On the impact of cross-ownership in a common property renewable resource oligopoly*

Miao Dai[†], Hassan Benchekroun[‡], Ilyass Dahmouni[§]

September 30, 2025

Abstract

We construct a Markov Perfect Nash Equilibrium of a dynamic Cournot oligopoly where firms jointly exploit a productive asset (renewable resource) and engage in rival cross-shareholdings. We show that there exists an interval of stocks where cross-ownership can (i) increase market output, (ii) be profitable, and (iii) increase social welfare, both in the short run and at the steady state. These effects are in stark contrast with those obtained in a static oligopoly framework with strategic substitutes.

Keywords: Cross-ownership, Renewable resource, Cournot Competition, Dynamic Oligopoly, Shareholdings, Productive Asset, Antitrust

JEL Codes: L13, L41, Q2, C73, D43

*We are grateful to participants at the McGill-CIREQ Lunch Seminar, Conference on Sustainable Resource Use and Economic Dynamics (SURED), 23rd Journées Louis-André Gérard-Varet (LAGV), 29th Annual Conference of the European Association of Environmental and Resource Economists (EAERE), CEE-M Environmental Economics Seminar, Annual meeting of the Association of Southern European Economic Theorists (ASSET) and Annual Conference of the European Association for Research in Industrial Economics (EARIE) for their insightful comments. The authors would like to thank the Canadian Social Sciences and Humanities Research Council (SSHRC) and Fonds de recherche du Québec - Société et culture (FRQSC) for their financial support. Miao Dai also acknowledges the financial support from the French Agence Nationale de la Recherche, grant ANR-21-CE03-0012 (ScarCyclET) and the Swiss National Science Foundation (SNSF) (Division 1, Grant No. 212632). We declare that we have no conflict of interest.

[†]Corresponding author. Department of Spatial Economics, Vrije Universiteit Amsterdam and Tinbergen Institute. E-mail: miao.dai@mail.mcgill.ca.

[‡]Department of Economics, McGill University and CIREQ. Email: hassan.benchekroun@mcgill.ca.

[§]Fisheries Economics Research Unit, Institute for the Oceans and Fisheries, University of British Columbia. Email: ilyass.dahmouni@ubc.ca.

1 Introduction

In a static Cournot oligopoly homogenous-product model, it is shown that when firms engage in rival cross-shareholdings, the equilibrium production of the participating firms as well as total industry output decreases, resulting in a lower consumer surplus, higher producer surplus and ultimately lower welfare compared to the equilibrium under no cross-ownership (Reynolds and Snapp, 1986; Bresnahan and Salop, 1986; Farrell and Shapiro, 1990; O'Brien and Salop, 2000). Moreover, analogous to the seminal merger paradox (Salant et al., 1983; Gaudet and Salant, 1991), there exists a cross-ownership paradox that rival cross-shareholdings are only profitable if they involve a substantial share of firms in the industry (Dai et al., 2022).

In this paper, we examine the impact of cross-ownership on the equilibrium production strategies, profitability and social welfare in the context of a common property productive asset (renewable resource) oligopoly. Firms compete as Cournot rivals in the output market, but a subset of firms engage in rival cross-shareholdings. We take into account the complex economic ties that exist in an industry characterized by rival cross-shareholdings, where the aggregate profits of a firm include not only the stream of profits generated from its own operations but also a share in its competitors' aggregate profits due to its direct and indirect ownership stakes in these firms (Flath, 1992; Gilo et al., 2006). Such cross-ownership activities are particularly prevalent in the renewable resource industries. For example, a Matis report commissioned by Seafish and the Grimsby seafood cluster sheds light on the intricate web of connections and dependencies in ownership within the largest seafood companies in Iceland.[1] In addition, the implementation of a tradeable quota system has incentivized large cross-ownership in the New Zealand fishing industry as firms try to circumvent restrictions on the maximum quotas individuals can own.[2] Abundant evidence has also been documented for the cooperation between fishermen who typically live in small communities and behave cooperatively by exchanging information, sharing costs and dividing labour, which entitles them to a share of the profits from other fishermen's catches (Colombo and Labrecciosa, 2018).

