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Advanced Computation:
Computational Electromagnetics

RCWA Extras

1

Outline

- Calculating internal fields
- One spatial harmonic: P=Q=1
- Simulation of 1D Gratings with 3D-RCWA
- Formulation of 2D-RCWA with fast Fourier factorization
- Danger of RCWA and convergence
- RCWA and curved structures
- Strategically truncating the set of spatial harmonics
- RCWA for generalized symmetries
- Modeling hexagonal gratings with rectangular RCWA
- Enhanced transmittance matrix approach

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Calculating Internal Fields

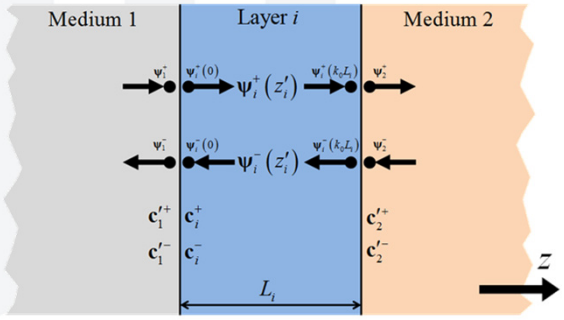
Information Needed

The field inside the i^{th} layer is calculated as:

$$\boldsymbol{\psi}_i(z_i') = \begin{bmatrix} \mathbf{s}_x(z_i') \\ \mathbf{s}_y(z_i') \\ \mathbf{u}_x(z_i') \\ \mathbf{u}_y(z_i') \end{bmatrix} = \begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ -\mathbf{V}_i & \mathbf{V}_i \end{bmatrix} \begin{bmatrix} e^{-\boldsymbol{\lambda}_i z_i'} & \mathbf{0} \\ \mathbf{0} & e^{\boldsymbol{\lambda}_i z_i'} \end{bmatrix} \begin{bmatrix} \mathbf{c}_i^+ \\ \mathbf{c}_i^- \end{bmatrix}$$

To do this, the parameters $\mathbf{W}_i$, $\mathbf{V}_i$ and $\boldsymbol{\lambda}_i$ must be recorded when analyzing the ith layer.

But how are $\mathbf{c}_i^+$ and $\mathbf{c}_i^-$ calculated?



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Calculating the Internal Mode Coefficients

The scattering matrices calculate mode coefficients, $\mathbf{c}_{i1}^{\pm}$ and $\mathbf{c}_{i2}^{\pm}$, that are external of the ith layer. However, it is the internal mode coefficients $\mathbf{c}_{i}^{\pm}$ that are needed.

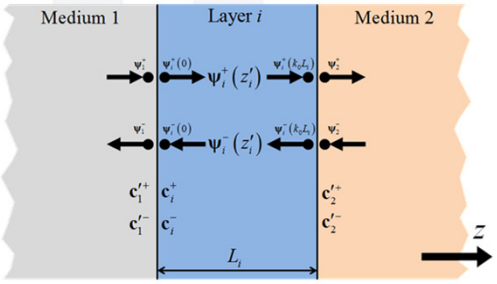
Boundary conditions require the fields to be continuous across an interface. Recall the boundary condition for the first interface is

$$\boldsymbol{\psi}_{i1} = \boldsymbol{\psi}_i(0)$$

$$\begin{bmatrix} \mathbf{W}_{i1} & \mathbf{W}_{i1} \\ -\mathbf{V}_{i1} & \mathbf{V}_{i1} \end{bmatrix} \begin{bmatrix} \mathbf{c}_{1i}^{+} \\ \mathbf{c}_{1i}^{-} \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{i} & \mathbf{W}_{i} \\ -\mathbf{V}_{i} & \mathbf{V}_{i} \end{bmatrix} \begin{bmatrix} \mathbf{c}_{i}^{+} \\ \mathbf{c}_{i}^{-} \end{bmatrix}$$

Solving for the internal mode coefficients gives

$$\begin{bmatrix} \mathbf{c}_{i}^{+} \\ \mathbf{c}_{i}^{-} \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{i} & \mathbf{W}_{i} \\ -\mathbf{V}_{i} & \mathbf{V}_{i} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{W}_{i1} & \mathbf{W}_{i1} \\ -\mathbf{V}_{i1} & \mathbf{V}_{i1} \end{bmatrix} \begin{bmatrix} \mathbf{c}_{1i}^{+} \\ \mathbf{c}_{1i}^{-} \end{bmatrix}$$



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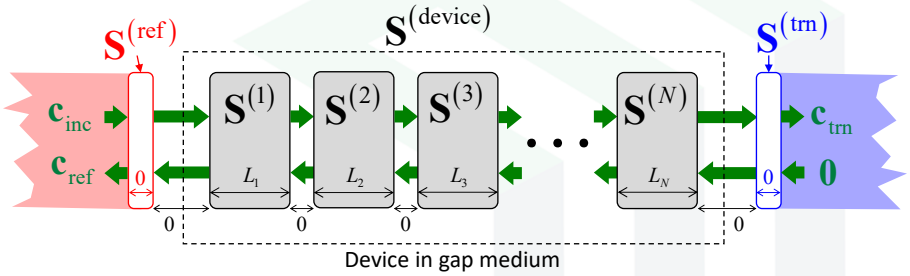
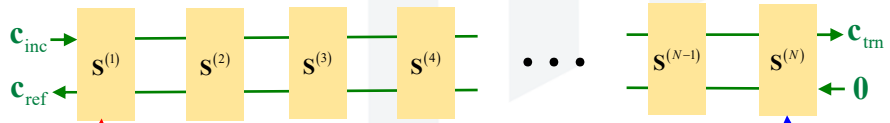
General Scattering Matrix Framework

The simulation is really just a series of scattering matrices.



Consider this in a simplified framework as



This is probably the reflection-region S-matrix.

This is probably the transmission-region S-matrix.

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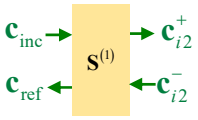
Algorithm: Forward Pass

Progressing from left to right, the global scattering matrix is built one layer at a time.



Record:

$\mathbf{W}_1$

$\mathbf{V}_1$

$\boldsymbol{\lambda}_1$

$\mathbf{S}^{(G1)} = \mathbf{S}^{(1)}$

Note, it is only necessary to store these intermediate parameters if the fields inside of this layer are to be calculated.

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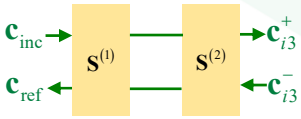
Algorithm: Forward Pass

Progressing from left to right, the global scattering matrix is built one layer at a time.



Record:

$\mathbf{W}_2$

$\mathbf{V}_2$

$\boldsymbol{\lambda}_2$

Note, it is only necessary to store these intermediate parameters if the fields inside of this layer are to be calculated.

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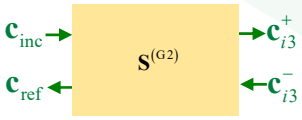
Algorithm: Forward Pass

Progressing from left to right, the global scattering matrix is built one layer at a time.



Record:

$$\mathbf{S}^{(G2)} = \mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)}$$

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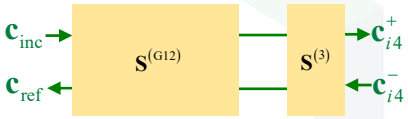
Algorithm: Forward Pass

Progressing from left to right, the global scattering matrix is built one layer at a time.



Record:

$\mathbf{W}_3$

$\mathbf{V}_3$

$\boldsymbol{\lambda}_3$

Note, it is only necessary to store these intermediate parameters if the fields inside of this layer are to be calculated.

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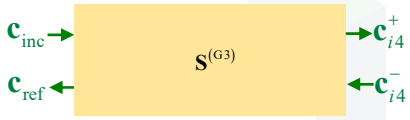
Algorithm: Forward Pass

Progressing from left to right, the global scattering matrix is built one layer at a time.



Record:

$$\mathbf{S}^{(G3)} = \mathbf{S}^{(G2)} \otimes \mathbf{S}^{(3)}$$

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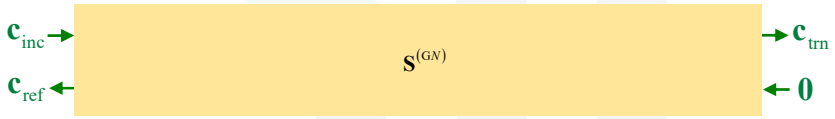
Algorithm: Forward Pass

The final global scattering matrix is

$$\begin{bmatrix} \mathbf{c}_{\text{ref}} \\ \mathbf{c}_{\text{trn}} \end{bmatrix} = \underbrace{\left(\begin{bmatrix} \mathbf{S}_{11}^{(1)} & \mathbf{S}_{12}^{(1)} \\ \mathbf{S}_{21}^{(1)} & \mathbf{S}_{22}^{(1)} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{S}_{11}^{(2)} & \mathbf{S}_{12}^{(2)} \\ \mathbf{S}_{21}^{(2)} & \mathbf{S}_{22}^{(2)} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} \mathbf{S}_{11}^{(N-1)} & \mathbf{S}_{12}^{(N-1)} \\ \mathbf{S}_{21}^{(N-1)} & \mathbf{S}_{22}^{(N-1)} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{S}_{11}^{(N)} & \mathbf{S}_{12}^{(N)} \\ \mathbf{S}_{21}^{(N)} & \mathbf{S}_{22}^{(N)} \end{bmatrix} \right)}_{\mathbf{S}^{(GN)}} \begin{bmatrix} \mathbf{c}_{\text{inc}} \\ \mathbf{0} \end{bmatrix}$$



Record:

$$\mathbf{S}^{(GN)} = \mathbf{S}^{[G(N-1)]} \otimes \mathbf{S}^{(N)}$$

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Algorithm: Backward Pass

Now the external mode coefficients, $\mathbf{c}_{N1}^{+}$ and $\mathbf{c}_{N1}^{-}$ can be calculated on the left side of layer N.

