Majorization theory for unitary control of optical absorption and emission: Supplemental Material

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I. MAJORIZATION

Here we provide a brief introduction to the theory of majorization. We refer readers to Refs. [1, 2] for more details. Majorization is a pre-order on the set of n-dimensional real vectors:

Definition I.1 (Majorization)**.** For $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}, \qquad k = 1, 2, \dots, n-1; \tag{S.1}$$

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i, \tag{S.2}$$

we say that $\boldsymbol{x}$ is *majorized* by $\boldsymbol{y}$, written as $\boldsymbol{x} \prec \boldsymbol{y}$.

Properties. Some simple properties of majorization follow from its definition:

1. The positions of the vector components are unimportant for majorization.

2. Majorization $\prec$ is a transitive binary relation:

$$\boldsymbol{x} \prec \boldsymbol{y}, \quad \boldsymbol{y} \prec \boldsymbol{z} \implies \boldsymbol{x} \prec \boldsymbol{z}. \tag{S.3}$$

3. $\boldsymbol{x} \not\prec \boldsymbol{y} \not\Longrightarrow \boldsymbol{y} \prec \boldsymbol{x}$, e.g., $\boldsymbol{x} = (6, 3, 2)$, $\boldsymbol{y} = (5, 5, 1)$.

Majorization has a simple geometric interpretation:

Theorem I.1 (Rado, 1952 [3])**.** *For a given $\boldsymbol{y} \in \mathbb{R}^n$, the set $\{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \prec \boldsymbol{y} \,\}$ is the convex hull of points obtained by permuting the components of $\boldsymbol{y}$.*

Majorization has important applications in matrix theory [1]:

Theorem I.2 (Schur, 1923 [4])**.** *Let $H \in M_n$ be Hermitian. Then*

$$\boldsymbol{d}(H) \prec \boldsymbol{\lambda}(H). \tag{S.4}$$

Theorem I.3 (Horn, 1954 [5])**.** *Let $\boldsymbol{d} = (d_1, \dots, d_n)$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. If $\boldsymbol{d} \prec \boldsymbol{\lambda}$, there exists a real symmetric matrix $H \in M_n$ with diagonal entries $\boldsymbol{d}$ and eigenvalues $\boldsymbol{\lambda}$.*

Remark. Theorem I.3 is stronger than the converse of Theorem I.2. It guarantees the existence of a real symmetric matrix rather than merely a Hermitian matrix.

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II. CONJUGATE-NORMAL MATRICES AND SELF-CONCOMMUTATORS

Here we review the concepts of conjugate-normal matrices and self-concommutators. First, we recall the more familiar concept of normal matrices:

Definition II.1 (Normal matrices [6])**.** A matrix $A \in M_n$ is normal if $AA^\dagger = A^\dagger A$.

The class of normal matrices includes the unitary, Hermitian, skew Hermitian, real orthogonal, real symmetric, and real skew-symmetric matrices.

Similarly, we introduce the concept of conjugate normal matrices:

Definition II.2 (Conjugate normal matrices [6])**.** A matrix $A \in M_n$ is conjugate normal if $AA^\dagger = (A^\dagger A)^*$, or equivalently, $A^* A^T = A^\dagger A$.

The class of conjugate normal matrices includes the unitary, complex symmetric, and complex skew-symmetric matrices. See Ref. [7] for a survey on conjugate normal matrices.

The self-concommutator of a matrix measures the degree of its conjugate abnormality:

Definition II.3 (Self-concommutators)**.** The self-concommutator of a matrix $A \in M_n$ is

$$A^\dagger A - A^* A^T. \tag{S.5}$$

A is conjugate normal if and only if the self-concommutator of A is a zero matrix. The concepts of conjugate normality and self-concommutator arise naturally in the study of unitary congruence: $A \to U^T A U$ with U unitary [6], because conjugate normality and the eigenvalues of self-concommutators are invariant under unitary congruence.

