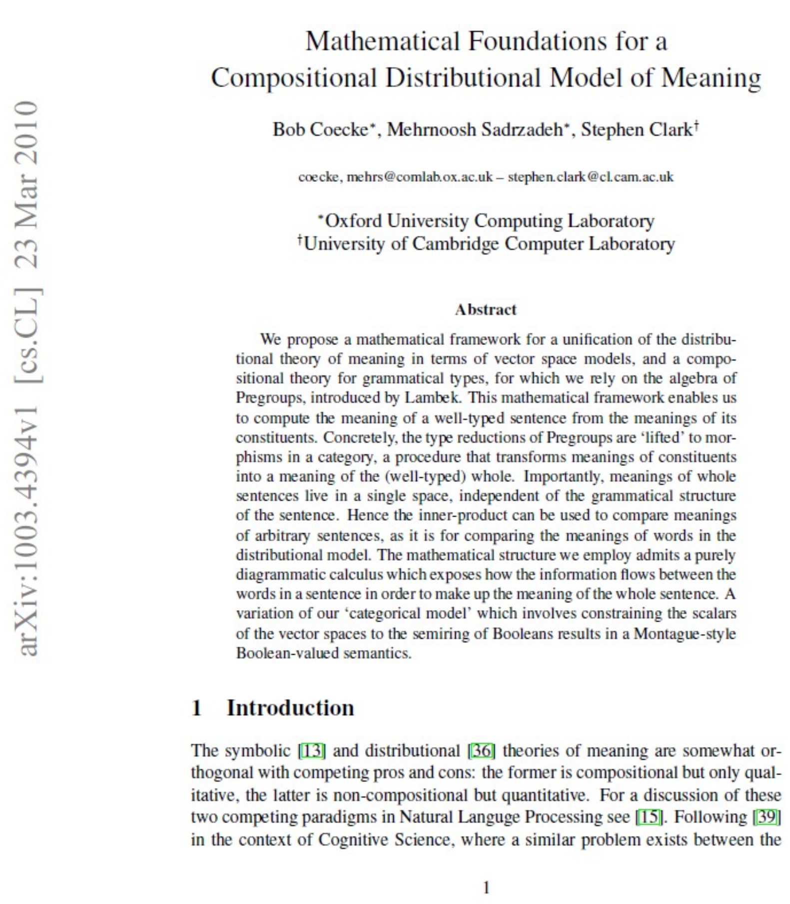
Brief History of Our References:



Our Motivation:



Original Example of sentence processing



arXiv:1003.4394v1 [cs.CL] 23 Mar 2010

Mathematical Foundations for a
Compositional Distributional Model of Meaning

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Abstract

We propose a mathematical framework for a unification of the distributional theory of meaning in terms of vector space models, and a compositional theory for grammatical types, for which we rely on the algebra of Pregroups, introduced by Lambek. This mathematical framework enables us to compute the meaning of a well-typed sentence from the meanings of its constituents. Concretely, the type reductions of Pregroups are 'lifted' to morphisms in a category, a procedure that transforms meanings of constituents into a meaning of the (well-typed) whole. Importantly, meanings of whole sentences live in a single space, independent of the grammatical structure of the sentence. Hence the inner-product can be used to compare meanings of arbitrary sentences, as it is for comparing the meanings of words in the distributional model. The mathematical structure we employ admits a purely diagrammatic calculus which exposes how the information flows between the words in a sentence in order to make up the meaning of the whole sentence. A variation of our 'categorical model' which involves constraining the scalars of the vector spaces to the semiring of Booleans results in a Montague-style Boolean-valued semantics.

1 Introduction

The symbolic [13] and distributional [36] theories of meaning are somewhat orthogonal with competing pros and cons: the former is compositional but only qualitative, the latter is non-compositional but quantitative. For a discussion of these two competing paradigms in Natural Language Processing see [15]. Following [39] in the context of Cognitive Science, where a similar problem exists between the

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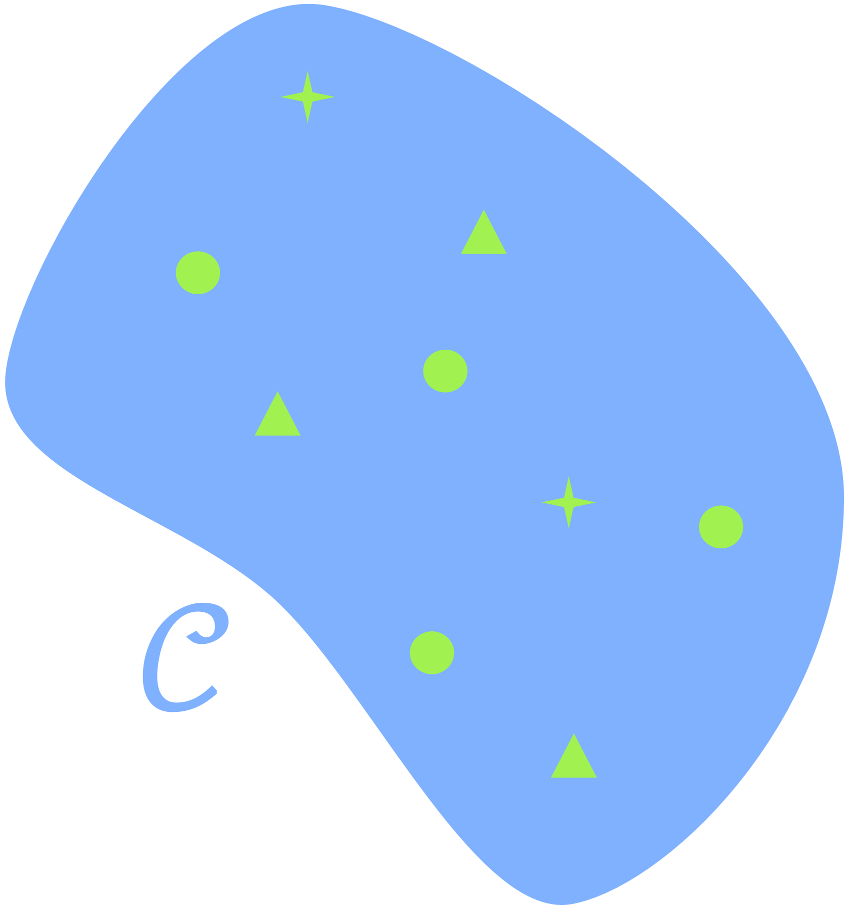
Categories

Category:



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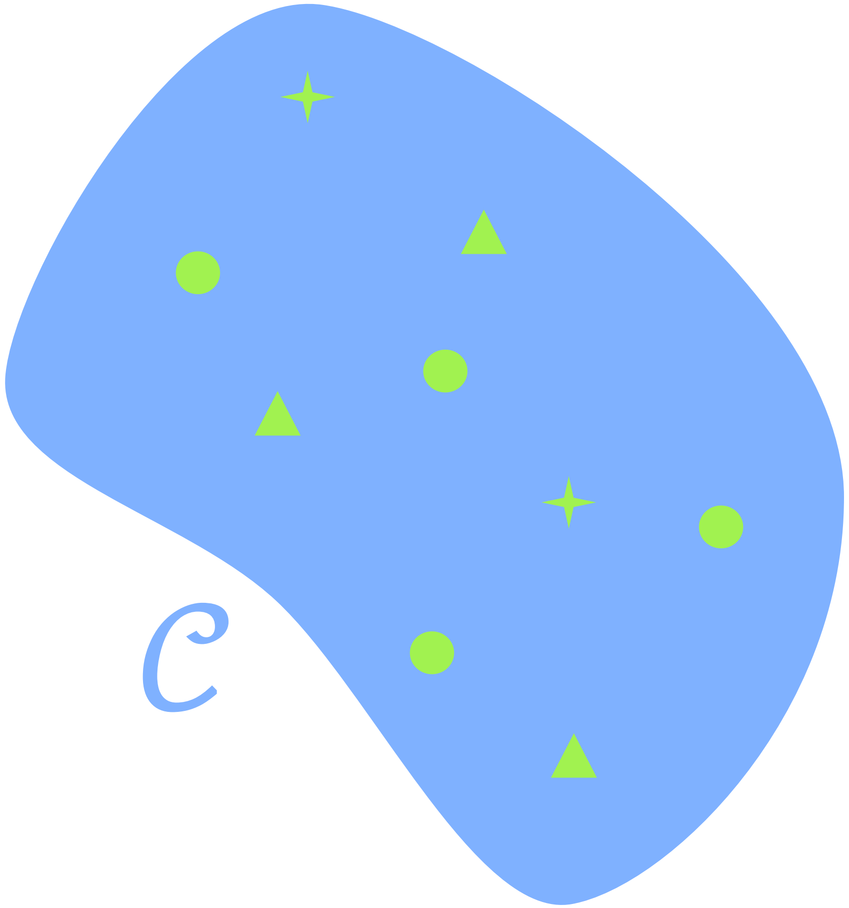
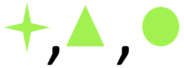


Objects: 

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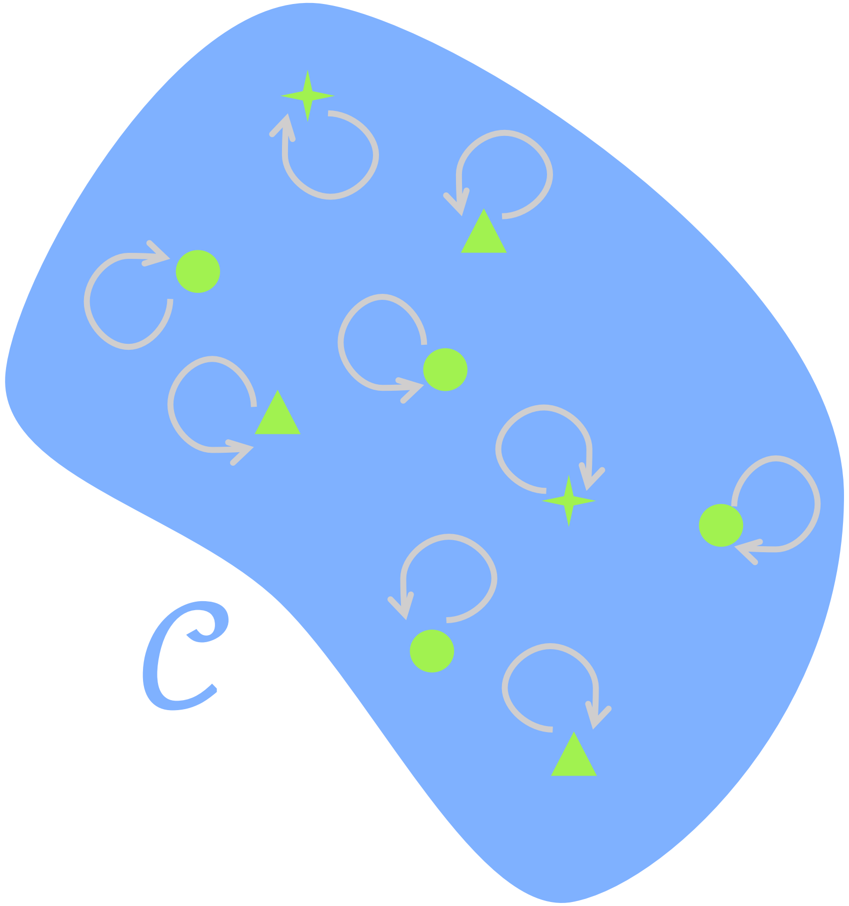


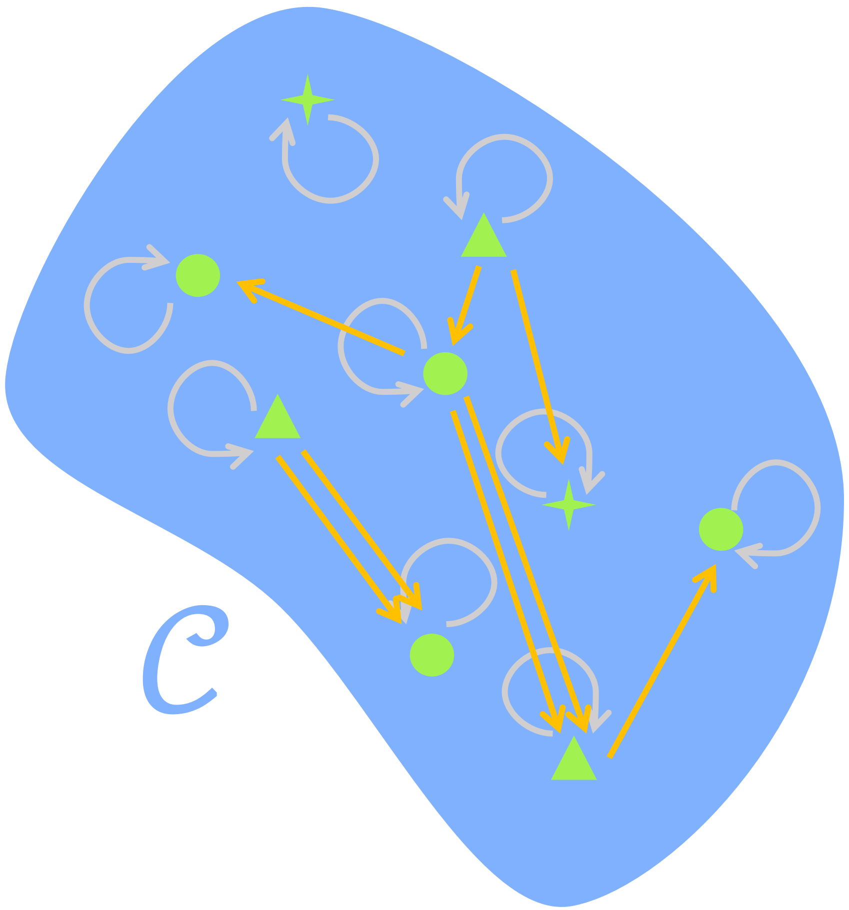
Objects: ✦, ▲, ●

Morphisms:

- Identity Morphism

  $id: c \mapsto c \in \mathcal{C}$

Category:

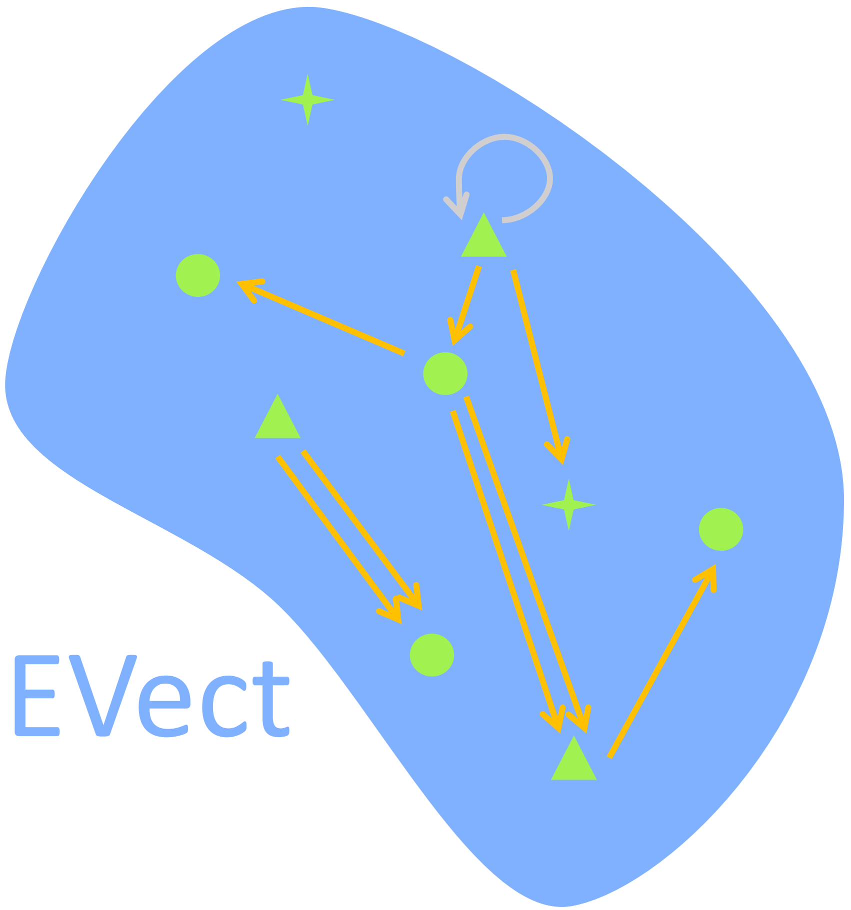


Objects: ✦, ▲, ●

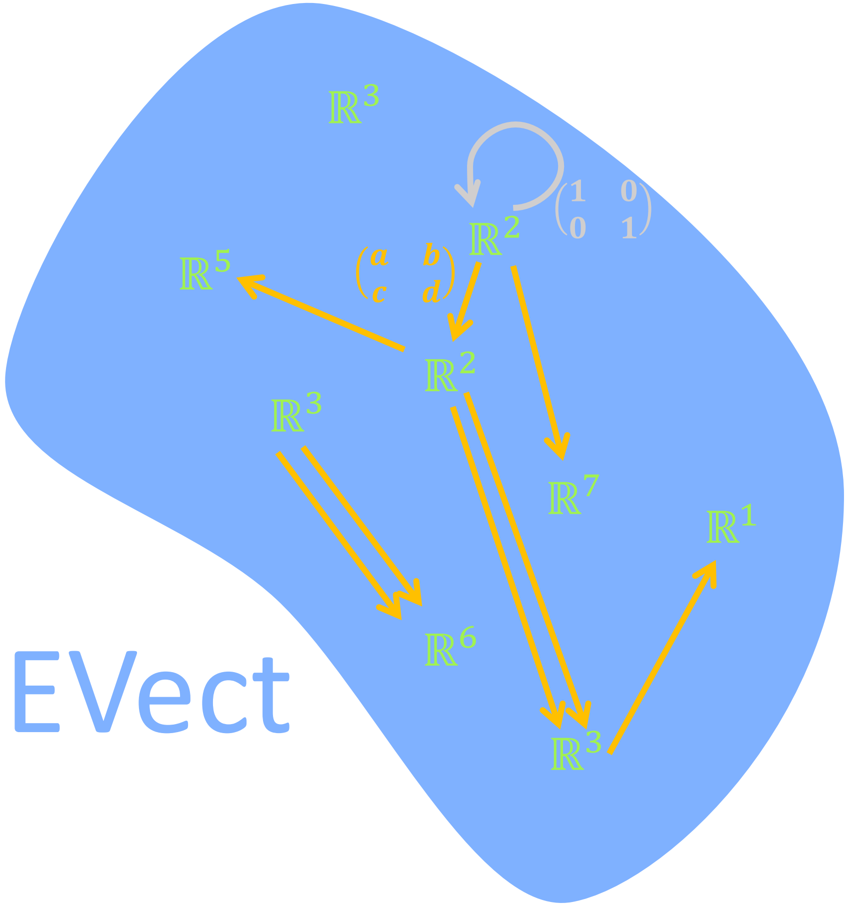
Morphisms:

