Majorization Theory for Unitary Control of Optical Absorption and Emission

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(Received 3 November 2022; accepted 22 March 2023; published 7 April 2023)

Unitary control changes the absorption and emission of an object by transforming the external light modes. It is widely used and underlies coherent perfect absorption. Yet two basic questions remain unanswered: For a given object under unitary control, what absorptivity α, emissivity e, and their contrast $\delta = e - \alpha$ are attainable? How to obtain a given α, e, or δ? We answer both questions using the mathematics of majorization. We show that unitary control can achieve perfect violation or preservation of Kirchhoff's law in nonreciprocal objects, and uniform absorption or emission for any object.

DOI: 10.1103/PhysRevLett.130.146202

Thermal radiation represents a ubiquitous aspect of nature [1–7]. The control of thermal radiation is therefore of fundamental interest and has wide applications in technologies such as renewable energy [8–17], imaging [18], and sensing [19,20].

Conventional thermal radiators are often approximated as blackbodies since their emission typically covers a broad frequency and angular range [1,4]. In recent years, however, with the advancement of nanophotonics concepts, it has become possible to design thermal photonic structures with absorption and emission properties that are drastically different from that of a blackbody [21–28]. For example, one can now achieve thermal radiators with emissivity and absorptivity spectra that are narrow in either spectral or angular domain [21]. There are also significant recent developments in nonreciprocal thermal radiators that strongly violate Kirchhoff's law [29–34]. In these nonreciprocal devices, the key figure of merit is the contrast between absorptivity and emissivity [15,30,35–37].

In addition to the design of a photonic structure itself, the absorption and emission properties can also be designed by controlling the external modes that the photonic structures interact with. In this Letter, we refer to such control as unitary control since mathematically the design process can be described by a unitary transformation in the space of the external modes. As practical examples that illustrate the concept of unitary control, Fresnel lenses can be used as concentrators to boost the absorption of a solar cell [36]. Curved reflecting mirrors can be used to redirect thermal emission, as exploited in fan heaters and more recently in radiative cooling experiments [2,38–42]. In these examples, the Fresnel lens or the curved mirror shapes the external modes to influence the absorption and emission properties of a device. The concept of unitary control is also directly connected to the effects of coherent perfect absorption [43–52], where the complete absorption of light can be achieved due to the interference of multiple incident waves.

Despite its fundamental and practical importance, there still lacks a systematic theory of unitary control. The theory should answer two basic questions: (i) Given an object, what are all the attainable absorptivity, emissivity, and nonreciprocal contrast for each mode under unitary control? (ii) How to obtain given absorptivity, emissivity, or nonreciprocal contrast for each mode via unitary control? The first question asks about the capability and limitation of unitary control. The second question asks for implementation. Here we provide complete answers to both questions using the mathematics of majorization [53].

We first briefly discuss the notations used in the mathematics of majorization [53]. For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we define $\boldsymbol{x}^\downarrow = (x_1^\downarrow, x_2^\downarrow, \ldots, x_n^\downarrow)$ and $\boldsymbol{x}^\uparrow = (x_1^\uparrow, x_2^\uparrow, \ldots, x_n^\uparrow)$, where $x_1^\downarrow \geq x_2^\downarrow \geq \cdots \geq x_n^\downarrow$ reorder the components of x in nonincreasing order, and $x_1^\uparrow \leq x_2^\uparrow \leq \cdots \leq x_n^\uparrow$ in nondecreasing order. For $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, if

$$\sum_{i=1}^{k} x_i^\downarrow \leq \sum_{i=1}^{k} y_i^\downarrow, \qquad k = 1, 2, \ldots, n-1; \tag{1}$$

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i, \tag{2}$$

we say that $\boldsymbol{x}$ is *majorized* by $\boldsymbol{y}$, written as $\boldsymbol{x} \prec \boldsymbol{y}$. See Supplemental Material [54], Sec. I for more details.