We perform our analysis in the context of a differential game (see Dockner et al. (2000), Long (2010) and Başar and Zaccour (2018) for concepts and applications), and we focus on closed-loop strategies, where firms' strategies are production rules that depend both on time and the asset's stock. A Markov Perfect Nash Equilibrium of the game is then characterized and used to contrast with the case without cross-ownership.

We first show that, in the short run, cross-ownership participants may increase their production and non-participants may lower their output with respect to the benchmark case where there is no cross-ownership. The former case occurs for some range of large initial stocks when all firms are involved in cross-ownership, while the total number of

[1]See https://www.seafish.org/document/?id=4d67242e-b529-43b9-81aa-477de5a9133e.
[2]See https://www.fao.org/3/Y2498E/y2498e0e.htm.

firms is larger than two and the ownership stake is small enough. The latter case occurs for some range of small initial resource stocks when only a subset of firms participate in cross-ownership. These results are in sharp contrast with the static oligopoly theory and the findings from comparative statics in oligopolies with strategic substitutes (see e.g., Gaudet and Salant (1991); Amir (1996); Amir and Rietzke (2023)). Indeed, when rival firms participate in cross-ownership, they have an incentive to compete less aggressively as one firm's gain may come at the loss of the other firms in which it has shareholdings. As such, each cross-ownership participant will reduce their output, but in terms of strategic substitutes in Cournot competition, firms that do not participate in cross-ownership will always respond by expanding their production. However, in our context where an oligopoly exploits a common property productive asset, there is an additional channel through which cross-ownership influences firms' extraction rates, beyond the typical static "market power" mechanism due to reduced competition in the output market. That is, cross-ownership also affects how firms interact with each other at the resource level.

When the initial resource stock is abundant enough, firms will behave as if they are not constrained by the resource stock, and thus resource scarcity plays no role. However, when the resource stock falls below a certain threshold, a positive resource rent arises, affecting firms' production strategies. The impact of rival cross-shareholdings on the resource rent manifests as a negative effect for relatively large stocks and a positive effect for relatively small stocks. In the former case, cross-owners will only slightly reduce their production due to a relatively low shareholding in the output market, but this results in a sufficiently abundant stock of the asset, which in turn reduces the marginal valuation of the resource stock by each cross-ownership participant. A smaller resource rent thus incentivizes each cross-owner to expand its production, which outweighs the production reduction brought by cross-ownership at the output market. In the latter case, the asset remains sufficiently scarce for the relatively low levels of stocks, leading to an increase in the marginal valuation of the resource stock for each cross-ownership non-participant. A higher resource rent thus provides an incentive for the outsiders to reduce their production, which dominates the static effect of production expansion induced by cross-ownership.

We then establish that these cases may indeed materialize, since firms may find it profitable to engage in rival cross-shareholdings. We conduct a detailed profitability analysis to understand the private incentives driving competing firms to participate in cross-ownership arrangements in the context of a common property productive asset oligopoly and compare it to the static case. We show that the cross-ownership paradox does not necessarily carry over to the case of a renewable resource industry. There always exists an interval of stocks for which a symmetric cross-ownership can be profitable, even though such rival cross-shareholdings are strictly unprofitable in the corresponding static equilibrium framework. The main intuition behind this result lies in the common property nature of renewable asset exploitation. When multiple players share a common resource, the rate at which it is exploited (the decision variable) is intrinsically linked to the available stock of the resource (the state variable). This interdependence means that any action by one player that alters the stock level will have a direct impact on the decisions made by all other players in the industry. But no such link exists in the corresponding static oligopoly with cross-ownership, where any given rate of production can be sustained forever. In the static game, the outsiders always respond aggressively by expanding their production when insiders decrease their output due to their ownership stakes. However, in the dynamic setting, the outsiders might respond more cautiously or even reduce their production in some instances. This moderated response occurs because, within certain stock ranges, cross-ownership can lead to an increased valuation by each player for the marginal unit of remaining resource stock. A consequence is that there is always an interval of initial stocks such that the profitability of cross-ownership is always positive.

