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Winding number criterion for the origin to belong to the numerical range of a matrix on a loop of matrices

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The numerical range of a matrix is a compact convex set in the complex plane that contains the convex hull of the spectrum of the matrix. A key question is whether the origin belongs to the convex hull of the spectrum or the numerical range. This question arises in many computational and engineering applications such as the stability analysis of dynamic systems and the study of scattering matrices. Yet, there lacks an analytical criterion for this problem. Many relevant applications involve parameter-dependent matrices. In this paper, we provide a simple sufficient analytical criterion for the zero-inclusion problem, for a loop of parameter-dependent matrices. We prove the following theorem: Let $A : [0, 1] \to GL(n, \mathbb{C})$ (the set of $n \times n$ invertible complex matrices) be continuous with $A(0) = A(1)$, thus the winding number of $\det A$ is well defined. If the winding number is not divisible by n, then the origin belongs to the convex hull of the spectrum, thus the numerical range of $A(\phi)$ for some $\phi \in [0, 1]$. Our criterion may find potential applications in physics, engineering, and computation.

Keywords: Numerical range; determinants; winding number; sectorial matrix.

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1. Introduction

Let $M \in M_n$ be an $n \times n$ complex matrix. The numerical range of M is the subset of $\mathbb{C}$ defined as

$$W(M) := \{x^*Mx : x \in \mathbb{C}^n, x^*x = 1\}, \tag{1.1}$$

where * denotes conjugate transpose.

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The numerical range of a matrix has been extensively studied in various contexts. The numerical range serves as a useful tool to estimate the eigenvalues of a matrix, as it contains the convex hull of the eigenvalues [29, 57]:

$$\operatorname{conv} \boldsymbol{\lambda}(M) \subseteq W(M), \tag{1.2}$$

where $\operatorname{conv} \boldsymbol{\lambda}(M)$ denotes the convex hull of the eigenvalues of M. The numerical range has also found applications in the stability analysis of dynamic systems [9, 47] and the study of the quantum state geometry [5]. There exist known algorithms to compute the boundary of the numerical range [36, 42, 58].

In the mathematical study of the numerical range, a key question is the zero-inclusion problem [6, 20], which asks whether the origin belongs to the numerical range. This question arises in many computational and engineering problems, including the sensitivity analysis of generalized eigenvalue problems [10, 11, 31, 54], the convergence analysis of Krylov subspace methods [17–19, 41], and the coercivity condition for the discretized variational problems [7, 13]. For example, consider the generalized eigenvalue problem

$$Bx = \lambda Cx, \tag{1.3}$$

where B and C are $n \times n$ Hermitian matrices. If the pair of Hermitian matrices (B, C) is definite [11, 54], i.e.

$$0 \notin W(B + iC), \tag{1.4}$$

then there exists an invertible matrix X such that X^*BX and X^*CX are both diagonal [54], which significantly simplifies the study of the generalized eigenvalue problem.

The set of matrices

$$\mathbb{W}_n := \{M \in M_n : 0 \notin W(M)\} \tag{1.5}$$

are called (rotational) sectorial matrices [1–4, 8, 15, 16, 21, 45, 46, 49, 50, 52, 56, 60, 64, 65]. These matrices exhibit many nice properties. For example, the geometric mean between positive definite matrices can be generalized to sectorial matrices [16]. Moreover, sectorial matrices have been used to define the phases of a matrix [34, 59, 60, 65], which can be used to angularly bound the eigenvalues by majorization-type inequalities [59]. It is therefore important to find criteria that detect whether $M \in \mathbb{W}_n$. This problem has been discussed in many previous works [6, 14, 22, 37, 39, 48, 63]. Despite many efforts, however, there still lacks an analytic criterion for this problem. In other words, there has not been a known analytic procedure that can determine whether a numerical range of a matrix contains zero starting from the entries of the matrix.

In this paper, we consider the zero-inclusion problem for a loop of parametrized matrices. Many relevant applications involve parameter-dependent matrices. For example, in band theory, the Hamiltonian is often considered as a function of the wavevector, which acts as a parameter. In a one-dimensional system, this wavevector forms a loop within the Brillouin zone. Similarly, the scattering matrix in certain systems can be viewed as a function of incident angles [25]. Variations in these angles, under specific circumstances, can also be represented as a loop in the geometric context.

In this paper, we provide a simple sufficient analytical criterion for the zero-inclusion problem, for a loop of parameter-dependent matrices. We establish that for a continuous loop $A : [0, 1] \to GL(n, \mathbb{C})$ with $A(0) = A(1)$, if the winding number of $\det A$ is not divisible by n, the origin is in the convex hull of $A(\phi)$'s spectrum and numerical range for some $\phi \in [0, 1]$. This finding is potentially useful for understanding how the properties of matrices depend on parameters. For instance, in scattering problems, whether the numerical range contains the origin or not indicates distinct scattering behaviors [24], an aspect our theorem effectively addresses across varying parameters.

The zero-inclusion problem for a loop of parameterized matrices has been previously considered in [53]. The criterion that we discuss here was not explicitly stated in [53] even though it can be inferred from [53, Theorem I] (see the discussions in [23]). The author of [53] then utilized this result to study the partial indices of Wiener–Hopf factorization. In contrast, this paper presents a more accessible, elementary proof. Moreover, our proof establishes a further result, when the criterion is satisfied, the origin is in fact contained in the convex hull of the eigenvalues. This is a stronger result since the convex hull is in general a strict subset of the numerical range for non-normal matrices. This further result has not been noted in the previous literature.

