Advanced Computation:
Computational Electromagnetics

Maxwell's Equations

1

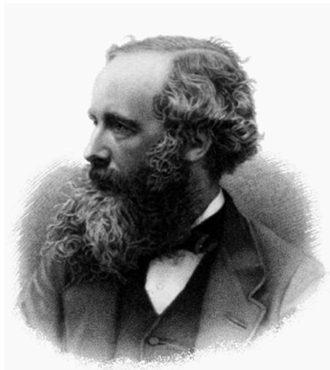
Outline

- Maxwell's equations
- Physical Boundary conditions
- Preparing Maxwell's equations for CEM
- Scaling properties of Maxwell's equations

Slide 2

2

Maxwell's Equations



James Clerk Maxwell

Born June 13,1831
Edinburgh, Scotland

Died November 5, 1879
Cambridge, England

Slide 3

3

Sign Conventions for Waves

To describe a wave propagating the positive z direction, we have two choices:

$$E(z,t) = A\cos(\omega t - kz)$$ Most common in engineering

$$E(z,t) = A\cos(-\omega t + kz)$$ Most common science and physics

Both are correct, but we must choose a convention and be consistent with it. For time-harmonic signals, this becomes

$$E(z) = A\exp(-jkz)$$ Negative sign convention

$$E(z) = A\exp(+jkz)$$ Positive sign convention

EMPossible

Slide 4

4

Time-Harmonic Maxwell's Equations

Time-Domain

$$\nabla \bullet \vec{D} = \rho_v \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Frequency-Domain (e^{+jkz} convention)

$$\nabla \bullet \vec{D} = \rho_v \qquad \nabla \times \vec{E} = j\omega\vec{B}$$

$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = \vec{J} - j\omega\vec{D}$$

Frequency-Domain (e^{-jkz} convention)

$$\nabla \bullet \vec{D} = \rho_v \qquad \nabla \times \vec{E} = -j\omega\vec{B}$$

$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = \vec{J} + j\omega\vec{D}$$

EMPossible

Slide 5

5

GOVERNING EQUATIONS FOR CLASSICAL ELECTROMAGNETICS

Pioneering 21st Century Electromagnetics and Photonics

http://emlab.utep.edu

	Integral Form	Differential Form	Name
Time-Domain	$Q_e(t) = \oiint_S \vec{D}(t) \bullet d\vec{s} = \iiint_V \rho_v(t)\,dv$	$\nabla \bullet \vec{D}(t) = \rho_v(t)$	Gauss' Law
	$\oiint_S \vec{B}(t) \bullet d\vec{s} = 0$	$\nabla \bullet \vec{B}(t) = 0$	No Magnetic Charge
	$V_{\text{emf}}(t) = \oint_L \vec{E}(t) \bullet d\vec{\ell} = -\iint_S \left[\frac{\partial \vec{B}(t)}{\partial t}\right] \bullet d\vec{s}$	$\nabla \times \vec{E}(t) = -\frac{\partial \vec{B}(t)}{\partial t}$	Faraday's Law
	$I(t) = \oint_L \vec{H}(t) \bullet d\vec{\ell} = \iint_S \left[\vec{J}(t) + \frac{\partial \vec{D}(t)}{\partial t}\right] \bullet d\vec{s}$	$\nabla \times \vec{H}(t) = \vec{J}(t) + \frac{\partial \vec{D}(t)}{\partial t}$	Ampere's Circuit Law
	$\oiint_S \vec{J} \bullet d\vec{s} = -\frac{\partial Q_e}{\partial t}$	$\nabla \bullet \vec{J} = -\frac{\partial \rho_v}{\partial t}$	Continuity of Current
	$\vec{D}(t) = [\varepsilon(t)] * \vec{E}(t)$ Electric Response $\vec{B}(t) = [\mu(t)] * \vec{H}(t)$ Magnetic Response		Constitutive Relations
Frequency-Domain	$Q_e = \oiint_S \vec{D} \bullet d\vec{s} = \iiint_V \rho_v\,dv$	$\nabla \bullet \vec{D} = \rho_v$	Gauss' Law
	$\oiint_S \vec{B} \bullet d\vec{s} = 0$	$\nabla \bullet \vec{B} = 0$	No Magnetic Charge
	$V_{\text{emf}} = \oint_L \vec{E} \bullet d\vec{\ell} = -\iint_S [j\omega\vec{B}] \bullet d\vec{s}$	$\nabla \times \vec{E} = -j\omega\vec{B}$	Faraday's Law
	$I = \oint_L \vec{H} \bullet d\vec{\ell} = \iint_S [\vec{J} + j\omega\vec{D}] \bullet d\vec{s}$	$\nabla \times \vec{H} = \vec{J} + j\omega\vec{D}$	Ampere's Circuit Law
	$\oiint_S \vec{J} \bullet d\vec{s} = -j\omega Q_e$	$\nabla \bullet \vec{J} = -j\omega\rho_v$	Continuity of Current
	$\vec{D} = [\varepsilon]\vec{E}$ Electric Response $\vec{B} = [\mu]\vec{H}$ Magnetic Response		Constitutive Relations

