Assignment 7

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Problem 1

Derive the solution to Merton's Portfolio problem for the case of the $\log(\cdot)$ Utility function. The goal is to find the optimal allocation and consumption at each time to maximize lifetime-aggregated expected utility of consumption. Assumption:

1. Current wealth is $W_0 > 0$ and we'll live for T more years.
2. We can invest in a risky assets and a riskless asset
3. The risky asset has known normal distribution of returns
4. We are allowed to long or short any fractional quantities of assets
5. We can trade in continuous time $0 \leq t < T$, with no transaction costs
6. We can consume any fractional amount of wealth at any time
7. We assume that the consumption utility has constant relative risk-aversion.

Notation:

- Riskless asset: $dR_t = r \cdot R_t \cdot dt$
- Risky asset: $dS_t = \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dz_t$ where z_t stands for Geometric Brownian
- $\mu > r > 0, \sigma > 0$
- Denote wealth at time t as $W_t > 0$
- Denote the fraction of wealth allocated to risky asset denoted by $\pi\left(t, W_t\right)$ and the fraction of wealth in riskless asset will then be $1 - \pi\left(t, W_t\right)$
- Denote the wealth consumption per unit time denoted by $c\left(t, W_t\right) \geq 0$
- Utility of Consumption function $U(x) = \log(x)$

Formal Problem Statement:

- Write π_t, c_t instead of $\pi\left(t, W_t\right), c\left(t, W_t\right)$
- Process for Wealth W_t

$$dW_t = \left(\left(\pi_t \cdot (\mu - r) + r\right) \cdot W_t - c_t\right) \cdot dt + \pi_t \cdot \sigma \cdot W_t \cdot dz_t$$

- At any time t, we determine optimal $\left[\pi\left(t, W_t\right), c\left(t, W_t\right)\right]$ to maximize:

$$\mathbb{E}\left[\int_t^T e^{-\rho(s-t)} \cdot log(c_s) \cdot ds + e^{-\rho(T-t)} \cdot B(T) \cdot log(W_T) \mid W_t\right]$$

- $\rho \geq 0$ is the utility discount rate, $B(T) = \epsilon^\gamma$ is the bequest function $0 < \epsilon \ll 1)$

We can think this as a continuous-time stochastic control problem.

- State at time t is (t, W_t)
- Action at time t is $[\pi_t, c_t]$
- Reward per unit time at time t is $U(c_t) = log(c_t)$
- Return at time t is the accumulated discounted Reward:
$$\int_t^T e^{-\rho(s-t)} \cdot log(c_s) \cdot ds$$
- We aim to find Policy : $(t, W_t) \rightarrow [\pi_t, c_t]$ that maximizes the Expected Return
- Note: $c_t \geq 0$, but π_t is unconstrained

Value Function for a State (under a given policy) is the Expected Return from the State (when following the given policy). We will focus on the optimal value function:

$$V^*(t, W_t) = \max_{\pi, c} \mathbb{E}_t \left[\int_t^T e^{-\rho(s-t)} \cdot log(c_s) \cdot ds + e^{-\rho(T-t)} \cdot \epsilon^\gamma \cdot log(W_T) \right]$$

For $0 \leq t < t_1 < T$

$$V^*(t, W_t) = \max_{\pi, c} \mathbb{E}_t \left[\int_t^{t_1} e^{-\rho(s-t)} \cdot log(c_s) \cdot ds + e^{-\rho(t_1-t)} \cdot V^*(t_1, W_{t_1}) \right]$$

$$\Rightarrow e^{-\rho t} \cdot V^*(t, W_t) = \max_{\pi, c} \mathbb{E}_t \left[\int_t^{t_1} e^{-\rho s} \cdot log(c_s) \cdot ds + e^{-\rho t_1} \cdot V^*(t_1, W_{t_1}) \right]$$

We rewrite in stochastic differential form and have the HJB formulation

$$\max_{\pi_t, c_t} \mathbb{E}_t \left[d\left(e^{-\rho t} \cdot V^*(t, W_t)\right) + e^{-\rho t} \cdot log(c_t) \cdot dt \right] = 0$$
$$\Rightarrow \max_{\pi_t, c_t} \mathbb{E}_t \left[dV^*(t, W_t) + log(c_t) \cdot dt \right] = \rho \cdot V^*(t, W_t) \cdot dt$$

We use Ito's Lemma on dV^*, remove the dz_t term since it's a martingale, and divide throughout by dt to produce the HJB Equation in PDE form:

$$\max_{\pi_t, c_t} \left[\frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W} \left((\pi_t(\mu - r) \quad + r)W_t - c_t \right) + \frac{\partial^2 V^*}{\partial W^2} \cdot \frac{\pi_t^2 \sigma^2 W_t^2}{2} + log(c_t) \right]$$
$$= \rho \cdot V^*(t, W_t)$$

For simplicity:

$$\max_{\pi_t, c_t} \Phi(t, W_t; \pi_t, c_t) = \rho \cdot V^*(t, W_t)$$

Note that we are working with the constraints $W_t > 0, c_t \geq 0$ for $0 \leq t < T$
We find optimal π_t^*, c_t^* by taking partial derivatives of $\Phi(t, W_t; \pi_t, c_t)$ w.r.t π_t and c_t, and equate to 0 (F.O.C for Φ).

- Partial derivative of Φ with respect to π_t :
$$(\mu - r) \cdot \frac{\partial V^*}{\partial W_t} + \frac{\partial^2 V^*}{\partial W_t^2} \cdot \pi_t \cdot \sigma^2 \cdot W_t = 0$$
$$\Rightarrow \pi_t^* = \frac{-\frac{\partial V^*}{\partial W_t} \cdot (\mu - r)}{\frac{\partial^2 V^*}{\partial W_t^2} \cdot \sigma^2 \cdot W_t}$$
- Partial derivative of Φ with respect to c_t :
$$-\frac{\partial V^*}{\partial W_t} + \frac{1}{c_t^*} = 0$$
$$\Rightarrow c_t^* = \left(\frac{\partial V^*}{\partial W_t} \right)^{-1}$$