2. Summary of Results

Question 2.1. Let $A : [0, 1] \to M_n$ be a continuous function with $A(0) = A(1)$, find a simple criterion to detect whether there exists $\phi \in [0, 1]$ such that

$$0 \in \operatorname{conv} \boldsymbol{\lambda}(A(\phi)) \subseteq W(A(\phi)). \tag{2.1}$$

This paper aims to provide a simple sufficient criterion for this question. Our criterion only needs the dimensionality and determinant of $A(t)$. (See [12, 35, 55, 61] for fast algorithms for computing matrix determinants.) We calculate $\det A(t)$ and check whether there exists $\phi \in [0, 1]$ such that

$$\det A(\phi) = 0. \tag{2.2}$$

(Throughout this paper, t denotes the parameter variable ranging over the interval $[0, 1]$, while ϕ represents a specific parameter value.) If true, then $0 \in \operatorname{conv}\boldsymbol{\lambda}(A(\phi)) \subseteq W(A(\phi))$. If false, then $\det A(t), t \in [0, 1]$ maps to a closed path in $\mathbb{C}\backslash\{0\}$. We calculate the winding number of $\det A$ around the origin:

$$\operatorname{wn}(A) := \operatorname{wn}(\det A) \in \mathbb{Z}. \tag{2.3}$$

We check whether n divides $\mathrm{wn}(A)$. We claim that if

$$n \nmid \mathrm{wn}(A), \tag{2.4}$$

then there must exist $\phi \in [0,1]$ such that $0 \in \mathrm{conv}\,\boldsymbol{\lambda}(A(\phi)) \subseteq W(A(\phi))$.

Criterion (2.4) is our main result. It is sufficient but not necessary: if

$$n \mid \mathrm{wn}(A), \tag{2.5}$$

there may or may not exist $\phi \in [0,1]$ such that $0 \in \mathrm{conv}\,\boldsymbol{\lambda}(A(\phi)) \subseteq W(A(\phi))$. For example, consider

$$A_k(t) = e^{i2\pi kt} A_0, \quad \tilde{A}_k(t) = e^{i2\pi kt} \tilde{A}_0, \quad k \in \mathbb{Z}, \quad t \in [0,1], \tag{2.6}$$

where A_0 and $\tilde{A}_0$ are invertible $n \times n$ matrices with

$$0 \in \mathrm{conv}\,\boldsymbol{\lambda}(A_0), \quad 0 \notin W(\tilde{A}_0). \tag{2.7}$$

As concrete examples, consider $n \times n$ diagonal matrices

$$A_0 = \mathrm{diag}(-1, 1, \ldots, 1), \quad \tilde{A}_0 = \mathrm{diag}(1, 1, \ldots, 1), \tag{2.8}$$

where $\mathrm{diag}(\boldsymbol{v})$ denotes a diagonal matrix with $\boldsymbol{v}$ being the vector of diagonal entries. One can show that

$$\mathrm{wn}(A_k) = \mathrm{wn}(\tilde{A}_k) = nk. \tag{2.9}$$

Hence

$$n \mid \mathrm{wn}(A_k), \quad n \mid \mathrm{wn}(\tilde{A}_k). \tag{2.10}$$

Nonetheless,

$$0 \in \mathrm{conv}\,\boldsymbol{\lambda}(A_k(t)) \subseteq W(A_k(t)), \quad 0 \notin W(\tilde{A}_k(t)). \tag{2.11}$$

In the case of $n|\mathrm{wn}(A)$, other conditions as provided in [6, 14, 22, 37, 39, 48, 63] can be used as supplements to address the zero-inclusion problem.

The rest of this paper provides detailed proof of our criterion (2.4), which will be articulated as the main theorem. It should be noted that [53] has previously demonstrated a weaker version of this result, showing that criterion (2.4) implies the existence of $\phi \in [0,1]$ such that $0 \in W(A(\phi))$.

3. Mathematical Background

We summarize the necessary background that will be useful in our proof.

First, we review some properties of the numerical range [26, 33, 62].

Proposition 3.1 (Toeplitz–Hausdorff [29, 57]). *Let $M \in M_n$. Then $W(M)$ is a compact convex subset of $\mathbb{C}$ that contains all the eigenvalues of M. Thus* $\mathrm{conv}\,\boldsymbol{\lambda}(M) \subseteq W(M)$.

Second, we review the functional continuity of matrix eigenvalues [40].

Proposition 3.2 (Kato [38]). *Let $A : [0,1] \to M_n$ be a continuous function. Then there exist n continuous functions $\lambda_1(t), \ldots, \lambda_n(t)$ from $[0,1]$ to $\mathbb{C}$ that parameterize the n eigenvalues* (*counted with algebraic multiplicities*) *of $A(t)$.*

Third, we review the concept of the winding number [28, 51].

Defintion 3.1. Let $\gamma : [0,1] \to \mathbb{C}\backslash\{0\}$ be a continuous path with $\gamma(0) = \gamma(1)$. The winding number of γ around the origin is

$$\mathrm{wn}(\gamma) := s(1) - s(0) \in \mathbb{Z}, \tag{3.1}$$

where (ρ, s) is the path written in polar coordinates, i.e. the lifted path through the covering map

$$p : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}\backslash\{0\} : (\rho_0, s_0) \mapsto \rho_0 e^{i2\pi s_0}. \tag{3.2}$$

In other words, $2\pi s(t)$ is the continuous function of the polar angle of $\gamma(t)$. Since $\gamma(1) = \gamma(0)$, $s(1) - s(0)$ must be an integer.