- Identity Morphism
  $id: c \mapsto c \in \mathcal{C}$
- Morphisms between pairs of objects

Example: EVect



Example: EVect

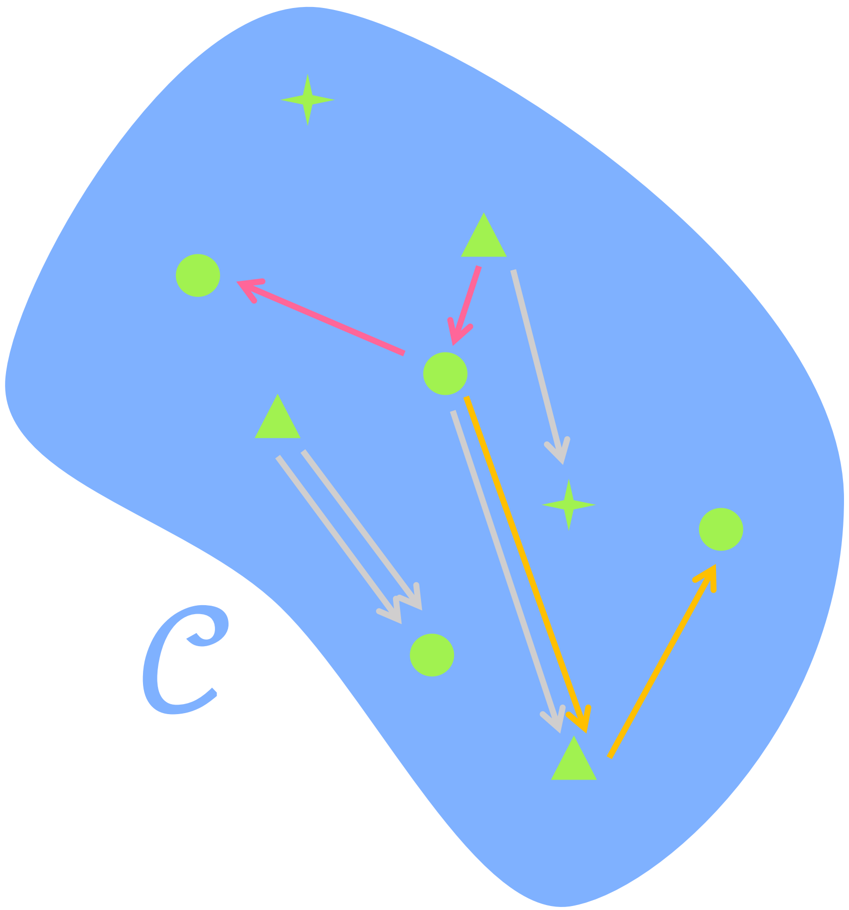


- Objects: Euclidian Vector Spaces ($\mathbb{R}^n$)
- Morphisms: matrices (Linear maps)

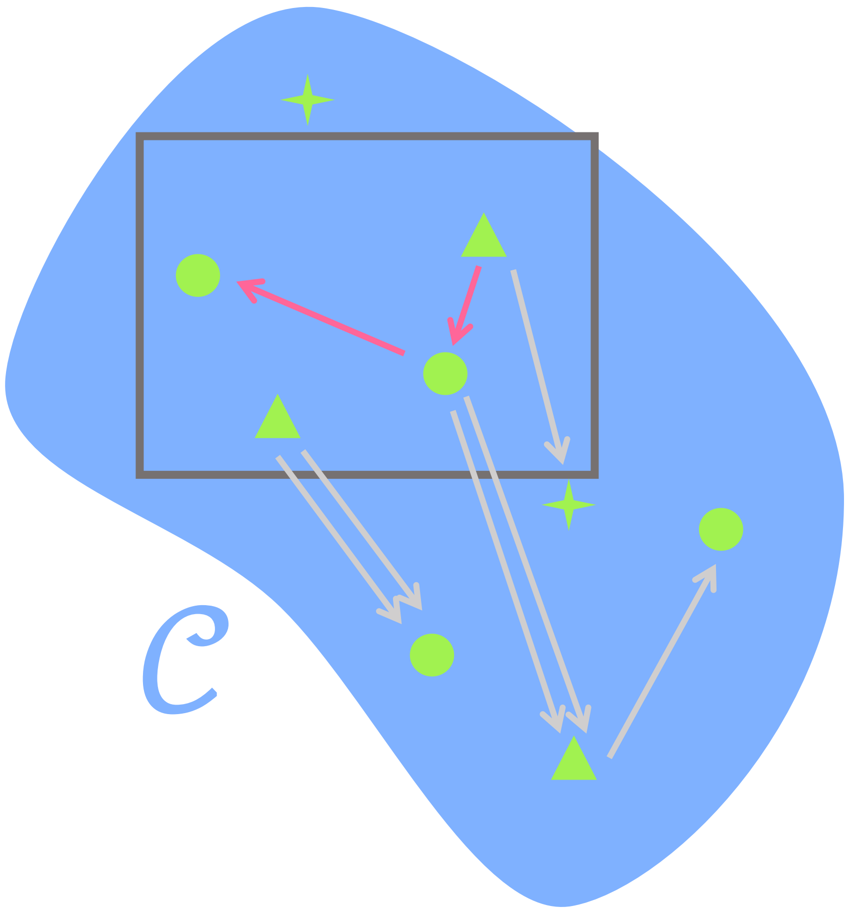
$$A = \begin{bmatrix} 4 & 6 \\ -3 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

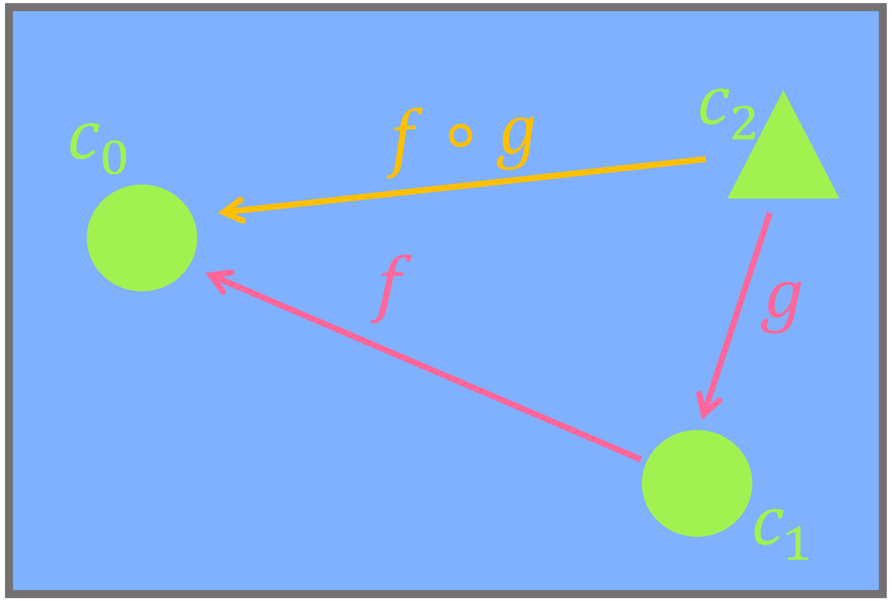
$$\mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$$

Category: Composition of Morphisms



Category: Composition of Morphisms

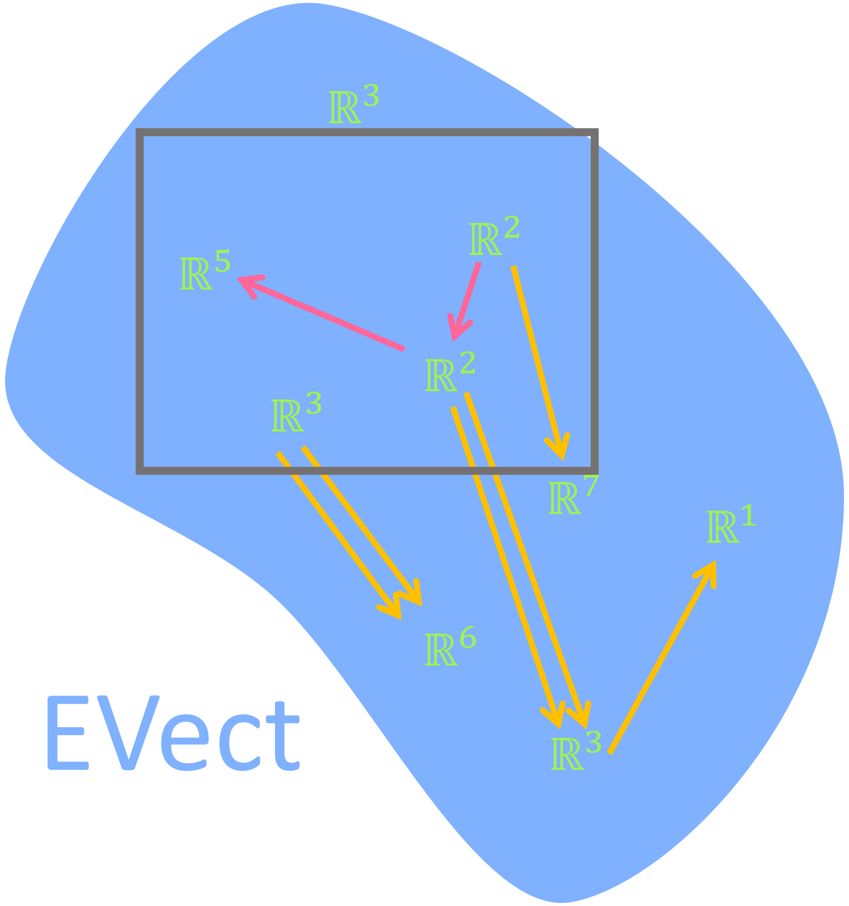


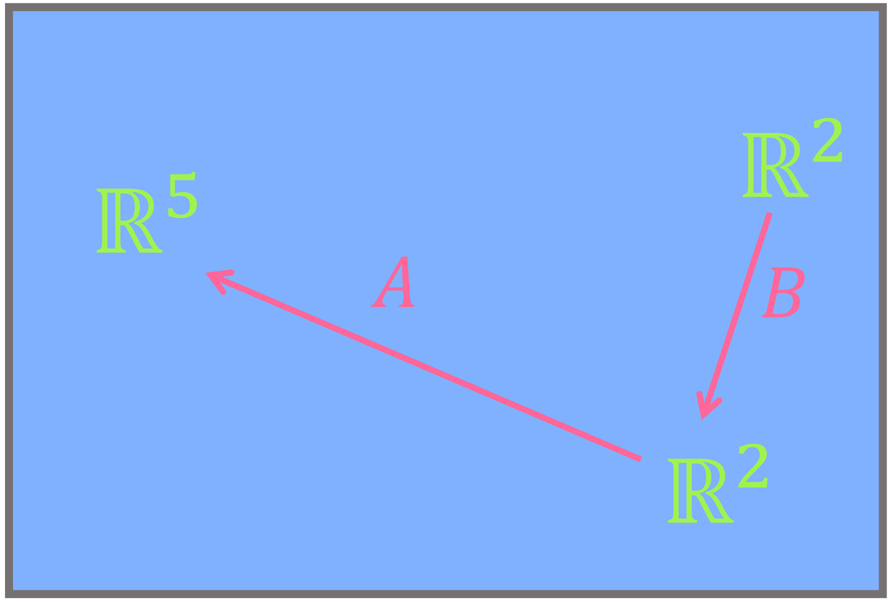


$$f : c_1 \mapsto c_0 \in \mathcal{C}$$
$$g : c_2 \mapsto c_1 \in \mathcal{C}$$

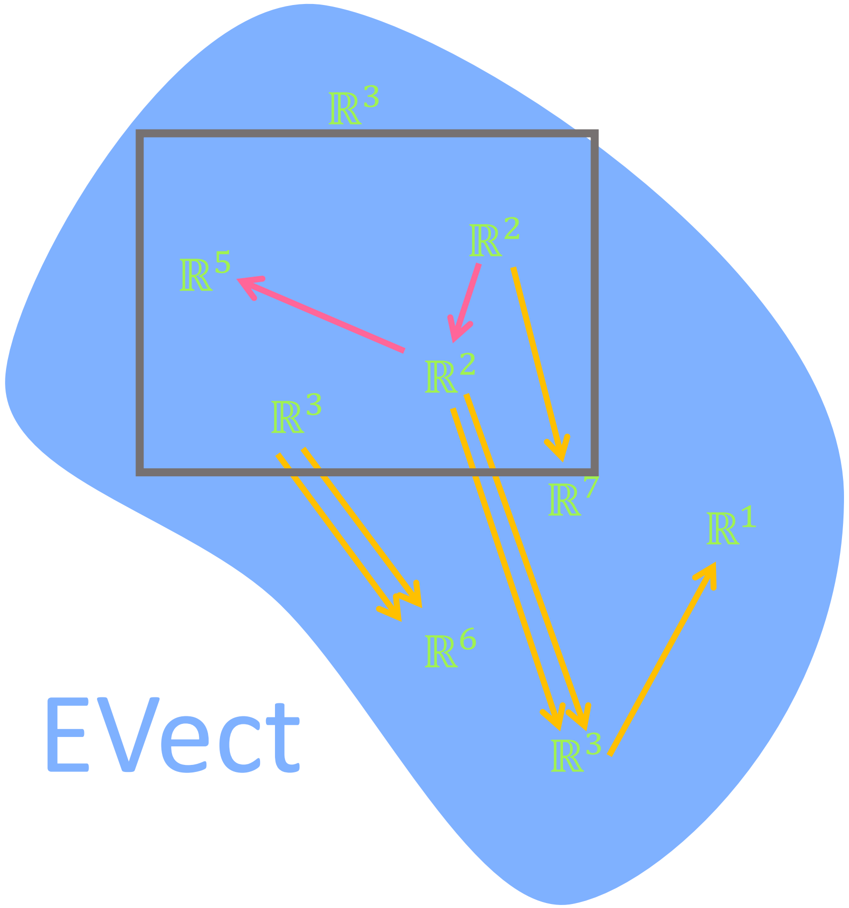
$$f \circ g : c_2 \mapsto c_0 \in \mathcal{C}$$