Parameter Definitions

Electric Field Intensity, E (V/m)
Electric Flux Density, D (C/m^2)
Magnetic Field Intensity, H (A/m)
Magnetic Flux Density, B (Wb/m^2)
Electric Current Density, J (A/m^2)
Volume Charge Density, ρ_v (C/m^3)
Permittivity, ε (F/m)
Permeability, μ (H/m)
Electrical Conductivity, σ (1/$\Omega\cdot$m)

Constants

Permittivity: $[\varepsilon] = \varepsilon_0[\varepsilon_r]$
$\varepsilon_0 = 8.8541878176 \times 10^{-12}$ (F/m)

Permeability: $[\mu] = \mu_0[\mu_r]$
$\mu_0 \approx 4\pi \times 10^{-7}$ (H/m)
$\mu_0 = 1.2566370614 \times 10^{-6}$ (H/m)

Impedance: $\eta_0 \approx 120\pi$ (Ω)
$\eta_0 = 376.73031346177$ (Ω)

Speed of Light: $c_0 = 299,792,458$ (m/s)

Lorentz Force Law

$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B})$

Sign Convention

e^{-jkz} For propagation in the $+z$ direction.

EMPossible

Slide 6

6

Lorentz Force Law

One additional equation is needed to completely describe classical electromagnetism...

$$\vec{F} = \underbrace{q\vec{E}}_{\text{Electric Force}} + \underbrace{q\vec{v} \times \vec{B}}_{\text{Magnetic Force}}$$

EMPossible

Slide 7

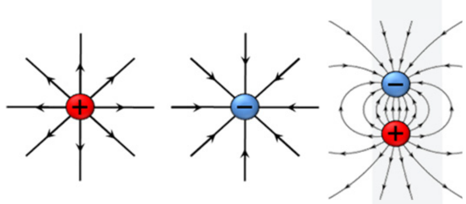
7

Gauss's Law

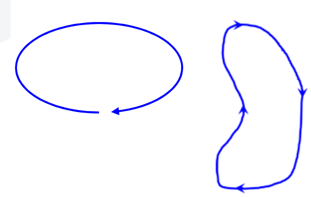
$$\boxed{\nabla \bullet \vec{D} = \rho_v}$$

$$\nabla \bullet \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Electric fields diverge from positive charges and converge on negative charges.



If there are no charges, electric fields must form loops.



EMPossible

Slide 8

8

Gauss's Law for Magnetism

$$\boxed{\nabla \bullet \vec{B} = 0}$$

$$\nabla \bullet \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

Magnetic fields always form loops.

EMPossible

Slide 9

9

Consequence of Zero Divergence

The divergence theorems force the $\vec{D}$ and $\vec{B}$ fields to be perpendicular to the propagation direction $\vec{k}$ of a plane wave.

