Assignment 6

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February 21, 2022

Problem 1

Assuem utility function $U(x) = x - \frac{\alpha x^2}{2}$ and $x \sim \mathcal{N}(\mu, \sigma^2)$, calculate:
1. Expected Utility $E[U(x)]$:

$$E[U(x)] = E[x - \frac{\alpha x^2}{2}] = E[x] - \frac{\alpha}{2}E[x^2] = \mu - \frac{\alpha}{2}(\sigma^2 + \mu^2)$$

2. Certainty-Equivalent Value: x_{CE}

$$x_{CE} = U^{-1}(E[U(x)]) = U^{-1}(\mu - \frac{\alpha}{2}(\sigma^2 \ + \mu^2)) \Rightarrow x_{CE} = \frac{1 \pm \sqrt{\alpha^2\mu^2 + \alpha^2\sigma^2 - 2\alpha\mu + 1}}{\alpha}$$

3. Absolute Risk-Premium π_A

$$\pi_A = E[x] - x_{CE} = \mu - \frac{1 \pm \sqrt{\alpha^2\mu^2 + \alpha^2\sigma^2 - 2\alpha\mu + 1}}{\alpha}$$

Invest z dollars in risky asset and $1 - z$ dollars in riskless asset. Let W denote the wealth in one year where $W \sim \mathcal{N}(1 + r + z(\mu - r), z^2\sigma^2)$. Our goal is to maximize $E[U(W)]$.

$$\max_z E(U(W)) = 1 + r + z(\mu - r) - \frac{\alpha}{2}(z^2\sigma^2 + (1 + r + z(\mu - r))^2)$$

F.O.C:

$$\mu - r - \frac{\alpha}{2}(2z\sigma^2 + 2(\mu - r)(1 + r + z(\mu - r))) = 0$$

$$\mu - r - \alpha z\sigma^2 - \alpha(\mu - r)(1 + r) - \alpha z(\mu - r)^2 = 0$$

$$z(-\alpha\sigma^2 - \alpha(\mu - r)^2) = (\mu - r)(\alpha + \alpha r - 1)$$

$$z = \frac{(\mu - r)(\alpha + \alpha r - 1)}{-\alpha\sigma^2 - \alpha(\mu - r)^2}$$

Problem 3

(a) Write down the two outcomes for wealth W at the end of your single bet of $f \cdot W_0$

i. $W = f \cdot W_0(1 + \alpha) + (1 - f) \cdot W_0 = f \cdot W_0 \cdot \alpha + W_0$

ii. $W = f \cdot W_0(1 - \beta) + (1 - f) \cdot W_0 = -f \cdot W_0 \cdot \beta + W_0$

(b) Write down the two outcomes for log (Utility) of W.

i. $log(W) = log(f \cdot W_0 \cdot \alpha + W_0)$

ii. $log(W) = log(-f \cdot W_0 \cdot \beta + W_0)$

(c) Write down $\mathbb{E}[\log(W)]$.

$$\mathbb{E}[\log(W)] = p \cdot log(f \cdot W_0 \cdot \alpha + W_0) + q \cdot log(-f \cdot W_0 \cdot \beta + W_0)$$

(d) Take the derivative of $\mathbb{E}[\log(W)]$ with respect to f.

$$p \cdot \frac{W_0 \cdot \alpha}{f \cdot W_0 \cdot \alpha + W_0} + q \cdot \frac{W_0 \cdot \beta}{f \cdot W_0 \cdot \beta + W_0}$$

(e) Set this derivative to 0 to solve for f^*. Verify that this is indeed a maxima by evaluating the second derivative at f^*. This formula for f^* is known as the Kelly Criterion.

$$p \cdot \frac{W_0 \cdot \alpha}{f^* \cdot W_0 \cdot \alpha + W_0} + q \cdot \frac{W_0 \cdot \beta}{f^* \cdot W_0 \cdot \beta + W_0} = 0$$

$$f^* \cdot p \cdot \alpha \cdot \beta \cdot W_0^2 + p \cdot \alpha \cdot W_0^2 = -f^* \cdot q \cdot \alpha \cdot \beta \cdot W_0^2 - q \cdot \beta \cdot W_0^2$$

$$f^* = \frac{-p \cdot \alpha W_0^2 - q \cdot \beta W_0^2}{W_0^2 \cdot \alpha \cdot \beta} = \frac{-p \cdot \alpha - q \cdot \beta}{\alpha \cdot \beta}$$

Second derivative:

$$-\frac{p \cdot (w_0 \cdot \alpha)^2}{(f \cdot W_0 \cdot \alpha + W_0)^2} - \frac{q \cdot (w_0 \cdot \beta)^2}{(f \cdot W_0 \cdot \beta + W_0)^2}$$

Since

$$-\frac{p \cdot (w_0 \cdot \alpha)^2}{(f^* \cdot W_0 \cdot \alpha + W_0)^2} - \frac{q \cdot (w_0 \cdot \beta)^2}{(f^* \cdot W_0 \cdot \beta + W_0)^2} < 0,$$

this is indeed a maxima.