If γ is piecewise differentiable, $\mathrm{wn}(\gamma)$ can be calculated via integration

$$\mathrm{wn}(\gamma) = \frac{1}{2\pi i} \int_0^1 \frac{1}{\gamma(t)} \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} \mathrm{d}t. \tag{3.3}$$

Defintion 3.2. Let $A : [0,1] \to GL(n, \mathbb{C})$ (the set of $n \times n$ invertible complex matrices) be continuous with $A(0) = A(1)$. The winding number of A is defined as the winding number of $\det A$:

$$\mathrm{wn}(A) := \mathrm{wn}(\det A) \in \mathbb{Z}. \tag{3.4}$$

If A is piecewise differentiable, $\mathrm{wn}(A)$ can be calculated via integration

$$\mathrm{wn}(A) = \frac{1}{2\pi i} \int_0^1 \frac{1}{\det A(t)} \frac{d \det A(t)}{dt} dt. \tag{3.5}$$

Or equivalently, using Jacobi's formula [44],

$$\mathrm{wn}(A) = \frac{1}{2\pi i} \int_0^1 \mathrm{Tr}\left[A^{-1}(t) \frac{dA(t)}{dt}\right] dt. \tag{3.6}$$

Finally, $\mathrm{wn}(A)$ has a simple topological interpretation: it labels the first homotopy class of A in $GL(n, \mathbb{C})$ since [27]

$$\pi_1[GL(n, \mathbb{C})] \cong \mathbb{Z}. \tag{3.7}$$

4. Main Theorem

Now we state and prove our main theorem.

Theorem 4.1. *Let* $A : [0,1] \to GL(n,\mathbb{C})$ *be continuous with* $A(0) = A(1)$*. If*

$$n \nmid \mathrm{wn}(A), \tag{4.1}$$

then there exists $\phi \in [0,1]$ *such that*

$$0 \in \operatorname{conv} \boldsymbol{\lambda}(A(\phi)) \subseteq W(A(\phi)). \tag{4.2}$$

Example 4.1. Consider

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{i2\pi t} \end{pmatrix}, \quad t \in [0,1], \tag{4.3}$$

which has $n = 2$ and $\mathrm{wn}(A) = 1$, hence satisfying condition (4.1). Indeed, $0 \in \operatorname{conv} \boldsymbol{\lambda}(A(0.5))$, as the latter is a line segment between -1 and 1.

Example 4.2. Condition (4.1) is sufficient but not necessary for conclusion (4.2). Consider

$$A(t) = \begin{pmatrix} \mathrm{e}^{i2\pi t} & 0 \\ 0 & \mathrm{e}^{i6\pi t} \end{pmatrix}, \quad t \in [0,1], \tag{4.4}$$

which has $n = 2$ and $\mathrm{wn}(A) = 4$, hence does not satisfy condition (4.1). Moreover, $0 \notin W(A(0))$. Still, $0 \in \operatorname{conv} \boldsymbol{\lambda}(A(0.25))$, as the latter is a line segment between $-i$ and i.

Proof. From Eq. (1.2), we obtain

$$\operatorname{conv} \boldsymbol{\lambda}(A(t)) \subseteq W(A(t)), \quad \forall\, t \in [0,1]. \tag{4.5}$$

It suffices to prove the existence of $\phi \in [0,1]$ such that

$$0 \in \operatorname{conv} \boldsymbol{\lambda}(A(\phi)). \tag{4.6}$$

We denote the n eigenvalues of $A(t)$ as

$$\boldsymbol{\lambda}(A(t)) = [\lambda_1(t), \ldots, \lambda_n(t)]^T. \tag{4.7}$$

According to Kato's theorem, we can choose $\lambda_1(t), \ldots, \lambda_n(t)$ to be continuous functions of t on $[0,1]$. Since $A(t) \in GL(n,\mathbb{C})$,

$$\lambda_j(t) \neq 0, \quad j = 1, \ldots, n. \tag{4.8}$$

Since $A(0) = A(1)$,

$$\lambda_j(1) = \lambda_{\alpha_j}(0), \tag{4.9}$$

where

$$\alpha(A) = \begin{pmatrix} 1 & 2 & \cdots & n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} \in S_n \tag{4.10}$$

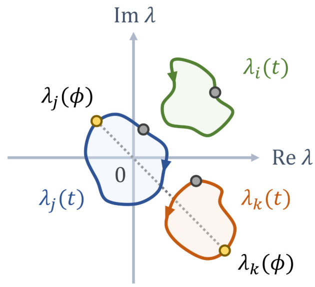

Fig. 1. Scheme for the $\alpha(A) = \text{id}$ case.

is a permutation. We will prove (4.6) first in the special case when

$$\alpha(A) = \text{id} := \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}, \tag{4.11}$$

then in the other cases when $\alpha(A) \neq \text{id}$.

Case I. $\alpha(A) = \text{id}$. (See Fig. 1 for a scheme.) Then each $\lambda_j : [0,1] \to \mathbb{C}\backslash\{0\}$ traces out a closed path, and thus has a well-defined winding number:

$$\text{wn}(\lambda_j) \in \mathbb{Z}, \quad j = 1, \ldots, n. \tag{4.12}$$

The winding number of a pointwise product of loops is the sum of winding numbers of each loop [51]:

$$\text{wn}(A) = \text{wn}(\det A) = \text{wn}\left(\prod_{j=1}^{n} \lambda_j\right) = \sum_{j=1}^{n} \text{wn}(\lambda_j). \tag{4.13}$$

Since

$$n \nmid \text{wn}(A), \tag{4.14}$$

there exist $1 \le j \neq k \le n$ such that

$$\text{wn}(\lambda_j) \neq \text{wn}(\lambda_k). \tag{4.15}$$

Otherwise, we would have

$$n | \text{wn}(A) = n \text{wn}(\lambda_1), \tag{4.16}$$

which contradicts (4.14). Now consider the line segment $\overline{\lambda_j(t)\lambda_k(t)}$. We claim that there exists $\phi \in [0,1]$ such that

$$0 \in \overline{\lambda_j(\phi)\lambda_k(\phi)} \subseteq \text{conv}\, \boldsymbol{\lambda}(A(\phi)), \tag{4.17}$$

which would complete the proof of (4.6) for the $\alpha(A) = \text{id}$ case.