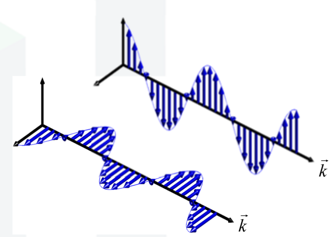
$$\nabla \bullet \vec{D} = 0$$

$$\nabla \bullet \left(\vec{d} e^{-j\vec{k}\bullet\vec{r}} \right) = 0$$

$$\underbrace{\cancel{\nabla \bullet \vec{d}}}_{\text{no charges}} - j\vec{k} \bullet \vec{d} = 0$$

$$\vec{k} \bullet \vec{d} = 0$$

$$\boxed{\vec{k} \perp \vec{D}}$$



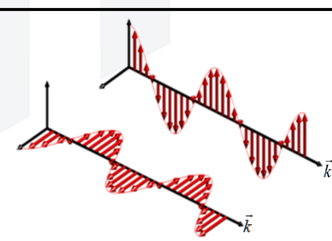
$$\nabla \bullet \vec{B} = 0$$

$$\nabla \bullet \left(\vec{b} e^{-j\vec{k}\bullet\vec{r}} \right) = 0$$

$$\underbrace{\cancel{\nabla \bullet \vec{b}}}_{\text{no charges}} - j\vec{k} \bullet \vec{b} = 0$$

$$\vec{k} \bullet \vec{b} = 0$$

$$\boxed{\vec{k} \perp \vec{B}}$$



EMPossible

Slide 10

10

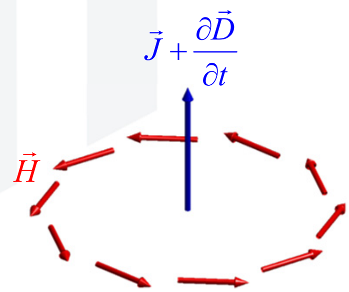
Ampere's Law with Maxwell's Correction

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

$$\nabla \times \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \hat{a}_z$$

Circulating magnetic fields induce currents and/or time varying electric fields.

Currents and/or time varying electric fields induce circulating magnetic fields.



EMPossible

Slide 11

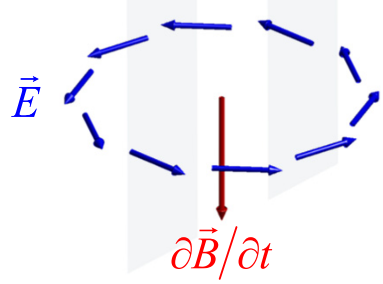
11

Faraday's Law of Induction

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

$$\nabla \times \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{a}_z$$

Circulating electric fields induce time varying magnetic fields.
Time varying magnetic fields induce circulating electric fields.



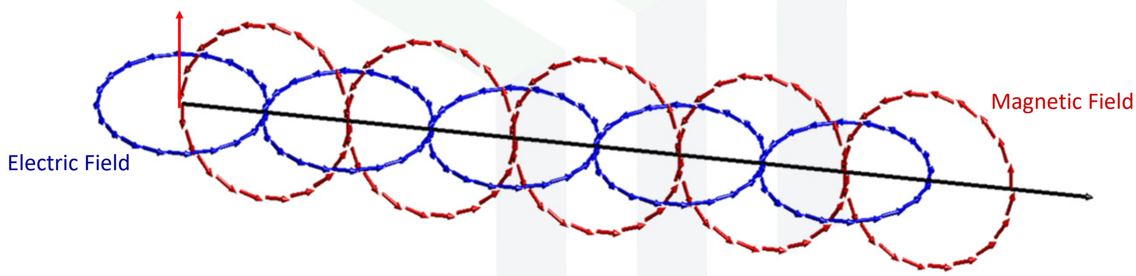
EMPossible

Slide 12

12

Consequence of Curl Equations

The curl equations predict electromagnetic waves!!



