Assignment 9

Miao-Chin Yen

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Problem 2

Temporary (LPT) Price Impact Model formulated by Bertsimas and Lo. The LPT model is described below for all $t = 0, 1, \ldots T-1$:

$$P_{t+1} = P_t \cdot e^{Z_t}$$
$$X_{t+1} = \rho \cdot X_t + \eta_t$$
$$Q_t = P_t \cdot (1 - \beta \cdot N_t - \theta \cdot X_t)$$

where Z_t are independent and identically distributed random variables with mean μ_Z and variance σ_Z^2 for all $t = 0, 1, \ldots, T-1$, η_t are independent and identically distributed random variables with mean 0 for all $t = 0, 1, \ldots, T-1$, Z_t and η_t are independent of each other for all $t = 0, 1, \ldots, T-1$, and ρ, β, θ are given constants. The model assumes no risk-aversion (Utility function is the identity function) and so, the objective is to maximize the Expected Total Sales Proceeds over the finitehorizon up to time T (discount factor is 1). In your derivation, use the same methodology as we followed for the Simple Linear Price Impact Model with no Risk-Aversion.

Sol.

Denote Value Function for policy π as:

$$V_t^{\pi}\left((P_t, X_t, R_t)\right) = \mathbb{E}_{\pi}\left[\sum_{i=t}^{T} N_i \cdot P_i\left(1 - \beta \cdot N_i - \theta X_i\right) \mid (P_t, X_t, R_t)\right]$$

Denote Optimal Value Function as $V_t^*\left((P_t, X_t, R_t)\right) = \max_{\pi} V_t^{\pi}\left((P_t, X_t, R_t)\right)$

Optimal Value Function satisfies the Bellman Equation ($\forall\ 0 \le t < T-1$) :

$$V_t^*\left((P_t, X_t, R_t)\right) = \max_{N_t}\left\{N_t P_t\left(1 - \beta \cdot N_t - \theta X_t\right) + \mathbb{E}\left[V_{t+1}^*\left((P_{t+1}, X_{t+1}, R_{t+1})\right)\right]\right\}$$

$$V_{T-1}^*\left((P_{T-1}, X_{T-1}, R_{T-1})\right) = N_{T-1}P_{T-1}\left(1 - \beta N_{T-1} - \theta \cdot X_{T-1}\right) = R_{T-1}P_{T-1}\left(1 - \beta R_{T-1} - \theta \cdot X_{T-1}\right)$$

From the above, we can infer $V_{T-2}^*\left((P_{T-2}, X_{T-2}, R_{T-2})\right)$ as:

$$\max_{N_{T-2}}\left\{N_{T-2}P_{T-2}\left(1 - \beta N_{T-2} - \theta \cdot X_{T-2}\right) + \mathbb{E}\left[R_{T-1}P_{T-1}\left(1 - \beta R_{T-1} - \theta \cdot X_{T-1}\right)\right]\right\}$$

$$= \max_{N_{T-2}}\left\{N_{T-2}P_{T-2}\left(1 - \beta N_{T-2} - \theta \cdot X_{T-2}\right) + \mathbb{E}\left[(R_{T-2} - N_{T-2})P_{T-1}\left(1 - \beta(R_{T-2} - N_{T-2}) - \theta \cdot X_{T-1}\right)\right]\right\}$$

Since $P_{T-1} = P_{T-2} \cdot e^{Z_{T-2}}$ and $X_{T-1} = \rho \cdot X_{T-2} + \eta_{T-2}$, our objective function can be written as follows:

$$= \max_{N_{T-2}}\{N_{T-2}P_{T-2}\left(1 - \beta N_{T-2} - \theta \cdot X_{T-2}\right) +$$
$$\mathbb{E}\left[(R_{T-2} - N_{T-2}) \cdot P_{T-2} \cdot e^{Z_{T-2}}\left(1 - \beta(R_{T-2} - N_{T-2}) - \theta \cdot (\rho \cdot X_{T-2} + \eta_{T-2})\right)\right]\}$$

Since $\mathbb{E}[e^{Z_{T-2}}] = e^{\mu_z + \frac{\sigma_z^2}{2}}$ and $\mathbb{E}[\eta_{T-2}] = 0$, our objective function is as follows (denote $e^{\mu_z + \frac{\sigma_z^2}{2}}$ as q):

$$\max_{N_{T-2}}\{(1-q)N_{T-2}P_{T-2} - \beta \cdot (1+q)N_{T-2}^2 P_{T-2} - \theta(1 - \rho \cdot q)X_{T-2}P_{T-2}N_{T-2}$$
$$+q \cdot R_{T-2}P_{T-2} - \beta q R_{T-2}^2 P_{T-2} + 2\beta q N_{T-2}R_{T-2}P_{T-2} - \rho \cdot \theta q X_{T-2}R_{T-2}P_{T-2}$$

