Pricing of Barrier Options

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1 Introduction

Barrier options are path-dependent options where the payoff relies on whether the underlying asset price hits a specified value before expiring. Since barrier options holders only receive payoffs when the underlying asset price of these options reaches a certain barrier, the holders face more risk. Therefore, barrier options are cheaper and thus, ubiquitous in financial markets. In this report, we introduce the valuation of barrier options. The report is organized as follows. Section 2 covers some financial theories used to price barrier options. Section 3 presents an overview of barrier options. Section 4 introduces several methods utilized to price the options. Section 5 compares numerical results obtained from these methods.

2 Preliminaries

We provide some useful concepts for understanding barrier options and their pricing methods.

2.1 Option

Options are contracts that give the holders the right to buy or sell underlying assets at specified prices and dates. They are also known as financial derivatives and contingent claims. Some useful option definitions:

- - Call option: a Call option is a contract that gives the holder the right to buy an underlying asset at a specified price and date.
  - Put option: a Put option is a contract that gives the holder the right to sell an underlying asset at a specified price and date.
- - European option: a European option is a contract that allows the holder to exercise the option only at its maturity.
  - American option: an American option is a contract that allows the holder to exercise the option at any time up to its maturity.
- - Vanilla option: a Vanilla option is a normal option with no special features, terms, or conditions.
  - Exotic option: an Exotic option is an option with features making it more complicated compared with a Vanilla option.

Barrier options studied in this report are exotic options.

2.2 Standard Brownian Motion

A standard Brownian motion $W(t)$ is a real-valued continuous stochastic process satisfying the following properties:

1. $W_0 = 0$
2. Independent Increment: For $t_{n-1} < t_n < t_{n+1}$, random variables $W_{t_{n+1}} - W_{t_n}$ and $W_{t_n} - W_{t_{n-1}}$ are independent.
3. $W_t - W_s \sim N(0, t-s)$ for $0 \leq s < t$
4. W_t is continuous in t.

2.3 Geometric Brownian Motion

A stochastic process S_t follows a geometric Brownian motion if it satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a standard Brownian motion, μ is a constant rate of return (drift term), and σ is a positive constant volatility.

2.4 Black-Scholes Equation

The Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

describes the price of an option over a term, where V is the value (price) of the option, S is the underlying asset price, t is time, r is risk-free rate, and σ is the volatility of the underlying asset.

2.5 Ito's Formula

Suppose function $g(t, X_t)$ is a twice continuously differentiable and bounded function that has bounded derivatives and

$$dX_t = \mu\left(t, X_t\right) dt + \sigma\left(t, X_t\right) dW_t$$

Ito's formula states that

$$dg(t, X_t) = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2$$

2.6 Quadratic Variation

If W_t is a standard Brownian motion,

$$(dW_t)^2 = dt$$

3 Barrier Options

To characterize different types of barrier options, we first provide some terminology and notations.

1. (a) in: contract is activated if the barrier is hit throughout the lifetime of the option.
   (b) out: contract is cancelled if the barrier is hit throughout the lifetime of the option.
2. (a) up: when the barrier is higher than the current underlying asset price
   (b) down: when the barrier is lower than the current underlying asset price

We further denote the barrier level as B, the maturity time as T, the strike price as K, the underlying asset price at time t as S_t, and the price of the barrier option at $t = 0$ as V.We then provide two examples.

- Up-and-in call option with $K = 20,\ S_0 = 25$, and $B = 30$:
  This barrier option becomes a call option with $K = 20$ if $S_t \geq 30$ for some t.
- Down-and-out put option with $K = 20,\ S_0 = 25$, and $B = 10$:
  This barrier option is a put option with $K = 20$ by default. Nevertheless, it becomes worthless if $S_t \leq 10$ for some t.

Remark

1. A combination of an in and an out option that are both either call or put options replicates a vanilla option.

- Holding both a down-and-out and a down-and-in put options is equivalent to holding a vanilla put option since one option is knocked in when the other is knocked out.