The Constitutive Relations

Electric Response

$$\vec{D} = \varepsilon \vec{E}$$

- Electric field intensity (V/m)
- Initial electric "push."
- Induced electric field.
- Electric energy in vacuum.

- Permittivity (F/m)
- Measure of how well a material stores electric energy.

- Electric flux density (C/m^2)
- Pretends as if all electric energy is displaced charge.
- Includes electric energy in vacuum and matter.

Magnetic Response

$$\vec{B} = \mu \vec{H}$$

- Magnetic field intensity (A/m)
- Initial magnetic "push."
- Induced magnetic field.
- Magnetic energy in vacuum.

- Permeability (H/m)
- Measure of how well a material stores magnetic energy.

- Magnetic flux density (Wb/m^2)
- Pretends as if all magnetic energy is tilted magnetic dipoles.
- Includes magnetic energy in vacuum and matter.

Material Classifications

Linear, isotropic and non-dispersive materials:

$$\vec{D}(t) = \varepsilon \vec{E}(t)$$

We will use this almost exclusively

Dispersive materials:

$$\vec{D}(t) = \varepsilon(t) * \vec{E}(t)$$

Anisotropic materials:

$$\vec{D}(t) = [\varepsilon] \vec{E}(t)$$

A key point is that you can wrap all of the complexities associated with modeling strange materials into this single equation. This will make your code more modular and easier to modify. It may not be as efficient as it could be though.

Nonlinear materials:

$$D(t) = \varepsilon_0 \chi_e^{(1)} E(t) + \varepsilon_0 \chi_e^{(2)} E^2(t) + \varepsilon_0 \chi_e^{(3)} E^3(t) + \cdots$$

EMPossible 15

15

All Together Now…

Divergence Equations

$$\nabla \bullet \vec{B} = 0$$

$$\nabla \bullet \vec{D} = \rho_v$$

Curl Equations

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

What produces fields

Constitutive Relations

$$\vec{D}(t) = [\varepsilon(t)] * \vec{E}(t)$$

$$\vec{B}(t) = [\mu(t)] * \vec{H}(t)$$

$*$ means convolution

[ ] means tensor

How fields interact with materials

EMPossible Slide 16

16

Maxwell's Equations in Cartesian Coordinates (1 of 4)

Vector Terms

$$\vec{E} = E_x\hat{a}_x + E_y\hat{a}_y + E_z\hat{a}_z \qquad \vec{H} = H_x\hat{a}_x + H_y\hat{a}_y + H_z\hat{a}_z \qquad \vec{J} = J_x\hat{a}_x + J_y\hat{a}_y + J_z\hat{a}_z$$

$$\vec{D} = D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z \qquad \vec{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$$

Divergence Equations

$$\nabla \bullet \vec{D} = 0 \qquad \nabla \bullet \vec{B} = 0$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0 \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

EMPossible

Slide 17

17

Maxwell's Equations in Cartesian Coordinates (2 of 4)

Constitutive Relations

$$\vec{D} = [\varepsilon]\vec{E}$$

$$D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z = \left(\varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z\right)\hat{a}_x + \left(\varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z\right)\hat{a}_y + \left(\varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \varepsilon_{zz}E_z\right)\hat{a}_z$$

$$D_x = \varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z$$
$$D_y = \varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z$$
$$D_z = \varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \varepsilon_{zz}E_z$$

$$\vec{B} = [\mu]\vec{H} \longrightarrow \begin{aligned} B_x &= \varepsilon_{xx}H_x + \varepsilon_{xy}H_y + \varepsilon_{xz}H_z \\ B_y &= \varepsilon_{yx}H_x + \varepsilon_{yy}H_y + \varepsilon_{yz}H_z \\ B_z &= \varepsilon_{zx}H_x + \varepsilon_{zy}H_y + \varepsilon_{zz}H_z \end{aligned}$$