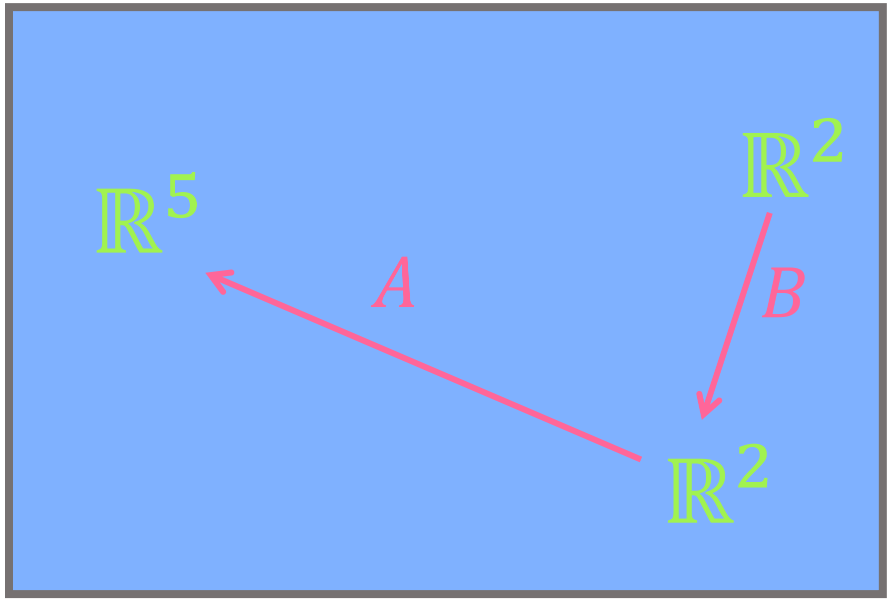
Example: Matrix Multiplication





Example: Matrix Multiplication

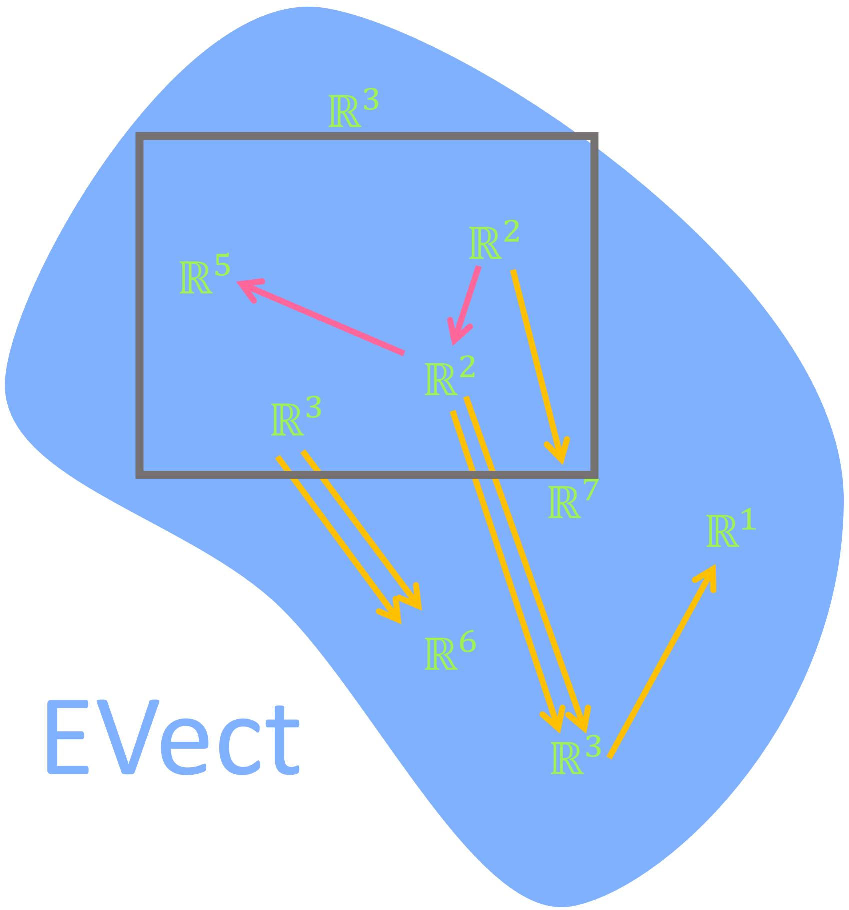


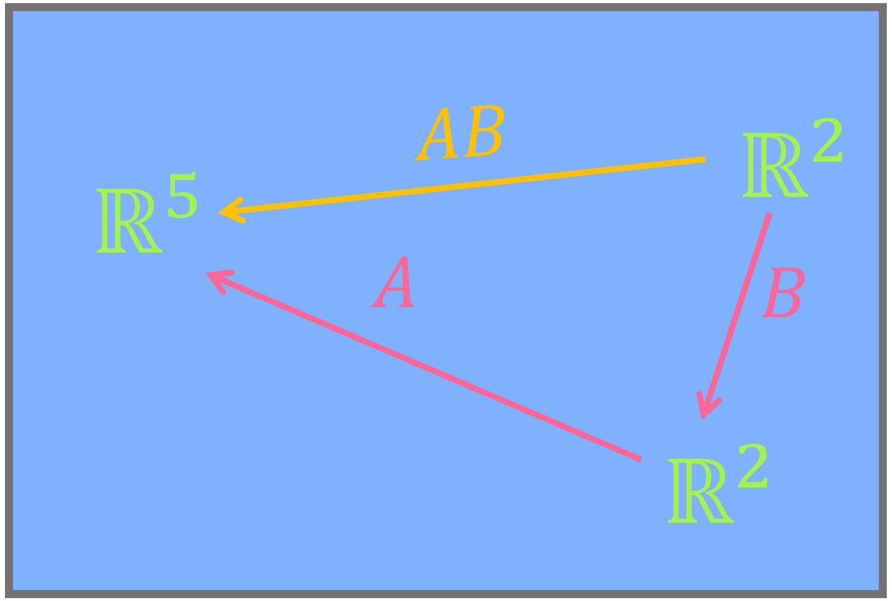


$A \in Mat_{5\times 2}$

$B \in Mat_{2\times 2}$

Example: Matrix Multiplication





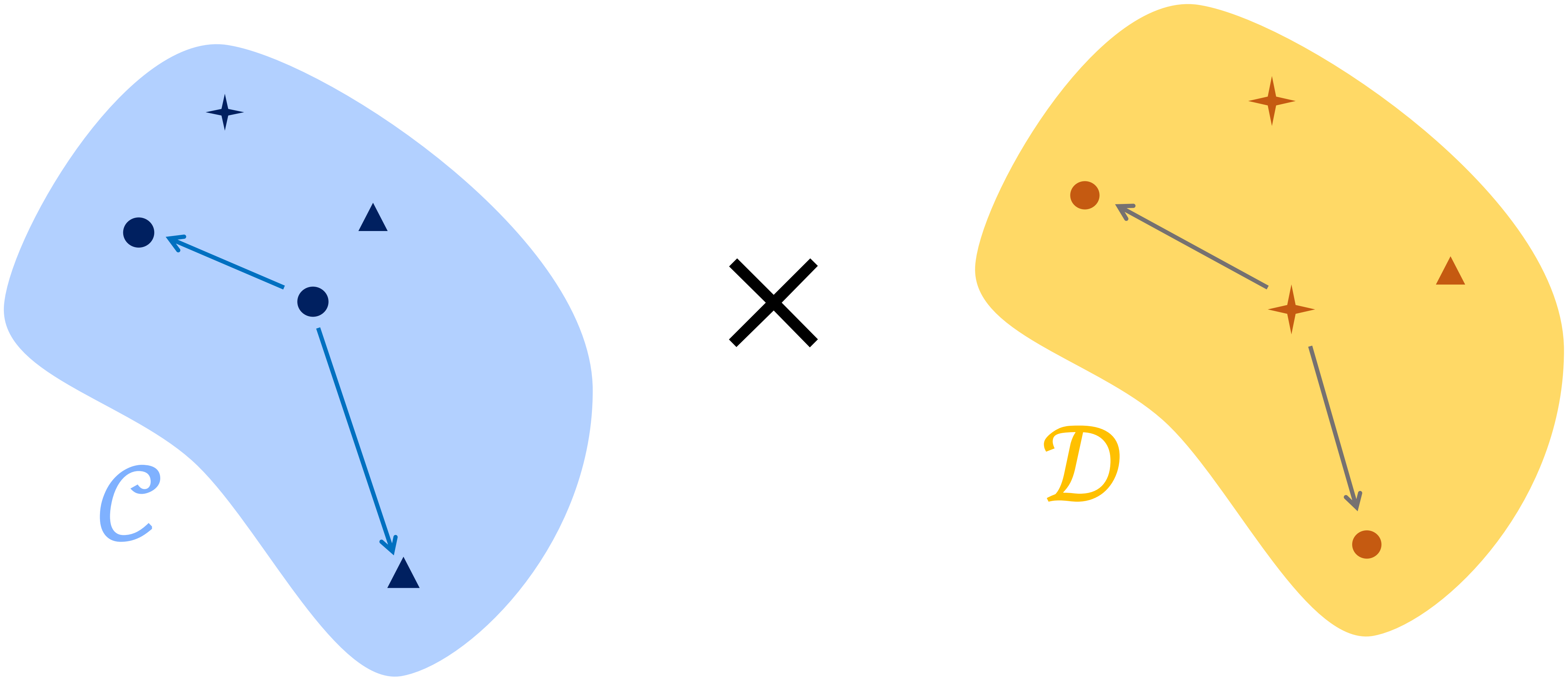
$$A \in Mat_{5\times 2}$$

$$B \in Mat_{2\times 2}$$

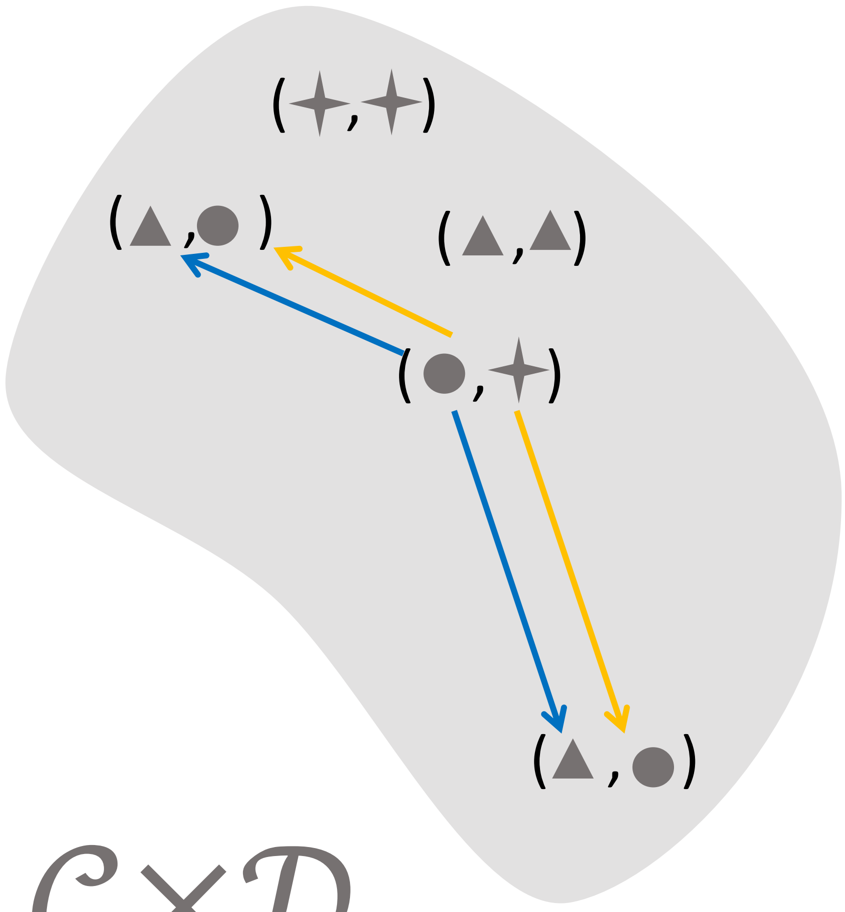
$$A \circ B = AB \in Mat_{5\times 2}$$

Product Categories

Product Category:

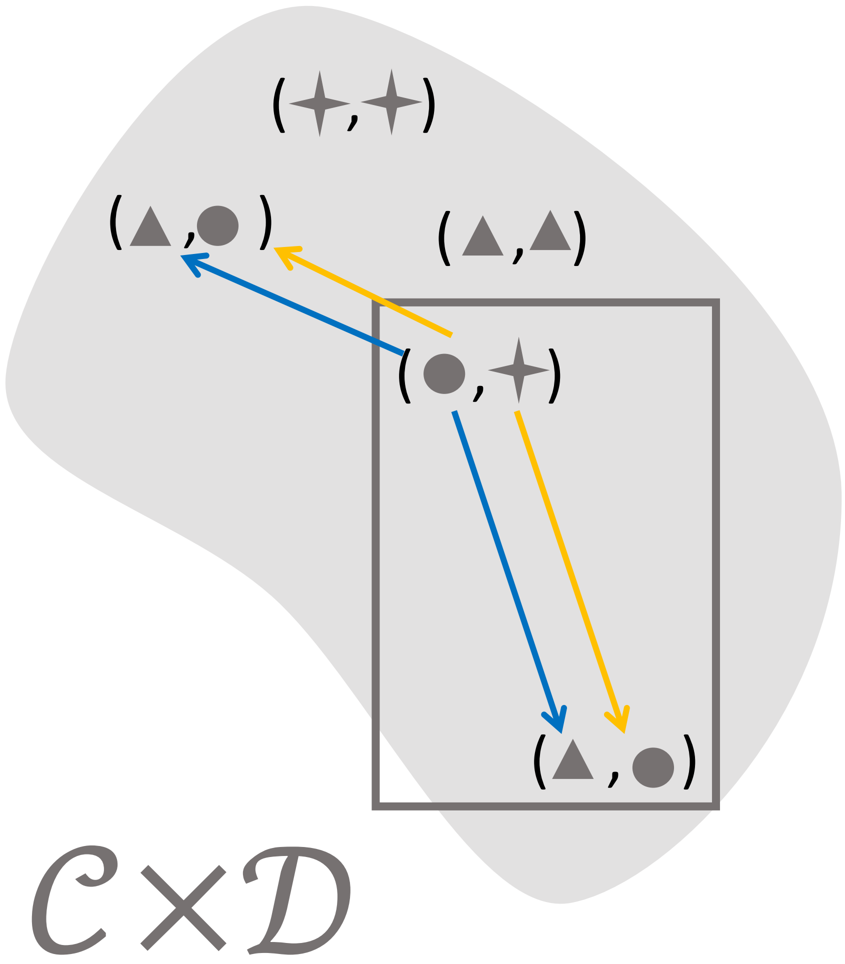


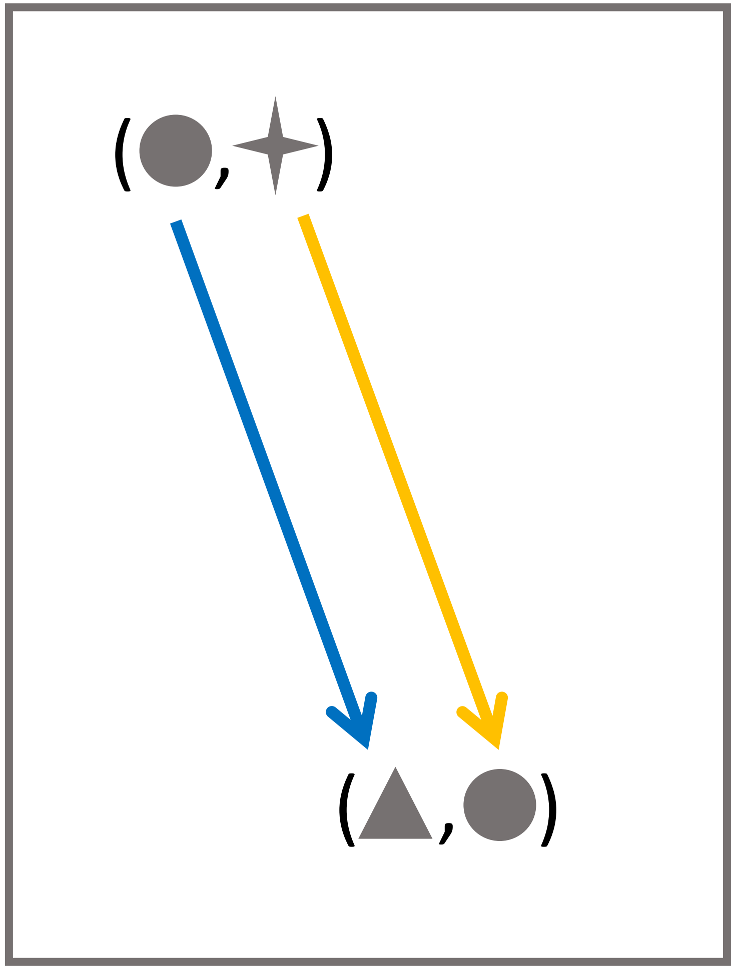
Product Category:



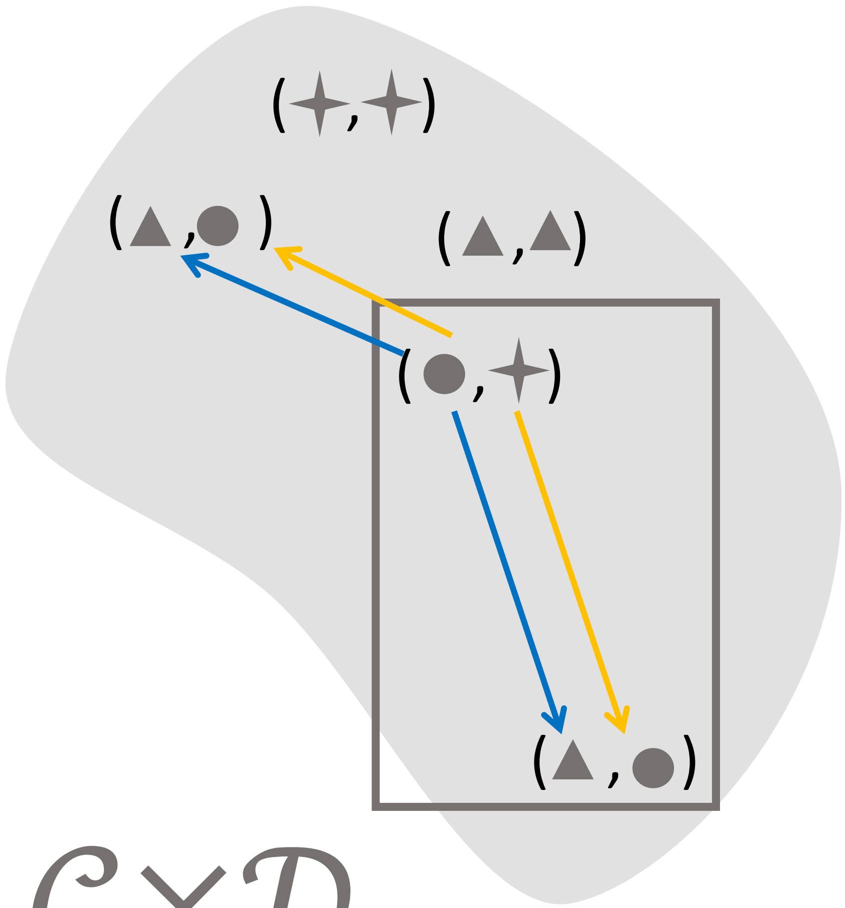
$\mathcal{C} \times \mathcal{D}$

Product Category:

