Theoretical constraints on reciprocal and non-reciprocal many-body radiative heat transfer

Cheng Guo [1] and Shanhui Fan [2,*]

[1] *Department of Applied Physics, Stanford University, Stanford, California 94305, USA*

[2] *Ginzton Laboratory and Department of Electrical Engineering, Stanford University, Stanford, California 94305, USA*

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We study the constraints on reciprocal and non-reciprocal many-body radiative heat transfer imposed by symmetry and the second law of thermodynamics. We show that the symmetry of such a many-body system in general forms a magnetic group, and the constraints of the magnetic group on the heat transfer can be derived using a generalized reciprocity theorem. We also show that the second law of thermodynamics provides additional constraints in the form of a nodal conservation law of heat flow at equilibrium. As an application, we provide a systematic approach to determine the existence of persistent heat current in arbitrary many-body systems.

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I. INTRODUCTION

Thermal radiation is important for both fundamental science and engineering applications [1–6]. The vast majority of the literature on radiative heat transfer assumes Lorentz reciprocity, which imposes a strong constraint on radiative heat transfer [7–9]. On the other hand, recently, there has been significant progress in studying radiative heat transfer using non-reciprocal materials such as magneto-optical materials [10–17] and magnetic Weyl semimetals [18–20]. These studies have led to the discoveries of novel phenomena in non-reciprocal many-body radiative heat transfer such as a persistent heat current [21] and the photon thermal Hall effect [20,22,23], which exemplifies the opportunities of exploring novel aspects of radiative heat transfer that can arise in complex reciprocal and non-reciprocal many-body systems [24,25].

In this paper, in order to provide theoretical guidance on the explorations of many-body radiative heat transfer, we consider the general theoretical constraints on such a process. Certainly, the heat transfer is constrained by the second law of thermodynamics. Moreover, for a non-reciprocal many-body system, where reciprocity is broken with either internal or external bias magnetic fields on at least some of the bodies, its symmetry consists of two classes of operations. The first class is the usual spatial operations, such as rotation and mirror operations, which transform the magnetic field bias on each of the bodies in the usual way of pseudovectors. The second class consists of operations that flip all the magnetic field bias in addition to the usual spatial operations. These two classes of operations together form the magnetic group of the many-body system. We show that the properties of many-body radiative heat transfer are strongly constrained by the structure of the magnetic group. The derivation in particular relies upon a generalized reciprocity theorem that relates the properties of two complementary systems. As an illustration of these theoretical constraints, we show these constraints can be used to identify many-body non-reciprocal systems that do not exhibit a persistent heat current.

The rest of this paper is organized as follows. In Sec. II we provide the theory. In Sec. III we apply our theory to determine the existence of a persistent heat current in arbitrary many-body systems. We conclude in Sec. IV.

II. THEORY

We consider a system consisting of N bodies that exchange heat via radiation with each other and an environment. We label the environment (env) and the bodies as $\{0 \equiv \text{env}, 1, 2, \dots, N\}$. In general, the system is an inhomogeneous dispersive bianisotropic medium, which can be described by a 6×6 constitutive matrix $C(\omega, \boldsymbol{r})$ [26]:

$$\begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{B} \end{pmatrix} = C \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \begin{pmatrix} \varepsilon & \zeta \\ \eta & \mu \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}, \tag{1}$$

where $\varepsilon, \mu, \zeta, \eta$ are 3×3 matrices of electric permittivity, magnetic permeability, electric-magnetic coupling strength, and magnetoelectric coupling strength, respectively. ω and $\boldsymbol{r}$ denote the frequencies and the spatial coordinates, respectively.

For radiative heat transfer, one considers the spectral heat flux to body j due to thermal noise sources in body i of temperature T_i:

$$S_{i\to j}(\omega) = \frac{\Theta(\omega, T_i)}{2\pi} F_{i\to j}(\omega), \tag{2}$$

where $\Theta(\omega, T_i) = \hbar\omega/[\exp(\hbar\omega/k_B T_i) - 1]$ and $F_{i\to j}(\omega)$ denotes the temperature-independent transmission coefficient from body i to j. A general theory of many-body radiative heat transfer was developed in Ref. [27] that allows us to calculate $F_{i\to j}(\omega)$ given $C(\omega, \boldsymbol{r})$. All the directional transmission coefficients $F_{i\to j}(\omega)$ can be arranged into an *exchange matrix*

*shanhui@stanford.edu

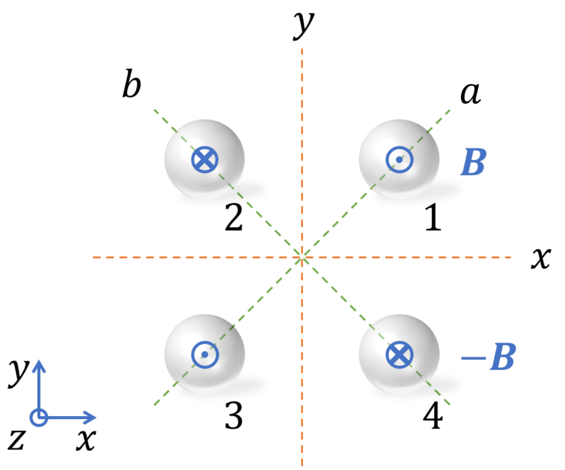

FIG. 1. Schematic of a system consisting of four gyrotropic spheres that exchange heat via radiation with each other and an environment. The centers of the spheres form a square on the x-y plane. There is a magnetic field along the z direction with alternating signs in its distribution in the x-y plane.

$\mathcal{F}$ of dimension $(N+1)\times(N+1)$:

$$\mathcal{F} = \begin{pmatrix} 0 & F_{0\to 1} & \dots & F_{0\to N} \\ F_{1\to 0} & 0 & \dots & F_{1\to N} \\ \vdots & \vdots & \ddots & \vdots \\ F_{N\to 0} & F_{N\to 1} & \dots & 0 \end{pmatrix}, \tag{3}$$

where we have suppressed the parameter ω for clarity. Our objective is to find the constraints imposed on $\mathcal{F}$ by the symmetry of the system as well as the second law of thermodynamics.