Moreover, we demonstrate that there exists a specific range of resource stocks in which not only is cross-ownership between rival firms profitable, but it also increases industry production. One direct implication of this result is that consumer surplus will increase following the profitable cross-shareholdings, which, at the same time, boosts industry profits, leading to a higher overall welfare in the short run. This outcome could never occur within a static framework: in a static Cournot game, unless there are substantial efficiency gains, cross-ownership always leads to a loss in social welfare (Reynolds and Snapp, 1986). This aspect holds significant relevance for discussions surrounding competition policies, as there is a growing call for more stringent regulations of these non-controlling minority shareholdings that are currently subject to a very lenient approach by antitrust authorities.[3] Our findings thus suggest that competition authorities should be cautious when ruling in the renewable resource sector, as cross-ownership may turn out to be welfare-improving.

We also study the effects of cross-ownership on the stock of resources, industry output, profitability, and social welfare in the long run, i.e., at the steady state. Our analysis shows that cross-ownership results in a larger steady-state level of the productive asset's stock, regardless of the initial resource stock. Additionally, we establish that cross-shareholdings between rival firms can result in an increase in the industry's output at the steady state when the implicit growth rate falls below a certain threshold or the initial resource stock is small enough. This is in stark contrast with static oligopoly theory, where cross-ownership leads to a decrease in industry output. The key insight here is that in our dynamic framework with a common productive asset, cross-ownership influences the industry's exploitation rate through two main channels: the output market and the interaction at the resource level. The former represents the traditional mechanism by which reduced competition in the output market due to ownership links leads to a decrease in industry output. The latter, unique to the renewable resource industry, suggests that cross-ownership can lead to a larger steady-state

[3] See for instance, Posner et al. (2016) and Commission et al. (2016).

stock of the asset, thereby enabling greater industry extraction. This interaction at the resource level thus significantly alters the dynamics of industry output and ultimately leads to increased industry production in the long run, challenging the conventional static perspective.

Furthermore, we show that the above-mentioned cases could occur, as firms will find it profitable to engage in cross-shareholdings in the transition to the steady state of the stocks. Consequently, the long-run expansion of industry production becomes a viable prospect, suggesting the potential for an increase in consumer surplus in the long run as a result of cross-ownership. We then show that producer surplus is also higher at the stationary equilibrium for these scenarios, which implies that welfare can increase due to cross-shareholdings in the long run as well. Therefore, antitrust authorities should exercise caution when regulating renewable resource industries, as strict policies that restrict cooperation among users of common pool renewable resources could ultimately harm consumers and society. Unintentionally, these measures might produce the exact opposite effect of what is intended.

Our paper contributes to several strands of literature. The first one is on the growing literature on cross-ownership (Reynolds and Snapp, 1986; Bresnahan and Salop, 1986; Farrell and Shapiro, 1990; Flath, 1991, 1992; Malueg, 1992; O'Brien and Salop, 2000; Dietzenbacher et al., 2000; Gilo et al., 2006; Brito et al., 2014a,b, 2018a,b; Benndorf and Odenkirchen, 2021; Dai et al., 2022; Benchekroun et al., 2022; Huse et al., 2024). However, most of these studies have focused on the static anticompetitive effects induced by cross-ownership, i.e., unilateral effects and coordinated effects, with the only exception of Dai et al. (2022), which considers a nonrenewable resource oligopoly with each firm owning a private resource stock. In that framework, the cumulative production of each firm is fixed since the resource owned by each firm is nonrenewable and is physically exhausted, it is shown that cross-ownership also results in an overall decrease in initial industry output (i.e. a slowdown of the resource extraction).[4] We consider a dynamic Cournot game in which firms exploit a common property productive asset in this paper, and we show that cross-ownership can result in an overall increase in industry output both in the short run and at the steady state. López and Vives (2019) also find that output can increase for high enough technological spillovers due to cost-reducing R&D investment in a Cournot oligopoly with overlapping ownership. However, the production expansion in their paper is mainly driven by the positive externality of R&D investment. In contrast, in our context, there are no positive externalities nor synergies resulting from cross-ownership.