Now we prove (4.17). Since $\lambda_j(t) \neq 0$ and $\lambda_k(t) \neq 0$, it suffices to show the existence of $\phi \in [0,1]$ such that

$$\arg[\lambda_j(\phi)] - \arg[\lambda_k(\phi)] = (2l+1)\pi, \quad l \in \mathbb{Z}. \tag{4.18}$$

Since $\lambda_j(t)$ and $\lambda_k(t)$ are continuous, we can choose $\arg[\lambda_j(t)]$ and $\arg[\lambda_k(t)]$ to be continuous functions of t. Then from Eq. (4.12),

$$\arg[\lambda_j(1)] - \arg[\lambda_j(0)] = 2\pi \cdot \mathrm{wn}(\lambda_j), \tag{4.19}$$
$$\arg[\lambda_k(1)] - \arg[\lambda_k(0)] = 2\pi \cdot \mathrm{wn}(\lambda_k). \tag{4.20}$$

Now consider the function

$$\Psi(t) := \arg[\lambda_j(t)] - \arg[\lambda_k(t)], \quad t \in [0,1]. \tag{4.21}$$

Ψ is continuous and

$$\Psi(1) - \Psi(0) = 2\pi\nu, \quad \nu := \mathrm{wn}(\lambda_j) - \mathrm{wn}(\lambda_k) \in \mathbb{Z}\backslash\{0\}. \tag{4.22}$$

We denote

$$\Psi(0) = 2\pi\chi + \xi, \quad \chi \in \mathbb{Z}, \quad \xi \in [-\pi, \pi). \tag{4.23}$$

Then

$$\Psi(1) = 2\pi(\nu + \chi) + \xi. \tag{4.24}$$

We know that the integer $\nu \neq 0$. If $\nu > 0$, then

$$\Psi(0) \leq 2\pi\chi + \pi \leq \Psi(1). \tag{4.25}$$

The intermediate value theorem implies that there exists $\phi \in [0,1]$ such that

$$\Psi(\phi) = 2\pi\chi + \pi. \tag{4.26}$$

We set $l = \chi$. If $\nu < 0$, then

$$\Psi(1) \leq 2\pi\chi - \pi \leq \Psi(0). \tag{4.27}$$

The intermediate value theorem implies that there exists $\phi \in [0,1]$ such that

$$\Psi(\phi) = 2\pi\chi - \pi. \tag{4.28}$$

We set $l = \chi - 1$. In any case, we obtain Eq. (4.18). This completes the proof of (4.17).

We summarized our result so far in its contrapositive form as a lemma.

Lemma 1. *Let $A : [0,1] \to GL(n, \mathbb{C})$ be continuous with $A(0) = A(1)$. If*

$$\alpha(A) = \mathrm{id} \quad \text{and} \quad 0 \notin \mathrm{conv}\, \boldsymbol{\lambda}[A(t)], \quad \forall\, t \in [0,1], \tag{4.29}$$

then

$$n \mid \mathrm{wn}(A). \tag{4.30}$$

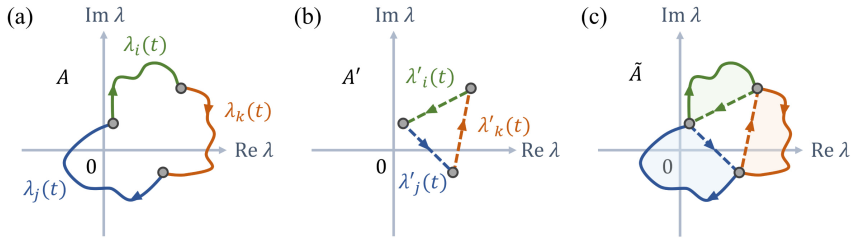

Fig. 2. Scheme for the $\alpha(A) \neq \mathrm{id}$ case.

Case II. $\alpha(A) \neq \mathrm{id}$. (See Fig. 2 for a scheme.) We first outline the proof. We prove this case by contradiction. Suppose that there exists $A : [0,1] \to GL(n,\mathbb{C})$ that is continuous with the following properties (Fig. 2(a)):

$$A(1) = A(0), \tag{4.31}$$
$$\lambda_j(1) = \lambda_{\alpha_j}(0), \quad \alpha(A) \neq \mathrm{id}, \tag{4.32}$$
$$0 \notin \mathrm{conv}\, \boldsymbol{\lambda}[A(t)], \quad \forall t \in [0,1], \tag{4.33}$$
$$n \nmid \mathrm{wn}(A). \tag{4.34}$$

Assumption (4.32) is opposite to what we want to prove (4.2). We will show that these assumptions lead to a contradiction with lemma. We construct a continuous and piecewise differentiable function $A' : [0,1] \to M_n$ with the following properties (Fig. 2(b)): (Note that $A'(t)$ does not denote the derivative of matrix-valued functions $\frac{\mathrm{d}}{\mathrm{d}t}A(t)$.)

$$A'(1) = A'(0) = A(1) = A(0), \tag{4.35}$$
$$\lambda'_j(0) = \lambda_j(1), \quad \lambda'_j(1) = \lambda_j(0), \tag{4.36}$$
$$\mathrm{conv}\, \boldsymbol{\lambda}[A'(t)] \subseteq \mathrm{conv}\, \boldsymbol{\lambda}[A(1)], \quad \forall\, t \in [0,1], \tag{4.37}$$
$$\det A'(t) \neq 0, \quad \forall\, t \in [0,1], \tag{4.38}$$
$$\mathrm{wn}(A') = 0. \tag{4.39}$$

Then we construct the concatenation of A and A' (Fig. 2(c)) defined as [28, 51]

$$\tilde{A}(t) := \begin{cases} A(2t), & t \in \left[0, \dfrac{1}{2}\right], \\ A'(2t-1), & t \in \left[\dfrac{1}{2}, 1\right]. \end{cases} \tag{4.40}$$

The winding number of a concatenation of loops is the sum of the winding numbers of each loop [51]:

$$\mathrm{wn}(\tilde{A}) = \mathrm{wn}(A) + \mathrm{wn}(A') = \mathrm{wn}(A). \tag{4.41}$$

Moreover, $\tilde{A}$ satisfies all the conditions of the lemma, and consequently,

$$n \mid \mathrm{wn}(\tilde{A}) = \mathrm{wn}(A), \tag{4.42}$$

which contradicts our premise (4.34). This means that assumption (4.33) is false, and our original conclusion (4.2) is true. This completes the proof of Case II.
Now we fill in the missing details of the proof outlined above.