To illustrate the possible symmetry of these systems, we first consider a concrete example, as shown in Fig. 1, which consists of a cluster of four gyrotropic spheres. These spheres are assumed to be made of the same materials but may be subjected to different local magnetic fields B_i (including those generated by internal magnetization). The dielectric permittivities of the spheres are

$$\varepsilon_i = \varepsilon(B_i) = \begin{pmatrix} \epsilon_x & -i\epsilon'(B_i) & 0 \\ i\epsilon'(B_i) & \epsilon_x & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix}, \tag{4}$$

where $\epsilon'(B) = -\epsilon'(-B)$ such that $\varepsilon(B)^T = \varepsilon(-B)$ per generalized Onsager reciprocal relations [7–9]. Here for illustration purposes we choose $B_1 = B_3 = -B_2 = -B_4$. For this system, there are two classes of symmetries: (1) the usual point group symmetry $D_{2h} = \{E, C_2(z), 2C_2'(x), i, \sigma_h(z), 2\sigma_v(x)\}$ and (2) the compound symmetry $R = \{2\mathcal{T}C_4(z), 2\mathcal{T}C_2''(a), 2\mathcal{T}S_4(z), 2\mathcal{T}\sigma_d(a)\}$, where $\mathcal{T}$ is a so-called antisymmetry operation that transforms $\varepsilon(\boldsymbol{r})$ to $\varepsilon^T(\boldsymbol{r})$, which is equivalent to reversing the direction of the local magnetic field. We use the standard Schoenflies notations [28], and for clarity, we indicate the rotation axis and the normal direction of the mirror plane in the parentheses. For example, $C_2(z)$ denotes a 180° rotation around the z axis, $\sigma_v(x)$ denotes the mirror operation with respect to the y-z plane, while $\mathcal{T}C_4(z)$ denotes a 90° rotation around the z axis combined with the antisymmetry operation. We note that each compound symmetry $R_n = \mathcal{T}A_n$ is a combination of $\mathcal{T}$ and the usual spatial operation A_n, but $\mathcal{T}$ and A_n are not the symmetry by themselves. All $\{A_n\}$ (not $\{R_n\}$!) together with the usual point group D_{2h} form a larger point group D_{4h}. The symmetry group of such a gyrotropic cluster is therefore a magnetic group:

$$\begin{aligned} \mathcal{M} =& \underline{4}/mmm = D_{4h}(D_{2h}) \equiv D_{2h} + \mathcal{T}(D_{4h} - D_{2h}) \\ =& \{E,\ 2\underline{C_4}(z),\ C_2(z),\ 2C_2'(x),\ 2\underline{C_2''}(a), \\ & i,\ 2\underline{S_4}(z),\ \sigma_h(z),\ 2\sigma_v(x),\ 2\underline{\sigma_d}(a)\}, \end{aligned} \tag{5}$$

where the underline denotes the compound elements that are combined with $\mathcal{T}$. A magnetic group contains the usual point group symmetry operations, as well as compound symmetry operations which contain an antisymmetry operator $\mathcal{T}$ [28–30].

The generalization of the discussion above to an arbitrary system is straightforward. We define a local operation of antisymmetry $\mathcal{T}$, which transforms between a general medium as described by $C(\omega, \boldsymbol{r})$ and its complementary medium as described by $\widetilde{C}(\omega, \boldsymbol{r})$ [31]:

$$C = \begin{pmatrix} \varepsilon & \zeta \\ \eta & \mu \end{pmatrix} \xleftrightarrow{\mathcal{T}} \widetilde{C} = \begin{pmatrix} \varepsilon^T & -\eta^T \\ -\zeta^T & \mu^T \end{pmatrix}. \tag{6}$$

$\mathcal{T}^2 = E$, where E is the identity operation. By this definition, for a gyrotropic plasma under an external DC magnetic field bias, its complementary medium is the same gyrotropic plasma, but with the direction of the magnetic field bias reversed. A medium is reciprocal if and only if it is self-complementary, i.e., invariant under $\mathcal{T}$. Since $\mathcal{T}$ acts on the constitutive matrix instead of the ordinary position coordinates, it commutes with all the ordinary spatial operations. The symmetry of a general system consists of ordinary spatial symmetry operations and their combination with $\mathcal{T}$. Following Ref. [28], we refer to the former type of operations as uncolored and the latter type as colored. The magnetic group of a system as described by $C(\omega, \boldsymbol{r})$ is the sets of all the symmetry operations that leave the system invariant.

Below we consider the constraints on $\mathcal{F}$ as imposed by the two different classes of symmetry operations:

(1) *Uncolored operation* A_l. A_l can be represented by the permutation of the bodies:

$$\mathbb{P}_{A_l} = \begin{pmatrix} 0 & 1 & 2 & \cdots & N \\ 0 & P_1 & P_2 & \cdots & P_N \end{pmatrix}. \tag{7}$$

We note the environment is invariant under the permutation of the bodies $(0 \to 0)$. Such a permutation leaves the system invariant; thus, it enforces the constraints

$$F_{P_i \to P_j} = F_{i\to j}, \quad i, j = 0, \dots, N. \tag{8}$$

In matrix form, Eq. (8) can be written as

$$\mathbf{P}_{A_l} \mathcal{F} \mathbf{P}_{A_l}^T = \mathcal{F}, \tag{9}$$

where $\mathbf{P}_{A_l}$ is the permutation matrix corresponding to $\mathbb{P}_{A_l}$.