Our paper also contributes to the large game-theoretic literature on the exploitation of renewable resources. Previous studies have focused either on the case in which agents behave non-cooperatively (Levhari and Mirman, 1980; Reinganum and Stokey,

[4]As opposed to Dai et al. (2022) where the cumulative output of each individual firm is fixed by its own stock and each firm's strategy consists of an extraction path, in this paper the cumulative extraction of each firm is not fixed and each firm chooses an extraction policy that is stock dependent, thus ensuring subgame perfectness of the equilibrium we characterize.

1985; Karp, 1992; Dockner and Sorger, 1996; Dawid and Kopel, 1997; Sorger, 1998; Benchekroun, 2003, 2008; Sandal and Steinshamn, 2004; Benchekroun and Gaudet, 2015; Colombo and Labrecciosa, 2013, 2015) or the case where agents act cooperatively (Benhabib and Radner, 1992; Kopel and Szidarovszky, 2006; Colombo and Labrecciosa, 2018). Despite such a well-established literature in dynamic resource games, no previous studies have examined how ownership links between any rival firms may affect market equilibrium outcomes. By distinguishing the situation in which all the firms in the industry participate in cross-ownership from the case where only a subset of firms do so, we illustrate how cross-ownership might affect the exploitation of a common property productive asset and its impact on social welfare.

Our paper is closely related to Colombo and Labrecciosa (2018), but differs significantly in several aspects. While they focus primarily on cooperative strategies in a duopoly context, we examine a broader oligopoly setting, exploring scenarios where all firms engage in cross-shareholding and cases where only a subset of firms do so. This more general approach has allowed us to analyze the effects of cross-ownership in a more varied and realistic market structure, providing some new insights into its impact on market behaviour and competition. In addition, we delve into the private incentives of rival firms to participate in cross-ownership and demonstrate that there exist situations where the industry output, consumer surplus, producer surplus and social welfare may increase following the profitable rival cross-shareholdings in both the short run and long run.

Our research also adds to the ongoing policy discussions on how to regulate cross-ownership. While horizontal mergers generally face considerable antitrust scrutiny and often encounter opposition from antitrust authorities, non-controlling minority shareholdings tend to escape similar levels of examination. As highlighted by Gilo (2000) and Gilo et al. (2006), partial cross-ownership arrangements often receive minimal attention from competition authorities and enjoy a de facto exemption from antitrust liability. According to Nain and Wang (2018), fewer than 1% of minority acquisitions are challenged by the Federal Trade Commission or the Department of Justice, and even fewer are blocked outright. In many jurisdictions, antitrust authorities do not even have the competence to investigate such cases (Fotis and Zevgolis, 2016). Therefore, firms might consider cross-ownership a more appealing corporate strategy and opt for it disproportionately, knowing that it generally lacks legal accountability (Jovanovic and Wey, 2014). This trend is particularly concerning given the minimal antitrust enforcement against non-controlling minority shareholdings, allowing firms to reap the benefits of cooperation with competitors while evading the heightened regulatory scrutiny typically associated with horizontal mergers. Regarding this, there have been growing calls for more stringent regulations to limit such rival cross-shareholdings.[5]

[5] For instance, the latest U.S. Merger Guidelines (2023) provide specific guidance on cross-ownership and common ownership – an issue not explicitly addressed in earlier versions. See https://www.justice.gov/atr/merger-guidelines.

Our findings, however, suggest that antitrust authorities should be careful in ruling in renewable resource industries, as cross-ownership may actually lead to a higher social welfare. Applying a "per se illegal" antitrust policy is misleading, as strict application of such a policy to renewable resources neglects the dynamics of the common resource stock that should be explicitly taken into account (Adler, 2004; Deacon, 2012; Benchekroun and Gaudet, 2015; Colombo and Labrecciosa, 2018).