2. Relationship between the option value and the barrier level
   - in option: if it is difficult for an in option to hit the barrier level, the probability of triggering the barrier option approaches 0. The barrier option would be worthless, and the value of the barrier option falls to 0.
   - out option: if it is difficult for an out option to hit the barrier level, the probability of ceasing the barrier option approaches 0. The barrier option would retain its worth and the value of the barrier option would tend to equal the value of a vanilla option.

Given the technical background, we can further price barrier options.

4 Methodology

Our objective is to solve the Black-Scholes partial differential equation (PDE) stated in Section 2.4.

4.1 Analytical Solution

To get an accurate closed form solution for the Black-Scholes PDE, we use the PDE theory. For simplicity, we only provide the derivation of European down-and-out call options with $K > B$ and annual dividend yield (q) being 0. In [1], Paul Wilmott, Sam Howison, and Jeff Dewynne transform the Black-Scholes PDE into a heat equation as follows:

$$V(S,t) \rightarrow u(x,\tau)$$

The authors use these change of variables:

$$S = Ke^{x}, \quad t = T - \tau/\frac{1}{2}\sigma^2, \text{ and } \quad V = Ke^{\alpha x + \beta\tau}u(x,\tau)$$

with $\alpha = -\frac{1}{2}(k-1), \beta = -\frac{1}{4}(k+1)^2$, and $k = r/\frac{1}{2}\sigma^2$. The barrier is then transformed into

$$b = \log(B/K),$$

and we can instead solve the heat problem

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

with

$$u(x,0) = \max\left(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right) = u_0(x), \quad x \geq b,$$
$$u(x,\tau) \sim e^{(1-\alpha)x - \beta\tau} \quad \text{as} \quad x \rightarrow \infty,$$

and

$$u\,(b,t) = 0$$

Finally, they derive the solution

$$V(S,t) = C(S,t) - \left(\frac{S}{B}\right)^{-(k-1)} C\left(B^2/S, t\right)$$

where $C(\cdot,\cdot)$ is the vanilla option with the same maturity time and strike price.

The pricing formula of vanilla options and the other types of barrier options are as follows:

- Vanilla Options
  1. Call Options
  $$C(S) = e^{-qT}SN(d_1) - Ke^{-rT}N(d_2)$$

2. Put Options

$$P(S) = Ke^{-rT}N(-d_2) - e^{-qT}SN(-d_1)$$

- In Options

1. Down-and-in call

$$\begin{cases} K \geq B: \ Se^{-qT}(B/S)^{2\lambda}N(y) - Ke^{-rT}(B/S)^{2\lambda-2}N(y-\sigma\sqrt{T}) \\ K < B: \ C(S) - (SN(x_1)e^{-qT} - Ke^{-rT}N\left(x_1 - \sigma\sqrt{T}\right) - Se^{-qT}(B/S)^{2\lambda}N(x_2) \\ \qquad\qquad +Ke^{-rT}(B/S)^{2\lambda-2}N\left(x_2 - \sigma\sqrt{T}\right)) \end{cases}$$

2. Up-and-in call

$$\begin{cases} K \geq B: \ C(S) \\ K < B: \ SN(x_1)e^{-qT} - Ke^{-rT}N\left(x_1 - \sigma\sqrt{T}\right) - Se^{-qT}(B/S)^{2\lambda}\left(N(-y) - N(-x_2)\right) \\ \qquad\qquad +Ke^{-rT}(B/S)^{2\lambda-2}\left(N(-y+\sigma\sqrt{T}) - N\left(-x_2 + \sigma\sqrt{T}\right)\right) \end{cases}$$