For subsequent use, we also summarize some notations related to matrices. We denote by M_n the set of $n \times n$ complex matrices and $U(n)$ the set of $n \times n$ unitary matrices. For $M \in M_n$, we denote by $\boldsymbol{d}(M) = [d_1(M), \ldots, d_n(M)]^T$, $\boldsymbol{\lambda}(M) = [\lambda_1(M), \ldots, \lambda_n(M)]^T$, and $\boldsymbol{\sigma}(M) = [\sigma_1(M), \ldots, \sigma_n(M)]^T$ the vectors of diagonal entries, eigenvalues, and singular values of M. (A singular value of M is

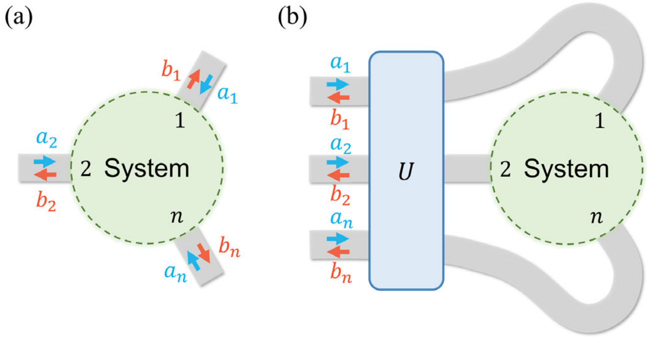

FIG. 1. (a) A general n-port passive system. (b) Illustration of unitary control.

the nonnegative square root of an eigenvalue of $M^\dagger M$ [60].) We also denote

$$\mathbf{1} - \boldsymbol{\sigma}^2(M) \equiv [1 - \sigma_1^2(M), \dots, 1 - \sigma_n^2(M)]^T. \tag{3}$$

We introduce the self-concommutator of M: $M^\dagger M - M^* M^T$, and denote its eigenvalues as

$$\boldsymbol{c}(M) \equiv \boldsymbol{\lambda}(M^\dagger M - M^* M^T). \tag{4}$$

See Supplemental Material [54], Sec. II for an introduction to the self-concommutator.

We consider a general passive linear time-invariant electromagnetic system [Fig. 1(a)]. It is connected to its exterior by n ports. Each port supports an input mode $|\phi_j^{(i)}\rangle$ and an output mode $|\phi_j^{(o)}\rangle$. We adopt the usual convention that the output mode is the time reversal of the input mode [61], i.e.,

$$|\phi_j^{(o)}\rangle = (|\phi_j^{(i)}\rangle)^*, \qquad j = 1, \dots, n. \tag{5}$$

Using these modes as bases, we describe the incoming and outgoing waves as

$$\boldsymbol{a} = (a_1, \dots, a_n)^T, \qquad \boldsymbol{b} = (b_1, \dots, b_n)^T, \tag{6}$$

where a_i and b_i are the input and output wave amplitudes in the ith port, respectively. The system is described by a scattering matrix $S \in M_n$ such that

$$\boldsymbol{b} = S\boldsymbol{a}, \tag{7}$$

where S_{ij} is the transport coefficient from the jth to the ith port. In general, S can be any complex matrix that satisfies the passivity condition: $I - S^\dagger S$ is positive semidefinite. If the system is reciprocal, S needs to be symmetric: $S = S^T$ [61–63].

Now we introduce unitary control. Unitary control refers to unitarily transforming the input and output modes [Fig. 1(b)]:

$$|\phi_j^{(i)}\rangle \to \sum_{k=1}^{n} U_{kj} |\phi_k^{(i)}\rangle, \qquad |\phi_j^{(o)}\rangle \to \sum_{k=1}^{n} U_{kj}^* |\phi_k^{(o)}\rangle, \tag{8}$$

where $U \in U(n)$. Under the new mode bases, the scattering matrix is modified by

$$S \to U^T S U. \tag{9}$$

Hence, unitary control corresponds to a unitary congruence [60] of an S matrix.

Unitary control also transforms the absorption and emission properties of a system. The absorption and emission properties are described by three real vectors:

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^T, \qquad \boldsymbol{e} = (e_1, \dots, e_n)^T, \tag{10}$$

$$\boldsymbol{\delta} \equiv \boldsymbol{e} - \boldsymbol{\alpha} = (\delta_1, \dots, \delta_n)^T, \tag{11}$$

where α_i and e_i are the absorptivity and emissivity in the ith port, respectively, and $\delta_i = e_i - \alpha_i$ is the nonreciprocal contrast that measures the violation of Kirchhoff's law. $\boldsymbol{\alpha}$, $\boldsymbol{e}$, and $\boldsymbol{\delta}$ can be determined from S. We define

$$A \equiv I - S^\dagger S, \qquad E \equiv I - S S^\dagger, \tag{12}$$

$$\Delta \equiv E^T - A = S^\dagger S - S^* S^T, \tag{13}$$

which are referred to as the absorptivity, emissivity, and nonreciprocal contrast matrices, respectively. All three matrices are Hermitian. Using the laws of thermodynamics, one can prove [31]

$$\boldsymbol{\alpha} = \boldsymbol{d}(A), \qquad \boldsymbol{e} = \boldsymbol{d}(E), \qquad \boldsymbol{\delta} = \boldsymbol{d}(\Delta). \tag{14}$$

Under the unitary control as defined in Eq. (9), the absorptivity, emissivity, and nonreciprocal contrast matrices are modified via unitary similarity [60]:

$$A \to U^\dagger A U, \qquad E \to U^T E U^*, \qquad \Delta \to U^\dagger \Delta U. \tag{15}$$

The absorptivity, emissivity, and nonreciprocal contrast vectors become

$$\boldsymbol{\alpha}[U] = \boldsymbol{d}(U^\dagger A U), \qquad \boldsymbol{e}[U] = \boldsymbol{d}(U^T E U^*), \tag{16}$$

$$\boldsymbol{\delta}[U] = \boldsymbol{d}(U^\dagger \Delta U), \tag{17}$$

which are all explicitly U dependent.

The concept of unitary control is closely related to the effect of coherent perfect absorption [43–52]. A coherent perfect absorber is a multiport system that can completely absorb incident light with an appropriate modal profile. In the Supplemental Material [54], Sec. III, we analyze a concrete example of a two-port coherent perfect absorber to illustrate the concept of unitary control. Our theory generalizes the study of coherent perfect absorbers by highlighting many other effects that are achievable through unitary control, including nonreciprocal effects. Also, the mathematics of majorization has not been applied in the previous study of coherent perfect absorbers.

Now we reformulate our key questions mathematically. Question 1: Given S, what are the sets

$$\{\boldsymbol{\alpha}\} \equiv \{\boldsymbol{\alpha}[U] | U \in U(n)\}, \tag{18}$$

$$\{\boldsymbol{e}\} \equiv \{\boldsymbol{e}[U] | U \in U(n)\}, \tag{19}$$

$$\{\boldsymbol{\delta}\} \equiv \{\boldsymbol{\delta}[U] | U \in U(n)\}? \tag{20}$$

Question 2: Given S, $\boldsymbol{\alpha}_0 \in \{\boldsymbol{\alpha}\}$, $\boldsymbol{e}_0 \in \{\boldsymbol{e}\}$, and $\boldsymbol{\delta}_0 \in \{\boldsymbol{\delta}\}$, what are the sets

$$\{U[\boldsymbol{\alpha}_0]\} \equiv \{U \in U(n) | \boldsymbol{\alpha}[U] = \boldsymbol{\alpha}_0\}, \tag{21}$$

$$\{U[\boldsymbol{e}_0]\} \equiv \{U \in U(n) | \boldsymbol{e}[U] = \boldsymbol{e}_0\}, \tag{22}$$

$$\{U[\boldsymbol{\delta}_0]\} \equiv \{U \in U(n) | \boldsymbol{\delta}[U] = \boldsymbol{\delta}_0\}? \tag{23}$$

We comment on the mathematical aspects of Eqs. (18)–(23). $\boldsymbol{\alpha}[U]$, $\boldsymbol{e}[U]$, and $\boldsymbol{\delta}[U]$ as defined in Eqs. (16) and (17) are different smooth maps from $U(n)$ to $\mathbb{R}^n$. Question 1 asks for the range of each map. Since $U(n)$ is compact, each range is a compact subset of $\mathbb{R}^n$. Question 2 asks for the preimage of each element of the range under each map. These maps are not one to one; for example, $\boldsymbol{\alpha}[U] = \boldsymbol{\alpha}[U']$, $\boldsymbol{e}[U] = \boldsymbol{e}[U']$, and $\boldsymbol{\delta}[U] = \boldsymbol{e}[U']$ with $U' = UD$, where D is any diagonal unitary matrix.