$$\begin{bmatrix}\mathbf{c}_{\text{ref}}\\ \mathbf{c}_{N1}^{+}\end{bmatrix}=\mathbf{S}^{\left[\mathrm{G}(N-1)\right]}\begin{bmatrix}\mathbf{c}_{\text{inc}}\\ \mathbf{c}_{N1}^{-}\end{bmatrix}\quad\rightarrow\quad\begin{aligned}\mathbf{c}_{N1}^{-}&=\left(\mathbf{S}_{12}^{\left[\mathrm{G}(N-1)\right]}\right)^{-1}\left(\mathbf{c}_{\text{ref}}-\mathbf{S}_{11}^{\left[\mathrm{G}(N-1)\right]}\mathbf{c}_{\text{inc}}\right)\\ \mathbf{c}_{N1}^{+}&=\mathbf{S}_{21}^{\left[\mathrm{G}(N-1)\right]}\mathbf{c}_{\text{inc}}+\mathbf{S}_{22}^{\left[\mathrm{G}(N-1)\right]}\mathbf{c}_{N1}^{-}\end{aligned}$$

The internal mode coefficients $\mathbf{c}_{N}^{+}$ and $\mathbf{c}_{N}^{-}$ of the Nth layer are then

$$\begin{bmatrix}\mathbf{c}_{N}^{+}\\ \mathbf{c}_{N}^{-}\end{bmatrix}=\begin{bmatrix}\mathbf{W}_{N} & \mathbf{W}_{N}\\ -\mathbf{V}_{N} & \mathbf{V}_{N}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{W}_{N1} & \mathbf{W}_{N1}\\ -\mathbf{V}_{N1} & \mathbf{V}_{N1}\end{bmatrix}\begin{bmatrix}\mathbf{c}_{N1}^{+}\\ \mathbf{c}_{N1}^{-}\end{bmatrix}$$

Note: for isotropic TMM

$\mathbf{W}_{N}=\mathbf{W}_{N1}=\mathrm{I}$

$\mathbf{V}_{N1}=\mathbf{V}_{g}$ for internal layers

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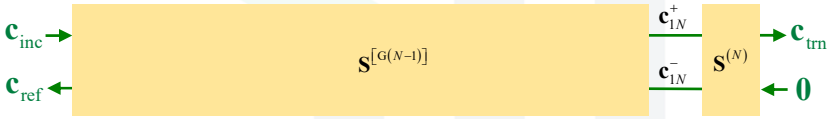
Algorithm: Backward Pass

The internal fields of the Nth layer can now be calculated at any position z'_N.

$$\boldsymbol{\psi}_{N}\left(z'_{N}\right)=\begin{bmatrix}\mathbf{s}_{x,N}\left(z'_{N}\right)\\ \mathbf{s}_{y,N}\left(z'_{N}\right)\\ \mathbf{u}_{x,N}\left(z'_{N}\right)\\ \mathbf{u}_{y,N}\left(z'_{N}\right)\end{bmatrix}=\begin{bmatrix}\mathbf{W}_{N} & \mathbf{W}_{N}\\ -\mathbf{V}_{N} & \mathbf{V}_{N}\end{bmatrix}\begin{bmatrix}e^{-\lambda_{N}z'_{N}} & \mathbf{0}\\ \mathbf{0} & e^{\lambda_{N}z'_{N}}\end{bmatrix}\begin{bmatrix}\mathbf{c}_{N}^{+}\\ \mathbf{c}_{N}^{-}\end{bmatrix}$$



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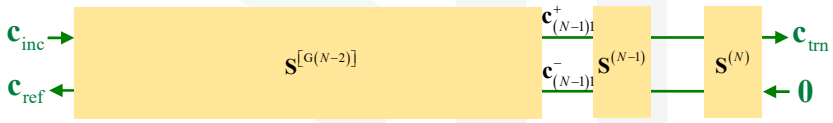
Algorithm: Backward Pass

It is now possible to calculate the mode coefficients inside layer N-1.

$$\begin{bmatrix} \mathbf{c}_{\text{ref}} \\ \mathbf{c}^{+}_{(N-1)1} \end{bmatrix} = \mathbf{S}^{[G(N-2)]} \begin{bmatrix} \mathbf{c}_{\text{inc}} \\ \mathbf{c}^{-}_{(N-1)1} \end{bmatrix} \rightarrow \begin{aligned} \mathbf{c}^{-}_{(N-1)1} &= \left(\mathbf{S}^{[G(N-2)]}_{12}\right)^{-1}\left(\mathbf{c}_{\text{ref}} - \mathbf{S}^{[G(N-2)]}_{11}\mathbf{c}_{\text{inc}}\right) \\ \mathbf{c}^{+}_{(N-1)1} &= \mathbf{S}^{[G(N-2)]}_{21}\mathbf{c}_{\text{inc}} + \mathbf{S}^{[G(N-2)]}_{22}\mathbf{c}^{-}_{(N-1)1} \end{aligned}$$



The internal mode coefficients $\mathbf{c}^{+}_{N-1}$ and $\mathbf{c}^{-}_{N-1}$ of the Nth layer are

$$\begin{bmatrix} \mathbf{c}^{+}_{N-1} \\ \mathbf{c}^{-}_{N-1} \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{N-1} & \mathbf{W}_{N-1} \\ -\mathbf{V}_{N-1} & \mathbf{V}_{N-1} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{W}_{(N-1)1} & \mathbf{W}_{(N-1)1} \\ -\mathbf{V}_{(N-1)1} & \mathbf{V}_{(N-1)1} \end{bmatrix} \begin{bmatrix} \mathbf{c}^{+}_{(N-1)1} \\ \mathbf{c}^{-}_{(N-1)1} \end{bmatrix}$$

Note: for isotropic TMM
$\mathbf{W}_{N-1} = \mathbf{W}_{(N-1)1} = \mathrm{I}$
$\mathbf{V}_{(N-1)1} = \mathbf{V}_{g}$ for internal layers

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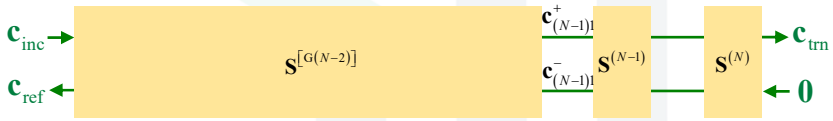
Algorithm: Backward Pass

The internal fields can now be calculated inside of the (N-1)th layer.

$$\boldsymbol{\psi}_{N-1}\left(z'_{N-1}\right) = \begin{bmatrix} \mathbf{s}_{x,N-1}\left(z'_{N-1}\right) \\ \mathbf{s}_{y,N-1}\left(z'_{N-1}\right) \\ \mathbf{u}_{x,N-1}\left(z'_{N-1}\right) \\ \mathbf{u}_{y,N-1}\left(z'_{N-1}\right) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{N-1} & \mathbf{W}_{N-1} \\ -\mathbf{V}_{N-1} & \mathbf{V}_{N-1} \end{bmatrix} \begin{bmatrix} e^{-\lambda_{N-1}z'_{N-1}} & \mathbf{0} \\ \mathbf{0} & e^{\lambda_{N-1}z'_{N-1}} \end{bmatrix} \begin{bmatrix} \mathbf{c}^{+}_{N-1} \\ \mathbf{c}^{-}_{N-1} \end{bmatrix}$$



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Summary of the Algorithm

Forward Pass:

Perform simulation as usual, building the global scattering matrix from left to right.

During this forward pass, record $\mathbf{W}_i$, $\mathbf{V}_i$, $\boldsymbol{\lambda}_i$ and $\mathbf{S}^{(Gi)}$ calculated for each layer where the internal field is to be calculated.

Backward Pass:

For each layer of interest, calculate the external mode coefficients.

$$\mathbf{c}_{i1}^{-} = \left(\mathbf{S}_{12}^{[G(i-1)]}\right)^{-1}\left(\mathbf{c}_{\text{ref}} - \mathbf{S}_{11}^{[G(i-1)]}\mathbf{c}_{\text{inc}}\right)$$

$$\mathbf{c}_{i1}^{+} = \mathbf{S}_{21}^{[G(i-1)]}\mathbf{c}_{\text{inc}} + \mathbf{S}_{22}^{[G(i-1)]}\mathbf{c}_{i1}^{-}$$

…and then the internal mode coefficients

$$\begin{bmatrix}\mathbf{c}_i^{+}\\ \mathbf{c}_i^{-}\end{bmatrix} = \begin{bmatrix}\mathbf{W}_i & \mathbf{W}_i\\ -\mathbf{V}_i & \mathbf{V}_i\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{W}_{i1} & \mathbf{W}_{i1}\\ -\mathbf{V}_{i1} & \mathbf{V}_{i1}\end{bmatrix}\begin{bmatrix}\mathbf{c}_{i1}^{+}\\ \mathbf{c}_{i1}^{-}\end{bmatrix}$$

…and finally the internal fields

$$\boldsymbol{\psi}_i(z_i) = \begin{bmatrix}\mathbf{s}_{x,i}(z_i)\\ \mathbf{s}_{y,i}(z_i)\\ \mathbf{u}_{x,i}(z_i)\\ \mathbf{u}_{y,i}(z_i)\end{bmatrix} = \begin{bmatrix}\mathbf{W}_i & \mathbf{W}_i\\ -\mathbf{V}_i & \mathbf{V}_i\end{bmatrix}\begin{bmatrix}e^{-k_0\lambda_i z_i} & \mathbf{0}\\ \mathbf{0} & e^{k_0\lambda_i z_i}\end{bmatrix}\begin{bmatrix}\mathbf{c}_i^{+}\\ \mathbf{c}_i^{-}\end{bmatrix}$$

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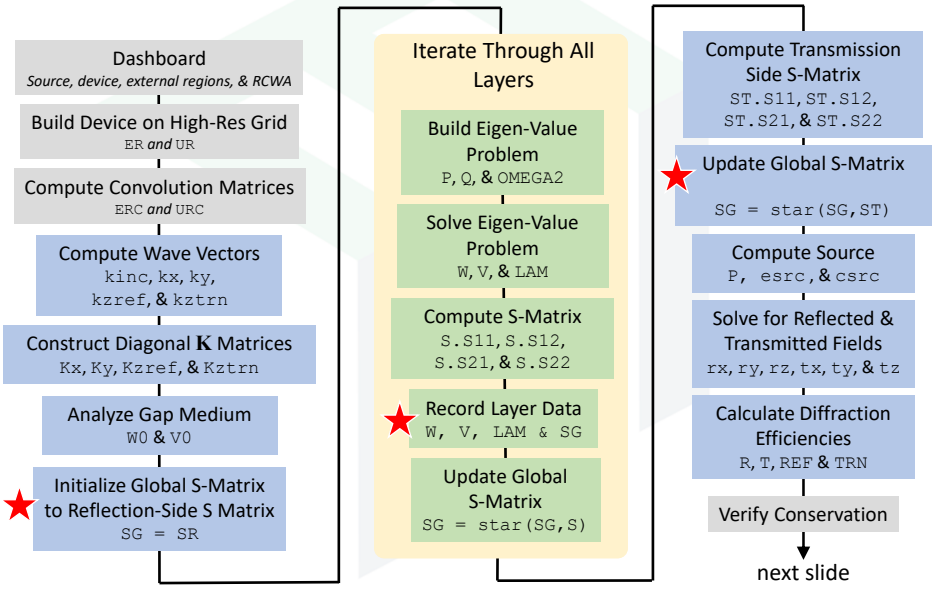
Modifications to RCWA Algorithm