EMPossible

Slide 18

18

Maxwell's Equations in Cartesian Coordinates (3 of 4)

Curl Equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{a}_z = -\frac{\partial}{\partial t}\left(B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z\right)$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{a}_z = -\frac{\partial B_x}{\partial t}\hat{a}_x - \frac{\partial B_y}{\partial t}\hat{a}_y - \frac{\partial B_z}{\partial t}\hat{a}_z$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \qquad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \qquad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t}$$

EMPossible

Slide 19

19

Maxwell's Equations in Cartesian Coordinates (4 of 4)

Curl Equations

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{a}_z = \left(J_x\hat{a}_x + J_y\hat{a}_y + J_z\hat{a}_z\right) + \frac{\partial}{\partial t}\left(D_x\hat{a}_x + D_y\hat{a}_y + D_z\hat{a}_z\right)$$

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{a}_z = \left(J_x + \frac{\partial D_x}{\partial t}\right)\hat{a}_x + \left(J_y + \frac{\partial D_y}{\partial t}\right)\hat{a}_y + \left(J_z + \frac{\partial D_z}{\partial t}\right)\hat{a}_z$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x + \frac{\partial D_x}{\partial t} \qquad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + \frac{\partial D_y}{\partial t} \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + \frac{\partial D_z}{\partial t}$$

EMPossible

Slide 20

20

Alternative Form of Maxwell's Equations in Cartesian Coordinates (1 of 2)

Alternate Curl Equations

$$\nabla \times \vec{H} = [\varepsilon] \frac{\partial \vec{E}}{\partial t}$$

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \hat{a}_z = \left(\varepsilon_{xx} \frac{\partial E_x}{\partial t} + \varepsilon_{xy} \frac{\partial E_y}{\partial t} + \varepsilon_{xz} \frac{\partial E_z}{\partial t} \right) \hat{a}_x + \left(\varepsilon_{yx} \frac{\partial E_x}{\partial t} + \varepsilon_{yy} \frac{\partial E_y}{\partial t} + \varepsilon_{yz} \frac{\partial E_z}{\partial t} \right) \hat{a}_y + \left(\varepsilon_{zx} \frac{\partial E_x}{\partial t} + \varepsilon_{zy} \frac{\partial E_y}{\partial t} + \varepsilon_{zz} \frac{\partial E_z}{\partial t} \right) \hat{a}_z$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \varepsilon_{xx} \frac{\partial E_x}{\partial t} + \varepsilon_{xy} \frac{\partial E_y}{\partial t} + \varepsilon_{xz} \frac{\partial E_z}{\partial t}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \varepsilon_{yx} \frac{\partial E_x}{\partial t} + \varepsilon_{yy} \frac{\partial E_y}{\partial t} + \varepsilon_{yz} \frac{\partial E_z}{\partial t}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \varepsilon_{zx} \frac{\partial E_x}{\partial t} + \varepsilon_{zy} \frac{\partial E_y}{\partial t} + \varepsilon_{zz} \frac{\partial E_z}{\partial t}$$

EMPossible

Slide 21

21

Alternative Form of Maxwell's Equations in Cartesian Coordinates (2 of 2)

Alternate Curl Equations

$$\nabla \times \vec{E} = -[\mu] \frac{\partial \vec{H}}{\partial t}$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{a}_z = -\left(\mu_{xx} \frac{\partial H_x}{\partial t} + \mu_{xy} \frac{\partial H_y}{\partial t} + \mu_{xz} \frac{\partial H_z}{\partial t} \right) \hat{a}_x - \left(\mu_{yx} \frac{\partial H_x}{\partial t} + \mu_{yy} \frac{\partial H_y}{\partial t} + \mu_{yz} \frac{\partial H_z}{\partial t} \right) \hat{a}_y - \left(\mu_{zx} \frac{\partial H_x}{\partial t} + \mu_{zy} \frac{\partial H_y}{\partial t} + \mu_{zz} \frac{\partial H_z}{\partial t} \right) \hat{a}_z$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu_{xx} \frac{\partial H_x}{\partial t} - \mu_{xy} \frac{\partial H_y}{\partial t} - \mu_{xz} \frac{\partial H_z}{\partial t}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu_{yx} \frac{\partial H_x}{\partial t} - \mu_{yy} \frac{\partial H_y}{\partial t} - \mu_{yz} \frac{\partial H_z}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu_{zx} \frac{\partial H_x}{\partial t} - \mu_{zy} \frac{\partial H_y}{\partial t} - \mu_{zz} \frac{\partial H_z}{\partial t}$$