3. Down-and-in put

$$\begin{cases} K \geq B: \ -SN(-x_1)e^{-qT} + Ke^{-rT}N\left(-x_1 + \sigma\sqrt{T}\right) + Se^{-qT}(B/S)^{2\lambda}\left(N(y) - N(x_2)\right) \\ \qquad\qquad -Ke^{-rT}(B/S)^{2\lambda-2}\left(N(y-\sigma\sqrt{T}) - N\left(x_2 - \sigma\sqrt{T}\right)\right) \\ K < B: \ P(S) \end{cases}$$

4. Up-and-in put

$$\begin{cases} K \geq B: \ P(S) - (-SN(-x_1)e^{-qT} + Ke^{-rT}N\left(-x_1 + \sigma\sqrt{T}\right) + Se^{-qT}(B/S)^{2\lambda}N(-x_2) \\ \qquad\qquad -Ke^{-rT}(B/S)^{2\lambda-2}N\left(-x_2 + \sigma\sqrt{T}\right)) \\ K < B: \ -Se^{-qT}(B/S)^{2\lambda}N(-y) + Ke^{-rT}(B/S)^{2\lambda-2}N(-y+\sigma\sqrt{T}) \end{cases}$$

- Out Options

1. Down-and-out call

$$\begin{cases} K \geq B: \ C(S) - (Se^{-qT}(B/S)^{2\lambda}N(y) - Ke^{-rT}(B/S)^{2\lambda-2}N(y-\sigma\sqrt{T})) \\ K < B: \ SN(x_1)e^{-qT} - Ke^{-rT}N\left(x_1 - \sigma\sqrt{T}\right) - Se^{-qT}(B/S)^{2\lambda}N(x_2) \\ \qquad\qquad +Ke^{-rT}(B/S)^{2\lambda-2}N\left(x_2 - \sigma\sqrt{T}\right) \end{cases}$$

2. Up-and-out call

$$\begin{cases} K \geq B: \ 0 \\ K < B: \ C(S) - (SN(x_1)e^{-qT} - Ke^{-rT}N\left(x_1 - \sigma\sqrt{T}\right) - Se^{-qT}(B/S)^{2\lambda}\left(N(-y) - N(-x_2)\right) \\ \qquad\qquad +Ke^{-rT}(B/S)^{2\lambda-2}\left(N(-y+\sigma\sqrt{T}) - N\left(-x_2 + \sigma\sqrt{T}\right)\right)) \end{cases}$$

3. Down-and-out put

$$\begin{cases} K \geq B: \ P(S) - (-SN(-x_1)e^{-qT} + Ke^{-rT}N\left(-x_1 + \sigma\sqrt{T}\right) + Se^{-qT}(B/S)^{2\lambda}\left(N(y) - N(y_1)\right) \\ \qquad\qquad -Ke^{-rT}(B/S)^{2\lambda-2}\left(N(y-\sigma\sqrt{T}) - N\left(x_2 - \sigma\sqrt{T}\right)\right)) \\ K < B: \ 0 \end{cases}$$

4. Up-and-out put

$$\begin{cases} K \geq B: \ -SN(-x_1)e^{-qT} + Ke^{-rT}N\left(-x_1 + \sigma\sqrt{T}\right) + Se^{-qT}(B/S)^{2\lambda}N(-x_2) \\ \qquad\qquad -Ke^{-rT}(B/S)^{2\lambda-2}N\left(-x_2 + \sigma\sqrt{T}\right) \\ K < B: \ P(S) - (-Se^{-qT}(B/S)^{2\lambda}N(-y) + Ke^{-rT}(B/S)^{2\lambda-2}N(-y+\sigma\sqrt{T})) \end{cases}$$

where

$$d_1 = \frac{\ln(S/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$\lambda = \frac{r - q + \sigma^2/2}{\sigma^2}$$

$$y = \frac{\ln\left(B^2/(SK)\right)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

$$x_1 = \frac{\ln(S/B)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

$$x_2 = \frac{\ln(B/S)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}.$$

Using the PDE theory to solve the Black-Scholes PDE is complex. Instead, we can derive an approximation solution of the Black-Scholes PDE. We use the finite difference method and the Monte Carlo method to approximate.