(2) *Colored operation* $R_n = \mathcal{T}A_n$. A_n permutates the bodies by

$$\mathbb{P}_{A_n} = \begin{pmatrix} 0 & 1 & 2 & \cdots & N \\ 0 & P_1' & P_2' & \cdots & P_N' \end{pmatrix}. \tag{10}$$

The resultant system is complementary to the original one, and an additional $\mathcal{T}$ operation maps the system back to the original one.

The generalized reciprocity theorem [31] of electromagnetism requires that the exchange matrices $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ of two complementary systems $C(\omega, \boldsymbol{r})$ and $\widetilde{C}(\omega, \boldsymbol{r})$ are the transpose of each other (see the proof in the Appendix):

$$\widetilde{\mathcal{F}} = \mathcal{F}^T, \quad i.e.\ \widetilde{F}_{i\to j} = F_{j\to i}. \tag{11}$$

Therefore, R_n, being a symmetry of the system, enforces the constraints

$$F_{P_i'\to P_j'} = F_{j\to i}, \quad i, j = 0, \ldots, N. \tag{12}$$

In matrix form, Eq. (12) can be written as

$$\mathbf{P}_{A_n}\, \mathcal{F}\, \mathbf{P}_{A_n}^T = \mathcal{F}^T, \tag{13}$$

where $\mathbf{P}_{A_n}$ is the permutation matrix corresponding to $\mathbb{P}_{A_n}$.

By considering all the symmetry elements $\{A_l, R_n\}$ in the magnetic group $\mathcal{M}$, we obtain *all* the constraints imposed by the symmetry on radiative heat transfer.

Magnetic groups, denoted as $\mathcal{M}$, can be classified into three types [28,30]:

(1) *Colorless group.* Here $\mathcal{M}$ is an ordinary point group $\mathcal{G}$ with no colored elements.

(2) *Gray group.* Here $\mathcal{M}$ is isomorphic to a direct product $\mathcal{G} \otimes \{E, \mathcal{T}\}$, where $\mathcal{G}$ is an ordinary point group. Being a direct product immediately implies that $\mathcal{T}$ commutes with all elements of the point group $\mathcal{G}$.

(3) *Black-and-white group.* Here $\mathcal{M} = \{A_l, R_n\}$, where half of the elements are colorless, forming the set $\{A_l\}$, and the other half are colored, forming the set $\{R_n = \mathcal{T}A_n\}$. Moreover, $\{A_l, A_n\}$ forms an ordinary point group $\mathcal{G}'$, and $\{A_l\} = \mathcal{H}$ forms a subgroup of $\mathcal{G}'$ with index 2. Thus, a black-and-white group has the form

$$\mathcal{M} = \mathcal{H} + \mathcal{T}(\mathcal{G}' - \mathcal{H}). \tag{14}$$

We denote $\mathcal{M} = \mathcal{G}'(\mathcal{H})$, following Ref. [30].

A reciprocal system is, by definition, invariant under $\mathcal{T}$. Since $\mathcal{T}$ is an element only of a gray group, a system is reciprocal if and only if its symmetry is a gray group; a system is non-reciprocal if and only its symmetry is a colorless or black-and-white group.

Finally, we consider the constraints of the second law of thermodynamics. Since we consider the bodies exchanging energy only by radiation, in the equilibrium case where all bodies as well as the environment have the same temperature, the energy flow into any body must balances that out of the body:

$$\sum_{j=0;\, j\neq i}^{N} F_{i\to j} = \sum_{j=0;\, j\neq i}^{N} F_{j\to i}; \tag{15}$$

that is, the $\mathcal{F}$ matrix has the same row sum and column sum. This represents a nodal conservation law of heat flow at equilibrium. In matrix form, Eq. (15) can be written as

$$(\mathcal{F} - \mathcal{F}^T)\vec{j} = 0, \tag{16}$$

where $\vec{j}$ is an all-one vector. Conversely, if Eq. (16) is satisfied, in equilibrium the net heat flow into any of the subsystems consisting of a few of bodies is zero. Thus, Eq. (16) is sufficient to impose the second law of thermodynamics in the many-body system.

The second law of thermodynamics can provide unique constraints beyond those from symmetry. For example, a system where radiative heat transfer occurs entirely between two bodies has an exchange matrix

$$\mathcal{F} = \begin{pmatrix} 0 & F_{1\to 2} \\ F_{2\to 1} & 0 \end{pmatrix}. \tag{17}$$

The second law of thermodynamics requires $F_{1\to 2} = F_{2\to 1}$, regardless of any symmetry [21].

The three sets of constraints, Eqs. (8) and (9), Eqs. (12) and (13), and Eqs. (15) and (16), are the main results of this paper. These are all the constraints on radiative heat transfer that can be stated from symmetry and the second law of thermodynamics.

Let us apply the general theory to the concrete example in Fig. 1. The magnetic group of the system is $\mathcal{M} = D_{4h}(D_{2h})$ [Eq. (5)]. The exchange matrix is

$$\mathcal{F} = \begin{pmatrix} 0 & F_{01} & F_{02} & F_{03} & F_{04} \\ F_{10} & 0 & F_{12} & F_{13} & F_{14} \\ F_{20} & F_{21} & 0 & F_{23} & F_{24} \\ F_{30} & F_{31} & F_{32} & 0 & F_{34} \\ F_{40} & F_{41} & F_{42} & F_{43} & 0 \end{pmatrix}, \tag{18}$$

where $F_{ij} \equiv F_{i\to j}$ for simplicity. We first study the constraints on $\mathcal{F}$ imposed by $\mathcal{M}$ by considering all the elements:

(1) For $2\underline{C_4}(z)$, $C_4(z)$ permutates the bodies by

$$\mathbb{P}_{C_4} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \end{pmatrix}. \tag{19}$$

Constraints from Eq. (12) are therefore

$$\begin{aligned} F_{01} &= F_{20} = F_{03} = F_{40}, \\ F_{02} &= F_{30} = F_{04} = F_{10}, \\ F_{12} &= F_{32} = F_{34} = F_{14}, \\ F_{23} &= F_{43} = F_{41} = F_{21}, \\ F_{13} &= F_{42} = F_{31} = F_{24}. \end{aligned} \tag{20}$$

(2) For $C_2(z)$, there are no new constraints since $C_2(z) = \underline{C_4}^2(z)$.