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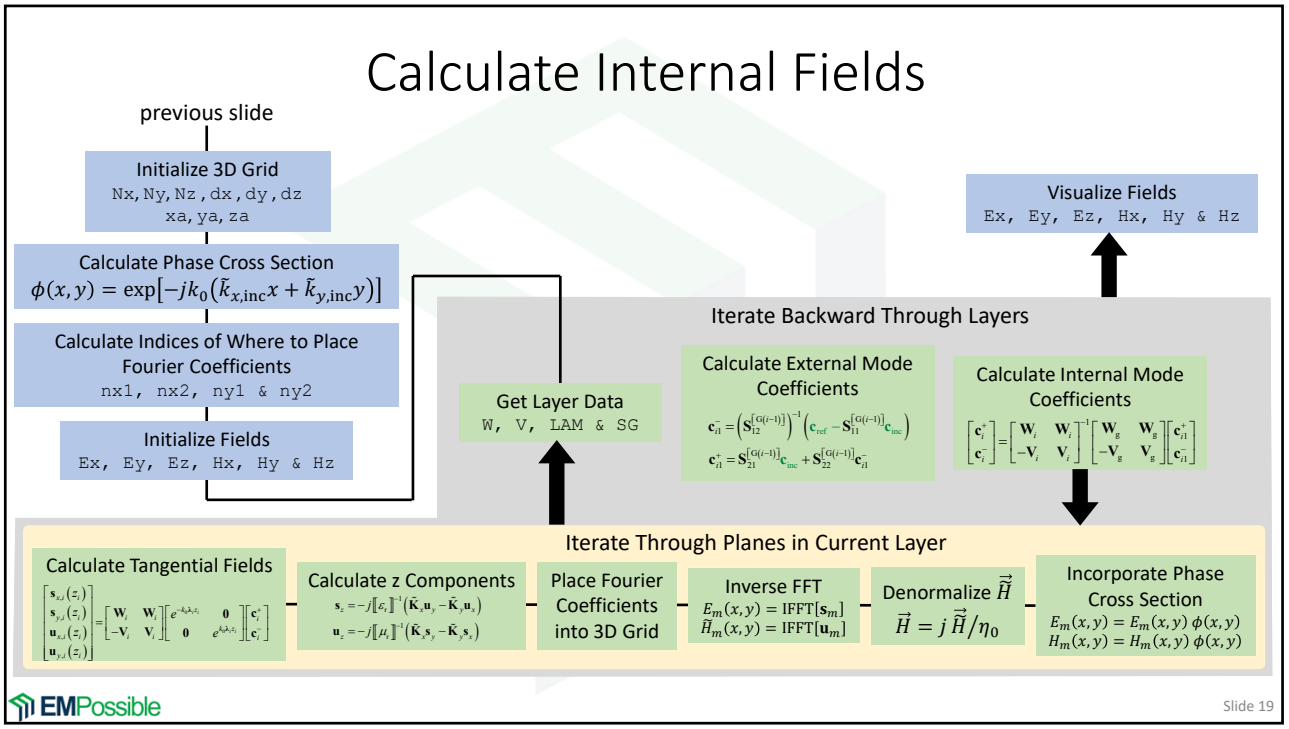

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One Spatial Harmonic

$$P = Q = 1$$

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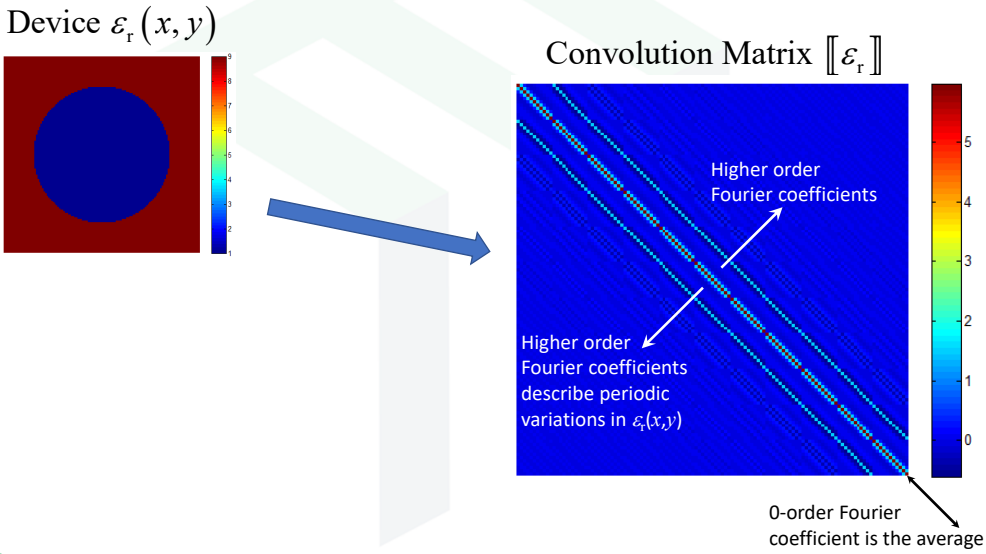
Anatomy of the Convolution Matrix



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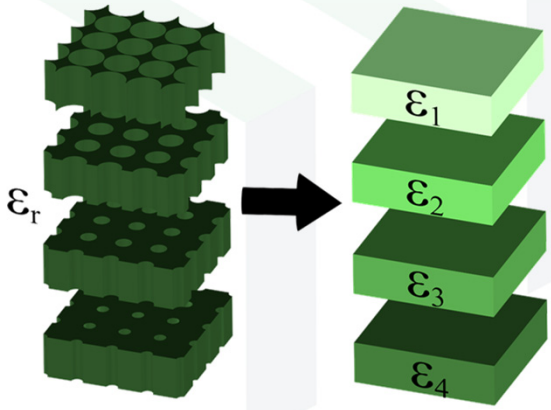
One Spatial Harmonic (P=Q=1)

When only one spatial harmonic is used, RCWA reduces to the 1D transfer matrix method, but uses the average value for the material properties in each layer. This is first-order "effective medium theory" and can make use of fast Fourier factorization so it is more sophisticated than the standard TMM.



$$\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4 < \varepsilon_r$$

Physical Device

Effective Medium Approximation in 1D

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Simulation of 1D Gratings with 3D-RCWA

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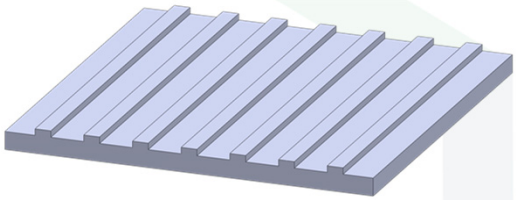
Grating Terminology

1D grating
Ruled grating



Requires a 2D simulation

2D grating
Crossed grating

Requires a 3D simulation

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3D-RCWA for 1D Gratings

Three-dimensional RCWA simulates all polarizations at the same time.

For 1D Gratings, Maxwell's equations decouple into the E mode and the H mode.

It is possible to reformulate RCWA specifically for 1D Gratings so that it will only simulate either the E mode or H mode, but not both. This approach will be several times faster due to smaller matrices.

It is my experience that 3D-RCWA is fast enough that few applications warrant formulating a 2D-RCWA code. Exceptions include when fast Fourier factorization is important or for running optimizations that require many thousands of simulations to be iterated.

There are some tricks that can be used when using RCWA to model 1D gratings to maximize the speed and efficiency.

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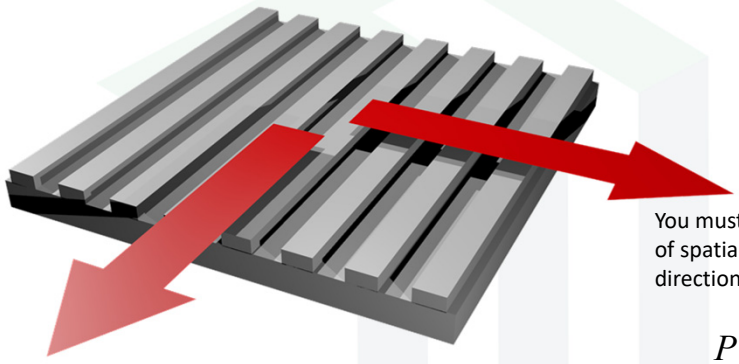
Number of Spatial Harmonics



You must use some number of spatial harmonics in this direction.

$$P \sim 7\frac{\Lambda}{\lambda}$$

$$N_x > 500$$

There is no contrast in this direction. Only one spatial harmonic is needed.

$$Q = 1$$

$$N_y = 1$$

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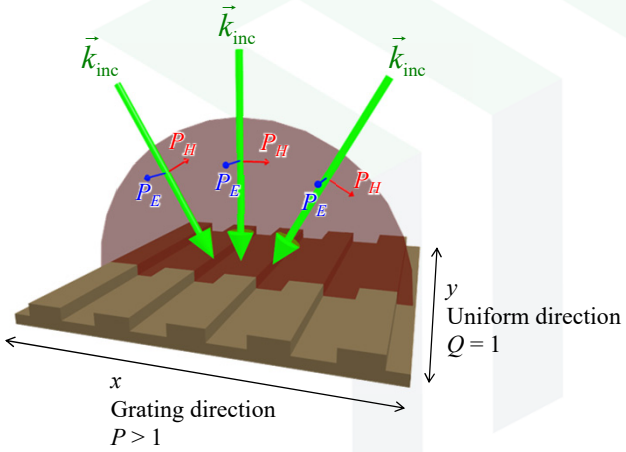
E and H Modes with 3D-RCWA

The field for both E and H modes are thought of completely in terms of the polarization vector for the electric field. If you have implemented your codes following these lectures, no changes to your code are needed outside of the dashboard.