EMPossible

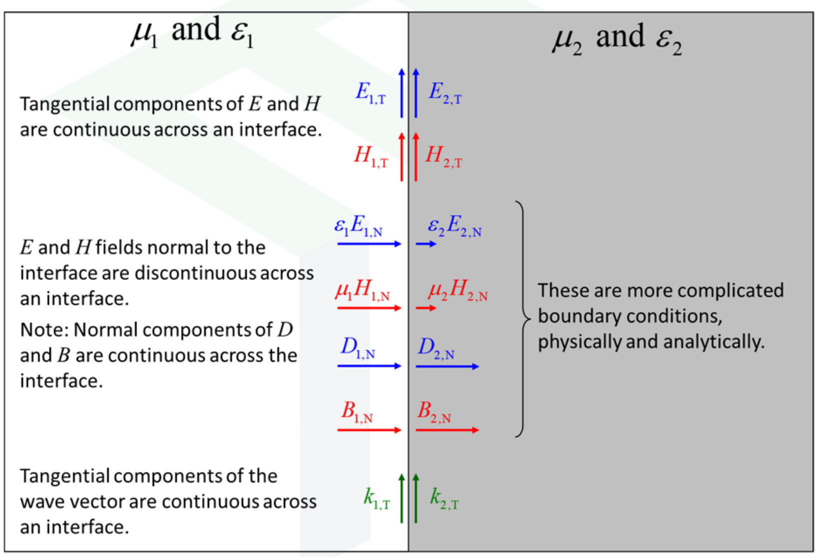
Slide 22

22

Physical Boundary Conditions

23

Physical Boundary Conditions



EMPossible

24

Preparing Maxwell's Equations for CEM

Slide 25

25

Simplifying Maxwell's Equations

1. Assume no charges or current sources: $\rho_v = 0,\ \vec{J} = 0$

$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = \partial \vec{D}/\partial t \qquad \vec{D}(t) = [\varepsilon(t)] * \vec{E}(t)$$
$$\nabla \bullet \vec{D} = 0 \qquad \nabla \times \vec{E} = -\partial \vec{B}/\partial t \qquad \vec{B}(t) = [\mu(t)] * \vec{H}(t)$$

2. Transform Maxwell's equations to frequency-domain:

$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = j\omega \vec{D} \qquad \vec{D} = [\varepsilon]\vec{E}$$
$$\nabla \bullet \vec{D} = 0 \qquad \nabla \times \vec{E} = -j\omega \vec{B} \qquad \vec{B} = [\mu]\vec{H}$$

Convolution becomes simple multiplication

Note: We have chosen to proceed with the negative sign convention.

3. Substitute constitutive relations into Maxwell's equations:

$$\nabla \bullet \left([\mu]\vec{H}\right) = 0 \qquad \nabla \times \vec{H} = j\omega[\varepsilon]\vec{E}$$
$$\nabla \bullet \left([\varepsilon]\vec{E}\right) = 0 \qquad \nabla \times \vec{E} = -j\omega[\mu]\vec{H}$$

Note: It is useful to retain μ and ε and not replace them with refractive index *n*.