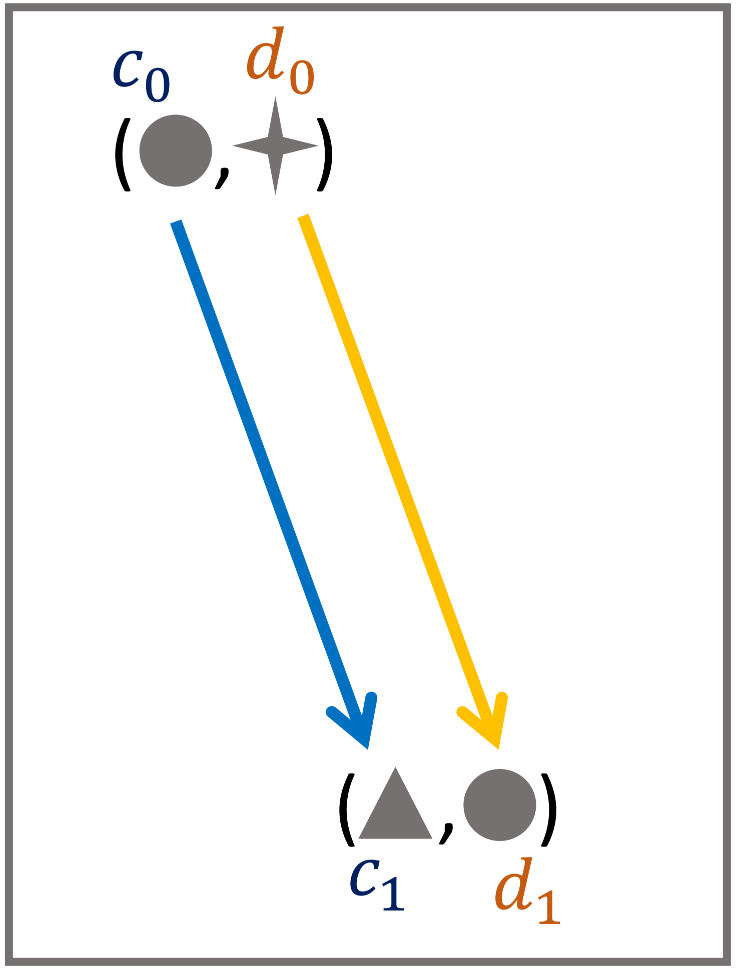




Product Category:



$\mathcal{C} \times \mathcal{D}$

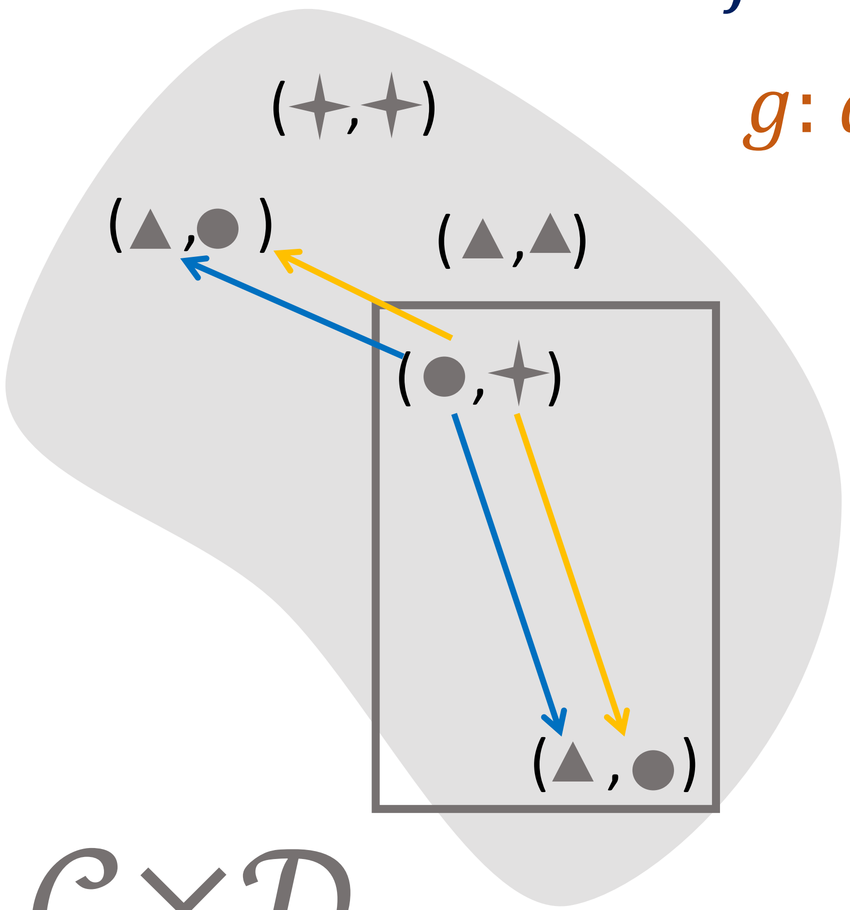


$(c_0, d_0) \in \mathcal{C} \times \mathcal{D}$ where $c_0 \in \mathcal{C}$ and $d_0 \in \mathcal{D}$

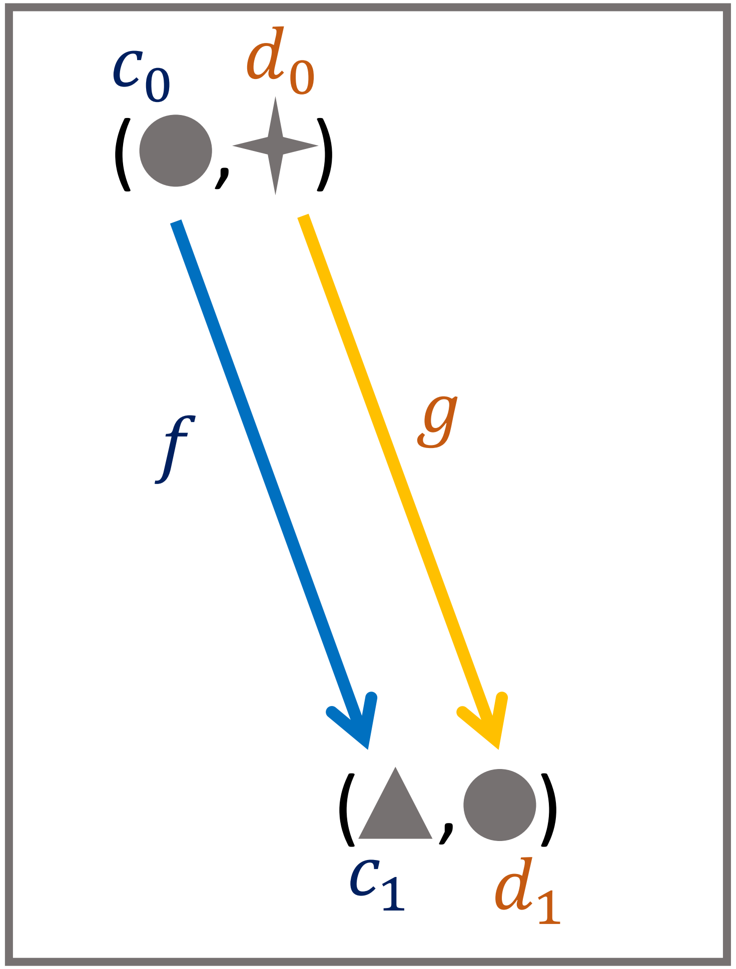
Product Category:

$f: c_0 \mapsto c_1 \in \mathcal{C}$

$g: d_0 \mapsto d_1 \in \mathcal{D}$



$\mathcal{C} \times \mathcal{D}$

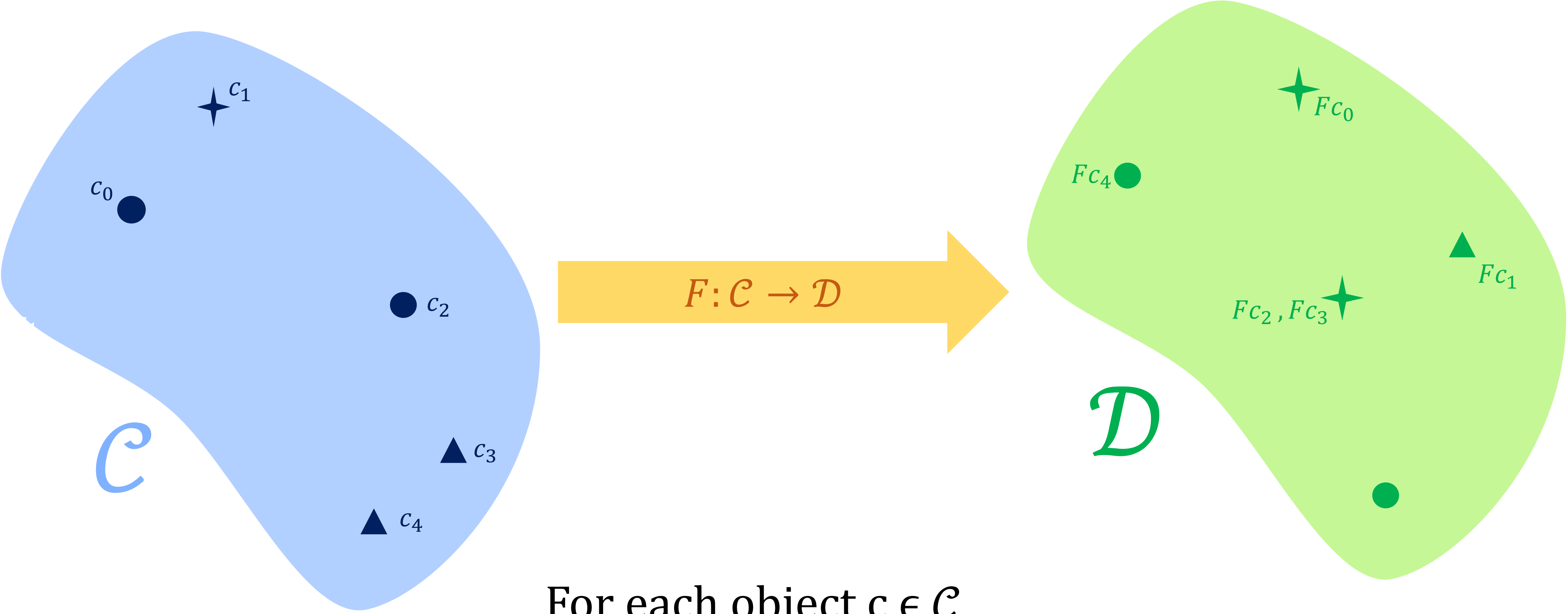


$$f \times g: (c_0, d_0) \mapsto (c_1, d_1) \in \mathcal{C} \times \mathcal{D}$$

Functor

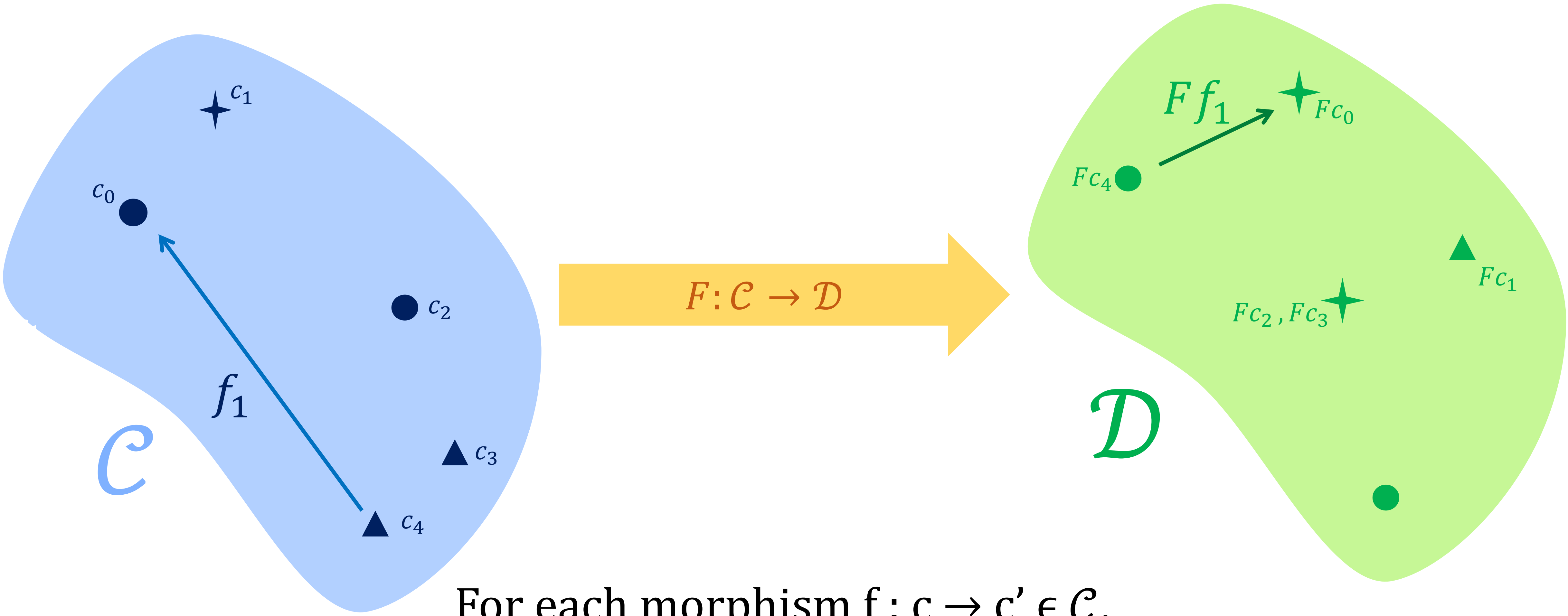
Assignments/Map between categories

Functor : Category → Category



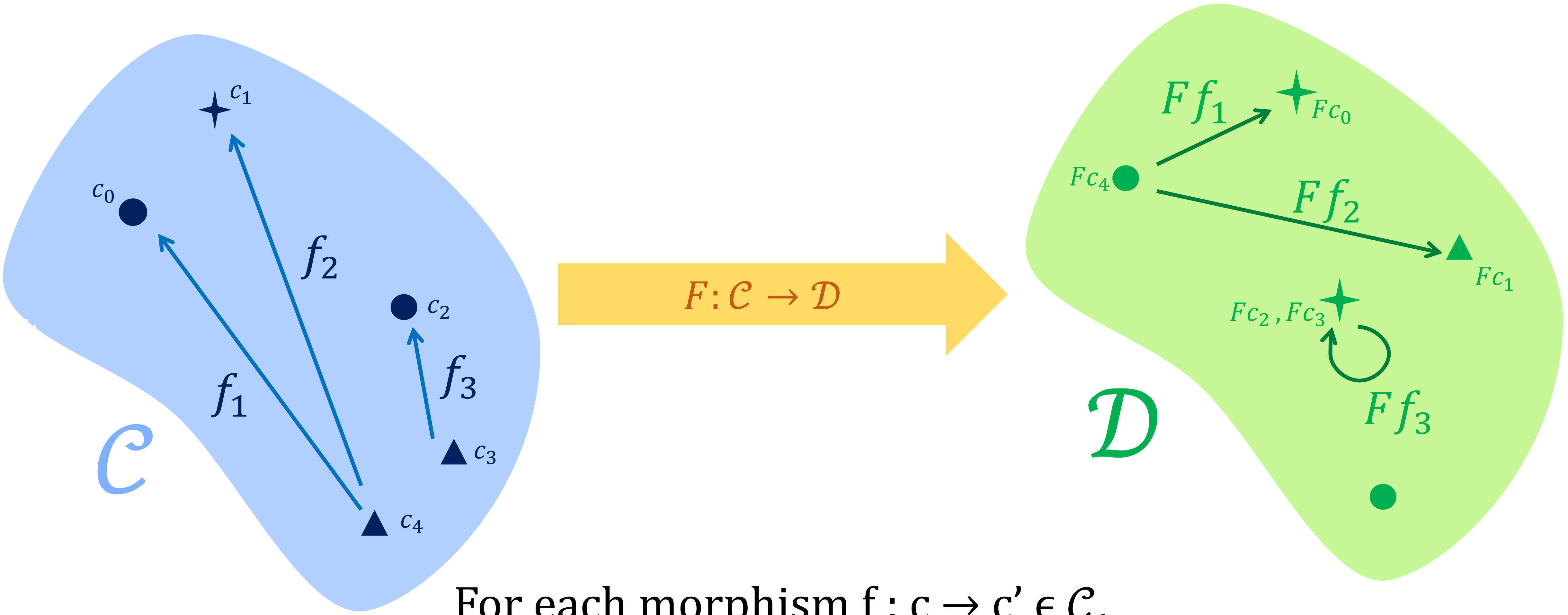
For each object c ϵ $\mathcal{C}$,
there is a corresponding Fc ϵ $\mathcal{D}$

Functor : Category → Category



For each morphism $f : c \to c' \in \mathcal{C}$,
there is a corresponding morphism
$Ff : Fc \to Fc' \in \mathcal{D}$

Functor : Category → Category

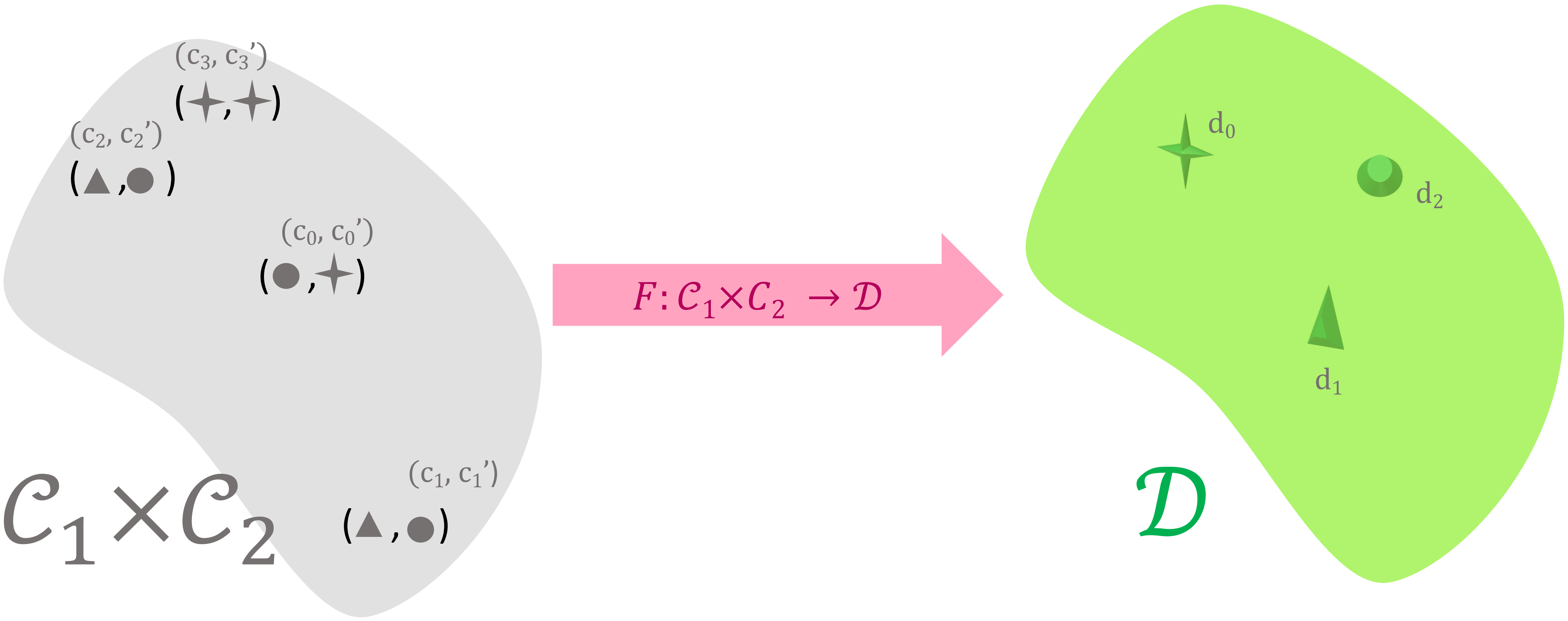


For each morphism $f : c \to c' \in \mathcal{C}$,
there is a corresponding morphism
$Ff : Fc \to Fc' \in \mathcal{D}$