1. Construction of A'.

We start with the Schur triangulation [32] of $A(0) = A(1)$:

$$A(0) = A(1) = U\Lambda U^* = V\Lambda' V^*, \tag{4.43}$$

where U, V are $m \times m$ unitary matrices, and

$$\Lambda = \begin{pmatrix} \lambda_1(0) & & \star \\ & \ddots & \\ 0 & & \lambda_n(0) \end{pmatrix}, \quad \Lambda' = \begin{pmatrix} \lambda_{\alpha_1}(0) & & \star \\ & \ddots & \\ 0 & & \lambda_{\alpha_n}(0) \end{pmatrix} \tag{4.44}$$

are upper triangular matrices. Here $\star$ denotes possibly nonzero elements. We recall that $\lambda_j(1) = \lambda_{\alpha_j}(0)$, $j = 1, \ldots, m$. We define two diagonal matrices

$$D = \begin{pmatrix} \lambda_1(0) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(0) \end{pmatrix}, \quad D' = \begin{pmatrix} \lambda_{\alpha_1}(0) & & 0 \\ & \ddots & \\ 0 & & \lambda_{\alpha_n}(0) \end{pmatrix}. \tag{4.45}$$

Now, we introduce a partition of $[0, 1]$:

$$0 = t_1 < t_2 < t_3 < t_4 < t_5 < t_6 = 1, \tag{4.46}$$

and set

$$A'(t_1) = V\Lambda' V^* = A(1), \tag{4.47}$$
$$A'(t_2) = \Lambda', \tag{4.48}$$
$$A'(t_3) = D', \tag{4.49}$$
$$A'(t_4) = D, \tag{4.50}$$
$$A'(t_5) = \Lambda, \tag{4.51}$$
$$A'(t_6) = U\Lambda U^* = A(0). \tag{4.52}$$

To connect these points in M_n, we define two skew-Hermitian matrices

$$J = \log U, \quad K = \log V. \tag{4.53}$$

(Every unitary matrix has a skew-Hermitian logarithm [30]. See [43] for an efficient algorithm to compute such a logarithm.) Then we set

$$A'(t) = \begin{cases} e^{K(t_2-t)/(t_2-t_1)} A'(t_2) e^{-K(t_2-t)/(t_2-t_1)}, & t_1 \le t < t_2, \\ \dfrac{t_3-t}{t_3-t_2} A'(t_2) + \dfrac{t-t_2}{t_3-t_2} A'(t_3), & t_2 \le t < t_3, \\ \dfrac{t_4-t}{t_4-t_3} A'(t_3) + \dfrac{t-t_3}{t_4-t_3} A'(t_4), & t_3 \le t < t_4, \\ \dfrac{t_5-t}{t_5-t_4} A'(t_4) + \dfrac{t-t_4}{t_5-t_4} A'(t_5), & t_4 \le t < t_5, \\ e^{J(t-t_5)/(t_6-t_5)} A'(t_5) e^{-J(t-t_5)/(t_6-t_5)}, & t_5 \le t \le t_6. \end{cases} \tag{4.54}$$

The evolution of $A'(t)$ is as follows: As t goes from $t_1 = 0$ to t_2, $A'(0) = A(1)$ is continuously deformed into its Schur triangulation Λ' by unitary similarity; from t_2 to t_3, Λ' is continuously reduced to D' by gradually diminishing the off-diagonal elements; from t_3 to t_4, D' is continuously deformed into D by linear interpolation, which leads to a permutation of the diagonal elements; from t_4 to t_5, D is continuously restored to another Schur triangulation Λ by gradually adding the off-diagonal elements; from t_5 to $t_6 = 1$, Λ is continuously deformed into $A'(1) = A(0)$ by unitary similarity.

2. Properties of A'.

We prove the claimed properties of A'. From the definition in Eq. (4.54), we confirm that $A' : [0,1] \to M_n$ is continuous and piecewise differentiable.

- *Proof of Eq.* (4.35). This is a direct result of Eqs. (4.31), (4.47) and (4.52).
- *Proof of Eq.* (4.36). We denote the n eigenvalues of $A'(t)$ as

$$\boldsymbol{\lambda}(A'(t)) = [\lambda'_1(t), \ldots, \lambda'_n(t)]^T. \tag{4.55}$$

By Kato's theorem, we can choose $\lambda'_1(t), \ldots, \lambda'_n(t)$ to be continuous function of t on $[0,1]$. Since $A'(0) = A(1)$, we can choose the ordering of λ'_j such that

$$\lambda'_j(0) = \lambda_j(1) = \lambda_{\alpha_j}(0). \tag{4.56}$$

From the definition of $A'(t)$ [Eq. (4.54)], we obtain

$$\lambda'_j(t) = \begin{cases} \lambda_{\alpha_j}(0), & t_1 \le t < t_3, \\ \dfrac{t_4-t}{t_4-t_3} \lambda_{\alpha_j}(0) + \dfrac{t-t_3}{t_4-t_3} \lambda_j(0), & t_3 \le t < t_4, \\ \lambda_j(0), & t_4 \le t \le t_6. \end{cases} \tag{4.57}$$

In particular,

$$\lambda'_j(1) = \lambda'_j(t_6) = \lambda_j(0). \tag{4.58}$$

This completes the proof of Eq. (4.36).