EMPossible

Slide 26

26

Isotropic Materials

For anisotropic materials, the permittivity and permeability terms are tensor quantities.

$$[\varepsilon]=\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}\end{bmatrix} \qquad [\mu]=\begin{bmatrix}\mu_{xx} & \mu_{xy} & \mu_{xz}\\ \mu_{yx} & \mu_{yy} & \mu_{yz}\\ \mu_{zx} & \mu_{zy} & \mu_{zz}\end{bmatrix}$$

For isotropic materials, the tensors reduce to a single scalar quantity.

$$[\varepsilon]=\begin{bmatrix}\varepsilon & 0 & 0\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon\end{bmatrix}=\varepsilon \qquad [\mu]=\begin{bmatrix}\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu\end{bmatrix}=\mu$$

Maxwell's equations can then be written as

$$\nabla\bullet\left(\mu_r\vec{H}\right)=0 \qquad \nabla\times\vec{H}=j\omega\varepsilon_0\varepsilon_r\vec{E}$$

$$\nabla\bullet\left(\varepsilon_r\vec{E}\right)=0 \qquad \nabla\times\vec{E}=-j\omega\mu_0\mu_r\vec{H}$$

ε_0 and μ_0 dropped from these equations because they are constants and do not vary spatially.

Expand Maxwell's Equations

Divergence Equations

$$\nabla\bullet\left(\mu_r\vec{H}\right)=0$$
$$\downarrow$$
$$\frac{\partial\left(\mu_r H_x\right)}{\partial x}+\frac{\partial\left(\mu_r H_y\right)}{\partial y}+\frac{\partial\left(\mu_r H_z\right)}{\partial z}=0$$

$$\nabla\bullet\left(\varepsilon_r\vec{E}\right)=0$$
$$\downarrow$$
$$\frac{\partial\left(\varepsilon_r E_x\right)}{\partial x}+\frac{\partial\left(\varepsilon_r E_y\right)}{\partial y}+\frac{\partial\left(\varepsilon_r E_z\right)}{\partial z}=0$$

Curl Equations

$$\nabla\times\vec{H}=j\omega\varepsilon_0\varepsilon_r\vec{E}$$
$$\downarrow$$
$$\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=j\omega\varepsilon_0\varepsilon_r E_x$$
$$\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=j\omega\varepsilon_0\varepsilon_r E_y$$
$$\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=j\omega\varepsilon_0\varepsilon_r E_z$$

$$\nabla\times\vec{E}=-j\omega\mu_0\mu_r\vec{H}$$
$$\downarrow$$
$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-j\omega\mu_0\mu_r H_x$$
$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-j\omega\mu_0\mu_r H_y$$
$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-j\omega\mu_0\mu_r H_z$$

Normalize the Magnetic Field

Standard form of "Maxwell's Curl Equations"

$$\nabla \times \vec{E} = -j\omega\mu_0\mu_r\vec{H} \qquad \nabla \times \vec{H} = j\omega\varepsilon_0\varepsilon_r\vec{E}$$

Normalized Magnetic Field

$$\frac{|\vec{E}|}{|\vec{H}|} \cong \frac{377}{n} \qquad \vec{\tilde{H}} = \underbrace{-j}\underbrace{\sqrt{\frac{\mu_0}{\varepsilon_0}}}\vec{H}$$

Note:
$k_0 = \omega\sqrt{\mu_0\varepsilon_0}$

- Eliminates $j\omega$
- No sign inconsistency
- Just have k_0

Equalizes E and H amplitudes

Normalized Maxwell's Equations

$$\nabla \times \vec{E} = k_0\mu_r\vec{\tilde{H}} \qquad \nabla \times \vec{\tilde{H}} = k_0\varepsilon_r\vec{E}$$

EMPossible

Slide 29

29

Starting Point for Most CEM

We arrive at the following set of equations that are the same regardless of the sign convention used.