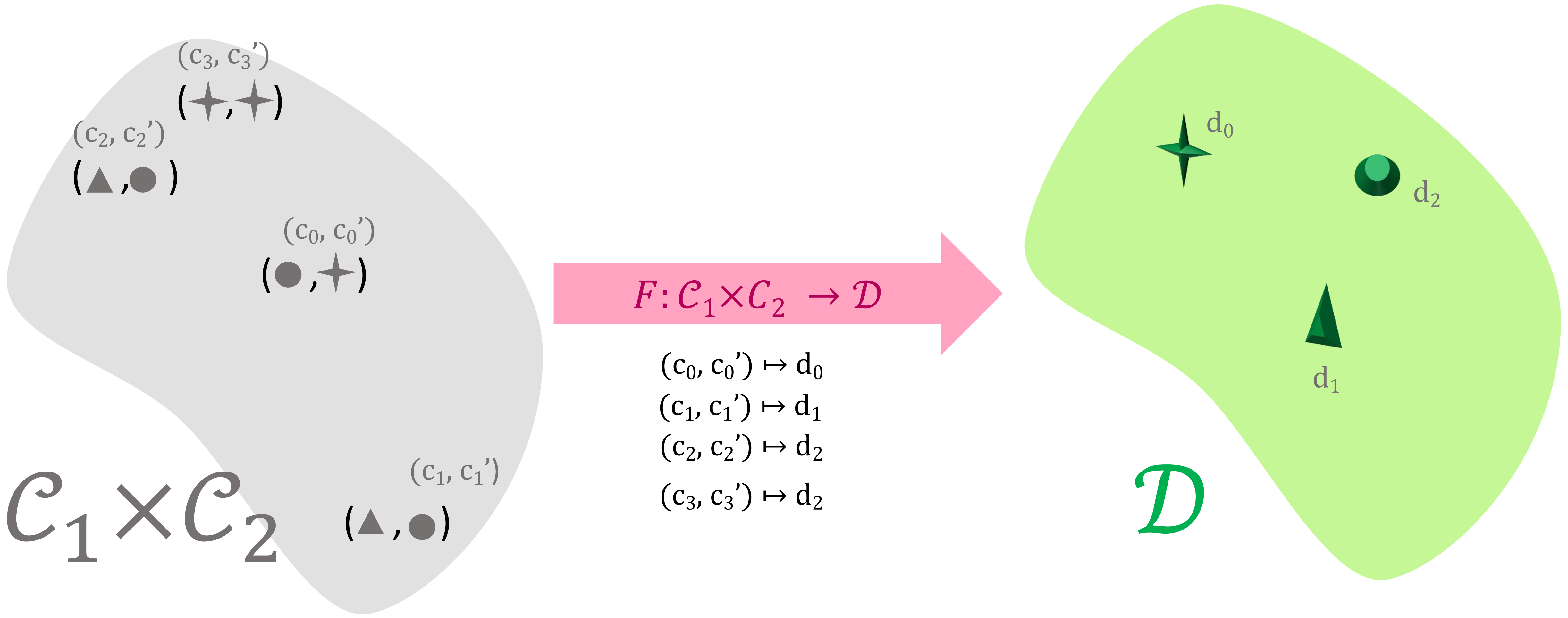
The most important functor in our story is the tensor product ($\otimes$), which is a

bifunctor.

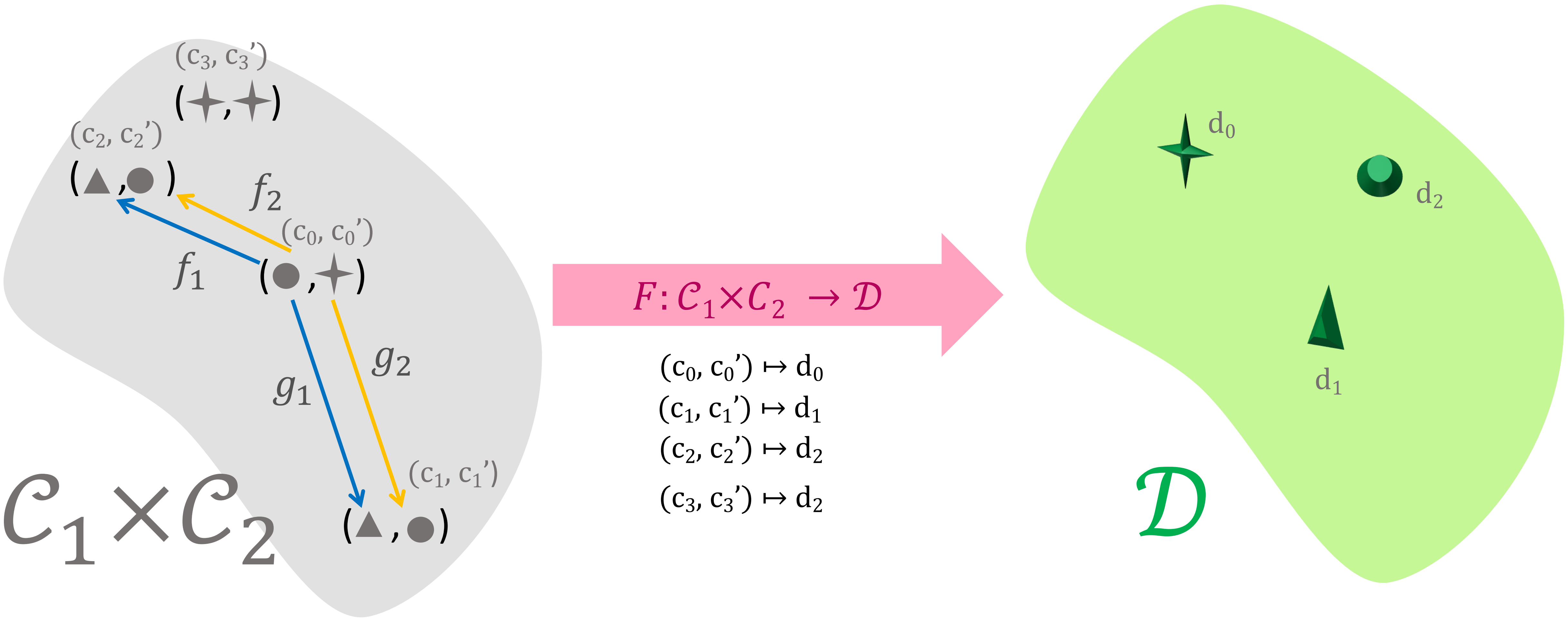
Bifunctor : Product Category → Category



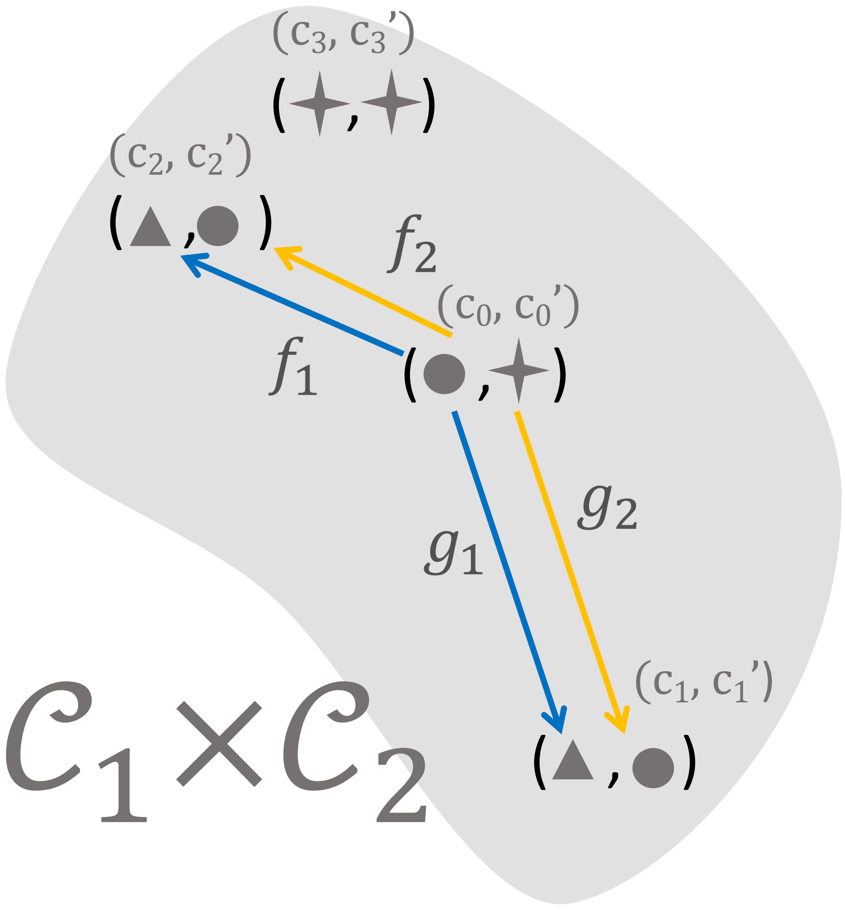
Bifunctor : Product Category → Category

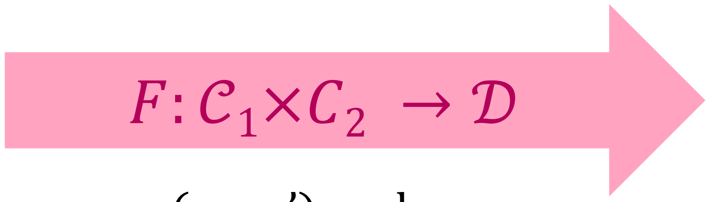


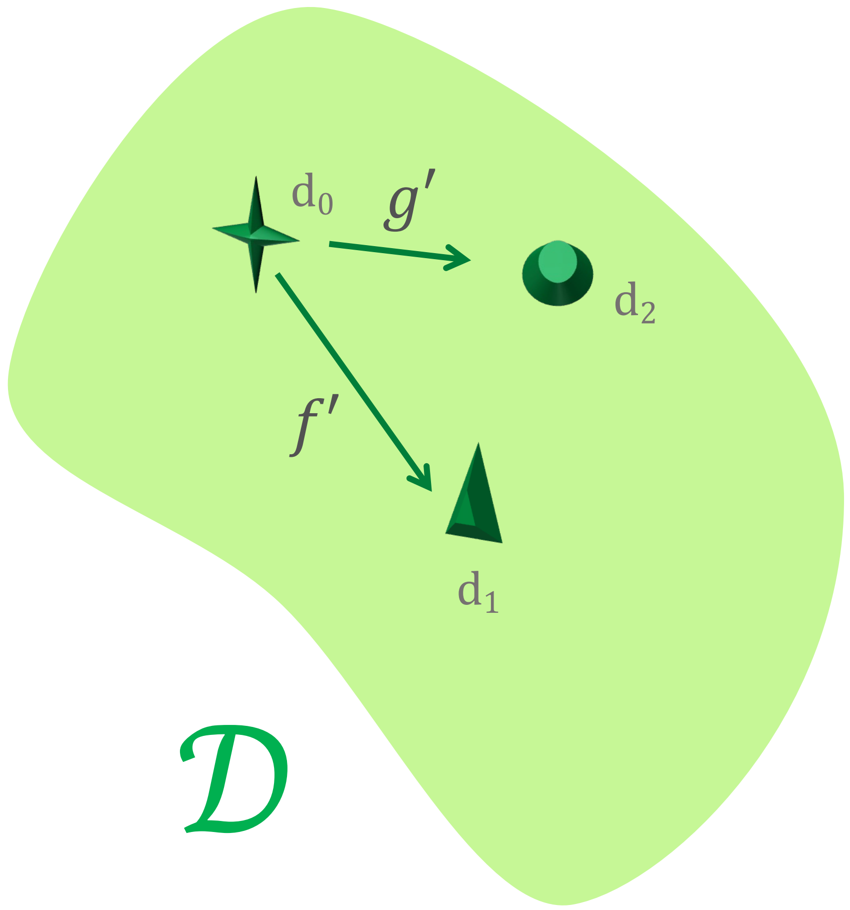
Bifunctor : Product Category → Category



Bifunctor : Product Category → Category







$(c_0, c_0') \mapsto d_0$

$(c_1, c_1') \mapsto d_1$

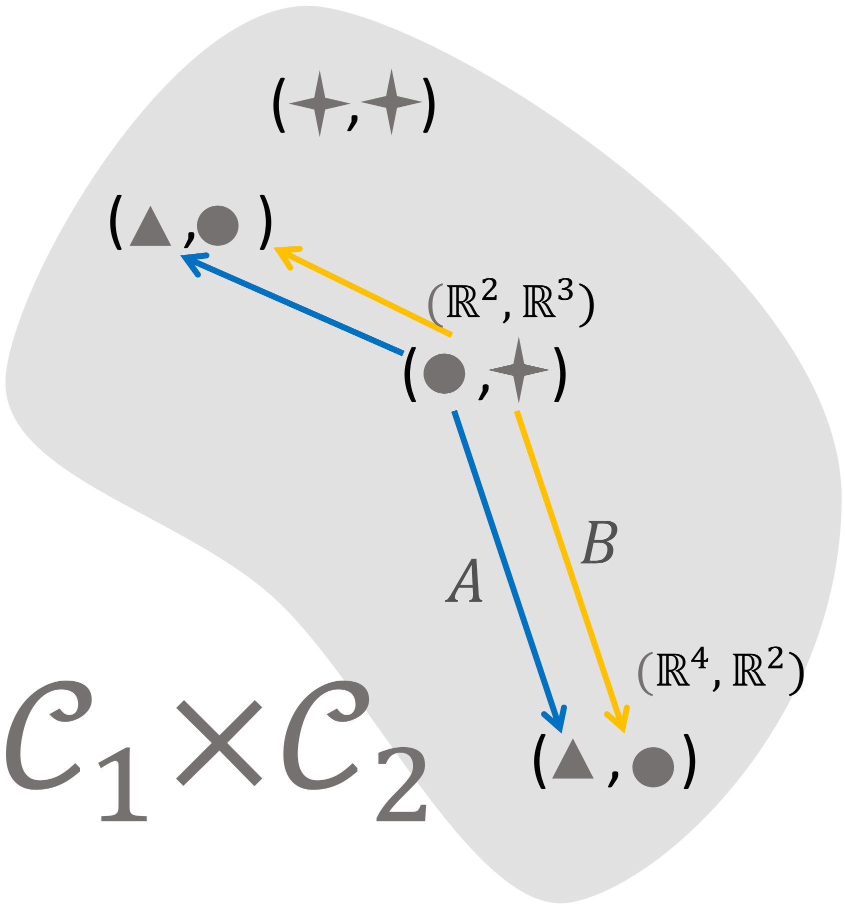
$(c_2, c_2') \mapsto d_2$

$(c_3, c_3') \mapsto d_2$

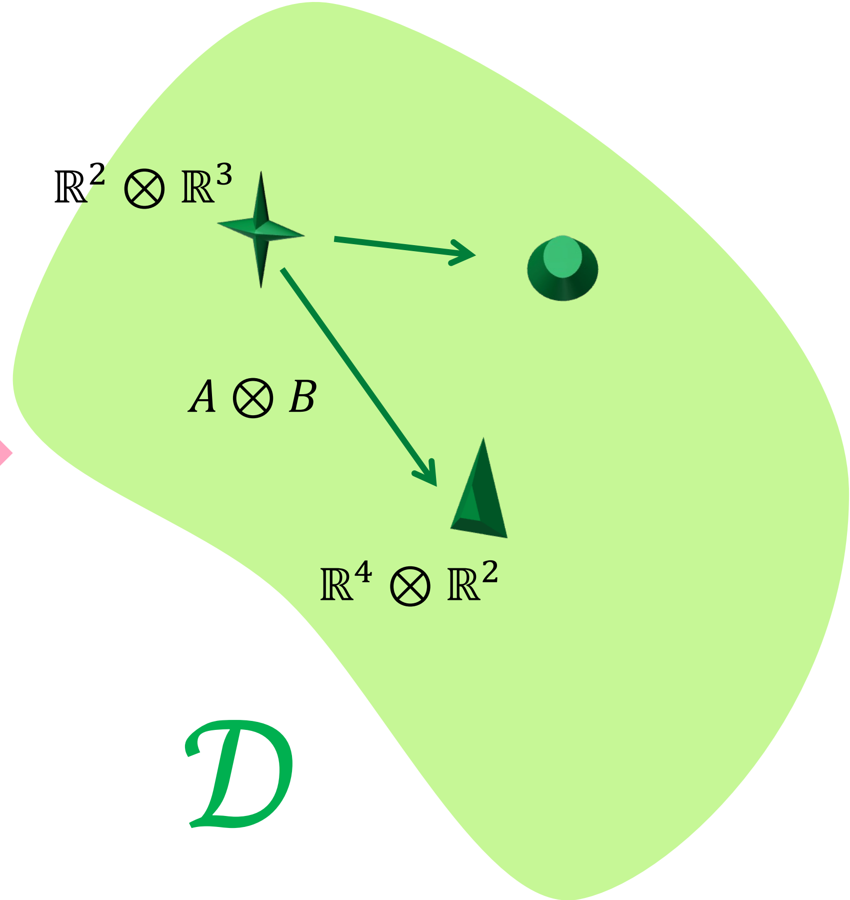
$(f_1, f_2) \mapsto f'$

$(g_1, g_2) \mapsto g'$

Tensor Product in EVect



$$EVect \times EVect \xrightarrow{\otimes} EVect$$



$$\mathbb{R}^n \otimes \mathbb{R}^m = \mathbb{R}^{n \cdot m}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \otimes \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} xa \\ xb \\ xc \\ ya \\ yb \\ yc \end{pmatrix}$$