III. A COHERENT PERFECT ABSORBER IN THE LANGUAGE OF UNITARY CONTROL

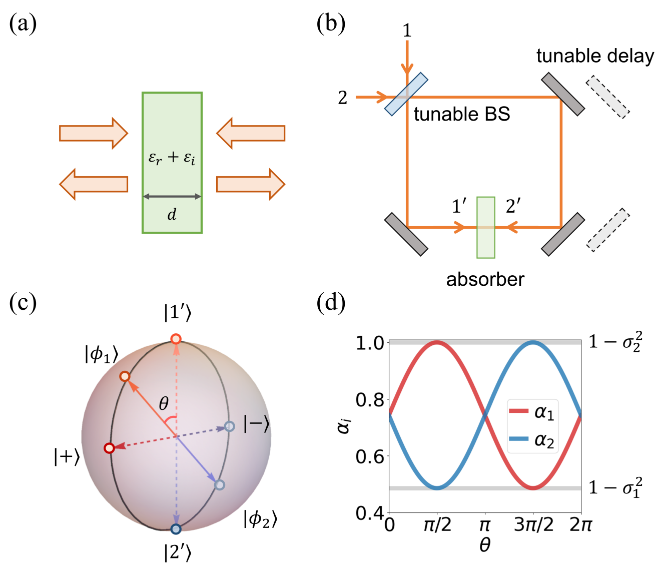

FIG. S1. A 2-port coherent perfect absorber. (a) A dielectric slab. (b) Setup scheme. (c) Bloch sphere representation. (d) Unitary control of absorptivity.

Here we illustrate the concept of unitary control with a simple 2-port coherent perfect absorber.

We consider a lossy dielectric slab with a thickness of $d = 1\,\mu\text{m}$ and a dielectric constant $\varepsilon = 6.0 + 0.658i$ (Fig. S1a). It is illuminated by linearly polarized light with a wavelength of $\lambda = 977.9\,\text{nm}$ in the normal direction. The structure is reciprocal and mirror-symmetric. It is characterized by a 2×2 scattering matrix

$$S = \begin{pmatrix} r & t \\ t & r \end{pmatrix} = \begin{pmatrix} -0.358 - 0.013i & -0.358 - 0.013i \\ -0.358 - 0.013i & -0.358 - 0.013i \end{pmatrix}, \tag{S.6}$$

which is determined using the transfer matrix method [8].

Now we study the unitary control of the absorptivity $\boldsymbol{\alpha}$. (Since the system is reciprocal, the emissivity $\boldsymbol{e} = \boldsymbol{\alpha}$ and the nonreciprocal contrast $\boldsymbol{\delta} = 0$.) Fig. S1b shows a standard experimental setup [9]. Light is incident from port 1 (or 2), split by a tunable beamsplitter (BS), redirected by movable mirrors, then incident on the slab from both sides with a modal profile

$$|\phi_i\rangle = \sum_{j=1}^{2} U_{ji} \, |j'\rangle \,, \quad i = 1, 2. \tag{S.7}$$

Here $U \in U(2)$, and $|1'\rangle$ and $|2'\rangle$ denote the modes at each side of the slab. One can change U by tuning the splitting ratio and the mirror delay. Coherent perfect absorption occurs if $|\phi_1\rangle$ or $|\phi_2\rangle$ is totally absorbed for a given U.

We can represent the unitary control scheme above on a Bloch sphere of the input state (Fig. S1c). Here, a pair of antipodal points on the sphere denotes a set of orthonormal bases. The effect of a unitary transformation U amounts to rotating the original set of bases $|1'\rangle$ and $|2'\rangle$ into another set of bases $|\phi_1\rangle$ and $|\phi_2\rangle$.

Here as concrete examples, we consider a specific family of unitary transformations:

$$U(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \qquad \theta \in [0, 2\pi] \,, \tag{S.8}$$

which corresponds to a rotation about the y-axis through the angle θ [10]. By Eq. (S.7),

$$|\phi_1(\theta)\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \qquad |\phi_2(\theta)\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}. \tag{S.9}$$

As θ runs over $[0, 2\pi]$, $|\phi_1(\theta)\rangle$ and $|\phi_2(\theta)\rangle$ encircle the great circle on the xz plane once. We notice two special cases:

$$|\phi_1(0)\rangle = |1'\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\phi_2(0)\rangle = |2'\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \tag{S.10}$$

$$|\phi_1(\frac{\pi}{2})\rangle \equiv |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |\phi_2(\frac{\pi}{2})\rangle \equiv |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{S.11}$$

Note that $|+\rangle$ is even while $|-\rangle$ is odd with respect to the mirror plane.