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0\mu_{xx}\tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z} = k_0\varepsilon_{xx}E_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = k_0\mu_{yy}\tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x} = k_0\varepsilon_{yy}E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = k_0\mu_{zz}\tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} = k_0\varepsilon_{zz}E_z$$

The manner in which the magnetic field is normalized does depend on the sign convention chosen.

$$\vec{\tilde{H}} = \begin{cases} -j\eta_0\vec{H} & \text{negative sign convnetion} \\ +j\eta_0\vec{H} & \text{positive sign convnetion} \end{cases}$$

EMPossible

Slide 30

30

Scaling Properties of Maxwell’s Equations

Scaling Properties of Maxwell’s Equations

There is no fundamental length scale in Maxwell’s equations.

Devices may be scaled to operate at different frequencies just by scaling the mechanical dimensions or material properties in proportion to the change in frequency.

This assumes it is physically possible to scale systems in this manner. In practice, building larger or smaller features may not be practical. Further, the properties of the materials may be different at the new operating frequency.

Scaling Dimensions

We start with the wave equation and write the parameters dependence on position explicitly.

$$\nabla \times \frac{1}{\mu_r(\vec{r})} \nabla \times \vec{E}(\vec{r}) = \omega^2 \mu_0 \varepsilon_0 \cdot \varepsilon_r(\vec{r}) \cdot \vec{E}(\vec{r})$$

Next, we scale the dimensions by a factor a.

$$(a\nabla) \times \frac{1}{\mu_r(\vec{r}/a)} (a\nabla) \times \vec{E}(\vec{r}/a) = \omega^2 \mu_0 \varepsilon_0 \cdot \varepsilon_r(\vec{r}/a) \cdot \vec{E}(\vec{r}/a)$$

$a > 1$ stretch dimensions
$a < 1$ compress dimensions

The scale factors multiplying the ∇ operators are moved to multiply the frequency term.

$$\nabla \times \frac{1}{\mu_r(\vec{r}\,')} \nabla \times \vec{E}(\vec{r}\,') = \left(\frac{\omega}{a}\right)^2 \mu_0 \varepsilon_0 \cdot \varepsilon_r(\vec{r}\,') \cdot \vec{E}(\vec{r}\,') \qquad \vec{r}\,' = \frac{\vec{r}}{a}$$

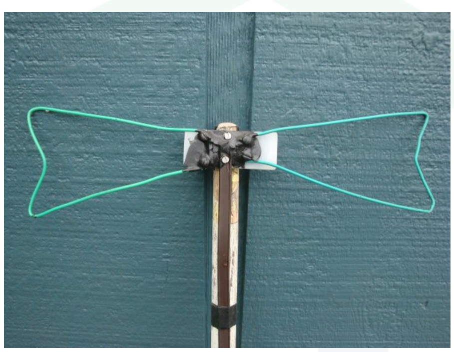
The effect of scaling the dimensions is just a shift in frequency.

EMPossible Slide 33

33

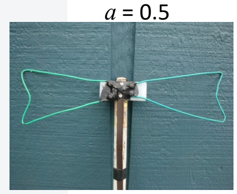
Visualization of Size Scaling

a = 1.0



f_c = 500 MHz

a = 0.5



f_c = 1000 MHz

EMPossible Slide 34

34

Scaling μ and ε

We apply separate scaling factors to μ and ε.

$$\nabla \times \frac{1}{\left(a_{\mu}\mu_{\mathrm{r}}\right)} \nabla \times \vec{E} = \omega^{2} \mu_{0} \varepsilon_{0} \cdot \left(a_{\varepsilon}\varepsilon_{\mathrm{r}}\right) \cdot \vec{E}$$

The scale factors are moved to multiply the frequency term.

$$\nabla \times \frac{1}{\mu_{\mathrm{r}}} \nabla \times \vec{E} = \left(\omega\sqrt{a_{\mu}a_{\varepsilon}}\right)^{2} \mu_{0} \varepsilon_{0} \cdot \varepsilon_{\mathrm{r}} \cdot \vec{E}$$

The effect of scaling the material properties is just a factor in frequency.

EMPossible

Slide 35

35