- *Proof of Eq.* (4.37). From Eq. (4.57), we obtain

$$\operatorname{conv}\boldsymbol{\lambda}[A'(t)] = \operatorname{conv}\boldsymbol{\lambda}[A(1)], \quad t_1 \le t < t_3, \tag{4.59}$$
$$\operatorname{conv}\boldsymbol{\lambda}[A'(t)] = \operatorname{conv}\boldsymbol{\lambda}[A(0)] = \operatorname{conv}\boldsymbol{\lambda}[A(1)], \quad t_4 \le t < t_6, \tag{4.60}$$

where we have used Eq. (4.31) in Eq. (4.60). We also note from Eq. (4.57) that

$$\lambda_j(t) \in \overline{\lambda_{\alpha_j}(0)\lambda_j(0)} \subseteq \operatorname{conv}\boldsymbol{\lambda}[A(1)], \quad j = 1, \ldots, m, \quad t_3 \le t < t_4. \tag{4.61}$$

By definition, $\operatorname{conv}\boldsymbol{\lambda}[A'(t)]$ is the smallest convex set on $\mathbb{C}$ that contains all $\lambda_j(t)$. Thus, Eq. (4.61) implies that

$$\operatorname{conv}\boldsymbol{\lambda}[A'(t)] \subseteq \operatorname{conv}\boldsymbol{\lambda}[A(1)], \quad t_3 \le t < t_4. \tag{4.62}$$

Combining (4.59), (4.60) and (4.62), we obtain

$$\operatorname{conv}\boldsymbol{\lambda}[A'(t)] \subseteq \operatorname{conv}\boldsymbol{\lambda}[A(1)], \quad \forall\, t \in [0,1]. \tag{4.63}$$

This completes the proof of (4.37).

- *Proof of Eq.* (4.38). From our assumption (4.32),

$$0 \notin \operatorname{conv}\boldsymbol{\lambda}[A(1)]. \tag{4.64}$$

Combining (4.64) and (4.37), we obtain

$$0 \notin \operatorname{conv}\boldsymbol{\lambda}[A'(t)], \quad \forall\, t \in [0,1], \tag{4.65}$$

which implies that

$$\lambda'_j(t) \neq 0, \quad \forall\, t \in [0,1], \quad j = 1, \ldots, m. \tag{4.66}$$

Therefore,

$$\det A'(t) = \prod_{j=1}^{n} \lambda'_j(t) \neq 0, \quad \forall\, t \in [0,1]. \tag{4.67}$$

This completes the proof of Eq. (4.38).

- *Proof of Eq.* (4.39). Since $0 \notin \operatorname{conv}\boldsymbol{\lambda}[A(1)]$ and the set $\operatorname{conv}\boldsymbol{\lambda}[A(1)]$ is compact and convex, we can take a branch $\log z$ of the logarithmic function such that this branch $\log z$ is analytic on an open neighborhood V of $\operatorname{conv}\boldsymbol{\lambda}[A(1)]$. Since $\lambda'_j(t) \in \operatorname{conv}\boldsymbol{\lambda}[A(1)] \subset V$ for all j and for all $t \in [0,1]$, then

$$\frac{d}{dt}\log\lambda'_j(t) = \frac{1}{\lambda'_j(t)}\frac{d\lambda'_j(t)}{dt}. \tag{4.68}$$

Since $A' : [0,1] \to M_n$ is continuous and piecewise differentiable with properties (4.35) and (4.38), it has a well-defined winding number

$$\mathrm{wn}(A') := \frac{1}{2\pi i}\int_0^1 \frac{1}{\det A'(t)}\frac{d\det A'(t)}{dt}dt = \frac{1}{2\pi i}\sum_{j=1}^{n}\int_0^1 \frac{1}{\lambda_j'(t)}\frac{d\lambda_j'(t)}{dt}dt, \tag{4.69}$$

$$= \frac{1}{2\pi i}\sum_{j=1}^{n}[\log\lambda_j'(1) - \log\lambda_j'(0)]$$

$$= \frac{1}{2\pi i}\left[\sum_{j=1}^{n}\log\lambda_j(0) - \sum_{j=1}^{n}\log\lambda_{\alpha_j}(0)\right] = 0 \tag{4.70}$$

since α is a permutation of $\{1,\ldots,n\}$, or $\{\lambda_1(0),\ldots,\lambda_n(0)\} = \{\lambda_{\alpha_1}(0),\ldots,\lambda_{\alpha_n}(0)\}$ (counting multiplicity). This completes the proof of Eq. (4.39). □

5. Conclusion

In conclusion, we have established a simple sufficient criterion for determining the zero-inclusion in the convex hull of the spectrum and numerical range for a parameter-dependent matrix loop. This criterion is based on the topological winding number of the matrix loop. It simplifies the assessment of this problem and may find potential applications in physics, engineering and computation. Future research could extend this approach to specific structured matrices and further practical applications.

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References

[1] M. Alakhrass, A note on sectorial matrices, *Linear Multilinear Algebra* **68**(11) (2020) 2228–2238, doi: 10.1080/03081087.2019.1575332.

[2] M. Alakhrass, On sectorial matrices and their inequalities, *Linear Algebra Appl.* **617** (2021) 179–189, doi: 10.1016/j.laa.2021.02.003.

[3] Yu. M. Arlinskiĭ and A. B. Popov, On sectorial matrices, *Linear Algebra Appl.* **370** (2003) 133–146, doi: 10.1016/S0024-3795(03)00388-4.

[4] C. Ballantine and C. Johnson, Accretive matrix products, *Linear Multilinear Algebra* **3**(3) (1975) 169–185, doi: 10.1080/03081087508817108.

[5] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement*, 2nd edn. (Cambridge University Press, 2017).

[6] P. S. Bourdon and J. H. Shapiro, When is zero in the numerical range of a composition operator? *Integr. Equ. Oper. Theory* **44**(4) (2002) 410–441, doi: 10.1007/BF01193669.

[7] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd edn., Texts in Applied Mathematics, Vol. 15 (Springer, New York, NY, 2008).

