16.4-1

We verify whether the 3 conditions from pp. 437 for a matroid hold.

1) S must be a finite set => already provided, thus true
2) The hereditary property:

   Assuming that we have B from I_k (including the empty set), where k is in the range [0, |S|], if we consider $A \subseteq B$ =>

   $|A| \leq |B|$ => A is also from I_k (i.e. for the same value of k)
3) The exchange property: we consider A and B from I_k where $|A|<|B|$.

   Then clearly there exist $x \in B-A$ and we can extend A□AU{x} such that:

   $|Aux| = |A|+1 \leq |B| = k$ => AU{x} remains in I_k

16.4-2

If we consider the columns of T being $c_1,\ldots,c_m$ and the subset $C=\{c_{i1},\ldots,c_{ik}\}$ being dependent then, as per the definition, it means there are k scalars $s_1,\ldots,s_k$ such that $Sum(s_i c_{ji})=0$, where i=1,k (of course, not all k scalars are equal to 0).

If we extend the subset C with more columns and set the corresponding scalar coefficient for them 0, then it is obvious the resulting new set of columns is also dependent.

Since it is “if and only if”, we must prove both directions.

=>

To prove this, we take the contrapositive: if a set is independent, then a subset of it must also be independent. This proves this direction.

<=

Now let’s consider X and Y two independent sets of columns such that $|X|<|Y|$. If we cannot extend X such that the new set of columns to remain independent => all elements of X are linear combinations of elements of Y. From here it would result that Y is a |X|-dimensional space, which cannot be case though. This means the initial made assumption is false, so we can extend X and since $|X|<|Y|$ and Y is in I => X is in I also => exchange property of matroid is satisfied => the system is a matroid as required.

16.4-4

The 1st condition for a matroid is satisfied because S is a finite set.

For 2nd property, let's consider A and B where B is in I and $A \subseteq B$.

In this case, for any integer i, we have that $A \cap S_i \subseteq B \cap S_i \Rightarrow |A \cap S_i| \leq |B \cap S_i| \leq 1$ (as per the way how the condition for I is defined in the hypothesis).

Since the last inequality holds for all i in the range [1,k] => M is closed under inclusion operation and property 2 is satisfied because A is in I as well.

For the exchange property: if we consider A and B such that $|B|=|A|+1$

Then $|B \cap S_x| = 1 = |A \cap S_x| + 1$ because $|A \cap S_x| = 0$ (for some index x).

Now let $b = B \cap S_x \Rightarrow$ b does not belong to A

Also, we have that $|(A \cup \{b\}) \cap S_y| = |A \cap S_y|$ for all $y \neq x$

But $|(A \cup \{b\}) \cap S_x| = 1 \Rightarrow A \cup \{b\}$ belongs to I

Since all 3 conditions are satisfied => M is a matroid

16.4-5

We take the function $f(x)=1+W-w(x)$, where W represents the biggest possible weight on an element.

Since w(x) is at most W => function f(x) is always strictly positive.

We assume that all maximal independent sets have size S (we use this below where we estimate w(A) against w(B) over function w(x)).

Now, let's say that A is maximal with respect to f(x) and B is minimal with respect to w(x).

We assume by contradiction that $w(A) > w(B)$ (because B is minimal).

Then, since $w(A) > w(B) \Rightarrow f(B) - (1+W)S > f(A) - (1+W)S \Rightarrow f(B) > f(A)$ which contradicts the assumption made A is maximal for f(x).

This means the initial assumption is false => $w(A)=w(B)$

This shows that the maximal independent set with the biggest weight in respect to f(x) must be a minimal independent set in respect to w(x).

17.3-1

Because we are given that $\Phi(D_i) \geq \Phi(D_0)\ \forall i \geq 0$, we just take $\mathbf{\Phi'(D_i) = \Phi(D_i) - \Phi(D_0)}$ which is 0 when i=0 and non-negative otherwise.

In this case, the amortized **cost** $\hat{c}_i' = \mathbf{c_i + \Phi'(D_i) - \Phi'(D_{i-1})}$

$$= \mathbf{c_i} + (\mathbf{\Phi(D_i) - \Phi(D_0)}) - (\mathbf{\Phi(D_{i-1}) - \Phi(D_0)})$$

$$= \mathbf{c_i + \Phi(D_i) - \Phi(D_0) - \Phi(D_{i-1}) + \Phi(D_0)}$$

$$= \mathbf{c_i + (\Phi(D_i) - \Phi(D_{i-1}))}$$

$$= \hat{c}_i \text{ as required}$$

17.3-3

If we choose the potential function to be **Sum(logi), where i=1,n**, then we notice that, because Sum(logi) =log1+log2+..+logn=log(n!) which is Big-Theta(nlogn), we can conclude that the insertion leads to a cost of 2logn (thus O(logn)), while extract-min has amortized cost 0. The amortized cost of the insertion can be explained by the fact that logn is spent to increase the size of the heap by 1 element. As for extract-min, the amortized cost is 0 because everything gets compensated (i.e. in what regards the total cost).

17.3-5

In this case, we assess the total cost for n operations.

First, using the usual identity, we can write that **cost** $\hat{c}_i = \mathbf{c_i + \Phi(D_i) - \Phi(D_{i-1})}$

From there results that $c_i = \hat{c}_i - \mathbf{\Phi(D_i) + \Phi(D_{i-1})}$

Now, when we write the identity from above for i=1,n and sum-up, after making the necessary simplifications, we obtain:

$\text{Sum}(c_i) = \text{Sum}(\hat{c}_i) - \mathbf{\Phi(D_n) + \Phi(D_0)}$ **where i=1,n in the two sums**

So, we obtain:

$\text{Sum}(c_i)$ = n-ct+b, where ct is a constant

But, since n=Big-Omega(b) => b=Big-O(n) => b<kn for a constant k whenever n>k

Therefore, $\text{Sum}(c_i)$ =n –ct+b < n+b < n+kn=(k+1)n =O(n) as required

17.4-3

First, if the loading factor α_i is in the range (1/3, ½], the amortized cost for the given potential function is $\hat{c}_i$ **=c_i+ Φ(i) - Φ(i-1) =1+ |2num_i-$size_i$| - |2num_{i-1}-$size_{i-1}$| =**

=1+|2num_i-$size_i$| -|2num_i+1+1-$size_i$| ≤ **1+|2num_i-$size_i$| -(|2num_i-$size_i$|** -2) =1+2=3

(we have applied the triangle inequality **|2num_i-$size_i$**+2|≥**|2num_i-$size_i$**|-2 =>

⇨ -**|2num_i-$size_i$**+2|≤ **-|2num_i-$size_i$**|+2)

This holds when no resizing is necessary. The same happens whenever the loading factor > ½.

Otherwise, when resizing is necessary, we have $size_{i-1}/3 = num_i+1$ because $num_{i-1} = num_i+1$ since a delete operation is done before moving the elements.

Because $size_{i-1}/3 = num_i+1$ => $size_i = 2/3 size_{i-1} = 2(num_i+1)$

Then the amortized cost $\hat{c}_i$ **=c_i+ Φ(i) - Φ(i-1) = (num_i+1)+ |2num_i-$size_i$| -|2num_{i-1}-$size_{i-1}$| =**

= (num_i+1)+ |2num_i – 2(num_i+1)| - |2num_i+2-3num_i-3| =

=(num_i+1)+ 2- |num_i-1| ≤**num_i+3 –num_i+1= 4** which is again upper bounded by a constant

(again, we have used the triangle inequality for |num_i-1|≥|num_i| -1 => -|num_i-1|≤ -num_i+1)