(3) $2C_2'(x)$ permutates the bodies by

$$\mathbb{P}_{C_2'} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 3 & 2 & 1 \end{pmatrix}. \tag{21}$$

Constraints from Eq. (8) are therefore

$$F_{01} = F_{04}, \quad F_{12} = F_{43}. \tag{22}$$

(4) The remaining elements impose no new constraints. Combining all the constraints (20) and (22),

$$\mathcal{F} = \begin{pmatrix} 0 & F_{01} & F_{01} & F_{01} & F_{01} \\ F_{01} & 0 & F_{12} & F_{13} & F_{12} \\ F_{01} & F_{12} & 0 & F_{12} & F_{13} \\ F_{01} & F_{13} & F_{12} & 0 & F_{12} \\ F_{01} & F_{12} & F_{13} & F_{12} & 0 \end{pmatrix}, \tag{23}$$

which has only three independent components, F_{01}, F_{12}, F_{13}. Also $\mathcal{F} = \mathcal{F}^T$, even though this system is non-reciprocal.

The second law of thermodynamics imposes no new constraints since $\mathcal{F} = \mathcal{F}^T$, and Eq. (16) is automatically satisfied.

III. APPLICATIONS

As an application of our theory, we study the persistent heat current in non-reciprocal radiative heat transfer. Persistent heat current is a phenomenon that can exist in some non-reciprocal many-body systems even at thermal equilibrium [21]. By definition, the persistent heat current exists between bodies i and j at equilibrium if and only if $F_{i\to j} \neq F_{j\to i}$. It has been proved that non-reciprocity is a necessary, but not sufficient, condition for the existence of the persistent heat current [21,23]. However, a systematic way to determine whether a given system can exhibit a persistent heat current is still lacking. Our theory can provide such a systematic approach.

From the definition, there is a persistent heat current in a system between at least one pair of bodies if and only if $\mathcal{F} \neq \mathcal{F}^T$. Since our theory provides all the general constraints on $\mathcal{F}$, we can check whether a system can support a persistent heat current by deducing the constrained form of $\mathcal{F}$ and then checking whether $\mathcal{F} = \mathcal{F}^T$. If $\mathcal{F} = \mathcal{F}^T$, there is no persistent heat current in such a system. Otherwise, there is no symmetry reason against the existence of a persistent heat current.

To demonstrate such a procedure, we consider three exemplary systems, as shown in Figs. 2(a), 2(c) and 2(e). The geometries are similar: all the systems consist of four gyrotropic spheres made of n-doped InSb. Each sphere has a radius of 100 nm. The centers of the four spheres are placed at the vertices of a square on the x-y plane. The side length of the square is 320 nm. These systems differ in the magnetic field configurations: the first system is under no magnetic field, the second is under spatially alternating fields ($B_1 = B_3 = -B_2 = -B_4 = 1$ T), and the last is under a uniform field ($B_1 = B_2 = B_3 = B_4 = 1$ T). The external magnetic field is perpendicular to the x-y plane. Under the magnetic field, n-doped InSb has a relative permittivity tensor

$$\bar{\bar{\epsilon}} = \epsilon_b \bar{\bar{I}} - \frac{\omega_p^2}{(\omega + i\Gamma)^2 - \omega_c^2} \times \begin{bmatrix} 1 + i\frac{\Gamma}{\omega} & -i\frac{\omega_c}{\omega} & 0 \\ i\frac{\omega_c}{\omega} & 1 + i\frac{\Gamma}{\omega} & 0 \\ 0 & 0 & \frac{(\omega+i\Gamma)^2-\omega_c^2}{\omega(\omega+i\Gamma)} \end{bmatrix}.$$

Here, the first term is the background permittivity as taken from Ref. [32]. The second term takes into account the free-carrier contribution, which is sensitive to external magnetic field. Γ is the free-carrier relaxation rate, $\omega_c = eB/m^*$ is the cyclotron frequency, and $\omega_p = \sqrt{n_e e^2/(m^* \epsilon_0)}$ is the plasma frequency. For calculation, we use $n_e = 1.36 \times 10^{19}\,\mathrm{cm}^{-3}$, $\Gamma = 10^{12}\,\mathrm{s}^{-1}$, and $m^* = 0.08 m_e$.

The three systems have different symmetries. The first system is reciprocal and has a gray group $\mathcal{M}_1 = D_{4h} \otimes \{E, \mathcal{T}\}$. The second system, which is identical to that in Fig. 1, has a black-and-white group $\mathcal{M}_2 = D_{4h}(D_{2h})$. The last system has a black-and-white group $\mathcal{M}_3 = D_{4h}(C_{4h})$. Consequently,