We calculate the absorptivity of $|\phi_1(\theta)\rangle$ and $|\phi_2(\theta)\rangle$ and plot in Fig. S1d. We observe that

$$\alpha_1(0) = \alpha_2(0) = 0.743; \qquad \alpha_1\left(\frac{\pi}{2}\right) = 1.000, \;\; \alpha_2\left(\frac{\pi}{2}\right) = 0.485. \tag{S.12}$$

So the system exhibits coherent perfect absorption at $\theta = \frac{\pi}{2}$ (and also $\theta = \frac{3\pi}{2}$).

These results are consistent with our majorization theory. From Eq. (S.6), we obtain

$$\boldsymbol{\sigma}(S) = (0.717, 0.000)^T, \quad \mathbf{1} - \boldsymbol{\sigma}^2(S) = (0.485, 1.000)^T \tag{S.13}$$

Our theory predicts that

$$\{\boldsymbol{\alpha}[U]\} = \{\, \boldsymbol{u} \in \mathbb{R}^n \mid \boldsymbol{u} \prec \mathbf{1} - \boldsymbol{\sigma}^2(S) \,\} \,, \tag{S.14}$$

that is,

$$0.485 \le \alpha_i(\theta) \le 1.00; \tag{S.15}$$

$$\alpha_1(\theta) + \alpha_2(\theta) = 1.485, \qquad \forall\, \theta \in [0, 2\pi] \,. \tag{S.16}$$

In Fig. S1d, we indicate the predicted bounds of α_i as the two horizontal gray lines. We confirm Eqs. (S.15) and (S.16). We also see that $\alpha_i(\theta)$ covers the whole interval of $[0.485, 1.000]$ as θ runs over $[0, 2\pi]$. All these results agree with our theoretical prediction.

IV. CHU'S ALGORITHM AND FICKUS' ALGORITHM

The initial proofs of Horn's Theorem are non-constructive [2, 5]. Constructing a matrix with prescribed diagonal entries and eigenvalues is a challenging inverse problem [11]. Chu (1995) provided an algorithm to obtain a real symmetric matrix with prescribed diagonal entries and eigenvalues [12]. Fickus (2013) provided an algorithm to obtain all Hermitian matrices with prescribed diagonal entries and eigenvalues [13]. Here we briefly review these algorithms:

Algorithm IV.1 (Chu, 1995 [12])**.** *One can construct a real symmetric matrix with prescribed eigenvalues $\boldsymbol{\lambda}$ and diagonal entries $\boldsymbol{d}$ by integrating the differential equations:*

$$\dot{X} = [X, [\mathrm{diag}(X) - \mathrm{diag}(\boldsymbol{d}), X]] \tag{S.17}$$

till equilibrium from a starting point $X_0 = Q^T \Lambda Q$ with Q a random orthogonal matrix and $\Lambda = \mathrm{diag}(\boldsymbol{\lambda})$. Here $[A, B] \equiv AB - BA$ is the Lie bracket, $\mathrm{diag}(X)$ is the diagonal matrix with the same diagonal entries of X, and $\mathrm{diag}(\boldsymbol{d})$ is the diagonal matrix with diagonal entries $\boldsymbol{d}$. This algorithm always converges to a valid solution.

Algorithm IV.2 (Fickus, 2013 [13])**.** *One can construct all Hermitian matrices with prescribed eigenvalues $\boldsymbol{\lambda}$ and diagonal entries $\boldsymbol{d}$ using finite frame theory. The explicit steps can be found in Ref. [13].*

V. PROOF OF THE COMPLETENESS OF ALGORITHM 1

Here we prove the completeness of Algorithm 1: It can obtain the whole sets of $\{\, U[\boldsymbol{\alpha}_0] \,\}$, $\{\, U[\boldsymbol{e}_0] \,\}$, and $\{\, U[\boldsymbol{\delta}_0] \,\}$. We need the following theorem from matrix analysis:

Theorem V.1 (Ref. [6], p. 134.)**.** *Let $A \in M_n$ be normal and have distinct eigenvalues $\lambda_1, \ldots, \lambda_d$ with respective multiplicities $n_1, \ldots, n_d$. Let $\Lambda = \lambda_1 I_{n_1} \oplus \ldots \oplus \lambda_d I_{n_d}$, and suppose that $A = U \Lambda U^\dagger$ with $U \in U(n)$. Then*
(a) $A = V \Lambda V^\dagger$ for some $V \in U(n)$ if and only if there are matrices $W_1, \ldots, W_d$ with each $W_i \in U(n_i)$ such that $U = V(W_1 \oplus \ldots \oplus W_d)$.
(b) Two normal matrices are unitarily similar if and only if they have the same eigenvalues.

Now we can prove the key claim on Eq. (32) in Algorithm 1 as a corollary:

Corollary V.1.1. *Let $A \in M_n$ be normal and have distinct eigenvalues $\lambda_1, \ldots, \lambda_d$ with respective multiplicities $n_1, \ldots, n_d$. Let $\Lambda = \lambda_1 I_{n_1} \oplus \ldots \oplus \lambda_d I_{n_d}$, and suppose that $A = V \Lambda V^\dagger$ with $V \in U(n)$. Let $B = \tilde{V} \Lambda \tilde{V}^\dagger$. Then $B = U^\dagger A U$ for some $U \in U(n)$ if and only if there are matrices $W_1, \ldots, W_d$ with each $W_i \in U(n_i)$ such that*

$$U = V(W_1 \oplus \ldots W_p)\tilde{V}^\dagger. \tag{S.18}$$

Proof.

$$B = U^\dagger A U \iff B = U^\dagger V \Lambda V^\dagger U \iff \tilde{V} = U^\dagger V(W_1 \oplus \ldots \oplus W_d) \tag{S.19}$$

$$\iff U = V(W_1 \oplus \ldots \oplus W_d)\tilde{V}^\dagger, \tag{S.20}$$

where we have used Theorem V.1 in Eq. (S.19). □

Finally, we can prove the completeness of Algorithm 1. We prove for $\{\, U[\boldsymbol{\alpha}_0] \,\}$. The proofs for $\{\, U[\boldsymbol{e}_0] \,\}$ and $\{\, U[\boldsymbol{\delta}_0] \,\}$ proceed along the same lines.

Proposition.

$$\{\, U[\boldsymbol{\alpha}_0] \,\} = \bigcup_i \{\, U_i \,\}\,. \tag{S.21}$$

Proof.

$$\{\, U[\boldsymbol{\alpha}_0] \,\} \equiv \{\, U \in U(n) \mid \boldsymbol{d}(U^\dagger A U) = \boldsymbol{\alpha}_0 \,\} \tag{S.22}$$

$$= \bigcup_i \{\, U_i \in U(n) \mid U_i^\dagger A U_i = H_i, \text{ where } \boldsymbol{d}(H_i) = \boldsymbol{\alpha}_0 \,\} \tag{S.23}$$

$$= \bigcup_i \{\, U_i \in U(n) \mid U_i = V(W_1 \oplus \ldots W_p)V_i^\dagger \,\} \tag{S.24}$$

$$\equiv \bigcup_i \{\, U_i \,\}, \tag{S.25}$$

where we have used the completeness of Fickus' theorem in Eq. (S.23) and Corollary V.1.1 in Eq. (S.24). □

VI. NUMERICAL DEMONSTRATION OF ALGORITHM 2

To illustrate Algorithm 2, we consider a random 5-port lossy system characterized by

$$S = \begin{pmatrix} -0.16+0.15i & -0.17+0.25i & 0.35+0.08i & 0.00-0.21i & -0.19+0.01i \\ -0.13-0.24i & 0.01+0.08i & 0.05-0.25i & -0.20-0.25i & -0.03-0.03i \\ 0.08+0.07i & -0.24+0.14i & -0.10-0.27i & -0.14-0.04i & -0.16-0.08i \\ 0.10-0.11i & 0.18+0.04i & 0.33+0.22i & -0.05+0.38i & 0.06-0.21i \\ 0.10-0.12i & -0.17-0.18i & 0.05+0.01i & 0.03-0.01i & -0.16+0.35i \end{pmatrix}, \tag{S.26}$$

with

$$\boldsymbol{\sigma}(S) = \big(0.78,\ 0.61,\ 0.49,\ 0.36,\ 0.33\big)^T, \tag{S.27}$$

$$\mathbf{1} - \boldsymbol{\sigma}^2(S) = \big(0.39,\ 0.63,\ 0.76,\ 0.87,\ 0.89\big)^T, \tag{S.28}$$

$$\boldsymbol{c}(S) = \big(0.44,\ 0.11,\ -0.01,\ -0.17,\ -0.36\big)^T. \tag{S.29}$$