Structures on Categories

Monoidal structures

A <u>monoidal category</u> $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ consist of a category $\mathcal{C}$ with the following structures:

- Bifunctor $- \otimes -: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- Unit object $1_{\mathcal{C}} \in \mathcal{C}$

For $a, b, c \in \mathcal{C}$, there exists natural isomorphisms

- Associator $\alpha_{\mathcal{C}}: (a \otimes b) \otimes c \xrightarrow{\sim} a \otimes (b \otimes c)$
- Unitor $u_{\mathcal{C}}: a \xrightarrow{\sim} a \otimes 1_{\mathcal{C}}$

A <u>monoidal category</u> $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ consist of a category $\mathcal{C}$ with the following structures:

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(example) in $(FVect, \otimes, \mathbb{R}^1)$

$$V \xrightarrow{\sim} V \otimes \mathbb{R}^1$$

<u>FVect Tensor Product</u>

$$FVect \times FVect \xrightarrow{\otimes} FVect$$
$$V \times W \quad \mapsto \quad (V \otimes W)$$

$V, W \in FVect$
$V \simeq \mathbb{R}^n$ and $W \simeq \mathbb{R}^m$
$V \otimes W \simeq \mathbb{R}^{n \cdot m}$

Unitor in a single Category

$$\mathcal{C}$$

$$c$$

Unitor in a single Category

$$\begin{tikzcd}
\mathcal{C} \arrow[ddr, "id_{\mathcal{C}}"'] & \\
c & \\
& \mathcal{C} \\
& c
\end{tikzcd}$$

Unitor in a single Category

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "i_{\mathcal{C}}"] \arrow[ddr, "id_{\mathcal{C}}"'] & \mathcal{C}\times\mathcal{C} \\
c & (c, 1_{\mathcal{C}}) \\
 & \mathcal{C} \\
 & c
\end{tikzcd}$$

Unitor in a single Category

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i_{\mathcal{C}}"] \arrow[dr, "id_{\mathcal{C}}"'] & \mathcal{C} \times \mathcal{C} \\ & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} c \arrow[r, mapsto] \arrow[dr, mapsto] & (c, 1_{\mathcal{C}}) \\ & c \end{tikzcd}$$

$i = (id \times cons) \circ \Delta$

Where

$$\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$$
$$c \mapsto (c, c)$$

$$cons : \mathcal{C} \to \mathcal{C}$$
$$c \mapsto 1_{\mathcal{C}}$$

Unitor in a single Category

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "i_{\mathcal{C}}"] \arrow[dr, "id_{\mathcal{C}}"'] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \\
 & \mathcal{C}
\end{tikzcd}
$$

c

$(c, 1_{\mathcal{C}})$

$c \xrightarrow{\sim} c \otimes 1_{\mathcal{C}}$

$i_{\mathcal{C}} := (id \times cons) \circ \Delta$

Unitor in a single Category

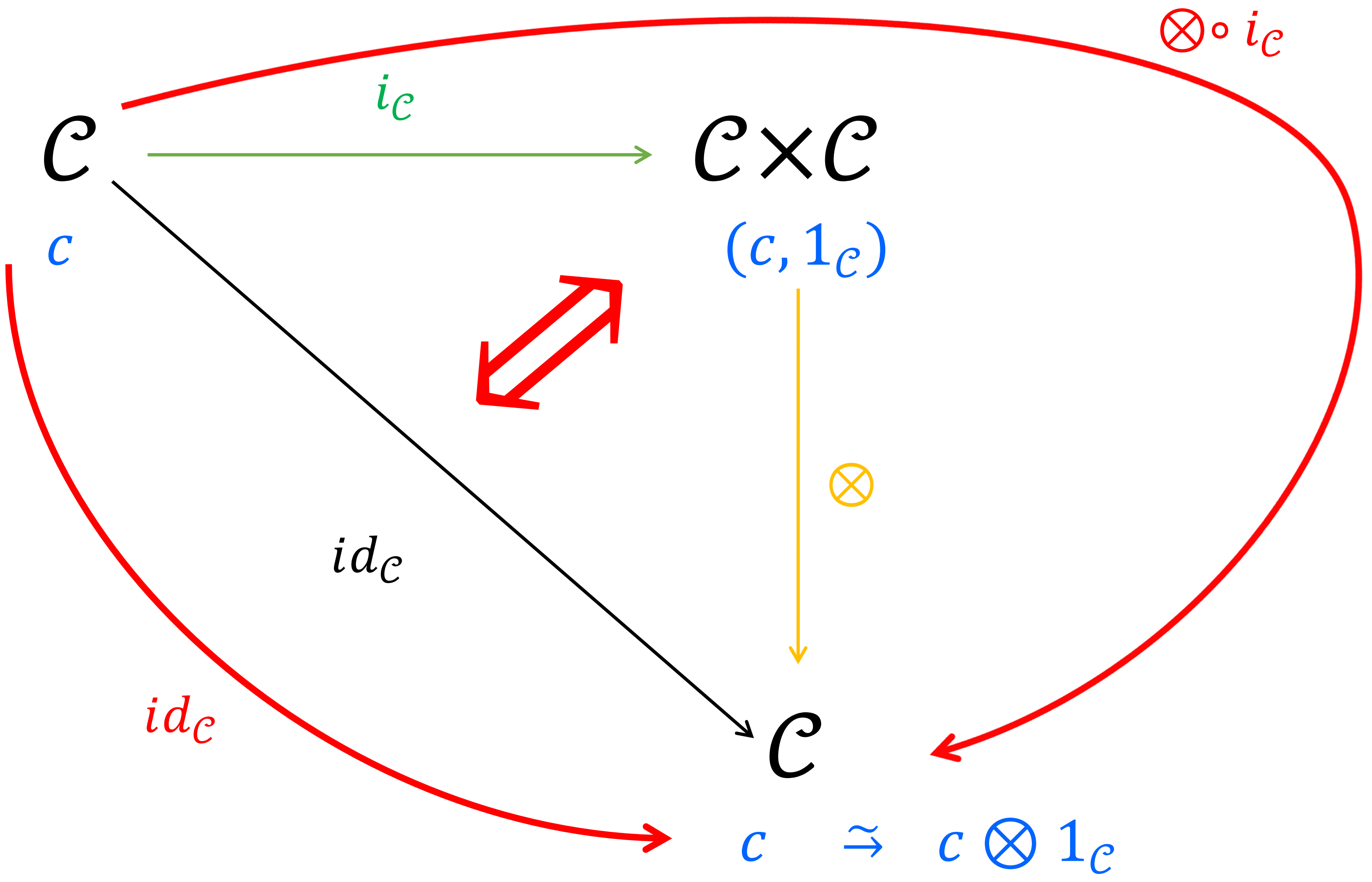
$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{C} \arrow[r, "i_{\mathcal{C}}"] \arrow[dr, "id_{\mathcal{C}}"'] \arrow[dr, bend left=70, "\otimes \circ i_{\mathcal{C}}"] \arrow[dr, bend right=50, "id_{\mathcal{C}}"'] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \\
 & \mathcal{C}
\end{tikzcd}$$

$c \mapsto (c, 1_{\mathcal{C}})$

$c \xrightarrow{\sim} c \otimes 1_{\mathcal{C}}$

$i_{\mathcal{C}} := (id \times cons) \circ \Delta$

Unitor in a single Category $u_{\mathcal{C}}: \otimes \circ i_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$



$i_{\mathcal{C}} := (id \times cons) \circ \Delta$

<u>Claim</u>: If $\mathcal{C}$ and $\mathcal{D}$ have a unitor, their product category $\mathcal{C}\times\mathcal{D}$ has a unitor

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr] & \mathcal{C}\times\mathcal{C} \arrow[d] \\ & \mathcal{C} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{D} \arrow[r] \arrow[dr] & \mathcal{D}\times\mathcal{D} \arrow[d] \\ & \mathcal{D} \end{tikzcd}$$

$$\Downarrow ??$$

$$\begin{tikzcd} (\mathcal{C}\times\mathcal{D}) \arrow[r] \arrow[dr] & (\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \arrow[d] \\ & (\mathcal{C}\times\mathcal{D}) \end{tikzcd}$$

Unitor in a product category

$$
\begin{tikzcd}
\mathcal{C}\times\mathcal{D} \arrow[rr, "i_{\mathcal{C}\times\mathcal{D}}"] \arrow[dr, bend right, "id_{\mathcal{C}\times\mathcal{D}}"'] & & (\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \arrow[dl, bend left, "\otimes_{\mathcal{C}\times\mathcal{D}}"] \\
& \mathcal{C}\times\mathcal{D} &
\end{tikzcd}
$$

(c, d)

$((c, d), 1_{\mathcal{C}\times\mathcal{D}})$

$(c, d) \xrightarrow{\sim} (c, d) \otimes 1_{\mathcal{C}\times\mathcal{D}}$

$u_{\mathcal{C}} : \otimes \circ i_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$

$i_{\mathcal{C}} := (id \times cons) \circ \Delta$

Unitor in a product category

$$\begin{tikzcd}
\mathcal{C}\times\mathcal{D} \arrow[rr, "i_{\mathcal{C}\times\mathcal{D}}"] \arrow[dr, bend right, "id_{\mathcal{C}}\times id_{\mathcal{D}}"] \arrow[dr, bend right=60, "id_{\mathcal{C}\times\mathcal{D}}"'] & & (\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \arrow[dl, bend left, "\otimes_{\mathcal{C}\times\mathcal{D}}"] \\
& \mathcal{C}\times\mathcal{D} &
\end{tikzcd}$$

(c, d)

$((c, d), 1_{\mathcal{C}\times\mathcal{D}})$

$(c, d) \xrightarrow{\sim} (c, d) \otimes 1_{\mathcal{C}\times\mathcal{D}}$

$u_{\mathcal{C}} : \otimes \circ i_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$

$i_{\mathcal{C}} := (id \times cons) \circ \Delta$

Unitor in a product category

$$
\begin{tikzcd}[column sep=large, row sep=large]
\begin{matrix} \mathcal{C}\times\mathcal{D} \\ (c,d) \end{matrix} \arrow[rr, "i_{\mathcal{C}\times\mathcal{D}}"] \arrow[dr, "i_{\mathcal{C}}\times i_{\mathcal{D}}"] \arrow[ddr, bend right=40, "id_{\mathcal{C}\times\mathcal{D}}"'] \arrow[ddr, bend right=20, "id_{\mathcal{C}}\times id_{\mathcal{D}}"] & & \begin{matrix} (\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \\ ((c,d),1_{\mathcal{C}\times\mathcal{D}}) \end{matrix} \arrow[ddl, bend left=40, "\otimes_{\mathcal{C}\times\mathcal{D}}"] \\
 & \begin{matrix} (\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D}) \\ (c,1_{\mathcal{C}})\times(d,1_{\mathcal{D}}) \end{matrix} & \\
 & \begin{matrix} \mathcal{C}\times\mathcal{D} \\ (c,d) \stackrel{\sim}{\rightarrow} (c,d)\otimes 1_{\mathcal{C}\times\mathcal{D}} \end{matrix} &
\end{tikzcd}
$$

$u_{\mathcal{C}} : \otimes \circ i_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$

$i_{\mathcal{C}} := (id\times cons)\circ \Delta$

Unitor in a product category

$$
\begin{tikzcd}[column sep=large, row sep=large]
\mathcal{C}\times\mathcal{D} \arrow[rr, "i_{\mathcal{C}\times\mathcal{D}}"] \arrow[dr, "i_{\mathcal{C}}\times i_{\mathcal{D}}"] \arrow[ddr, bend right=20, "id_{\mathcal{C}}\times id_{\mathcal{D}}"'] \arrow[ddr, bend right=40, "id_{\mathcal{C}\times\mathcal{D}}"'] & & (\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \arrow[ddl, bend left=40, "\otimes_{\mathcal{C}\times\mathcal{D}}"] \\
& (\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D}) \arrow[d, "\otimes_{\mathcal{C}}\times\otimes_{\mathcal{D}}"] & \\
& \mathcal{C}\times\mathcal{D} &
\end{tikzcd}
$$

(c,d)

$((c,d),1_{\mathcal{C}\times\mathcal{D}})$

$(c,1_{\mathcal{C}})\times(d,1_{\mathcal{D}})$

$(c,d)\ \overset{\sim}{\rightarrow}(c,d)\otimes 1_{\mathcal{C}\times\mathcal{D}} \overset{\sim}{\rightarrow}(c\otimes 1_{\mathcal{C}}, d\otimes 1_{\mathcal{D}})$

$u_{\mathcal{C}} : \otimes\circ\, i_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$

$i_{\mathcal{C}} := (id\times cons)\circ\Delta$

Unitor in a product category

$$
\begin{tikzcd}
\mathcal{C}\times\mathcal{D} \arrow[rr, "i_{\mathcal{C}\times\mathcal{D}}"] \arrow[dr, "i_{\mathcal{C}}\times i_{\mathcal{D}}"] \arrow[ddr, bend right, "id_{\mathcal{C}}\times id_{\mathcal{D}}"'] \arrow[ddr, bend right=60, "id_{\mathcal{C}\times\mathcal{D}}"'] & & (\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \arrow[dl, dashed, leftrightarrow, "?"'] \arrow[ddl, bend left, "\otimes_{\mathcal{C}\times\mathcal{D}}"] \\
& (\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D}) \arrow[d, "\otimes_{\mathcal{C}}\times\otimes_{\mathcal{D}}"] & \\
& \mathcal{C}\times\mathcal{D} &
\end{tikzcd}
$$

(c, d)

$((c, d), 1_{\mathcal{C}\times\mathcal{D}})$

$(c, 1_{\mathcal{C}})\times(d, 1_{\mathcal{D}})$

$(c, d) \xrightarrow{\sim} (c, d) \otimes 1_{\mathcal{C}\times\mathcal{D}} \xrightarrow{\sim} (c \otimes 1_{\mathcal{C}}, d \otimes 1_{\mathcal{D}})$

$u_{\mathcal{C}} : \otimes \circ i_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$

$i_{\mathcal{C}} := (id \times cons) \circ \Delta$

Unitor in a product category

$$\begin{tikzcd}
\mathcal{C}\times\mathcal{D} \arrow[rr, "i_{\mathcal{C}\times\mathcal{D}}"] \arrow[dr, "i_{\mathcal{C}}\times i_{\mathcal{D}}"] \arrow[ddr, bend right, "id_{\mathcal{C}}\times id_{\mathcal{D}}"'] \arrow[ddr, bend right=60, "id_{\mathcal{C}\times\mathcal{D}}"'] & & (\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \arrow[ddl, bend left, "\otimes_{\mathcal{C}\times\mathcal{D}}"] \\
& (\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D}) \arrow[ur, "T^{-1}"] \arrow[d, "\otimes_{\mathcal{C}}\times\otimes_{\mathcal{D}}"] & \\
& \mathcal{C}\times\mathcal{D} &
\end{tikzcd}$$

(c,d)

$((c,d),1_{\mathcal{C}\times\mathcal{D}}) \xrightarrow{\sim} (c,d)\times(1_{\mathcal{C}},1_{\mathcal{D}})$

$(c,1_{\mathcal{C}})\times(d,1_{\mathcal{D}})$

$(c,d) \xrightarrow{\sim} (c,d)\otimes 1_{\mathcal{C}\times\mathcal{D}} \xrightarrow{\sim} (c\otimes 1_{\mathcal{C}}, d\otimes 1_{\mathcal{D}})$

$T: (\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \to (\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D})$

$u_{\mathcal{C}} : \otimes\circ i_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$

$i_{\mathcal{C}} := (id\times cons)\circ\Delta$

Unitor in a product category

$$
\begin{tikzcd}
\mathcal{C}\times\mathcal{D} \arrow[rr, "i_{\mathcal{C}\times\mathcal{D}}"] \arrow[dr, "i_{\mathcal{C}}\times i_{\mathcal{D}}"] \arrow[ddr, bend right, "id_{\mathcal{C}}\times id_{\mathcal{D}}"] \arrow[ddr, bend right=70, "id_{\mathcal{C}\times\mathcal{D}}"'] & & (\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \arrow[ddl, bend left, "\otimes_{\mathcal{C}\times\mathcal{D}}"] \\
& (\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D}) \arrow[ur, "T^{-1}"] \arrow[d, "\otimes_{\mathcal{C}}\times\otimes_{\mathcal{D}}"] \arrow[u, phantom, "="] \arrow[l, phantom, "\Leftrightarrow u_{\mathcal{C}}\times u_{\mathcal{D}}"] \arrow[dr, phantom, "="] & \\
& \mathcal{C}\times\mathcal{D} &
\end{tikzcd}
$$

(c, d)

$((c, d), 1_{\mathcal{C}\times\mathcal{D}}) \overset{\sim}{\rightarrow} (c, d)\times(1_{\mathcal{C}}, 1_{\mathcal{D}})$