We start with Question 1. To hint at the solution, we perform two numerical experiments. In the first experiment, we consider a random 2×2 scattering matrix:

$$S = \begin{pmatrix} 0.2 - 0.2i & 0.3 + 0.7i \\ 0.1 + 0.3i & 0.5 - 0.1i \end{pmatrix}; \tag{24}$$

$$\boldsymbol{\sigma}(S) = \begin{pmatrix} 0.92 \\ 0.41 \end{pmatrix}, \qquad \boldsymbol{c}(S) = \begin{pmatrix} 0.60 \\ -0.60 \end{pmatrix}. \tag{25}$$

We generate 1000 random $U_i \in U(2)$ and calculate $\boldsymbol{\alpha}[U_i]$, $\boldsymbol{e}[U_i]$, and $\boldsymbol{\delta}[U_i]$ by Eq. (16). Figures 2(a)–2(c) show the results. We see that $\{\boldsymbol{\alpha}\} = \{\boldsymbol{e}\}$ is a line segment with end points obtained by permuting the coordinates of $\mathbf{1} - \boldsymbol{\sigma}^2(S) = (0.15, 0.83)^T$, while $\{\boldsymbol{\delta}\}$ is a line segment with end points obtained by permuting the coordinates of $\boldsymbol{c}(S)$.

In the second experiment, we consider a random 3×3 scattering matrix:

$$S = \begin{pmatrix} 0.06 - 0.15i & 0.21 + 0.12i & 0.15 + 0.01i \\ 0.12 + 0.35i & -0.02 + 0.04i & 0.21 + 0.55i \\ 0.09 - 0.43i & -0.15 - 0.09i & 0.20 + 0.21i \end{pmatrix}; \tag{26}$$

$$\boldsymbol{\sigma}(S) = \begin{pmatrix} 0.78 \\ 0.48 \\ 0.25 \end{pmatrix}, \qquad \boldsymbol{c}(S) = \begin{pmatrix} 0.39 \\ 0.05 \\ -0.44 \end{pmatrix}. \tag{27}$$

We generate 100 000 random $U_i \in U(3)$ and calculate $\boldsymbol{\alpha}[U_i]$, $\boldsymbol{e}[U_i]$, and $\boldsymbol{\delta}[U_i]$ by Eq. (16). Figures 3(a)–3(c) show the results. We see that $\{\boldsymbol{\alpha}\} = \{\boldsymbol{e}\}$ is a convex hexagon with vertices obtained by permuting the coordinates of $\mathbf{1} - \boldsymbol{\sigma}^2(S) = (0.39, 0.77, 0.93)^T$, while $\{\boldsymbol{\delta}\}$ is a convex hexagon with vertices obtained by permuting the coordinates of $\boldsymbol{c}(S)$.

The numerical results above suggest the following observation on the geometry of $\{\boldsymbol{\alpha}\}$, $\{\boldsymbol{e}\}$, and $\{\boldsymbol{\delta}\}$: For an n-port system, each set is a convex subset of an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$. $\{\boldsymbol{\alpha}\} = \{\boldsymbol{e}\}$ is the convex hull spanned by the $n!$ points obtained by permuting the coordinates of $\mathbf{1} - \boldsymbol{\sigma}^2(S)$, while $\{\boldsymbol{\delta}\}$ is the convex hull spanned by the $n!$ points obtained by permuting the coordinates of $\boldsymbol{c}(S)$. (The convex hull of a set is the smallest convex set that contains it.) We show that this observation is true as a result of the main theorem of our Letter:

Theorem.—Given a passive scattering matrix $S \in M_n$,

$$\{\boldsymbol{\alpha}\} = \{\boldsymbol{e}\} = \{\boldsymbol{u} \in \mathbb{R}^n | \boldsymbol{u} \prec \mathbf{1} - \boldsymbol{\sigma}^2(S)\}, \tag{28}$$

$$\{\boldsymbol{\delta}\} = \{\boldsymbol{v} \in \mathbb{R}^n | \boldsymbol{v} \prec \boldsymbol{c}(S)\}. \tag{29}$$