Source Wave Vector

$$\vec{k}_{inc} = k_0 n_{inc} \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix}$$

Source Polarization Vector

$$\vec{P}_E = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ E Mode

$$\vec{P}_H = \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix}$$ H Mode

The concept of E and H modes only apply when: (1) LHI or diagonally anisotropic materials, (2) device is uniform in y direction, and (3) propagation is restricted to x-z plane.

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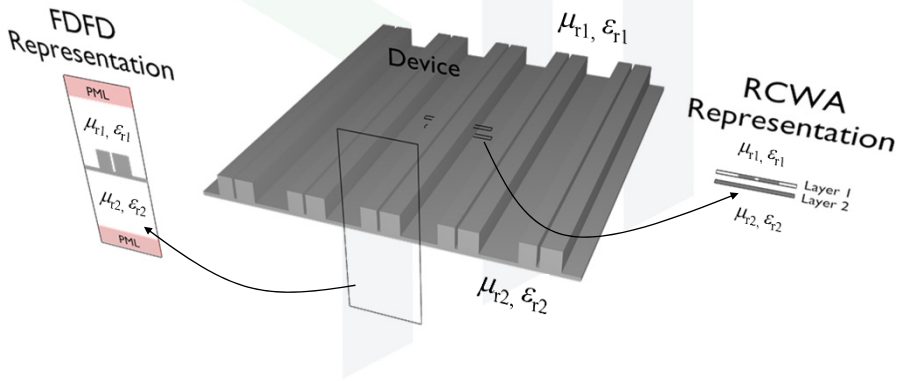
FDFD Vs. RCWA Representations

FDFD represents a device as the side view of a single unit cell.

In contrast, RCWA represents a device as a top view of each layer of a single unit cell.



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Formulation of 2D RCWA with FFF (1D Gratings)

Starting Point for Derivation

We will start with the semi-analytical matrix form of Maxwell's equations in Fourier-space. These are rigorous and valid even for 3D devices.

$$-j\tilde{\mathbf{K}}_y \mathbf{u}_z - \frac{d}{d\tilde{z}}\mathbf{u}_y = [\![\varepsilon_r]\!]\mathbf{s}_x$$

$$\frac{d}{d\tilde{z}}\mathbf{u}_x + j\tilde{\mathbf{K}}_x \mathbf{u}_z = [\![\varepsilon_r]\!]\mathbf{s}_y$$

$$\tilde{\mathbf{K}}_x \mathbf{u}_y - \tilde{\mathbf{K}}_y \mathbf{u}_x = j[\![\varepsilon_r]\!]\mathbf{s}_z$$

$$-j\tilde{\mathbf{K}}_y \mathbf{s}_z - \frac{d}{d\tilde{z}}\mathbf{s}_y = [\![\mu_r]\!]\mathbf{u}_x$$

$$\frac{d}{d\tilde{z}}\mathbf{s}_x + j\tilde{\mathbf{K}}_x \mathbf{s}_z = [\![\mu_r]\!]\mathbf{u}_y$$

$$\tilde{\mathbf{K}}_x \mathbf{s}_y - \tilde{\mathbf{K}}_y \mathbf{s}_x = j[\![\mu_r]\!]\mathbf{u}_z$$

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Reduction to Two Dimensions

When devices are uniform in the y-direction and no wave propagation occurs in this direction, we have

$$\tilde{\mathbf{K}}_y = \mathbf{0}$$

Our matrix equations reduce to

$$-\frac{d}{d\tilde{z}}\mathbf{u}_y = [\![\varepsilon_r]\!]\mathbf{s}_x \qquad -\frac{d}{d\tilde{z}}\mathbf{s}_y = [\![\mu_r]\!]\mathbf{u}_x$$

$$\frac{d}{d\tilde{z}}\mathbf{u}_x + j\tilde{\mathbf{K}}_x\mathbf{u}_z = [\![\varepsilon_r]\!]\mathbf{s}_y \qquad \frac{d}{d\tilde{z}}\mathbf{s}_x + j\tilde{\mathbf{K}}_x\mathbf{s}_z = [\![\mu_r]\!]\mathbf{u}_y$$

$$\tilde{\mathbf{K}}_x\mathbf{u}_y = j[\![\varepsilon_r]\!]\mathbf{s}_z \qquad \tilde{\mathbf{K}}_x\mathbf{s}_y = j[\![\mu_r]\!]\mathbf{u}_z$$

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Two Independent Modes

We see that Maxwell's equations have decoupled into two independent modes.

$$-\frac{d}{d\tilde{z}}\mathbf{u}_y = [\![\varepsilon_r]\!]\mathbf{s}_x \qquad -\frac{d}{d\tilde{z}}\mathbf{s}_y = [\![\mu_r]\!]\mathbf{u}_x$$

$$\frac{d}{d\tilde{z}}\mathbf{u}_x + j\tilde{\mathbf{K}}_x\mathbf{u}_z = [\![\varepsilon_r]\!]\mathbf{s}_y \qquad \frac{d}{d\tilde{z}}\mathbf{s}_x + j\tilde{\mathbf{K}}_x\mathbf{s}_z = [\![\mu_r]\!]\mathbf{u}_y$$

$$\tilde{\mathbf{K}}_x\mathbf{u}_y = j[\![\varepsilon_r]\!]\mathbf{s}_z \qquad \tilde{\mathbf{K}}_x\mathbf{s}_y = j[\![\mu_r]\!]\mathbf{u}_z$$

E Mode

$$\frac{d}{d\tilde{z}}\mathbf{u}_x + j\tilde{\mathbf{K}}_x\mathbf{u}_z = [\![\varepsilon_r]\!]\mathbf{s}_y$$

$$-\frac{d}{d\tilde{z}}\mathbf{s}_y = [\![\mu_r]\!]\mathbf{u}_x$$

$$\tilde{\mathbf{K}}_x\mathbf{s}_y = j[\![\mu_r]\!]\mathbf{u}_z$$

H Mode

$$\frac{d}{d\tilde{z}}\mathbf{s}_x + j\tilde{\mathbf{K}}_x\mathbf{s}_z = [\![\mu_r]\!]\mathbf{u}_y$$

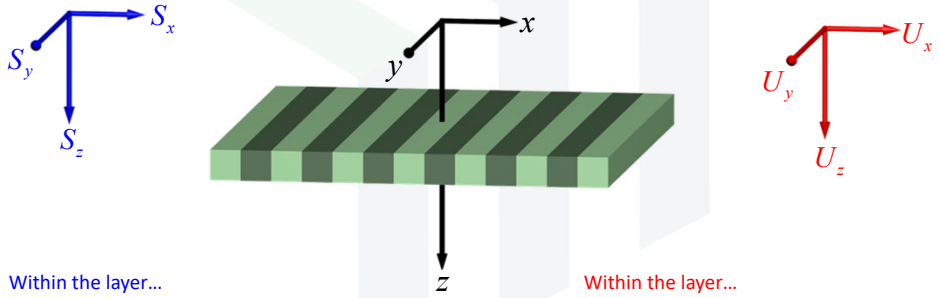
$$-\frac{d}{d\tilde{z}}\mathbf{u}_y = [\![\varepsilon_r]\!]\mathbf{s}_x$$

$$\tilde{\mathbf{K}}_x\mathbf{u}_y = j[\![\varepsilon_r]\!]\mathbf{s}_z$$

EMPossible

Orientation of the Field Components

For 1D gratings, the orientation of the field components relative to the interfaces is fixed. In this case, it is straightforward to incorporate fast Fourier factorization into the formulation.



Within the layer…

S_x is always perpendicular to interfaces.
S_y is always parallel to interfaces.
S_z is always parallel to interfaces.

Within the layer…

U_x is always perpendicular to interfaces.
U_y is always parallel to interfaces.
U_z is always parallel to interfaces.

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Incorporating Fast Fourier Factorization

For 1D gratings, it is very straightforward to incorporate fast Fourier factorization rules. This will improve convergence rates.

E Mode

$$\frac{d}{d\tilde{z}}\mathbf{u}_x + j\tilde{\mathbf{K}}_x\mathbf{u}_z = [\![\varepsilon_r]\!]\mathbf{s}_y$$

$$-\frac{d}{d\tilde{z}}\mathbf{s}_y = [\![1/\mu_r]\!]^{-1}\mathbf{u}_x$$

$$\tilde{\mathbf{K}}_x\mathbf{s}_y = j[\![\mu_r]\!]\mathbf{u}_z$$

$\mathbf{s}_y$ is always parallel to interfaces so $[\![\varepsilon_r]\!]$ is the standard convolution matrix.

$\mathbf{u}_x$ is always perpendicular so FFF rules are used to construct $[\![1/\mu_r]\!]^{-1}$.

$\mathbf{u}_z$ is always parallel to interfaces so $[\![\mu_r]\!]$ is the standard convolution matrix.

H Mode

$$\frac{d}{d\tilde{z}}\mathbf{s}_x + j\tilde{\mathbf{K}}_x\mathbf{s}_z = [\![\mu_r]\!]\mathbf{u}_y$$

$$-\frac{d}{d\tilde{z}}\mathbf{u}_y = [\![1/\varepsilon_r]\!]^{-1}\mathbf{s}_x$$

$$\tilde{\mathbf{K}}_x\mathbf{u}_y = j[\![\varepsilon_r]\!]\mathbf{s}_z$$

$\mathbf{u}_y$ is always parallel to interfaces so $[\![\mu_r]\!]$ is the standard convolution matrix.

$\mathbf{s}_x$ is always perpendicular so FFF rules are used to construct $[\![1/\varepsilon_r]\!]^{-1}$.

$\mathbf{s}_z$ is always parallel to interfaces so $[\![\varepsilon_r]\!]$ Is the standard convolution matrix.

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Note: FFF in *E*-mode with non-magnetic materials makes no difference.

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Eliminate Longitudinal Components

We solve the third equation for $\mathbf{u}_z$ and the sixth equation for $\mathbf{s}_z$ to obtain

$$\mathbf{u}_z = -j[\![\mu_r]\!]^{-1}\tilde{\mathbf{K}}_x\mathbf{s}_y \qquad \mathbf{s}_z = -j[\![\varepsilon_r]\!]^{-1}\tilde{\mathbf{K}}_x\mathbf{u}_y$$

We substitute these expressions into the remaining Maxwell's equations.