$(c, 1_{\mathcal{C}})\times(d, 1_{\mathcal{D}})$

$(c, d) \overset{\sim}{\rightarrow} (c, d) \otimes 1_{\mathcal{C}\times\mathcal{D}} \overset{\sim}{\rightarrow} (c \otimes 1_{\mathcal{C}}, d \otimes 1_{\mathcal{D}})$

$T: (\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \to (\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D})$

$u_{\mathcal{C}} : \otimes\circ\, i_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$

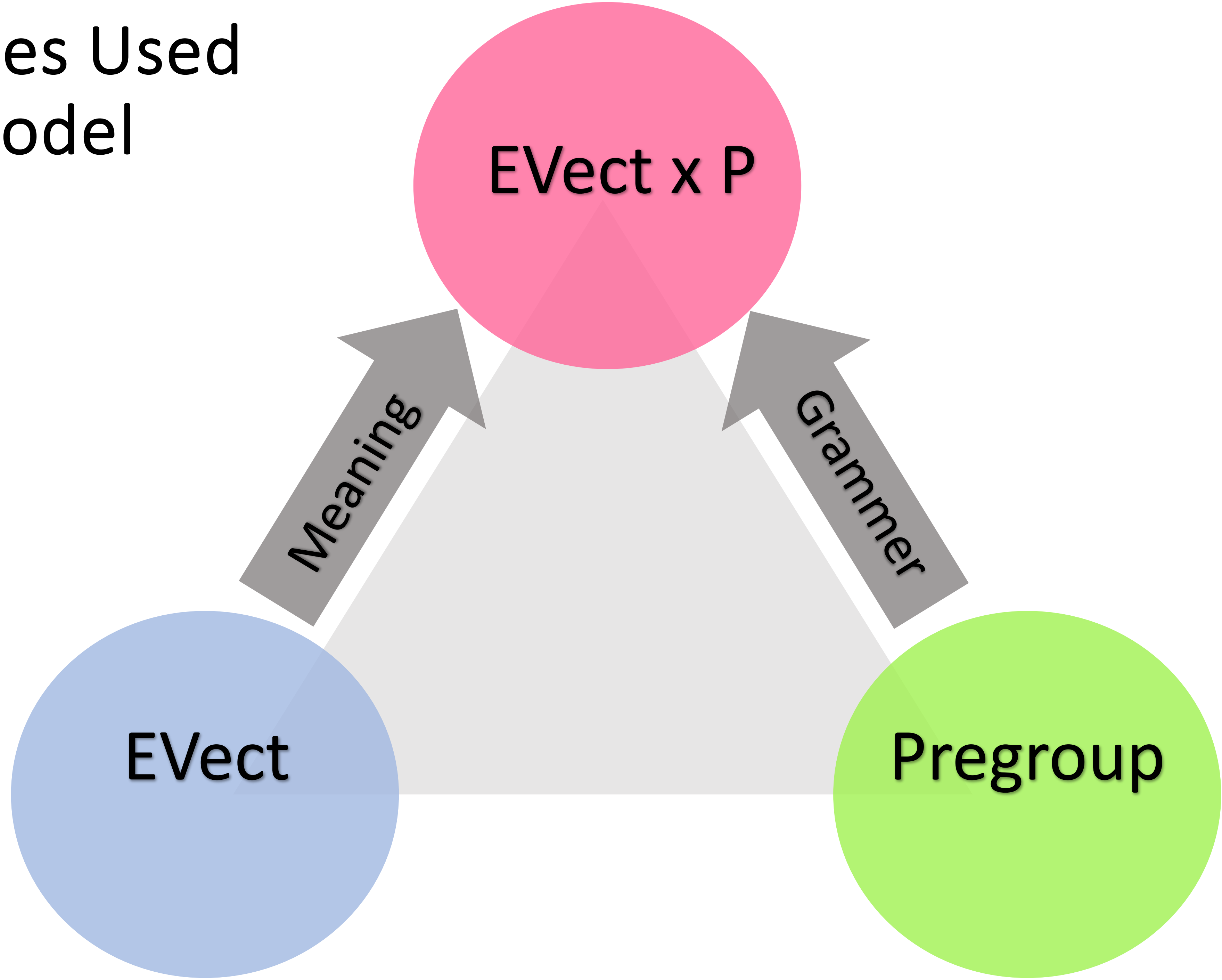
$i_{\mathcal{C}} := (id\times cons)\circ\Delta$

Associator in a product category

$$
\begin{tikzcd}
 & ((\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}))\times(\mathcal{C}\times\mathcal{D}) \arrow[rrrr, "\alpha_{CAT}"] \arrow[d, "T\times id_{\mathcal{C}\times\mathcal{D}}"] \arrow[dddd, bend right=60, "\otimes_{\mathcal{C}\times\mathcal{D}}\times id_{\mathcal{C}\times\mathcal{D}}"'] \arrow[dddd, phantom, bend right=30, "="] & & & & (\mathcal{C}\times\mathcal{D})\times((\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D})) \arrow[d, "id_{\mathcal{C}\times\mathcal{D}}\times T"'] \arrow[dddd, bend left=60, "id_{\mathcal{C}\times\mathcal{D}}\times\otimes_{\mathcal{C}\times\mathcal{D}}"] \arrow[dddd, phantom, bend left=30, "="] & \\
 & ((\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D}))\times(\mathcal{C}\times\mathcal{D}) \arrow[dr, "T"] \arrow[ddd, "(\otimes_{\mathcal{C}}\times\otimes_{\mathcal{D}})\times id_{\mathcal{C}\times\mathcal{D}}"'] \arrow[rrrr, phantom, "="] & & & & (\mathcal{C}\times\mathcal{D})\times((\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D})) \arrow[dl, "T"'] \arrow[ddd, "id_{\mathcal{C}\times\mathcal{D}}\times(\otimes_{\mathcal{C}}\times\otimes_{\mathcal{D}})"] & \\
 & & ((\mathcal{C}\times\mathcal{C})\times\mathcal{C})\times((\mathcal{D}\times\mathcal{D})\times\mathcal{D}) \arrow[rr, "\alpha_{CAT}\times\alpha_{CAT}"] \arrow[d, "(\otimes_{\mathcal{C}}\times id_{\mathcal{C}})\times(\otimes_{\mathcal{D}}\times id_{\mathcal{D}})"] & & (\mathcal{C}\times(\mathcal{C}\times\mathcal{C}))\times(\mathcal{D}\times(\mathcal{D}\times\mathcal{D})) \arrow[d, "(\otimes_{\mathcal{C}}\times id_{\mathcal{C}})\times(\otimes_{\mathcal{D}}\times id_{\mathcal{D}})"'] & & \\
 & & (\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D}) \arrow[dr, "\otimes_{\mathcal{C}}\times\otimes_{\mathcal{D}}"] \arrow[rr, phantom, "\Leftrightarrow\ \alpha_{\mathcal{C}}\times\alpha_{\mathcal{D}}"] & & (\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D}) \arrow[dl, "\otimes_{\mathcal{C}}\times\otimes_{\mathcal{D}}"'] & & \\
 & (\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \arrow[ur, "T"] \arrow[rr, "\otimes_{\mathcal{C}\times\mathcal{D}}"'] & & \mathcal{C}\times\mathcal{D} & & (\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \arrow[ul, "T"'] \arrow[ll, "\otimes_{\mathcal{C}\times\mathcal{D}}"] &
\end{tikzcd}
$$

Model in Sentence Structure

Categories Used in the Model



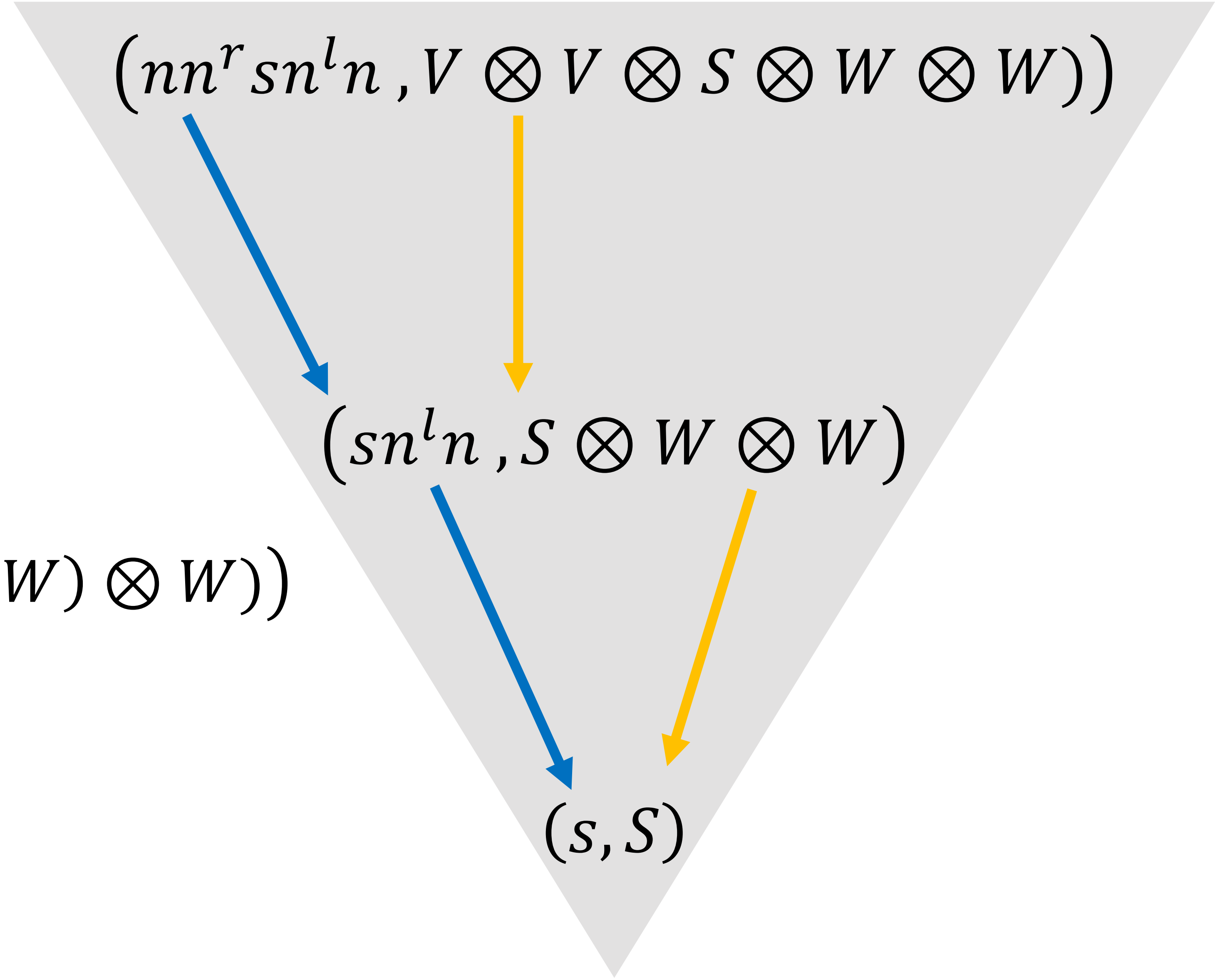
Application

The Sentence

ex: *John likes Marie*

- Noun One: (n, V)
- Verb: $(n^r s n^l, V \otimes S \otimes W)$
- Noun Two: (n, W)

$$\left(n\left(n^r s n^l\right)n\,, V \otimes (V \otimes S \otimes W) \otimes W\right)$$



Morphisms:
- (ϵ^r, ϵ): $(n n^r \to 1, V \otimes V \ \to \mathbb{R})$
- $\left(\epsilon^l, \epsilon\right)$: $n^l n \to 1, W \otimes W \ \to \mathbb{R})$

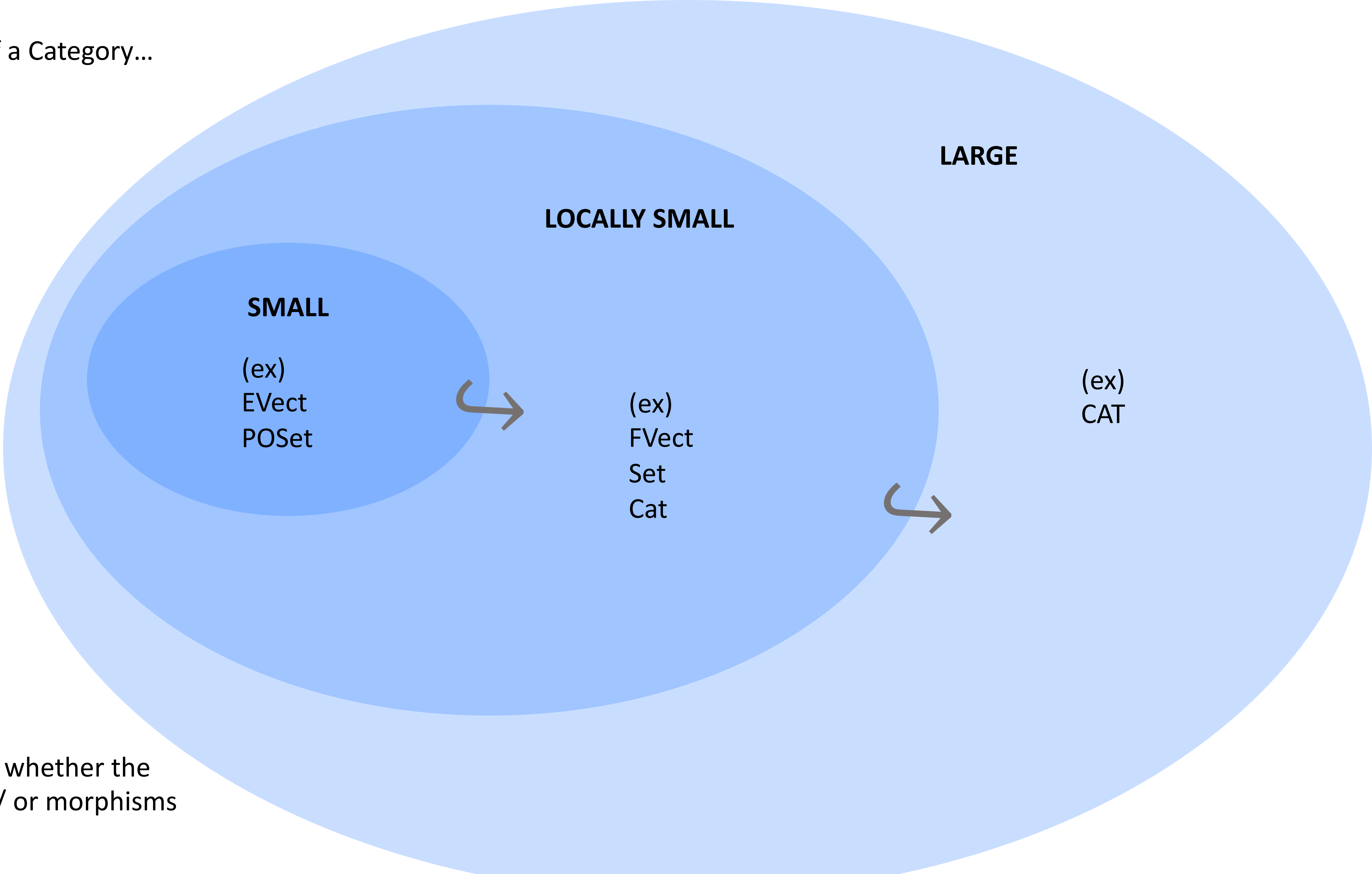
Thank You! Any Questions?

Mia Goldstein and Emily Herbert
SUNY New Paltz

Thanks again to RSCA, NCUWM, and Dr. Glass

Additional Information

The size of a Category…



depends on whether the objects and/ or morphisms form sets

Cat and CAT

A category of categories has:

- Categories as objects
- Functors as morphisms
- Natural transformations as functors

Objects in Cat are small while Cat is locally small

Objects in CAT are locally small while CAT is large

Tensor Product

Universal Property:

$$\begin{tikzcd} \mathbb{V} \times \mathbb{W} \arrow[r, "\eta"] \arrow[dr, "b"'] & \mathbb{V} \otimes \mathbb{W} \arrow[d, "f"] \\ & \mathbb{Z} \end{tikzcd}$$

Concrete Model:

$$\begin{tikzcd} \mathbb{R}^n \times \mathbb{R}^m \arrow[r, "\phi"] \arrow[dr, "b(A \times B)"] & \mathbb{R}^n \otimes \mathbb{R}^m \arrow[d, "A \otimes B"] \\ & \mathbb{R}^{nm} \end{tikzcd}$$

The vector space $V \otimes W$ whose elements are ich satisfy three properties:

$$(\alpha \cdot v, w) = (v, \alpha \cdot w) = \alpha \cdot (v, w)$$
$$(v_1 + v_2, w) = (v_1, w) + (v_2, w)$$
$$(v, w_1 + w_2) = (v, w_1) + (v, w_2)$$

If $\mathcal{B}_1 = \{v_1, \ldots, v_n\}$ is the basis for V and $\mathcal{B}_2 = \{w_1, \ldots, w_n\}$ for W, then $\mathcal{B}_1 \otimes \mathcal{B}_2 = \{v_i \otimes w_j | v_i \in \mathcal{B}_1, w_j \in \mathcal{B}_2\}$ is the basis for $V \otimes W$.

Given linear maps $V \xrightarrow{A} X$ and $W \xrightarrow{B} Y$ there exists a unique linear map: $V \otimes W \xrightarrow{A \otimes B} X \otimes Y$.

Pregroup

Pregroup: $(\mathrm{P}, \leq, \cdot, 1, p^r, p^l)\ \forall\, p \in P$

Pregroups are partially ordered monoids.

They hold the following properties:

$$p \leq q \Rightarrow p \cdot r \leq q \cdot r \quad \text{and} \quad r \cdot p \leq r \cdot q$$

$$p \cdot p^r \leq 1 \leq p^r \cdot p \quad \text{and} \quad p^l \cdot p \leq 1 \leq p \cdot p^l$$

There can be no more than 1 morphism between any 2 objects.