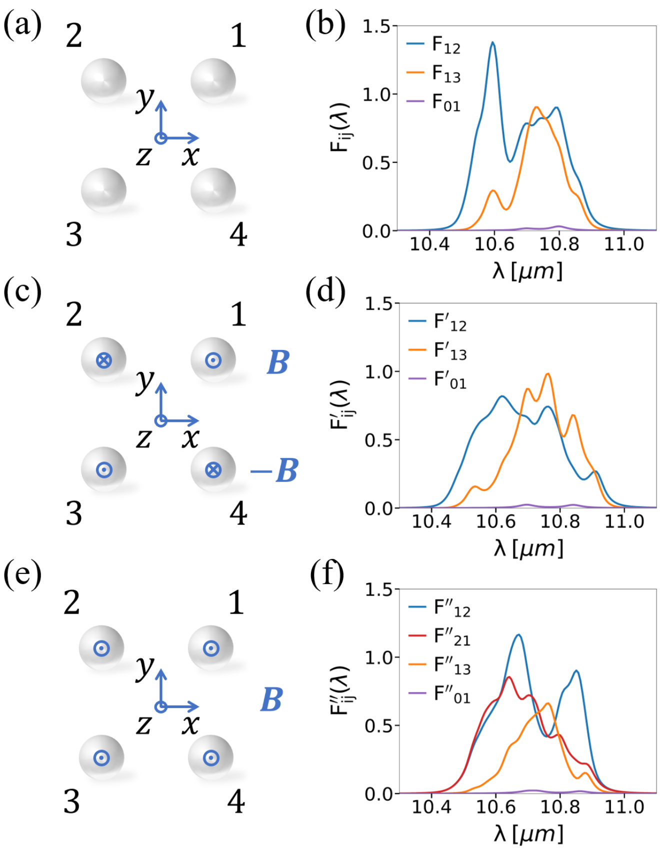

FIG. 2. (a) A reciprocal system that exhibits no persistent heat current. It consists of four InSb spheres with centers placed at the vertices of a square on the x-y plane under no external magnetic field. (b) The calculated transmission coefficient spectra $F_{ij}(\lambda)$ for the system in (a). Only the independent components are plotted. (c) A non-reciprocal system that exhibits no persistent heat current. It consists of the same spheres as (a), but under an external magnetic field along the z direction with alternating strength in the x-y plane: $B_1 = B_3 = -B_2 = -B_4 = 1$ T. (d) The calculated independent transmission coefficient spectra $F'_{ij}(\lambda)$ for the system in (c). (e) A non-reciprocal system that exhibits persistent heat current. It consists of the same spheres as (a), but under a uniform external magnetic field along the z direction: $B_1 = B_2 = B_3 = B_4 = 1$ T. (f) The calculated independent transmission coefficient spectra $F''_{ij}(\lambda)$ for the system in (e).

the three constrained $\mathcal{F}$ matrices have the following forms, respectively:

$$\mathcal{F}_1 = \begin{pmatrix} 0 & F_{01} & F_{01} & F_{01} & F_{01} \\ F_{01} & 0 & F_{12} & F_{13} & F_{12} \\ F_{01} & F_{12} & 0 & F_{12} & F_{13} \\ F_{01} & F_{13} & F_{12} & 0 & F_{12} \\ F_{01} & F_{12} & F_{13} & F_{12} & 0 \end{pmatrix}, \quad (24)$$

$$\mathcal{F}_2 = \begin{pmatrix} 0 & F'_{01} & F'_{01} & F'_{01} & F'_{01} \\ F'_{01} & 0 & F'_{12} & F'_{13} & F'_{12} \\ F'_{01} & F'_{12} & 0 & F'_{12} & F'_{13} \\ F'_{01} & F'_{13} & F'_{12} & 0 & F'_{12} \\ F'_{01} & F'_{12} & F'_{13} & F'_{12} & 0 \end{pmatrix}, \quad (25)$$

$$\mathcal{F}_3 = \begin{pmatrix} 0 & F''_{01} & F''_{01} & F''_{01} & F''_{01} \\ F''_{01} & 0 & F''_{12} & F''_{13} & F''_{21} \\ F''_{01} & F''_{21} & 0 & F''_{12} & F''_{13} \\ F''_{01} & F''_{13} & F''_{21} & 0 & F''_{12} \\ F''_{01} & F''_{12} & F''_{13} & F''_{21} & 0 \end{pmatrix}. \qquad (26)$$

We make several observations. The first system in Fig. 2(a) is reciprocal and thus cannot exhibit a persistent heat current as expected ($\mathcal{F}_1 = \mathcal{F}_1^T$). Moreover, $\mathcal{F}_1$ has only three independent components, F_{01}, F_{12}, and F_{13}, as required by symmetry. The second system in Fig. 2(c), even though it is non-reciprocal, cannot exhibit a persistent heat current either since $\mathcal{F}_2 = \mathcal{F}_2^T$. Interestingly, $\mathcal{F}_2$ has exactly the same form as $\mathcal{F}_1$. This highlights the possibility that many-body systems with completely different symmetries can exhibit the same qualitative behavior in radiative heat transfer. The last system in Fig. 2(e) is non-reciprocal and can hold a persistent heat current as $\mathcal{F}_3 \neq \mathcal{F}_3^T$. $\mathcal{F}_3$ has four independent components $F''_{01}, F''_{12}, F''_{21}$ and F''_{13}.

We numerically verify these observations by calculating the transmission coefficient spectra $F_{ij}(\lambda)$ in Figs. 2(b), 2(d) and 2(f), corresponding to the systems in Figs. 2(a), 2(c) and 2(e), respectively. We plot all the independent components of the $\mathcal{F}$ matrices and verify that the other components indeed obey the relations in Eqs. (24)–(26). We see there is no persistent heat (e.g., $F_{12} = F_{21}$ and $F'_{12} = F'_{21}$) in the first and second systems, while there is persistent heat current in the last system ($F''_{12} \neq F''_{21}$).

As another application of our theory, we have the following proposition: if a system has a colored symmetry $R = \mathcal{T}A$ where $\mathbb{P}_A = \mathbb{I}$ is an identity permutation, it cannot exhibit a persistent heat current. This is because if there is such an element R, its constraint on $\mathcal{F}$ [Eq. (13)] is

$$\mathbf{I}\mathcal{F}\mathbf{I} = \mathcal{F}^T, \qquad (27)$$

where $\mathbf{I}$ is the identity matrix. Therefore, $\mathcal{F} = \mathcal{F}^T$, which precludes persistent heat current.