[8] W. Chen, D. Wang, S. Z. Khong and L. Qiu, A phase theory of multi-input multi-output linear time-invariant systems, *SIAM J. Control Optim.* **62**(2) (2024) 1235–126, doi: 10.1137/22M148968X.

[9] W. Chen, D. Wang, S. Z. Khong and L. Qiu, Phase analysis of MIMO LTI systems, in *IEEE 58th Conf. Decision and Control (CDC)* (IEEE, 2019), pp. 6062–6067.

[10] S. H. Cheng and N. J. Higham, The nearest definite pair for the Hermitian generalized eigenvalue problem, *Linear Algebra Appl.* **302–303** (1999) 63–76, doi: 10.1016/S0024-3795(99)00026-9.

[11] C. R. Crawford, A stable generalized eigenvalue problem, *SIAM J. Numer. Anal.* **13**(6) (1976) 854–860, doi: 10.1137/0713067.

[12] L. Csanky, Fast parallel matrix inversion algorithms, in *16th Annual Symp. Foundations of Computer Science (SFCS 1975)* (IEEE, 1975), pp. 11–12.

[13] L. F. Demkowicz, *Mathematical Theory of Finite Elements* (Society for Industrial and Applied Mathematics, Philadelphia, PA, 2023).

[14] C. Diogo, Algebraic properties of the set of operators with 0 in the closure of the numerical range, *Oper. Matrices* **1** (2015) 83–93, doi: 10.7153/oam-09-04.

[15] S. Drury, Principal powers of matrices with positive definite real part, *Linear Multilinear Algebra* **63**(2) (2015) 296–301, doi: 10.1080/03081087.2013.865732.

[16] S. Drury and M. Lin, Singular value inequalities for matrices with numerical ranges in a sector, *Oper. Matrices* **8**(4) (2014) 1143–1148, doi: 10.7153/oam-08-64.

[17] M. Eiermann, Fields of values and iterative methods, *Linear Algebra Appl.* **180** (1993) 167–197, doi: 10.1016/0024-3795(93)90530-2.

[18] M. Eiermann and O. G. Ernst, Geometric aspects of the theory of Krylov subspace methods, *Acta Numer.* **10** (2001) 251–312, doi: 10.1017/S0962492901000046.

[19] S. C. Eisenstat, H. C. Elman and M. H. Schultz, Variational iterative methods for nonsymmetric systems of linear equations, *SIAM J. Numer. Anal.* **20**(2) (1983) 345–357, doi: 10.1137/0720023.

[20] C. Finet and H. Queffélec, Numerical range of composition operators on a Hilbert space of Dirichlet series, *Linear Algebra Appl.* **377** (2004) 1–10, doi: 10.1016/j.laa.2003.05.003.

[21] S. Furtado and C. R. Johnson, Perturbation of matrices diagonalizable under congruence, *SIAM J. Matrix Anal. Appl.* **28**(1) (2006) 81–88, doi: 10.1137/S0895479803 43775X.

[22] H.-L. Gau and P. Y. Wu, Numerical ranges of nilpotent operators, *Linear Algebra Appl.* **429**(4) (2008) 716–726, doi: 10.1016/j.laa.2008.03.029.

[23] I. Gohberg, M. A. Kaashoek and I. M. Spitkovsky, An overview of matrix factorization theory and operator applications, in *Factorization and Integrable Systems*, eds. I. Gohberg, N. Manojlovic and A. F. dos Santos, Operator Theory: Advances and Applications (Birkhäuser, Basel, 2003), pp. 1–102.

[24] C. Guo and S. Fan, Topological winding guaranteed coherent orthogonal scattering, *Phys. Rev. A* **109**(6) (2024) L061503, doi: 10.1103/PhysRevA.109.L061503.

[25] C. Guo, J. Li, M. Xiao and S. Fan, Singular topology of scattering matrices, *Phys. Rev. B* **108**(15) (2023) 155418, doi: 10.1103/PhysRevB.108.155418.

[26] K. E. Gustafson and D. K. M. Rao, *Numerical Range: The Field of Values of Linear Operators and Matrices*, Universitext (Springer, New York, 1997).

[27] B. C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, 2nd edn., Graduate Texts in Mathematics, Vol. 222 (Springer, Cham, 2015).

[28] A. Hatcher, *Algebraic Topology* (Cambridge University Press, Cambridge, 2002).

[29] F. Hausdorff, Der Wertvorrat einer Bilinearform, *Math. Z.* **3**(1) (1919) 314–316, doi: 10.1007/BF01292610.

[30] N. J. Higham, *Functions of Matrices: Theory and Computation* (Society for Industrial and Applied Mathematics, Philadelphia, PA, 2008).

[31] N. J. Higham, F. Tisseur and P. M. Van Dooren, Detecting a definite Hermitian pair and a hyperbolic or elliptic quadratic eigenvalue problem, and associated nearness problems, *Linear Algebra Appl.* **351–352** (2002) 455–474, doi: 10.1016/S0024-3795(02)00281-1.

[32] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd edn. (Cambridge University Press, Cambridge, 2012).

[33] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis* (Cambridge University Press, Cambridge, 2008).

[34] A. Horn and R. Steinberg, Eigenvalues of the unitary part of a matrix, *Pac. J. Math.* **9**(2) (1959) 541–550.

[35] J. Jaja, *An Introduction to Parallel Algorithms*, 1st edn. (Addison-Wesley Professional, Reading, MA, 1992).

[36] C. R. Johnson, Numerical determination of the field of values of a general complex matrix, *SIAM J. Numer. Anal.* **15**(3) (1978) 595–602, doi: 10.1137/0715039.

[37] C. R. Johnson and C.-K. Li, Matrices for which $0 \notin F(A^m)$, *Linear Multilinear Algebra* **27**(1) (1990) 57–62, doi: 10.1080/03081089008817993.