Proof.—We prove $\{\boldsymbol{\alpha}\} = \{\boldsymbol{u} \in \mathbb{R}^n | \boldsymbol{u} \prec \mathbf{1} - \boldsymbol{\sigma}^2(S)\}$. First, we show $\boldsymbol{\alpha}[U] \in \{\boldsymbol{\alpha}\} \Rightarrow \boldsymbol{\alpha}[U] \prec \mathbf{1} - \boldsymbol{\sigma}^2(S)$. We use Schur's theorem [64] (see the Supplemental Material [54], Theorem I.2):

$$\boldsymbol{\alpha}[U] = \boldsymbol{d}(U^\dagger A U) \prec \boldsymbol{\lambda}(U^\dagger A U) = \boldsymbol{\lambda}(A) = \mathbf{1} - \boldsymbol{\sigma}^2(S). \tag{30}$$

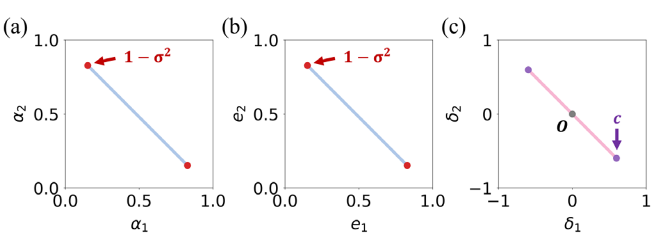

FIG. 2. Numerical experiment for $n = 2$. Shown here are scatter plots for (a) $\{\boldsymbol{\alpha}[U_i]\}$, (b) $\{\boldsymbol{e}[U_i]\}$, (c) $\{\boldsymbol{\delta}[U_i]\}$.

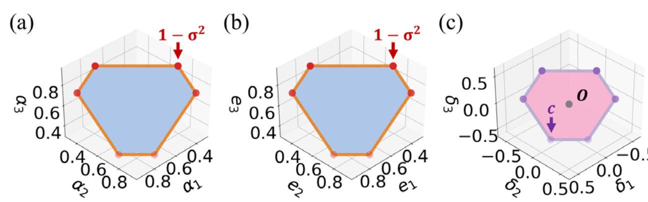

FIG. 3. Numerical experiment for $n = 3$. Shown here are scatter plots for (a) $\{\boldsymbol{\alpha}[U_i]\}$, (b) $\{\boldsymbol{e}[U_i]\}$, (c) $\{\boldsymbol{\delta}[U_i]\}$.

Second, we show $\boldsymbol{u} \prec \mathbf{1} - \boldsymbol{\sigma}^2(S) \Rightarrow \boldsymbol{u} \in \{\boldsymbol{\alpha}\}$, i.e., there exists $U \in U(n)$ such that $\boldsymbol{\alpha}[U] = \boldsymbol{u}$. We use Horn's theorem [65] (see the Supplemental Material [54], Theorem I.3): As $\boldsymbol{u} \prec \mathbf{1} - \boldsymbol{\sigma}^2(S)$, there exists a Hermitian matrix H with $\boldsymbol{d}(H) = \boldsymbol{u}$ and $\boldsymbol{\lambda}(H) = \mathbf{1} - \boldsymbol{\sigma}^2(S)$. Since $\lambda(A) = \mathbf{1} - \boldsymbol{\sigma}^2(S) = \lambda(H)$, H and A are unitarily similar [see the Supplemental Material [54], Theorem V.1(b)]. Hence there exists $U \in U(n)$ such that $H = U^\dagger A U$. Now we can check

$$\boldsymbol{\alpha}[U] \equiv \boldsymbol{d}(U^\dagger A U) = \boldsymbol{d}(H) = \boldsymbol{u}. \tag{31}$$

This completes the proof for $\{\boldsymbol{\alpha}\}$. The proofs for $\{\boldsymbol{e}\}$ and $\{\boldsymbol{\delta}\}$ are similar. ■

The geometric observation above is a direct consequence of our theorem. We use Rado's theorem [66] (see the Supplemental Material [54], Theorem I.1), which states that for a given $\boldsymbol{y} \in \mathbb{R}^n$, the set $\{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{x} \prec \boldsymbol{y}\}$ is the convex hull of points obtained by permuting the components of $\boldsymbol{y}$. The geometric observation is an application of Rado's theorem to Eqs. (28) and (29).