As an example, we consider clusters of gyrotropic spheres with their centers lying on a plane under an external magnetic field parallel to that plane. The spheres can be of different sizes, and the magnetic field can be inhomogeneous. One typical system is depicted in Fig. 3(a). Such systems have $\mathcal{T}m$ symmetry, where m is the mirror operation with respect to the plane passing through the centers, and $\mathbb{P}_m = \mathbb{I}$. Therefore, such systems cannot exhibit a persistent heat current. In contrast, clusters of randomly positioned gyrotropic particles subjected to magnetic field with random magnitudes and directions in general do not have $\mathcal{T}m$ symmetry and can therefore exhibit persistent heat current.

We now provide numerical evidence. For the convenience of simulation, instead of Fig. 3(a), we consider the system in Fig. 3(b), where the four InSb spheres are the same size, with a radius of 100 nm, and the magnetic field is inhomogeneous but along the same (y) direction. The centers of the spheres are placed on the x-y plane with randomly chosen coordinates (units are nanometers): $(484, -146)$, $(-167, 313)$, $(174, -252)$, and $(-303, -41)$. The spheres are under randomly assigned magnetic fields $B_1 = -0.256$ T, $B_2 = -1.896$ T, $B_3 = 0.199$ T, and $B_4 = -0.259$ T. Such a system has a black-and-white magnetic group $C_{1h}(C_1) = \{E, \mathcal{T}m\}$. The only constraints that can be deduced are $F_{ij} = F_{ji}$. Thus, there are ten independent components of the $\mathcal{F}$ matrix. We numerically calculate the transmission coefficient spectra $F_{ij}(\lambda)$. Figure 3(c) plots the transmission coefficients between bodies. Here we use a logarithmic scale to accommodate the different magnitudes. Figure 3(d) plots the transmission coefficients between the bodies and the environment. These two plots confirm that $F_{ij}(\lambda) = F_{ji}(\lambda)$, $0 \leqslant i < j \leqslant 4$, and there are indeed ten independent components.

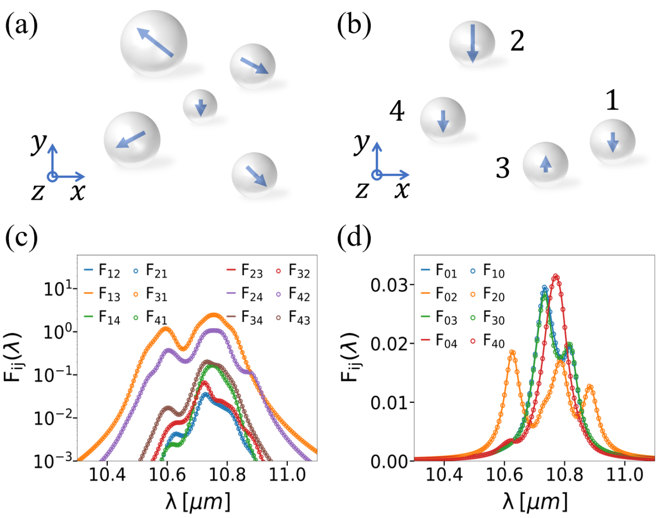

FIG. 3. (a) A cluster of magneto-optical spheres with their centers placed on the x-y plane under an inhomogeneous external magnetic field parallel to the x-y plane. Such a system exhibits no persistent heat current. (b) A cluster of four InSb spheres with their centers placed on the x-y plane. The spheres are the same size, with a radius of 100 nm. The spheres experience different magnetic fields along the y direction: $B_1 = -0.256$ T, $B_2 = -1.896$ T, $B_3 = 0.199$ T, $B_4 = -0.259$ T. (c) The transmission coefficient spectra between the spheres. The logarithmic scale is used to accommodate the different magnitudes. (d) The transmission coefficient spectra between the spheres and the environment. (c) and (d) show $F_{ij}(\lambda) = F_{ji}(\lambda)$; thus, there is no persistent heat current.

IV. DISCUSSION AND CONCLUSION

Throughout the paper, we have used clusters of spherical particles subjected to local magnetic field as concrete examples to illustrate the theory. Our theory, which is based on the symmetry argument alone, is not restricted to either spherical particles or subwavelength particles but is applicable to arbitrary many-body systems, which can include non-spherical objects or objects with sizes comparable to or larger than the relevant thermal wavelengths.

In conclusion, we have studied the constraints on many-body radiative heat transfer imposed by symmetry and the second law of thermodynamics. We show that the symmetry of these systems in general can be described by a magnetic group. And the constraints of the magnetic group on heat transfer can be derived using the generalized reciprocity theorem. We also showed that the second law of thermodynamics provides additional constraints in the form of a nodal conservation law of heat flow at equilibrium. As an application of the theory, we provided a systematic approach to determine the existence of a persistent heat current in arbitrary many-body systems. Our work should be useful in providing theoretical guidance for exploring novel effects of radiative heat transfer in complex many-body systems and networks.

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APPENDIX: PROOF OF EQUATION (11)

We prove Eq. (11) using the generalized reciprocity theorem [31]. First, we briefly review Lorentz reciprocity [33]. Consider two sources $\boldsymbol{J_a}$ and $\boldsymbol{J_b}$, which produce fields $\boldsymbol{E}_a$ and $\boldsymbol{E}_b$, respectively. The Lorentz reciprocity theorem states that for a reciprocal medium that satisfies $C = \widetilde{C}$,

$$\iiint_V dV \boldsymbol{E}_a \cdot \boldsymbol{J}_b = \iiint_V dV \boldsymbol{E}_b \cdot \boldsymbol{J}_a, \tag{A1}$$

where the integration is over the volume that contains sources a and b.