Take derivative with respect to N_{T-2}:

$$
\begin{aligned}
P_{T-2}(1-q) - 2\beta(1+q)N^*_{T-2}P_{T-2} - \theta X_{T-2}P_{T-2}(1-\rho q) + 2\beta q R_{T-2}P_{T-2=0} \\
\Rightarrow 2\beta(1+q)N^*_{T-2} = (1-q) + 2\beta q R_{T-2} - \theta(1-\rho q)X_{T-2} \\
\Rightarrow N^*_{T-2} = \frac{(1-q) + 2\beta q R_{T-2} - \theta(1-\rho q)X_{T-2}}{2\beta(1+q)} \\
= \frac{1-q}{2\beta(1+q)} + \frac{q}{1+q}R_{T-2} - \frac{\theta(1-\rho q)}{2\beta(1+q)}X_{T-2} \\
= c^{(1)}_{T-2} + c^{(2)}_{T-2}R_{T-2} + c^{(3)}_{T-2}X_{T-2}
\end{aligned}
$$

We then substitute N^*_{T-2} into $V^*_{T-2}((P_{T-2}, X_{T-2}, R_{T-2}))$

$$
\begin{aligned}
V^*_{T-2}\left(P_{T-2}, X_{T-2}, R_{T-2}\right) =& qP_{T-2}\left[a_{T-2} + b_{T-2}X_{T-2} + c_{T-2}X^2_{T-2}\right. \\
& \left. + d_{T-2}X_{T-2}R_{T-2} + e_{T-2}R_{T-2} + f_{T-2}R^2_{T-2}\right]
\end{aligned}
$$

where

$$
\begin{aligned}
a_{T-2} &= c^{(1)}_{T-2}\left(1+\beta c^{(1)}_{T-2}\right) - qc^{(1)}_{T-2}\left(1-\beta c^{(1)}_{T-2}\right) \\
b_{T-2} &= (1-q)c^{(3)}_{T-2} \\
c_{T-2} &= c^{(3)}_{T-2}\left(\beta c^{(3)}_{T-2} + \theta\right) - qc^{(3)}_{T-2}\left(\theta\rho - \beta c^{(3)}_{T-2}\right) \\
d_{T-2} &= \theta(1+\rho)c^{(2)}_{T-2} \\
e_{T-2} &= 2c^{(2)}_{T-2} \\
f_{T-2} &= \beta c^{(2)}_{T-2}
\end{aligned}
$$

Using backward induction,

$$
N^*_{T-k} = c^{(1)}_{T-k} + c^{(2)}_{T-k}R_{T-k} + c^{(3)}_{T-k}X_{T-k}
$$

where

$$
c^{(3)}_{T-k} = \frac{q\rho d_{k-1} - \theta}{2\left(\beta + qf_{k-1}\right)}, \quad c^{(2)}_{T-k} = \frac{qf_{k-1}}{\beta + qf_{k-1}}, \quad c^{(1)}_{T-k} = \frac{qe_{k-1} - 1}{2\left(\beta + qf_{k-1}\right)}.
$$

Hence

$$
\begin{aligned}
V_{T-k}\left(P_{T-k}, X_{T-k}, R_{T-k}\right) =& qP_{T-k}\left[a_{T-k} + b_{T-k}X_{T-k} + c_{T-k}X^2_{T-k}\right. \\
& \left. + d_{T-k}X_{T-k}R_{T-k} + e_{T-k}R_{T-k} + f_{T-k}R^2_{T-k}\right]
\end{aligned}
$$

where

$$
\begin{aligned}
a_{T-k} &= c^{(1)}_{T-k}\left(1+\beta c^{(1)}_{T-k}\right) + q\left(a_{T-k-1} + \sigma^2_\eta c_{T-k-1}\right) - qc^{(1)}_{T-k}\left(e_{T-k-1} - c^{(1)}_{T-k}f_{T-k-1}\right), \\
b_{T-k} &= q\rho b_{T-k-1} - c^{(3)}_{T-k}\left(qe_{T-k-1} - 1\right), \\
c_{t-k} &= c^{(3)}_{T-k}\left(\beta c^{(3)}_{T-k} + \theta\right) + q\rho^2 c_{T-k-1} - qc^{(3)}_{T-k}\left(\rho d_{T-k-1} - c^{(3)}_{T-k}f_{k-1}\right), \\
d_{T-k} &= \theta c^{(2)}_{T-k} + q\rho d_{T-k-1}\left(1 - c^{(2)}_{T-k}\right), \\
e_{T-k} &= c^{(2)}_{T-k} + q\left(1 - c^{(2)}_{T-k}\right)e_{T-k-1}, \\
f_{T-k} &= \beta c^{(2)}_{T-k}.
\end{aligned}
$$

Reference optimal_order_execution_LPT.py