Equations (28) and (29) are our first main results. Equation (28) shows that $\{\boldsymbol{\alpha}\}$ and $\{\boldsymbol{e}\}$ are completely determined by $\boldsymbol{\sigma}(S)$, which is invariant under unitary control: $\boldsymbol{\sigma}(U^T S U) = \boldsymbol{\sigma}(S)$. We can classify all passive linear time-invariant systems by their $\boldsymbol{\sigma}$. Two systems exhibit the same $\{\boldsymbol{\alpha}\}$ and $\{\boldsymbol{e}\}$ if and only if they belong to the same $\boldsymbol{\sigma}$ class. Equation (29) shows that $\{\boldsymbol{\delta}\}$ is completely determined by $\boldsymbol{c}(S)$, which is also invariant under unitary control: $\boldsymbol{c}(U^T S U) = \boldsymbol{c}(S)$. We can classify all passive linear time-invariant systems by their $\boldsymbol{c}$. Two systems exhibit the same $\{\boldsymbol{\delta}\}$ if and only if they belong to the same $\boldsymbol{c}$ class.

We now turn to Question 2. For illustrative purposes, we consider only the problem of $\{U[\boldsymbol{\alpha}_0]\}$. The solution to the problem of $\{U[\boldsymbol{e}_0]\}$ and $\{U[\boldsymbol{\delta}_0]\}$ is similar. The problem corresponds to the following physical scenario. Suppose we have a photonic structure characterized by a scattering matrix S and therefore an absorptivity matrix $A = I - S^\dagger S$. Given an absorptivity vector $\boldsymbol{\alpha}_0 \prec \mathbf{1} - \boldsymbol{\sigma}^2(S)$, how do we construct the set of all possible unitary control schemes as described by unitary matrices $\{U[\boldsymbol{\alpha}_0]\}$ that achieve $\boldsymbol{\alpha}_0$? Or, alternatively, a simpler question, how to construct one unitary control scheme as described by a unitary matrix $U[\boldsymbol{\alpha}_0]$ that achieves $\boldsymbol{\alpha}_0$?

These two problems can be solved by the following algorithms. We first perform a preparatory step that is common in both algorithms: Suppose A have p distinct eigenvalues $\lambda_1, \ldots, \lambda_p$, with, respectively, multiplicities $n_1, \ldots, n_p$. Let $\Lambda = \lambda_1 I_{n_1} \oplus \ldots \oplus \lambda_p I_{n_p}$. We find a $V \in U(n)$ such that $A = V \Lambda V^\dagger$. Now we provide the two algorithms:

Algorithm 1. [Constructing $\{U[\boldsymbol{\alpha}_0]\}$].

1.Use Fickus's algorithm [67] (see the Supplemental Material [54], Algorithm IV.2) to construct all Hermitian matrices H_i with eigenvalues $\lambda(A)$ and diagonal entries $\boldsymbol{\alpha}_0$. For each H_i, find a $V_i \in U(n)$ such that $H_i = V_i \Lambda V_i^\dagger$.
2.We claim that $U_i \in U(n)$ such that $H_i = U_i^\dagger A U_i$ if and only if

$$U_i = V(W_1 \oplus \ldots W_p) V_i^\dagger, \tag{32}$$

where $W_k \in U(n_k)$, $k = 1, \ldots, p$, are arbitrary. Denote the set of all such U_i as $\{U_i\}$.
3.We claim that $\{U[\boldsymbol{\alpha}_0]\} = \cup_i \{U_i\}$. (See the Supplemental Material [54], Sec. V for proof of the two claims.)

Algorithm 2. [Constructing a $U[\boldsymbol{\alpha}_0]$].

1.Use Chu's algorithm [68] (see the Supplemental Material [54], Algorithm IV.1) to construct a Hermitian matrix A' with eigenvalues $\lambda(A)$ and diagonal elements $\boldsymbol{\alpha}_0$. Find a $V' \in U(n)$ such that $A' = V' \Lambda V'^\dagger$.
2.We obtain a $U[\boldsymbol{\alpha}_0] = V V'^\dagger$.

Algorithms 1 and 2 are our second main results.

To illustrate the usage of our algorithms, we provide a numerical example in the Supplemental Material [54], Sec. VI. We consider a five-port lossy system characterized by a random S matrix. Our task is to construct a $U[\boldsymbol{\alpha}_0]$, $U[\boldsymbol{e}_0]$, and $U[\boldsymbol{\delta}_0]$ with randomly assigned goals $\boldsymbol{\alpha}_0$, $\boldsymbol{e}_0$, and $\boldsymbol{\delta}_0$. We use Algorithm 2 and complete the task. Importantly, our algorithms allow us to achieve the prescribed absorptivity (or emissivity, or nonreciprocal contrast) in *all* ports with a *single* unitary matrix that performs unitary control.