E Mode

$$\frac{d}{d\tilde{z}}\mathbf{u}_x = [\![\varepsilon_r]\!]\mathbf{s}_y - \tilde{\mathbf{K}}_x[\![\mu_r]\!]^{-1}\tilde{\mathbf{K}}_x\mathbf{s}_y$$

$$\frac{d}{d\tilde{z}}\mathbf{s}_y = -[\![1/\mu_r]\!]^{-1}\mathbf{u}_x$$

H Mode

$$\frac{d}{d\tilde{z}}\mathbf{s}_x = [\![\mu_r]\!]\mathbf{u}_y - \tilde{\mathbf{K}}_x[\![\varepsilon_r]\!]^{-1}\tilde{\mathbf{K}}_x\mathbf{u}_y$$

$$\frac{d}{d\tilde{z}}\mathbf{u}_y = -[\![1/\varepsilon_r]\!]^{-1}\mathbf{s}_x$$

Standard P and Q Form

E Mode

$$\frac{d}{d\tilde{z}}\mathbf{s}_y = \mathbf{P}\mathbf{u}_x$$

$$\frac{d}{d\tilde{z}}\mathbf{u}_x = \mathbf{Q}\mathbf{s}_y$$

$$\mathbf{P} = -[\![1/\mu_r]\!]^{-1}$$

$$\mathbf{Q} = [\![\varepsilon_r]\!] - \tilde{\mathbf{K}}_x[\![\mu_r]\!]^{-1}\tilde{\mathbf{K}}_x$$

H Mode

$$\frac{d}{d\tilde{z}}\mathbf{u}_y = \mathbf{P}\mathbf{s}_x$$

$$\frac{d}{d\tilde{z}}\mathbf{s}_x = \mathbf{Q}\mathbf{u}_y$$

$$\mathbf{P} = -[\![1/\varepsilon_r]\!]^{-1}$$

$$\mathbf{Q} = [\![\mu_r]\!] - \tilde{\mathbf{K}}_x[\![\varepsilon_r]\!]^{-1}\tilde{\mathbf{K}}_x$$

We see that FFF is incorporated solely into the matrix $\mathbf{P}$.

Matrix Wave Equations

Now that we have our equations in standard **P** and **Q** form, we derive the wave equations in the same manner as before.

E Mode

$$\frac{d^2}{d\tilde{z}^2}\mathbf{s}_y - \mathbf{\Omega}^2\mathbf{s}_y = \mathbf{0}$$

$$\mathbf{\Omega}^2 = \mathbf{PQ}$$

H Mode

$$\frac{d^2}{d\tilde{z}^2}\mathbf{u}_y - \mathbf{\Omega}^2\mathbf{u}_y = \mathbf{0}$$

$$\mathbf{\Omega}^2 = \mathbf{PQ}$$

…and now you know the rest of the story.

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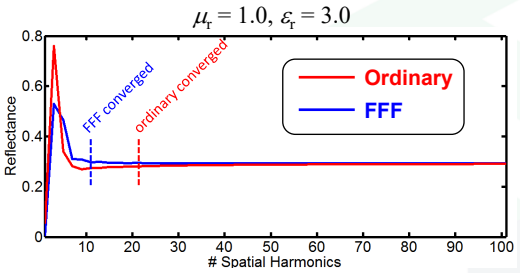
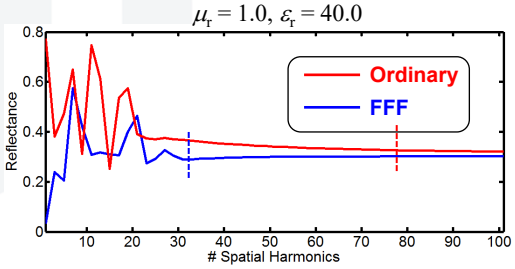
Convergence Study for 1D Gratings



H-Mode



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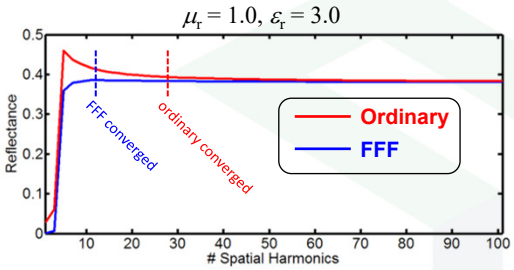
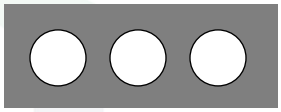
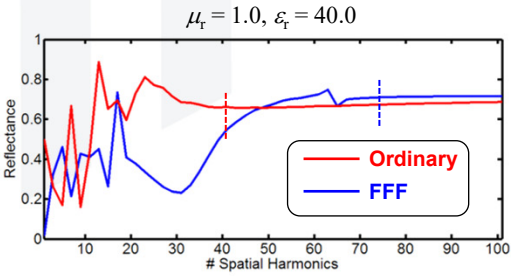
Convergence Study for 1D Curved Structures





H-Mode

?



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Danger of RCWA and Convergence

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Danger of RCWA

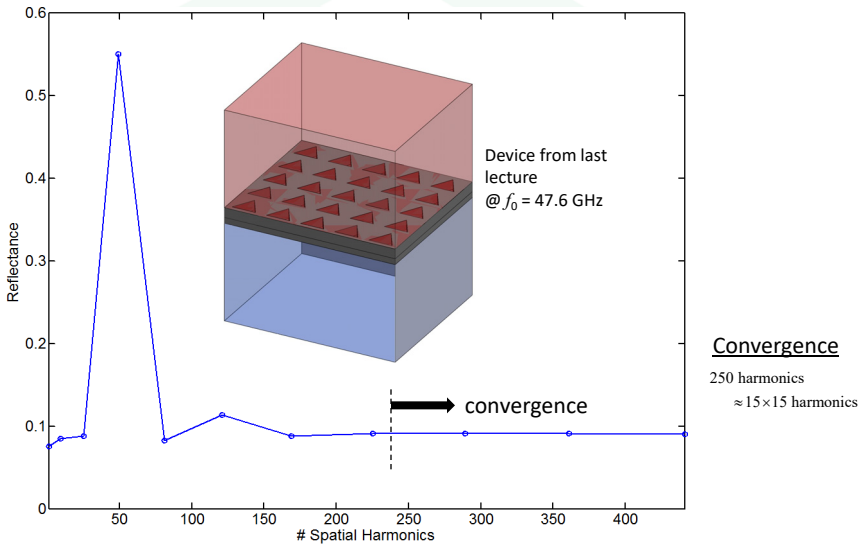
In real-space, poor grid resolution led to fluctuations in conservation of power and other very recognizable signs that things are wrong.

The danger of RCWA is that results can "look" correct even with very few spatial harmonics.

Conservation of power will always be obeyed in RCWA even using just one spatial harmonic.

It must become habit to look for convergence, as there are few other signs that more spatial harmonics are needed.

Typical Convergence Plot



RCWA and Curved Structures

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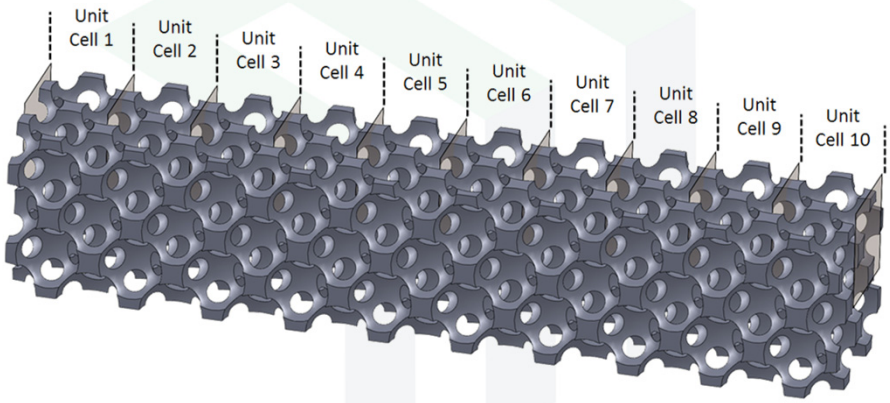
EBG Material



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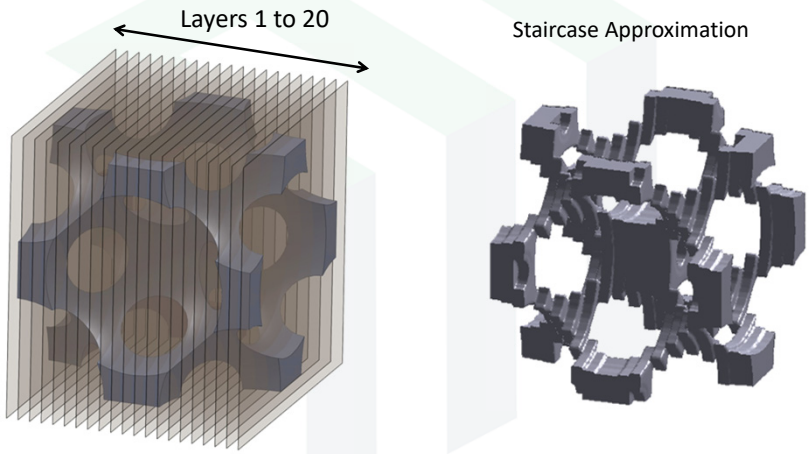
Divide into Thin Layers



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Strategically Truncating the Set of Spatial Harmonics

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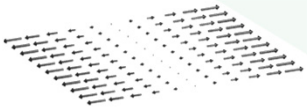
Notes on Truncating the Set of Spatial Harmonics

- The choice of which spatial harmonics to include in the expansion is arbitrary.
- Improper choice can lead to slow convergence and inaccurate results.
- We chose directions consistent with the physics of diffraction and a rectangular Fourier-space grid for simplicity.
- The number of harmonics retained in a particular direction determines the spatial resolution of structures with contrast in that direction.
- It seems optimal to keep the number of spatial harmonics uniform in all directions.