The task is to construct a $U[\boldsymbol{\alpha}_0]$, $U[\boldsymbol{e}_0]$, and $U[\boldsymbol{\delta}_0]$, with the randomly assigned goals

$$\boldsymbol{\alpha}_0 = \big(0.69,\ 0.75,\ 0.82,\ 0.66,\ 0.62\big)^T, \tag{S.30}$$

$$\boldsymbol{e}_0 = \big(0.65,\ 0.66,\ 0.78,\ 0.73,\ 0.72\big)^T, \tag{S.31}$$

$$\boldsymbol{\delta}_0 = \big(0.15,\ -0.02,\ -0.02,\ -0.27,\ 0.16\big)^T. \tag{S.32}$$

First, we check that

$$\boldsymbol{\alpha}_0 \prec \mathbf{1} - \boldsymbol{\sigma}^2(S), \quad \boldsymbol{e}_0 \prec \mathbf{1} - \boldsymbol{\sigma}^2(S), \quad \boldsymbol{\delta}_0 \prec \boldsymbol{c}(S), \tag{S.33}$$

so they are all attainable via unitary control. We use Algorithm 2 to obtain:

$$U[\boldsymbol{\alpha}_0] = \begin{pmatrix} 0.72+0.00i & 0.28+0.00i & 0.36+0.00i & -0.02+0.00i & -0.52+0.00i \\ -0.15-0.02i & -0.33-0.06i & 0.21+0.48i & -0.43-0.52i & -0.23+0.29i \\ 0.01+0.55i & -0.12-0.35i & 0.29-0.40i & 0.32-0.32i & 0.14+0.31i \\ 0.07+0.31i & -0.14+0.24i & -0.51+0.06i & 0.08+0.29i & -0.33+0.60i \\ -0.22-0.06i & 0.38+0.67i & 0.00-0.30i & 0.07-0.49i & -0.09+0.09i \end{pmatrix},$$

$$U[\boldsymbol{e}_0] = \begin{pmatrix} -0.15+0.00i & 0.06+0.00i & -0.01+0.00i & -0.91+0.00i & -0.38+0.00i \\ -0.09-0.51i & -0.06-0.26i & 0.26-0.24i & 0.19+0.27i & -0.44-0.49i \\ -0.58-0.41i & -0.15-0.14i & 0.06+0.59i & -0.04+0.02i & 0.31+0.09i \\ 0.15+0.08i & -0.43-0.73i & 0.19-0.26i & -0.15-0.16i & 0.22+0.23i \\ -0.24-0.35i & 0.34+0.21i & 0.21-0.62i & -0.11-0.02i & 0.41+0.24i \end{pmatrix},$$

$$U[\boldsymbol{\delta}_0] = \begin{pmatrix} 0.06+0.00i & 0.02-0.39i & -0.35-0.07i & 0.15+0.74i & -0.36+0.07i \\ -0.45-0.10i & 0.32+0.65i & 0.06+0.19i & 0.16+0.19i & -0.40+0.03i \\ 0.21-0.24i & 0.28-0.13i & 0.41+0.59i & 0.04+0.34i & 0.34-0.25i \\ 0.28-0.32i & 0.35-0.18i & -0.35+0.17i & 0.14-0.47i & -0.42-0.31i \\ 0.32+0.63i & -0.26+0.07i & 0.28+0.30i & -0.07-0.02i & -0.47-0.18i \end{pmatrix}.$$

We verify that these unitary transformations lead to desired $\boldsymbol{\alpha}_0$, $\boldsymbol{e}_0$, and $\boldsymbol{\delta}_0$, respectively.

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