Finally, we discuss physical applications of our theory.

First, we provide the criterion for k-fold degenerate coherent perfect absorption [52], i.e., the effect that a system exhibits coherent perfect absorption for k independent input modes. From Eq. (28), we obtain a necessary and sufficient condition: $\boldsymbol{\sigma}_1^\uparrow = \ldots = \boldsymbol{\sigma}_k^\uparrow = 0$.

Second, we propose the concept of *unitary perfect violation of Kirchhoff's law*, which refers to the effect that a system exhibits complete violation of Kirchhoff's law ($|\delta_i| = 1$ for some i) under some unitary control. From Eq. (29), we obtain the necessary and sufficient condition for unitary perfect violation of Kirchhoff's law for k independent inputs: $|\boldsymbol{c}|_1^\downarrow = \ldots = |\boldsymbol{c}|_k^\downarrow = 1$. As an example, consider a nonreciprocal two-port system with

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \boldsymbol{c}(S) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \tag{33}$$

$$\boldsymbol{\alpha} = \boldsymbol{e} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \qquad \boldsymbol{\delta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{34}$$

With this modal basis, Kirchhoff's law is satisfied, i.e., the absorptivities and emissivities are equal in every port. However, the system is nonreciprocal since its S matrix is not symmetric. We show that with the appropriate modal basis, this system can exhibit a perfect violation of Kirchhoff's law. We apply a unitary transformation

$$S \to U^T S U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{with} \quad U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{35}$$

Then

$$\boldsymbol{\alpha}[U] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{e}[U] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{\delta}[U] = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \tag{36}$$

the system exhibits perfect violation of Kirchhoff's law.

Third, we propose the concepts of *unitary uniform absorption*, *unitary uniform emission*, and *unitary detailed balance*, which refer to the effects that a system exhibits uniform absorptivity ($\alpha_i = \text{const}$), uniform emissivity ($e_i = \text{const}$), and zero nonreciprocal contrast ($\delta_i = 0$), respectively, under some unitary control. We claim that *any* n-port lossy system exhibits unitary uniform absorption, unitary uniform emission, and unitary detailed balance. We prove this by showing that for any $S \in M_n$, there exist $U, U', U'' \in U(n)$ such that

$$\boldsymbol{\alpha}[U] = \boldsymbol{e}[U'] = (a, \ldots, a)^T, \qquad \boldsymbol{\delta}[U''] = (0, \ldots, 0)^T, \tag{37}$$

where

$$a = 1 - \frac{1}{n} \sum_{i=1}^{n} \sigma_i^2(S). \tag{38}$$

This is because for any $(x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, we have

$$(\bar{x}, \bar{x}, \ldots, \bar{x})^T \prec (x_1, x_2, \ldots, x_n)^T, \qquad \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i. \tag{39}$$

Thus, for any $S \in M_n$, we have

$$(a, \ldots, a)^T \prec \mathbf{1} - \boldsymbol{\sigma}^2(S), \qquad (0, \ldots, 0)^T \prec \boldsymbol{c}(S), \tag{40}$$

hence they are attainable under unitary control. So, any system can exhibit uniform absorption or emission over any number of ports under suitable unitary control. This fact can be useful in applications such as thermal camouflage. Any nonreciprocal thermal emitter can behave just like a reciprocal emitter under some mode bases. Therefore, to maximize the nonreciprocal response of a nonreciprocal thermal emitter, one must carefully shape the external modes in addition to optimizing the photonic structure itself.

In conclusion, we provide a systematic theory for unitary control of optical absorption and emission. We reveal that majorization theory provides the mathematical structure to describe the physics of unitary control. Our results deepen the understanding of unitary control of absorption and emission and provide practical guidelines for its implementation.

C. G. thanks Dr. Zhaoyou Wang for helpful discussions. This work is funded by a Simons Investigator in Physics grant from the Simons Foundation (Grant No. 827065) and by a Multidisciplinary University Research Initiative (MURI) grant from the U.S. Army Research Office (Grant No. W911NF-19-1-0279).

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