The reciprocity theorem stated above can be generalized to arbitrary media [31,34]. Consider two sources $\boldsymbol{J_a}$ and $\boldsymbol{J_b}$, producing fields $\boldsymbol{E}_a$ and $\boldsymbol{E}_b$ in the original medium C and fields $\widetilde{\boldsymbol{E}}_a$ and $\widetilde{\boldsymbol{E}}_b$ in the complementary medium $\widetilde{C}$, respectively. Then the generalized reciprocity theorem states

$$\iiint_V dV \boldsymbol{E}_a \cdot \boldsymbol{J}_b = \iiint_V dV \widetilde{\boldsymbol{E}}_b \cdot \boldsymbol{J}_a, \tag{A2}$$

integrating over the volume that contains sources a and b. Note it reduces to the ordinary reciprocity theorem for the reciprocal medium.

The generalized reciprocity theorem requires the dyadic Green's functions for the corresponding bodies in the original and complementary systems to be the transpose of each other:

$$\widetilde{\mathbb{G}}_\alpha(\boldsymbol{r}, \boldsymbol{r}') = \mathbb{G}_\alpha^T(\boldsymbol{r}', \boldsymbol{r}), \tag{A3}$$

where body α can be a composite consisting of multiple bodies.

Consequently, the $\mathbb{T}$ operators for the corresponding bodies in the two systems are also the transpose of each other:

$$\widetilde{\mathbb{T}}_\alpha(\boldsymbol{r}, \boldsymbol{r}') = \mathbb{T}_\alpha^T(\boldsymbol{r}', \boldsymbol{r}). \tag{A4}$$

This follows from the definition of $\mathbb{T}$ [27,35],

$$\mathbb{G}_\alpha = \mathbb{G}_0 + \mathbb{G}_0 \mathbb{T}_\alpha \mathbb{G}_0, \tag{A5}$$

$$\widetilde{\mathbb{G}}_\alpha = \mathbb{G}_0 + \mathbb{G}_0 \widetilde{\mathbb{T}}_\alpha \mathbb{G}_0, \tag{A6}$$

where $\mathbb{G}_0$, being the free-space Green's function, is symmetric, i.e. $\mathbb{G}_0(\boldsymbol{r}, \boldsymbol{r}') = \mathbb{G}_0^T(\boldsymbol{r}', \boldsymbol{r})$.

Now we proceed to prove that the exchange matrices $\mathcal{F}$ and $\mathcal{F}'$ of the original and complementary systems are the transpose of each other [Eq. (11)]. Without loss of generality, we consider heat exchange between bodies 1 and 2 in the original many-body system, while body 3 includes all bodies other than 1 and 2. We label the corresponding bodies in the complementary system $1', 2', 3'$. In the derivation below, we follow a procedure similar to that in the Supplemental Material of Ref. [21] and suppress the parameters of the operators for clarity.

The spectral transmission coefficient for heat transfer to body 2 due to thermal noise of body 1 is

$$F_{1\to 2}(\omega) = 4\,\mathrm{Tr}[\mathbb{Q}_2 \mathbb{W}_{21} \mathbb{R}_1 \mathbb{W}_{21}^\dagger], \tag{A7}$$

while that to body $2'$ due to thermal noise of body $1'$ is

$$\begin{aligned}\widetilde{F}_{2'\to 1'}(\omega) &= 4\,\mathrm{Tr}[\widetilde{\mathbb{Q}}_1 \widetilde{\mathbb{W}}_{12} \widetilde{\mathbb{R}}_2 \widetilde{\mathbb{W}}_{12}^\dagger] \\ &= 4\,\mathrm{Tr}\left[\widetilde{\mathbb{W}}_{12}^* \widetilde{\mathbb{R}}_2^T \widetilde{\mathbb{W}}_{12}^T \widetilde{\mathbb{Q}}_1^T\right] \\ &= 4\,\mathrm{Tr}\left[\widetilde{\mathbb{R}}_2^T \widetilde{\mathbb{W}}_{12}^T \widetilde{\mathbb{Q}}_1^T \widetilde{\mathbb{W}}_{12}^*\right],\end{aligned} \tag{A8}$$

where we have performed the transposition of the matrix product in the second line and cyclic permutation in the third line. Here

$$\mathbb{R}_\alpha = \mathbb{G}_0\left[\frac{\mathbb{T}_\alpha - \mathbb{T}_\alpha^\dagger}{2i} - \mathbb{T}_\alpha \,\mathrm{Im}(\mathbb{G}_0)\mathbb{T}_\alpha^\dagger\right]\mathbb{G}_0^\dagger, \tag{A9}$$

$$\begin{aligned}\widetilde{\mathbb{R}}_\alpha^T &= \left\{\mathbb{G}_0\left[\frac{\widetilde{\mathbb{T}}_\alpha - \widetilde{\mathbb{T}}_\alpha^\dagger}{2i} - \widetilde{\mathbb{T}}_\alpha \,\mathrm{Im}(\mathbb{G}_0)\widetilde{\mathbb{T}}_\alpha^\dagger\right]\mathbb{G}_0^\dagger\right\}^T \\ &= \mathbb{G}_0^\dagger\left[\frac{\mathbb{T}_\alpha - \mathbb{T}_\alpha^\dagger}{2i} - \mathbb{T}_\alpha^\dagger \,\mathrm{Im}(\mathbb{G}_0)\mathbb{T}_\alpha\right]\mathbb{G}_0,\end{aligned} \tag{A10}$$

$$\mathbb{Q}_\alpha = \mathbb{G}_0^\dagger\left[\frac{\mathbb{T}_\alpha - \mathbb{T}_\alpha^\dagger}{2i} - \mathbb{T}_\alpha^\dagger \,\mathrm{Im}(\mathbb{G}_0)\mathbb{T}_\alpha\right]\mathbb{G}_0, \tag{A11}$$