The P used to model grammar is defined by Lambek grammar.

$$(\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \longrightarrow (\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D})$$

$\alpha : (\mathcal{C}\times\mathcal{C})\times\mathcal{C} \to \mathcal{C}\times(\mathcal{C}\times\mathcal{C})$

$\big((c_0, c_1), c_2\big) \mapsto (c_0, (c_1, c_2))$

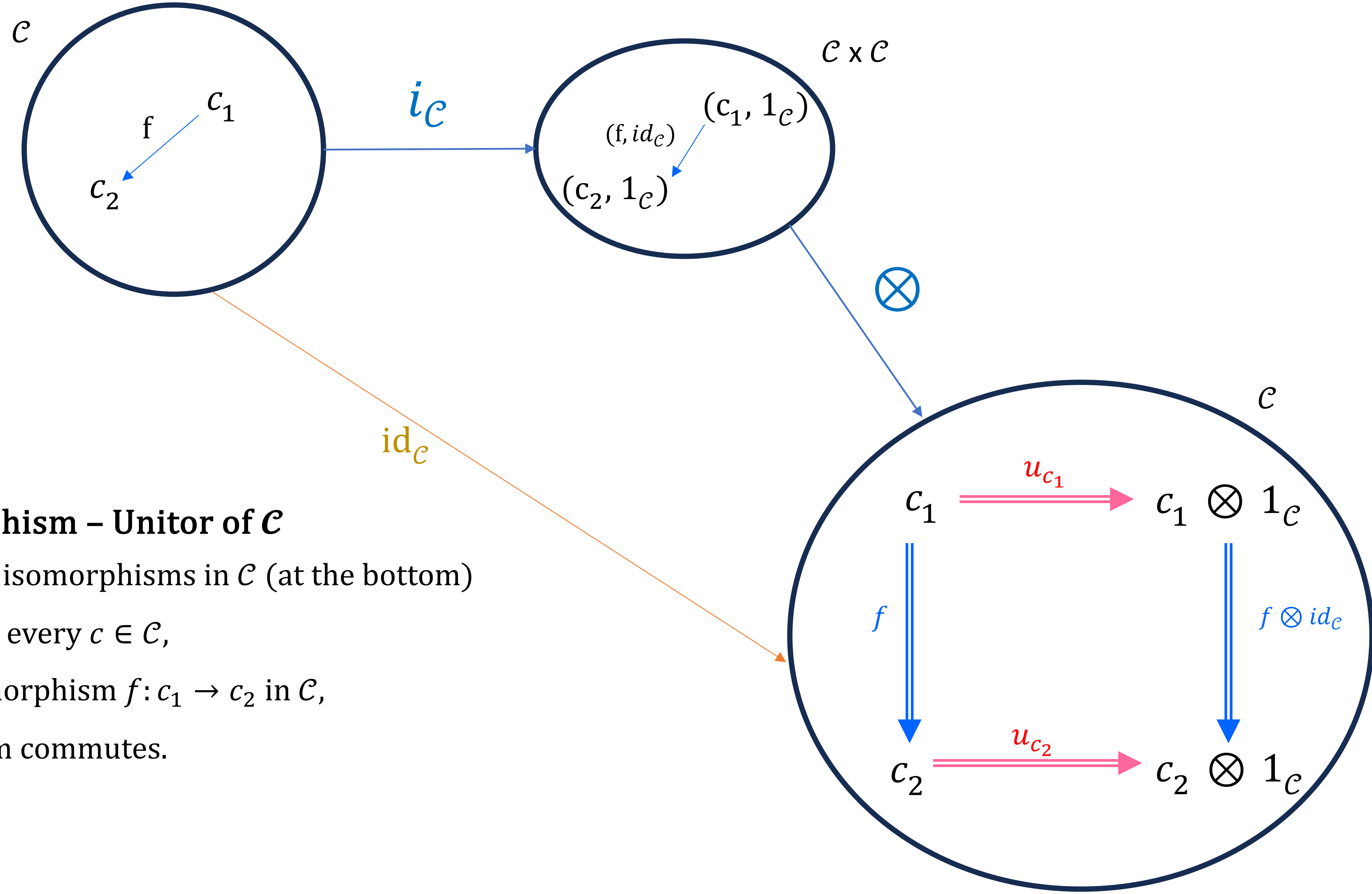
$\gamma : \mathcal{C}\times\mathcal{C} \to \mathcal{C}\times\mathcal{C}$

$(c_0, c_1) \mapsto (c_1, c_0)$

$$\begin{tikzcd}
(\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \arrow[d, "\alpha"] \arrow[ddddd, bend right=60, "T"'] \\
\mathcal{C}\times(\mathcal{D}\times(\mathcal{C}\times\mathcal{D})) \arrow[d, "\gamma"] \\
\mathcal{C}\times(\mathcal{D}\times(\mathcal{D}\times\mathcal{C})) \arrow[d, "\alpha"] \\
\mathcal{C}\times((\mathcal{D}\times\mathcal{D})\times\mathcal{C}) \arrow[d, "\gamma"] \\
\mathcal{C}\times(\mathcal{C}\times(\mathcal{D}\times\mathcal{D})) \arrow[d, "\alpha"] \\
(\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D}) \arrow[uuuuu, bend right=60, "T^{-1}"']
\end{tikzcd}$$

$$(\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \longrightarrow (\mathcal{C}\times\mathcal{D})$$

$$\begin{tikzcd}
(\mathcal{C}\times\mathcal{D})\times(\mathcal{C}\times\mathcal{D}) \arrow[d, "T"'] \arrow[dd, bend right=60, "\bigotimes_{\mathcal{C}\times\mathcal{D}}"'] \\
(\mathcal{C}\times\mathcal{C})\times(\mathcal{D}\times\mathcal{D}) \arrow[d, "\otimes_{\mathcal{C}}\times\otimes_{\mathcal{D}}"'] \\
\mathcal{C}\times\mathcal{D}
\end{tikzcd}$$



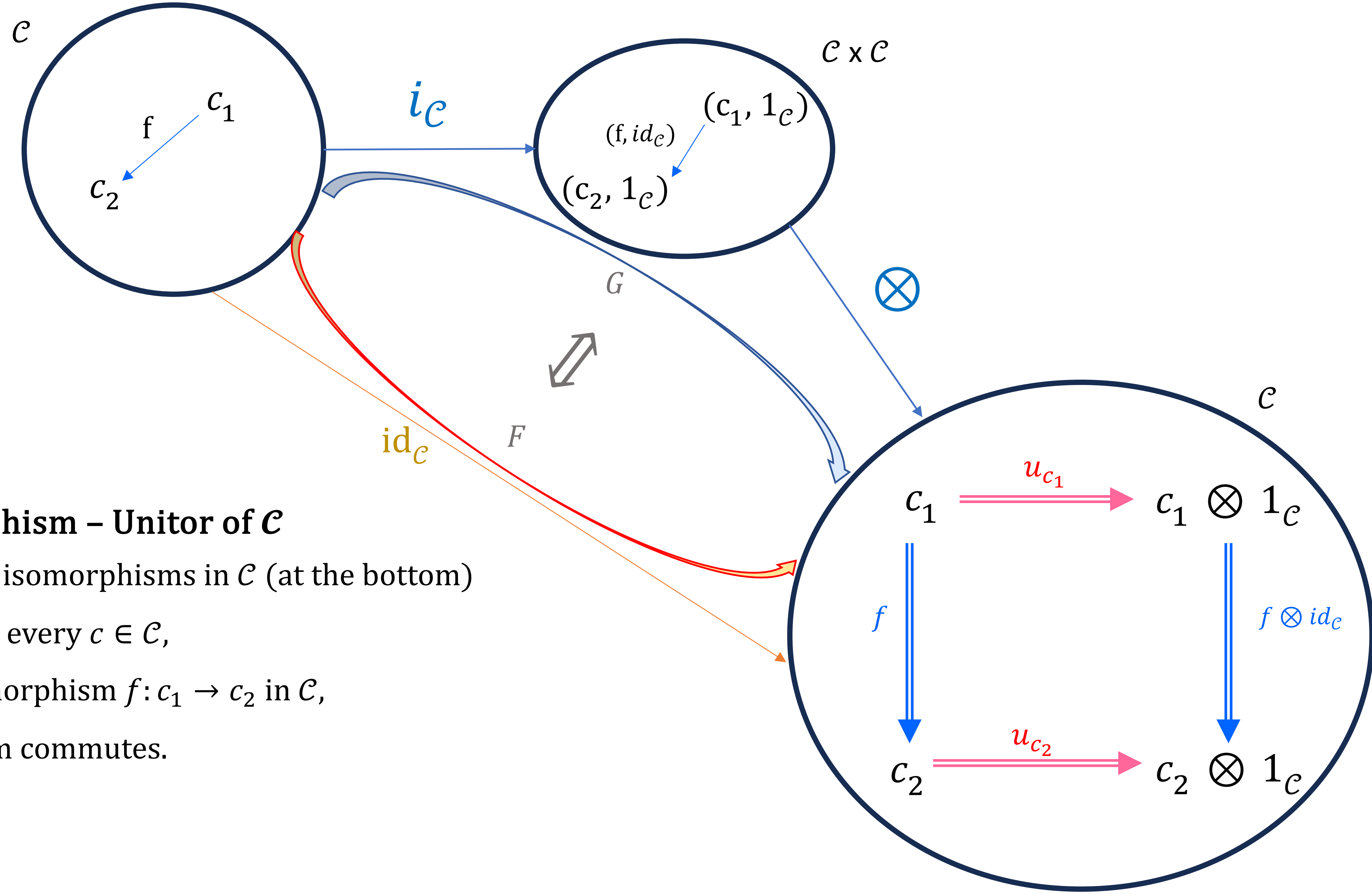
Natural Isomorphism – Unitor of $\mathcal{C}$

is the collection of isomorphisms in $\mathcal{C}$ (at the bottom)

$u_{\mathcal{C}}: c \to c \otimes 1_{\mathcal{C}}$ for every $c \in \mathcal{C}$,

such that for any morphism $f: c_1 \to c_2$ in $\mathcal{C}$,

the square diagram commutes.



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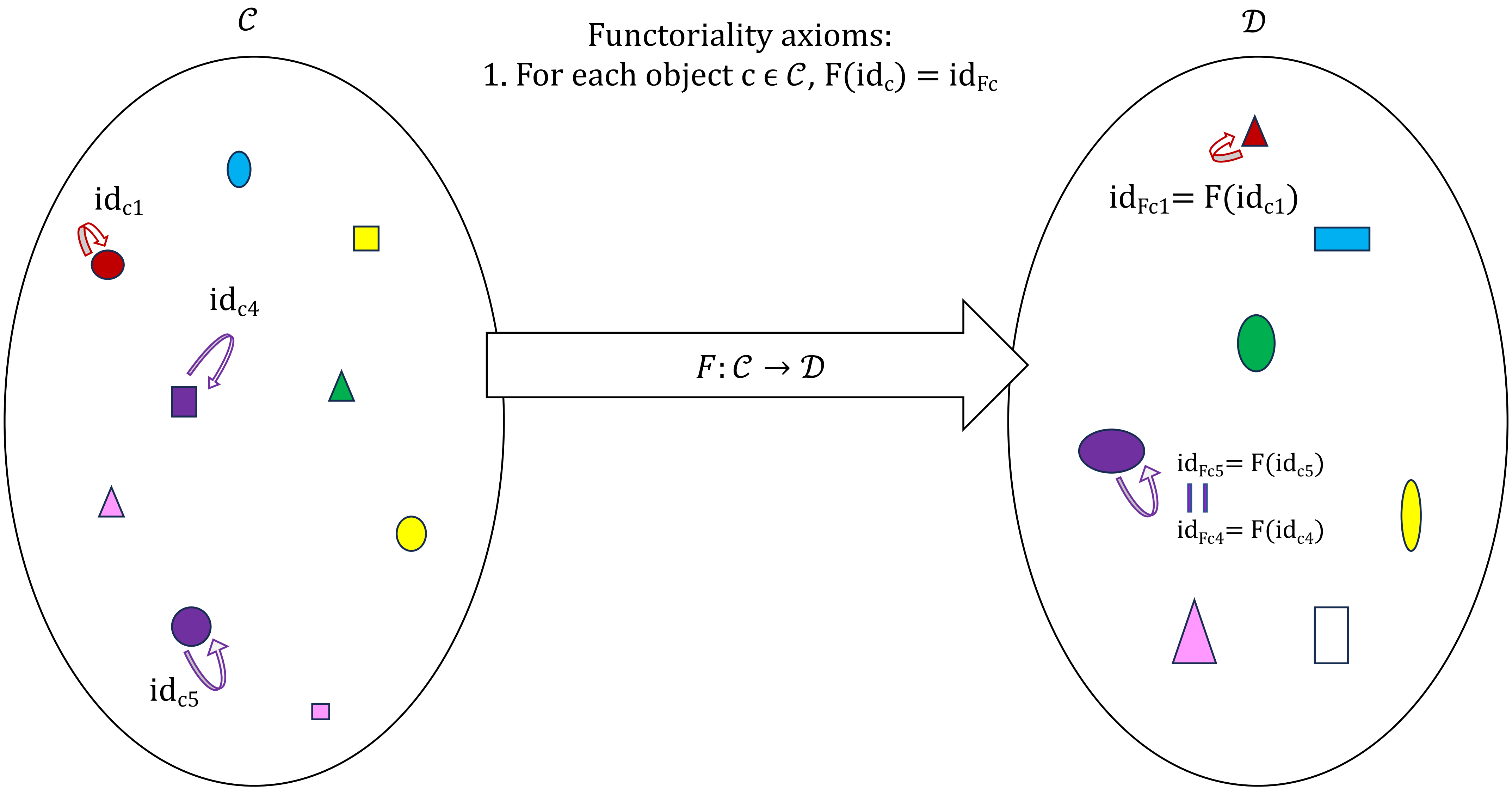
such that for any morphism $f: c_1 \to c_2$ in $\mathcal{C}$,

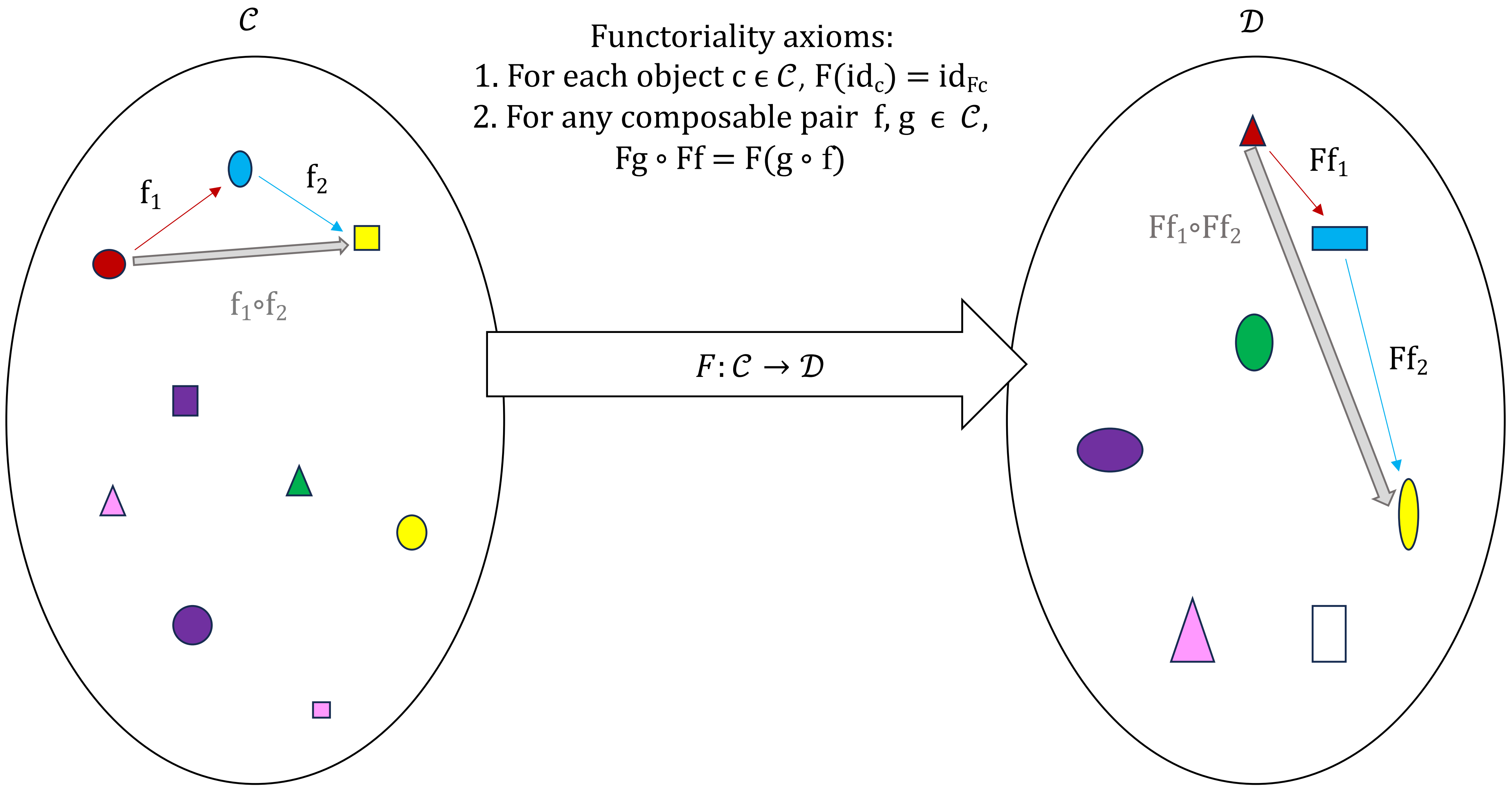
the square diagram commutes.

Functoriality Axioms

Functor : Category → Category



Functor : Category → Category



Functor : Category → Category