4.2 Finite Difference Method

The main idea of finite difference method is to approximate a derivative with a finite difference. For a first order derivative:

$$\frac{u(x, \tau + \Delta\tau) - u(x, \tau)}{\Delta\tau} - \frac{\partial u(x, \tau)}{\partial \tau} = O(\Delta\tau) \to 0 \text{ as } \Delta\tau \to 0$$

For a second order derivative, we use the central term method to approximate:

$$\frac{\frac{u(x+\Delta x, \tau) - u(x, \tau)}{\Delta x} - \frac{u(x, \tau) - u(x - \Delta x, \tau)}{\Delta x}}{\Delta x} - \frac{\partial^2 u(x, \tau)}{\partial x^2} = O((\Delta x)^2) \to 0 \text{ as } (\Delta x)^2 \to 0$$

$$\Rightarrow \frac{u(x + \Delta x, \tau) - 2u(x, \tau) + u(x - \Delta x, \tau)}{(\Delta x)^2} - \frac{\partial^2 u(x, \tau)}{\partial x^2} = O((\Delta x)^2) \to 0 \text{ as } (\Delta x)^2 \to 0$$

Hence,

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{u(x, \tau + \Delta\tau) - u(x, \tau)}{\Delta\tau} - O(\Delta\tau) = \frac{u(x + \Delta x, \tau) - 2u(x, \tau) + u(x - \Delta x, \tau)}{(\Delta x)^2} - O\left((\Delta x)^2\right)$$

Therefore, we approximate $u(x, \tau + \Delta\tau)$ using $u(x + \Delta x, \tau)$, $u(x, \tau)$ and $u(x - \Delta x, \tau)$

$$u(x, \tau + \Delta\tau) \approx \frac{\Delta\tau}{(\Delta x)^2} u(x + \Delta x, \tau) - (2\frac{\Delta\tau}{(\Delta x)^2} - 1)u(x, \tau) + \frac{\Delta\tau}{(\Delta x)^2} u(x - \Delta x, \tau)$$

4.3 Monte Carlo Method

Monte Carlo simulation techniques can be used to manage path dependent payoffs with no analytic expression. We follow three steps to price the barrier options using the Monte Carlo method.

- Step 1. Simulate n sample paths of the underlying asset price over the interval
- Step 2. Calculate the payoff of the option for each path
- Step 3. Average the discounted payoffs over the sample paths

To simulate a sample path, we select a stochastic differential equation to describe the price dynamics. We consider the price dynamics of the underlying asset $(S_t)_t$ following geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Then, we use Taylor expansion to the second order term and derive:

$$d\log S_t = \frac{dS_t}{S_t} - \frac{1}{2}\left(\frac{dS_t}{S_t}\right)^2$$

$$\left(\frac{dS_t}{dt}\right)^2 = (\mu dt + \sigma dW_t)^2 = \mu^2(dt)^2 + 2\mu\sigma(dt)(dW_t) + \sigma^2(dW_t)^2 = \sigma^2 dt$$

$$\Rightarrow d\log S_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$$

$$\Rightarrow \log S_t = \log S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t$$

$$\Rightarrow S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

5 Simulation Results

We compare the valuations between analytical PDE solution and Monte Carlo method. We run the Monte Carlo simulation for 5000 times.