Fourier-Space Grid Notation

The components of the wave vector expansion look like:

$\tilde{k}_x(m)$

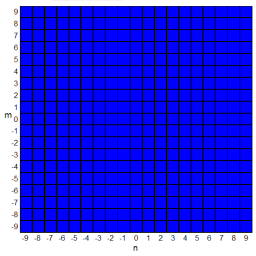


$\tilde{k}_y(n)$

$\tilde{k}_z(m,n) = \left(\sqrt{\mu_r^* \varepsilon_r^* - \tilde{k}_x^2(m) - \tilde{k}_y^2(m)}\right)^*$

We visualized it this way:

A simpler view of our 2D Fourier-space grid is



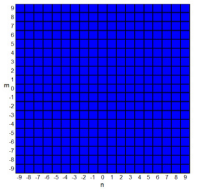
Simple Grid Truncation Scheme

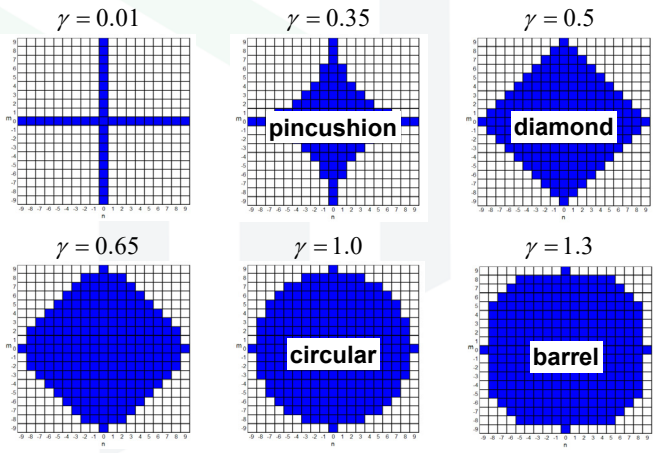
We will retain all spatial harmonics with indices that satisfy the following equation:

$$\left|\frac{m}{M}\right|^{2\gamma} + \left|\frac{n}{N}\right|^{2\gamma} \leq 1$$

Conventional

$|n| \leq N$ and $|m| \leq M$

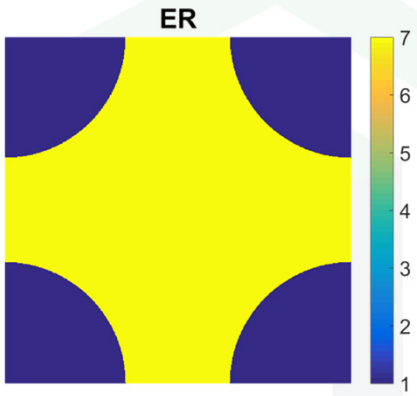




L. Li, "New formulation of the Fourier modal method for crossed surface-relief gratings," J. Opt. Soc. Am. A **14**(10), 2758-2767, 1997.

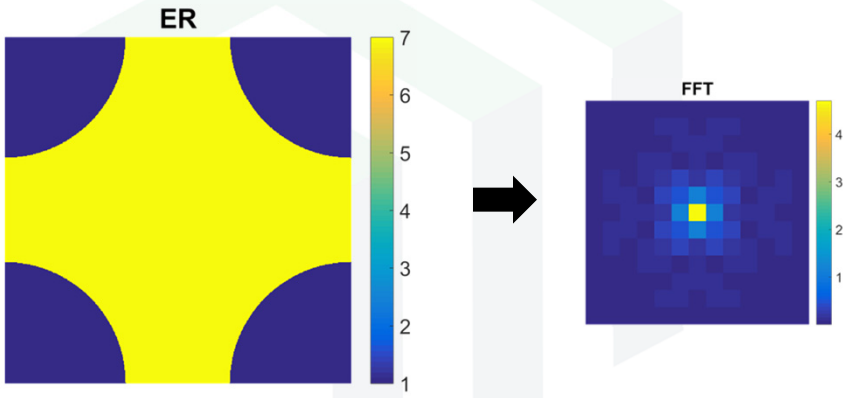
Implementation

Step 1 – Build Unit Cell on High-Resolution Grid



Implementation

Step 2 – Calculate Fourier Coefficients Using FFT Technique



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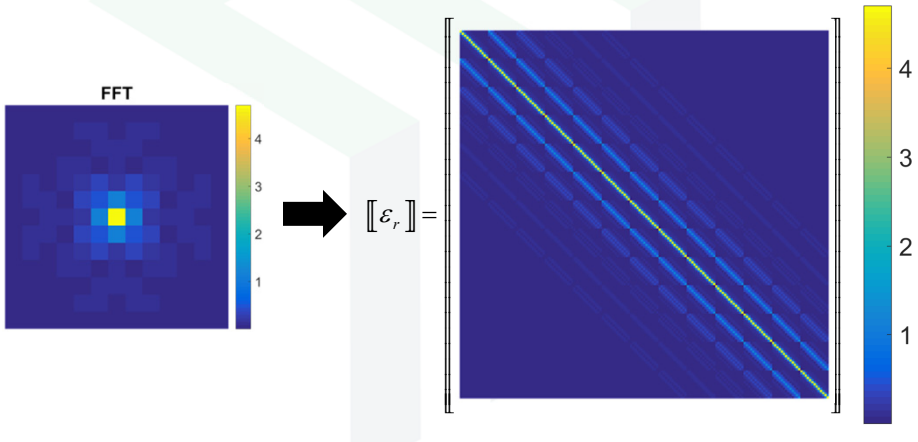
Implementation

Step 3 – Assemble Standard Convolution Matrix From Fourier Coefficients



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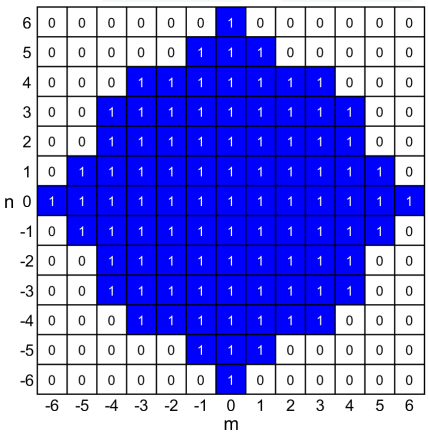
Implementation
Step 4 – Build Truncation Map




```
% Construct Truncation Map
TMAP = abs(m/M).^(2*p) …
     + abs(n/N).^(2*p);
TMAP = (TMAP <= 1);
```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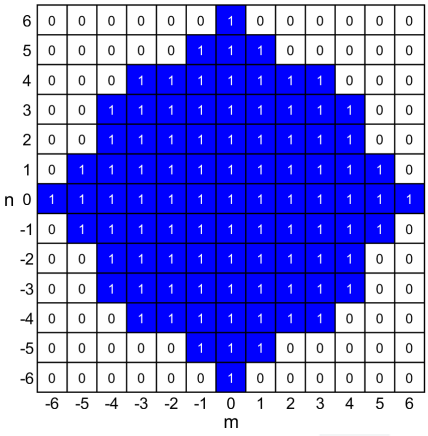
Implementation
Step 5 – Extract Indices of Spatial Harmonics to Retain




```
% Construct Truncation Map
TMAP = abs(m/M).^(2*p) …
     + abs(n/N).^(2*p);
TMAP = (TMAP <= 1);

% Extract Array Indices
ind = find(TMAP(:));
```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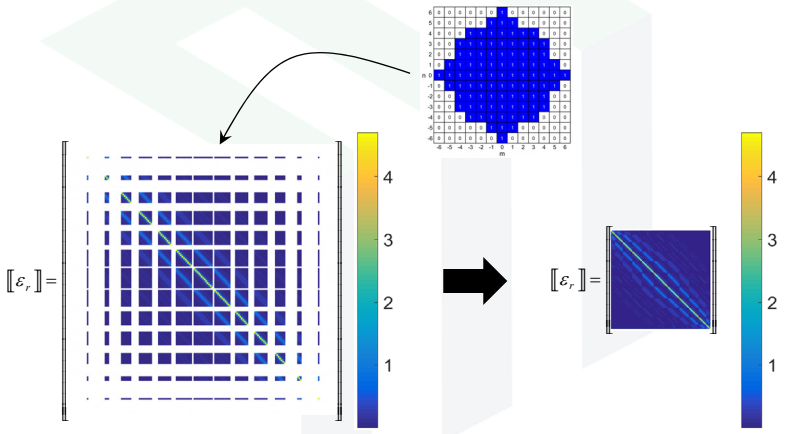
Implementation
Step 6 – Truncate Convolution Matrix




```
% Truncate Convolution Matrix
ERCT = ERC(ind,ind);
```


Implementation
Step 7 – Perform RCWA

The rest of the RCWA algorithm remains virtually unchanged.

You will need to consider your truncation again:

1. When you calculate the source.
2. If you calculate the fields from the eigen-modes.

RCWA for Generalized Symmetries

Revised Fourier Transforms

The materials…

$$\varepsilon_r(\vec{r}) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_{m,n} e^{j(m\vec{T}_1+n\vec{T}_2)\bullet\vec{r}} \qquad \mu_r(\vec{r}) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_{m,n} e^{j(m\vec{T}_1+n\vec{T}_2)\bullet\vec{r}}$$

$$a_{m,n} = \frac{1}{A}\iint_{\text{unit cell}} \varepsilon_r(\vec{r}) e^{-j(m\vec{T}_1+n\vec{T}_2)\bullet\vec{r}}\, dA \qquad b_{m,n} = \frac{1}{A}\iint_{\text{unit cell}} \mu_r(\vec{r}) e^{-j(m\vec{T}_1+n\vec{T}_2)\bullet\vec{r}}\, dA$$

The fields…

$$E_x(\vec{r},z) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} S_x(m,n;z)\cdot e^{-j\vec{k}(m,n)\bullet\vec{r}} \qquad \tilde{H}_x(\vec{r},z) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} U_x(m,n;z)\cdot e^{-j\vec{k}(m,n)\bullet\vec{r}}$$

$$E_y(\vec{r},z) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} S_y(m,n;z)\cdot e^{-j\vec{k}(m,n)\bullet\vec{r}} \qquad \tilde{H}_y(\vec{r},z) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} U_y(m,n;z)\cdot e^{-j\vec{k}(m,n)\bullet\vec{r}}$$

$$E_z(\vec{r},z) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} S_z(m,n;z)\cdot e^{-j\vec{k}(m,n)\bullet\vec{r}} \qquad \tilde{H}_z(\vec{r},z) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} U_z(m,n;z)\cdot e^{-j\vec{k}(m,n)\bullet\vec{r}}$$