$$\begin{aligned}\widetilde{\mathbb{Q}}_\alpha^T &= \left\{\mathbb{G}_0^\dagger\left[\frac{\widetilde{\mathbb{T}}_\alpha - \widetilde{\mathbb{T}}_\alpha^\dagger}{2i} - \widetilde{\mathbb{T}}_\alpha^\dagger \,\mathrm{Im}(\mathbb{G}_0)\widetilde{\mathbb{T}}_\alpha\right]\mathbb{G}_0\right\}^T \\ &= \mathbb{G}_0\left[\frac{\mathbb{T}_\alpha - \mathbb{T}_\alpha^\dagger}{2i} - \mathbb{T}_\alpha \,\mathrm{Im}(\mathbb{G}_0)\mathbb{T}_\alpha^\dagger\right]\mathbb{G}_0^\dagger,\end{aligned} \tag{A12}$$

where $\alpha = 1, 2$, and we have used Eq. (A4) to simplify Eqs. (A10) and (A12). Therefore,

$$\widetilde{\mathbb{R}}_\alpha^T = \mathbb{Q}_\alpha, \quad \widetilde{\mathbb{Q}}_\alpha^T = \mathbb{R}_\alpha, \tag{A13}$$

and

$$\mathbb{W}_{21} = \mathbb{G}_0^{-1}\frac{1}{1 - \mathbb{G}_0\mathbb{T}_3\mathbb{G}_0\mathbb{T}_2}(1 + \mathbb{G}_0\mathbb{T}_3)\frac{1}{1 - \mathbb{G}_0\mathbb{T}_1[(1 + \mathbb{G}_0\mathbb{T}_2)\frac{1}{1 - \mathbb{G}_0\mathbb{T}_3\mathbb{G}_0\mathbb{T}_2}(1 + \mathbb{G}_0\mathbb{T}_3) - 1]}, \tag{A14}$$

$$\widetilde{\mathbb{W}}_{12} = \mathbb{G}_0^{-1}\frac{1}{1 - \mathbb{G}_0\widetilde{\mathbb{T}}_3\mathbb{G}_0\widetilde{\mathbb{T}}_1}(1 + \mathbb{G}_0\widetilde{\mathbb{T}}_3)\frac{1}{1 - \mathbb{G}_0\widetilde{\mathbb{T}}_2[(1 + \mathbb{G}_0\widetilde{\mathbb{T}}_1)\frac{1}{1 - \mathbb{G}_0\widetilde{\mathbb{T}}_3\mathbb{G}_0\widetilde{\mathbb{T}}_1}(1 + \mathbb{G}_0\widetilde{\mathbb{T}}_3) - 1]} \tag{A15}$$

$$= \mathbb{G}_0^{-1} \frac{1}{1 - [(1 + \mathbb{G}_0 \widetilde{\mathbb{T}}_3) \frac{1}{1 - \mathbb{G}_0 \widetilde{\mathbb{T}}_2 \mathbb{G}_0 \widetilde{\mathbb{T}}_3} (1 + \mathbb{G}_0 \widetilde{\mathbb{T}}_2) - 1] \mathbb{G}_0 \widetilde{\mathbb{T}}_1} (1 + \mathbb{G}_0 \widetilde{\mathbb{T}}_3) \frac{1}{1 - \mathbb{G}_0 \widetilde{\mathbb{T}}_2 \mathbb{G}_0 \widetilde{\mathbb{T}}_3}$$

$$= \frac{1}{1 - [(1 + \widetilde{\mathbb{T}}_3 \mathbb{G}_0) \frac{1}{1 - \widetilde{\mathbb{T}}_2 \mathbb{G}_0 \widetilde{\mathbb{T}}_3 \mathbb{G}_0} (1 + \widetilde{\mathbb{T}}_2 \mathbb{G}_0) - 1] \widetilde{\mathbb{T}}_1 \mathbb{G}_0} (1 + \widetilde{\mathbb{T}}_3 \mathbb{G}_0) \frac{1}{1 - \widetilde{\mathbb{T}}_2 \mathbb{G}_0 \widetilde{\mathbb{T}}_3 \mathbb{G}_0} \mathbb{G}_0^{-1},$$

$$\widetilde{\mathbb{W}}_{12}^T = \mathbb{G}_0^{-1} \frac{1}{1 - \mathbb{G}_0 \mathbb{T}_3 \mathbb{G}_0 \mathbb{T}_2} (1 + \mathbb{G}_0 \mathbb{T}_3) \frac{1}{1 - \mathbb{G}_0 \mathbb{T}_1 [(1 + \mathbb{G}_0 \mathbb{T}_2) \frac{1}{1 - \mathbb{G}_0 \mathbb{T}_3 \mathbb{G}_0 \mathbb{T}_2} (1 + \mathbb{G}_0 \mathbb{T}_3) - 1]}, \tag{A16}$$

where in Eq. (A15) we have used Eq. (11) in the Supplemental Material of [21] to get the second line and rearranged the terms to get the third line. We transpose Eq. (A15) to obtain Eq. (A16). Therefore,

$$\widetilde{\mathbb{W}}_{12}^T = \mathbb{W}_{21}, \quad \widetilde{\mathbb{W}}_{12}^* = \mathbb{W}_{21}^\dagger. \tag{A17}$$

Using Eq. (A13) and Eq. (A17), Eq. (A8) becomes

$$\begin{aligned} \widetilde{F}_{2' \to 1'}(\omega) &= 4\,\mathrm{Tr}\left[\widetilde{\mathbb{R}}_2^T \widetilde{\mathbb{W}}_{12}^T \widetilde{\mathbb{Q}}_1^T \widetilde{\mathbb{W}}_{12}^*\right] \\ &= 4\,\mathrm{Tr}[\mathbb{Q}_2 \mathbb{W}_{21} \mathbb{R}_1 \mathbb{W}_{21}^\dagger]. \end{aligned} \tag{A18}$$

Comparing Eq. (A18) and Eq. (A7), we get

$$\widetilde{F}_{2' \to 1'}(\omega) = F_{1 \to 2}(\omega). \tag{A19}$$

Since bodies 1 and 2 are arbitrarily chosen, we have proved

$$\widetilde{\mathcal{F}} = \mathcal{F}^T. \tag{A20}$$

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