[38] T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics (Springer, Berlin, 1995).

[39] L. Knockaert, Necessary and sufficient conditions for the origin to belong to the numerical range of a matrix, *WSEAS Trans. Math.* **5** (2006) 1350–1352.

[40] C.-K. Li and F. Zhang, Eigenvalue continuity and Gersgorin's theorem, *Electron. J. Linear Algebra* **35** (2019) 619–625, doi: 10.13001/ela.2019.5179.

[41] J. Liesen and P. Tichý, Convergence analysis of Krylov subspace methods, *GAMM-Mitteilungen* **27**(2) (2004) 153–173, doi: 10.1002/gamm.201490008.

[42] S. Loisel and P. Maxwell, Path-following method to determine the field of values of a matrix with high accuracy, *SIAM J. Matrix Anal. Appl.* **39**(4) (2018) 1726–1749, doi: 10.1137/17M1148608.

[43] T. A. Loring, Computing a logarithm of a unitary matrix with general spectrum, *Numer. Linear Algebra Appl.* **21**(6) (2014) 744–760, doi: 10.1002/nla.1927.

[44] J. R. Magnus and H. Neudecker, *Matrix Differential Calculus with Applications in Statistics and Econometrics*, 3rd edn., Wiley Series in Probability and Statistics (Wiley, Hoboken, NJ, 2019).

[45] X. Mao, L. Qiu, A. Ringh and D. Wang, Some counterexamples related to sectorial matrices and matrix phases, *Examples Counterexamples* **1** (2021) 100019, doi: 10.1016/j.exco.2021.100019.

[46] R. Mathias, Matrices with positive definite Hermitian part: Inequalities and linear systems, *SIAM J. Matrix Anal. Appl.* **13**(2) (1992) 640–654, doi: 10.1137/0613038.

[47] D. H. Owens, The numerical range: A tool for robust stability studies? *Syst. Control Lett.* **5**(3) (1984) 153–158, doi: 10.1016/S0167-6911(84)80096-1.

[48] P. J. Psarrakos and M. J. Tsatsomeros, Numerical range: (in) a matrix nutshell, Notes, National Technical University, Athens, Greece (2004).
[49] M. Raïssouli, M. S. Moslehian and S. Furuichi, Relative entropy and Tsallis entropy of two accretive operators, *C.R. Math.* **355**(6) (2017) 687–693, doi: 10.1016/j.crma.2017.05.005.
[50] A. Ringh and L. Qiu, Finsler geometries on strictly accretive matrices, *Linear Multilinear Algebra* **70**(21) (2022) 6753–6771, doi: 10.1080/03081087.2021.1968781.
[51] J. Roe, *Winding Around: The Winding Number in Topology, Geometry, and Analysis*, Student Mathematical Library, Vol. 76 (American Mathematical Society, Providence, RI, 2015).
[52] T. Sano and K. Sugawara, Operator means for operators with positive definite real part, *Adv. Oper. Theory* **5**(3) (2020) 640–646, doi: 10.1007/s43036-019-00017-w.
[53] I. M. Spitkovsky, Stability of partial indices of a Riemann boundary-value problem with a strictly nonsingular matrix, *Dokl. Akad. Nauk* **218**(1) (1974) 46–49.
[54] G. W. Stewart, Perturbation bounds for the definite generalized eigenvalue problem, *Linear Algebra Appl.* **23** (1979) 69–85, doi: 10.1016/0024-3795(79)90094-6.
[55] V. Strassen, Gaussian elimination is not optimal, *Numer. Math.* **13**(4) (1969) 354–356, doi: 10.1007/BF02165411.
[56] F. Tan and A. Xie, An extension of the AM–GM–HM inequality, *Bull. Iran. Math. Soc.* **46**(1) (2020) 245–251, doi: 10.1007/s41980-019-00253-z.
[57] O. Toeplitz, Das algebraische Analogon zu einem Satze von Fejér, *Math. Z.* **2**(1) (1918) 187–197, doi: 10.1007/BF01212904.
[58] F. Uhlig, Zhang Neural Networks for fast and accurate computations of the field of values, *Linear Multilinear Algebra* **68**(9) (2020) 1894–1910, doi: 10.1080/03081087.2019.1648375.
[59] D. Wang, W. Chen, S. Z. Khong and L. Qiu, On the phases of a complex matrix, *Linear Algebra Appl.* **593** (2020) 152–179, doi: 10.1016/j.laa.2020.01.035.
[60] D. Wang, X. Mao, W. Chen and L. Qiu, On the phases of a semi-sectorial matrix and the essential phase of a Laplacian, *Linear Algebra Appl.* **676** (2023) 441–458, doi: 10.1016/j.laa.2023.07.014.
[61] V. V. Williams, Y. Xu, Z. Xu and R. Zhou, New bounds for matrix multiplication: From alpha to omega, in *Proc. 2024 Annual ACM-SIAM Symp. Discrete Algorithms (SODA)* (Society for Industrial and Applied Mathematics, 2024), pp. 3792–3835.
[62] P. Y. Wu, *Numerical Ranges of Hilbert Space Operators*, Encyclopedia of Mathematics and Its Applications, Vol. 179 (Cambridge University Press, Cambridge, 2021).
[63] X. H. Wu, J. J. Huang and Alatancang, On the origin and numerical range of bounded operators, *Acta Math. Sin., Engl. Ser.* **35**(10) (2019) 1715–1722, doi: 10.1007/s10114-019-8478-2.
[64] F. Zhang, A matrix decomposition and its applications, *Linear Multilinear Algebra* **63** (10) (2015) 2033–2042, doi: 10.1080/03081087.2014.933219.
[65] D. Zhao, A. Ringh, L. Qiu and S. Z. Khong, Low phase-rank approximation, *Linear Algebra Appl.* **639** (2022) 177–204, doi: 10.1016/j.laa.2022.01.003.