Wave vector expansion…

$$\vec{k}_t(m,n) = \vec{k}_{\text{inc}} - m\vec{T}_1 - n\vec{T}_2$$

$$k_z(m,n) = \left(\sqrt{k_0^2\mu_r^*\varepsilon_r^* - \left|\vec{k}_t(m,n)\right|^2}\right)^*$$

$$m = -\infty,\cdots,-2,-1,0,1,2,\cdots,\infty$$

$$n = -\infty,\cdots,-2,-1,0,1,2,\cdots,\infty$$

$\vec{T}_1, \vec{T}_2 \equiv$ reciprocal lattice vectors of the unit cell

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Modeling Hexagonal Gratings with Rectangular RCWA

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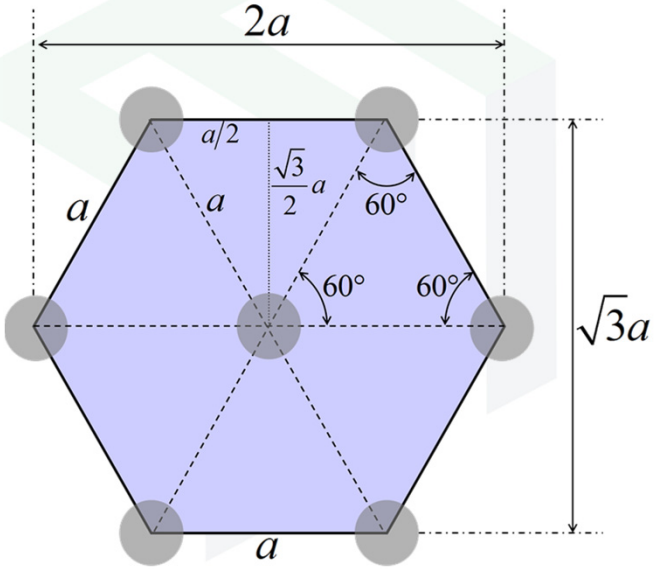
Geometry of a Hexagon



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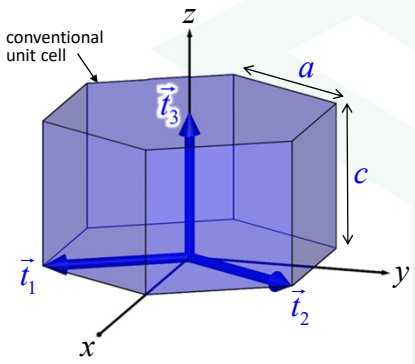
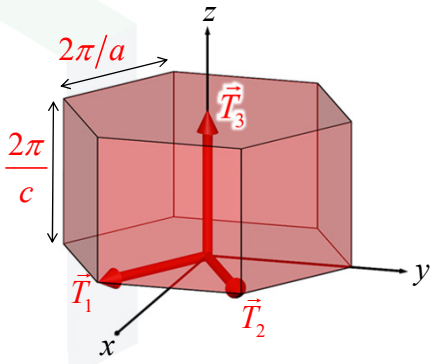
Grating Vectors of Hexagonal Structures

Direct Lattice



$$\vec{t}_1 = \frac{a}{2}\hat{x} - \frac{a\sqrt{3}}{2}\hat{y} \qquad \vec{t}_2 = \frac{a}{2}\hat{x} + \frac{a\sqrt{3}}{2}\hat{y} \qquad \vec{t}_3 = c\hat{z}$$

Reciprocal Lattice



$$\vec{T}_1 = \frac{2\pi}{a}\hat{x} - \frac{2\pi}{a\sqrt{3}}\hat{y} \qquad \vec{T}_2 = \frac{2\pi}{a}\hat{x} + \frac{2\pi}{a\sqrt{3}}\hat{y} \qquad \vec{T}_3 = (2\pi/c)\hat{z}$$

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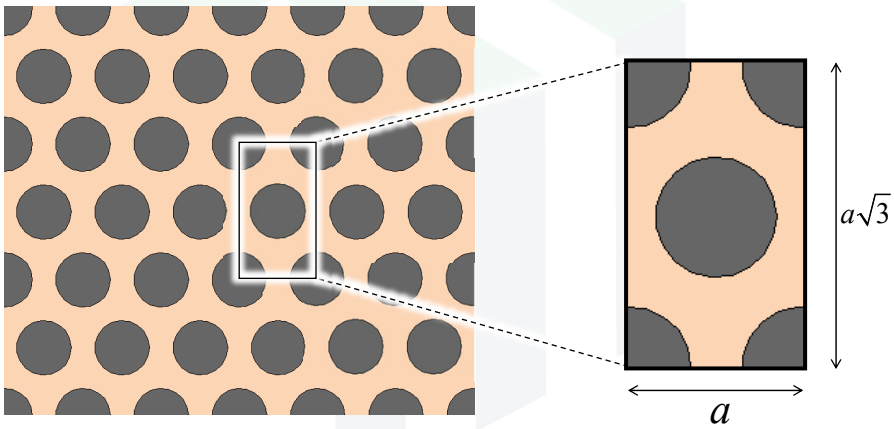
Rectangular Unit Cell in Hexagonal Array

We must identify a rectangular unit cell that reconstructs a hexagonal array.



This implies that we will need more spatial harmonics along the y direction than x.

```
Q = round(P*Sy/Sx);
```

EMPossible

Enhanced Transmittance Matrix Approach

M. G. Moharam, Drew A. Pommet, Eric B. Grann, "Stable implementation of the rigorous coupled-wave analysis for surface-relief gratings: enhanced transmittance matrix approach," J. Opt. Soc. Am. A, Vol. 12, No. 5, pp. 1077-1086, May 1995.

Motivation

The enhanced transmittance matrix (ETM) method involves less matrix manipulations so it is much faster than using scattering matrices. Maybe ~10×. It also provides easier computation of internal fields.

ETM works by first stepping backward through each layer (backward analysis) and then stepping forward (forward analysis). Intermediate parameters must be stored for each layer during the backward analysis that are recalled during the forward analysis. This leads to severe memory limitations when many layers are used.

Conclusion → Unless the device is composed of prohibitively large number of layers or if other features of scattering matrices are not needed, use ETM.

The Problem

The source of the instability is the following matrix.

$$\mathbf{X}_i^+ = e^{-\boldsymbol{\Omega}_i k_0 L_i}$$

$$\mathbf{X}_i^- = e^{\boldsymbol{\Omega}_i k_0 L_i}$$ ← Growing exponentials!!!

The enhanced transmittance matrix (ETM) method was the first technique applied to RCWA that fixed the instability.

ETM is much faster than scattering matrices, but is much less memory efficient because it requires parameters to be stored for all layers at the same time. Scattering matrices can proceed one layer at a time, forgetting everything about previous layers.

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Boundary Conditions

First interface:

$$\mathbf{s} + \mathbf{A}\mathbf{r} = \mathbf{F}_1 \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_1 \end{bmatrix} \mathbf{c}_1$$

unknown

Intermediate interfaces:

$$\mathbf{F}_i \begin{bmatrix} \mathbf{X}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{c}_i = \mathbf{F}_{i+1} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_{i+1} \end{bmatrix} \mathbf{c}_{i+1}$$

Last interface:

$$\mathbf{F}_N \begin{bmatrix} \mathbf{X}_N & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{c}_N = \mathbf{B}\mathbf{t}$$

unknown

$$\mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ -\frac{j}{\mu_\mathrm{I}} \frac{\tilde{\mathbf{K}}_x \tilde{\mathbf{K}}_y}{\tilde{\mathbf{K}}_{z,\mathrm{I}}} & -\frac{j}{\mu_\mathrm{I}} \frac{\tilde{\mathbf{K}}_y^2 + \tilde{\mathbf{K}}_{z,\mathrm{I}}^2}{\tilde{\mathbf{K}}_{z,\mathrm{I}}} \\ \frac{j}{\mu_\mathrm{I}} \frac{\tilde{\mathbf{K}}_x^2 + \tilde{\mathbf{K}}_{z,\mathrm{I}}^2}{\tilde{\mathbf{K}}_{z,\mathrm{I}}} & \frac{j}{\mu_\mathrm{I}} \frac{\tilde{\mathbf{K}}_x \tilde{\mathbf{K}}_y}{\tilde{\mathbf{K}}_{z,\mathrm{I}}} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \frac{j}{\mu_\mathrm{II}} \frac{\tilde{\mathbf{K}}_x \tilde{\mathbf{K}}_y}{\tilde{\mathbf{K}}_{z,\mathrm{II}}} & \frac{j}{\mu_\mathrm{II}} \frac{\tilde{\mathbf{K}}_y^2 + \tilde{\mathbf{K}}_{z,\mathrm{II}}^2}{\tilde{\mathbf{K}}_{z,\mathrm{II}}} \\ -\frac{j}{\mu_\mathrm{II}} \frac{\tilde{\mathbf{K}}_x^2 + \tilde{\mathbf{K}}_{z,\mathrm{II}}^2}{\tilde{\mathbf{K}}_{z,\mathrm{II}}} & -\frac{j}{\mu_\mathrm{II}} \frac{\tilde{\mathbf{K}}_x \tilde{\mathbf{K}}_y}{\tilde{\mathbf{K}}_{z,\mathrm{II}}} \end{bmatrix}$$

$$\mathbf{s} = \begin{bmatrix} p_x \boldsymbol{\delta}_{0,pq} \\ p_y \boldsymbol{\delta}_{0,pq} \\ \frac{j}{\mu_\mathrm{I}} \left(k_{z,\mathrm{inc}} p_y - k_{y,\mathrm{inc}} p_z \right) \boldsymbol{\delta}_{0,pq} \\ \frac{j}{\mu_\mathrm{I}} \left(k_{x,\mathrm{inc}} p_z - k_{z,\mathrm{inc}} p_x \right) \boldsymbol{\delta}_{0,pq} \end{bmatrix}$$

$$\mathbf{F}_i = \begin{bmatrix} \mathbf{W}_i & \mathbf{W}_i \\ -\mathbf{V}_i & \mathbf{V}_i \end{bmatrix}$$

$$\mathbf{X}_i = e^{-\boldsymbol{\Omega}_i k_0 L_i}$$

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Work Backward Through Layers (1 of 4)

The goal is to solve for $\mathbf{r}$ and $\mathbf{t}$ without using $\mathbf{X}^{-1}$.