1. Up-and-out call options with $r = 0.03, q = 0, K = 30, T = 1$ year, and $\sigma = 0.4$

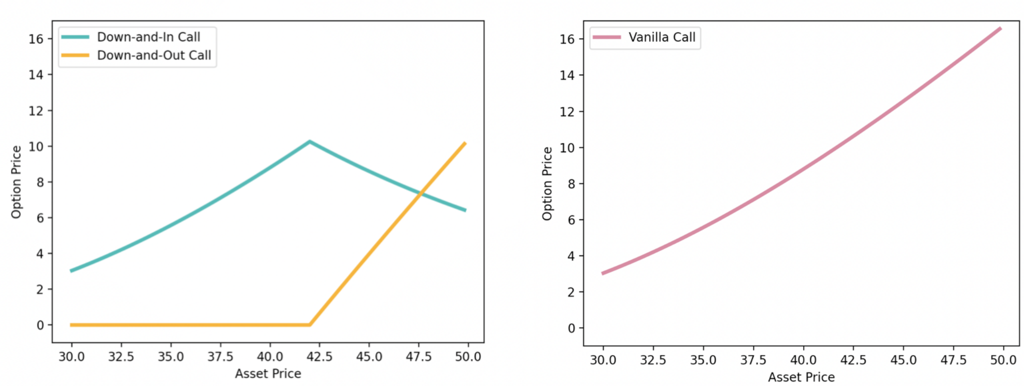
S_0	B	B-S PDE	Monte Carlo	S_0	B	B-S PDE	Monte Carlo
30	50	1.7043	1.7958	30	40	0.3067	0.3503
35	50	1.7897	1.9264	30	45	0.9162	0.9721
40	50	1.4378	1.5606	30	55	2.4894	2.7170

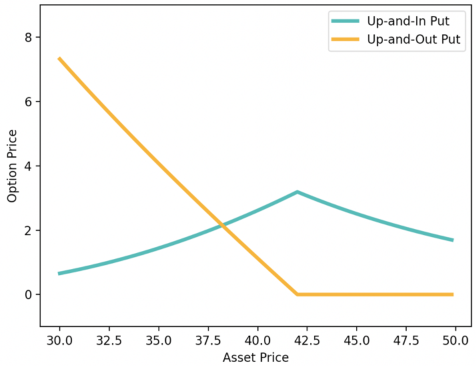
2. Down-and-in put options with $r = 0.03, q = 0, K = 50, T = 1$ year, and $\sigma = 0.4$

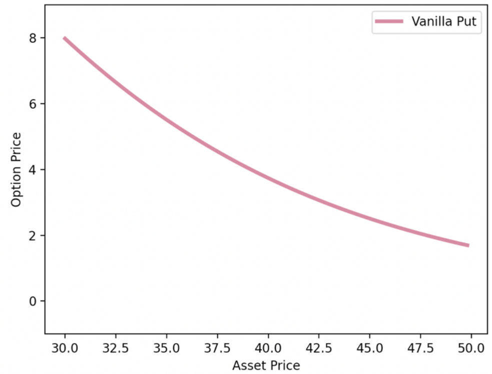
S_0	B	B-S PDE	Monte Carlo	S_0	B	B-S PDE	Monte Carlo
55	30	2.9960	2.9942	50	45	7.0800	6.8837
40	30	9.8900	9.6218	50	40	6.8921	6.8216
35	30	14.1137	14.0834	50	35	6.1252	6.0754

From the tables above, we notice that for knock-in options, increasing the absolute difference between the underlying asset initial price and the barrier reduces the option value, and it's opposite for knock-out options.

We then set $r = 0.03, q = 0, K = 36, B = 42, T = 1$ year$, \sigma = 0.4$ and $S_0 \in [30, 50]$ to observe the relationship between barrier options and vanilla options.







From the figures above, we find that a combination of a down-and-in and a down-and-out call option replicates a vanilla call option. Similarly, a combination of an up-and-in and an up-and-out put option replicates a vanilla put option. Furthermore, a down-and-out option with $S_0 \leq B$ and an up-and-out option with $S_0 \geq B$ are both valueless.

References

[1] P. Wilmott, J. Dewynne, and S. Howison(1993). *Option pricing: mathematical models and computation.* Oxford, Oxford Financial Press.

[2] J.C. Hull: *Options, Futures, And Other Derivatives*