We start by solving the equation at the last interface for $\mathbf{c}_N$.

$$\mathbf{F}_N \begin{bmatrix} \mathbf{X}_N & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{c}_N = \mathbf{Bt} \quad \rightarrow \quad \mathbf{c}_N = \begin{bmatrix} \mathbf{X}_N^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{F}_N^{-1} \mathbf{Bt}$$

We write this as

$$\mathbf{c}_N = \begin{bmatrix} \mathbf{X}_N^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{a}_N \\ \mathbf{b}_N \end{bmatrix} \mathbf{t}_N \qquad \begin{bmatrix} \mathbf{a}_N \\ \mathbf{b}_N \end{bmatrix} = \mathbf{F}_N^{-1} \mathbf{B}$$

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Work Backward Through Layers (2 of 4)

To eliminate the potentially ill conditioned matrix $\mathbf{X}^{-1}$, we introduce an intermediate transmittance matrix parameter $\mathbf{t}_N$ defined as

$$\mathbf{t} = \mathbf{a}_N^{-1} \mathbf{X}_N \mathbf{t}_N \qquad \longleftarrow \quad \mathbf{t} \text{ and } \mathbf{t}_N \text{ remain unknown.}$$

Our equation for $\mathbf{c}_N$ becomes

$$\mathbf{c}_N = \begin{bmatrix} \mathbf{X}_N^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{a}_N \\ \mathbf{b}_N \end{bmatrix} \mathbf{t}_N \quad \rightarrow \quad \mathbf{c}_N = \begin{bmatrix} \mathbf{I} \\ \mathbf{b}_N \mathbf{a}_N^{-1} \mathbf{X}_N \end{bmatrix} \mathbf{t}_N$$

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Work Backward Through Layers (3 of 4)

The boundary condition equation at the second-to-last interface is

$$\mathbf{F}_{N-1}\begin{bmatrix}\mathbf{X}_{N-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\mathbf{c}_{N-1}=\mathbf{F}_N\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_N\end{bmatrix}\mathbf{c}_N$$

Substituting our expression for $\mathbf{c}_N$ into the equation yields

$$\mathbf{F}_{N-1}\begin{bmatrix}\mathbf{X}_{N-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\mathbf{c}_{N-1}=\mathbf{F}_N\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_N\end{bmatrix}\begin{bmatrix}\mathbf{I}\\ \mathbf{b}_N\mathbf{a}_N^{-1}\mathbf{X}_N\end{bmatrix}\mathbf{t}_N$$

Solving this for $\mathbf{c}_{N-1}$ leads to

$$\mathbf{c}_{N-1}=\begin{bmatrix}\mathbf{X}_{N-1}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\mathbf{F}_{N-1}^{-1}\mathbf{F}_N\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_N\end{bmatrix}\begin{bmatrix}\mathbf{I}\\ \mathbf{b}_N\mathbf{a}_N^{-1}\mathbf{X}_N\end{bmatrix}\mathbf{t}_N$$

We have now worked backward by one layer while avoiding $\mathbf{X}_N^{-1}$.

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Work Backward Through Layers (4 of 4)

This process continues through all the layers.

$$\mathbf{c}_{N-1}=\begin{bmatrix}\mathbf{X}_{N-1}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\mathbf{F}_{N-1}^{-1}\mathbf{F}_N\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_N\end{bmatrix}\begin{bmatrix}\mathbf{I}\\ \mathbf{b}_N\mathbf{a}_N^{-1}\mathbf{X}_N\end{bmatrix}\mathbf{t}_N$$

$$\mathbf{c}_{N-1}=\begin{bmatrix}\mathbf{X}_{N-1}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{a}_{N-1}\\ \mathbf{b}_{N-1}\end{bmatrix}\mathbf{t}_N \qquad \begin{bmatrix}\mathbf{a}_{N-1}\\ \mathbf{b}_{N-1}\end{bmatrix}=\mathbf{F}_{N-1}^{-1}\mathbf{F}_N\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_N\end{bmatrix}\begin{bmatrix}\mathbf{I}\\ \mathbf{b}_N\mathbf{a}_N^{-1}\mathbf{X}_N\end{bmatrix}$$

$$\mathbf{c}_{N-1}=\begin{bmatrix}\mathbf{I}\\ \mathbf{b}_{N-1}\mathbf{a}_{N-1}^{-1}\mathbf{X}_{N-1}\end{bmatrix}\mathbf{t}_{N-1}$$

$$\mathbf{t}_N=\mathbf{a}_{N-1}^{-1}\mathbf{X}_{N-1}\mathbf{t}_{N-1}$$ ← $\mathbf{T}_{N-1}$ and $\mathbf{T}_N$ remain unknown.

$$\vdots$$

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Solve for Reflected and Transmitted Fields

After working through all interfaces, we are left with

$$\mathbf{s}+\mathbf{A}\mathbf{r}=\mathbf{F}_1\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_1\end{bmatrix}\begin{bmatrix}\mathbf{I}\\ \mathbf{b}_1\mathbf{a}_1^{-1}\mathbf{X}_1\end{bmatrix}\mathbf{t}_1$$

We solve this matrix equation for $\mathbf{r}$ and $\mathbf{t}_1$.

$$\begin{bmatrix}\mathbf{r}\\ \mathbf{t}_1\end{bmatrix}=\begin{bmatrix}-\mathbf{A} & \mathbf{B}'\end{bmatrix}^{-1}\mathbf{S} \qquad \mathbf{B}'=\mathbf{F}_1\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_1\end{bmatrix}\begin{bmatrix}\mathbf{I}\\ \mathbf{b}_1\mathbf{a}_1^{-1}\mathbf{X}_1\end{bmatrix}\mathbf{t}_1$$

Now that we know $\mathbf{t}_1$, we can work forward through the layers to solve for $\mathbf{t}$.

$$\mathbf{t}_2=\mathbf{a}_1^{-1}\mathbf{X}_1\mathbf{t}_1$$
$$\mathbf{t}_3=\mathbf{a}_2^{-1}\mathbf{X}_2\mathbf{t}_2$$
$$\vdots$$
$$\mathbf{t}_N=\mathbf{a}_{N-1}^{-1}\mathbf{X}_{N-1}\mathbf{t}_{N-1}$$
$$\mathbf{t}=\mathbf{a}_N^{-1}\mathbf{X}_N\mathbf{t}_N$$

Note: we must store $\mathbf{a}$ and $\mathbf{X}$ for each layer. This leads to poor memory efficiency.

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Calculating the Diffraction Efficiencies

First, we calculate the longitudinal field components.

$$\mathbf{r}_z=-\tilde{\mathbf{K}}_{z,\text{ref}}^{-1}\left(\tilde{\mathbf{K}}_x\mathbf{r}_x+\tilde{\mathbf{K}}_y\mathbf{r}_y\right) \qquad \mathbf{t}_z=-\tilde{\mathbf{K}}_{z,\text{ref}}^{-1}\left(\tilde{\mathbf{K}}_x\mathbf{t}_x+\tilde{\mathbf{K}}_y\mathbf{t}_y\right)$$

Second, we calculate the diffraction efficiencies.

$$\mathbf{R}=\frac{\text{Re}\left[-\tilde{\mathbf{K}}_{z,\text{ref}}/\mu_{\text{r,inc}}\right]}{\text{Re}\left[k_{z,\text{inc}}/\mu_{\text{r,inc}}\right]}\left|\vec{\mathbf{r}}\right|^2 \qquad \left|\vec{\mathbf{r}}\right|^2=\left|\mathbf{r}_x\right|^2+\left|\mathbf{r}_y\right|^2+\left|\mathbf{r}_z\right|^2$$

$$\mathbf{T}=\frac{\text{Re}\left[\tilde{\mathbf{K}}_{z,\text{trn}}/\mu_{\text{r,trn}}\right]}{\text{Re}\left[k_{z,\text{inc}}/\mu_{\text{r,inc}}\right]}\left|\vec{\mathbf{t}}\right|^2 \qquad \left|\vec{\mathbf{t}}\right|^2=\left|\mathbf{t}_x\right|^2+\left|\mathbf{t}_y\right|^2+\left|\mathbf{t}_z\right|^2$$

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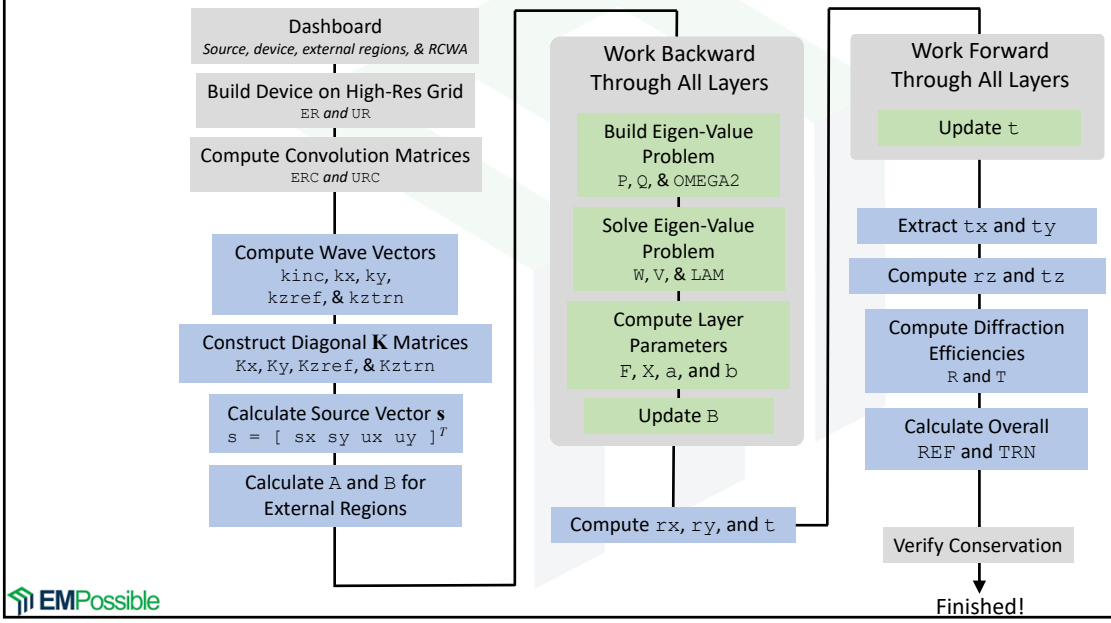
Block Diagram of ETM



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