The remainder of the paper is structured as follows. Section 2 presents the model. Section 3 investigates the impact of cross-ownership on firms' production and the profitability of cross-ownership in the short run. Section 4 studies the effects of cross-ownership in the long run. Finally, Section 5 concludes.

2 The model and preliminary analysis

Let S denote the stock of some renewable assets, for instance, a fish population. We assume that in the absence of exploitation, the stock of the asset evolves according to the following dynamics (see e.g., Benchekroun (2008)):

$$\frac{dS}{dt} = F(S) = \begin{cases} \delta S & \text{for } S \leq S_y \\ \delta S_y \left(\frac{\bar{S}-S}{\bar{S}-S_y} \right) & \text{for } S > S_y \end{cases}, \quad S(0) = S_0 \geq 0,$$

where S_0 is the initial stock of the asset, $\delta > 0$ is the intrinsic growth rate, $\bar{S}$ is the maximum carrying capacity and δS_y is the maximum sustainable yield of the asset. This specification of $F(S)$ can be thought of as a linearization of the classical logistic growth function in natural resource economics (Clark, 2010; Conrad, 2010). When the stock is very small (i.e., $S \leq S_y$), there is no habitat constraint and the asset grows at an exponential rate; however, beyond S_y, the asset grows at a decreasing rate facing limited availability of food and space. Without loss of generality, we set $\bar{S} = 1$ in what follows.

The access to this asset is shared by $J = \{1, 2, \cdots, n\}$ firms, indexed by j, where each firm exploits the asset to produce an output to sell in an oligopolistic market. For simplicity, we assume that one unit of the asset is transformed into one unit of the output at zero cost. Let $q_j(t)$ denote firm j's output at time t, and the inverse demand function for the output at time t is given by

$$p(t) = a - bQ(t) = a - b \sum_{j=1} q_j(t).$$

Suppose that a subset of k firms ($2 \leq k \leq n$) engage in rival cross-shareholdings. Following Dai et al. (2022), we consider a k-symmetric cross-ownership structure in which each of the k firms has an equal silent financial stake v in the other firms, while the remaining $n - k$ firms stay independent. We use the subsets $I = \{1, 2, \cdots, k\}$, indexed by i and $O = \{k+1, \cdots, n\}$, indexed by o, referring, respectively, to the insiders and outsiders to the cross-ownership. In an industry characterized by symmetric rival cross-shareholdings, the aggregate profits of firm j at time t is:

$$\Pi_j(t) = \pi_j(t) + v\sum_{i\neq j}\Pi_i(t) = p(t)q_j(t) + v\sum_{i\neq j}\Pi_i(t),$$

where $\pi_j(t) = p(t)q_j(t) = (a - b\sum_{j=1} q_j(t))q_j(t)$ denotes firm j's operating profit and $v \in (0, \frac{1}{k-1})$ represents firm j's fractional shareholdings in firm i for any $i \neq j$.[6]

Let $\mathbf{\Pi}$ and $\boldsymbol{q}$ denote the $n \times 1$ vectors of aggregate profits and outputs at time t, and $\boldsymbol{D}$ denote the $n \times n$ cross-shareholding matrix, then the aggregate profit functions can be expressed in matrix form as

$$\mathbf{\Pi} = p\boldsymbol{q} + \boldsymbol{D}\mathbf{\Pi},$$

where $\boldsymbol{D} = \begin{bmatrix} \boldsymbol{A}_{kk} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-k} \end{bmatrix}$, and $\boldsymbol{A}_{kk}$ is a $k \times k$ matrix with element 0 in the diagonal and v off-diagonal. This set of n equations implicitly defines the aggregate profit for each firm at time t. Then $\boldsymbol{I} - \boldsymbol{D} = \begin{bmatrix} \boldsymbol{B}_{kk} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{n-k} \end{bmatrix}$, where $\boldsymbol{B}_{kk}$ is a $k \times k$ matrix with element 1 in the diagonal and $-v$ off-diagonal, and $\boldsymbol{I}_{n-k}$ denote the $(n-k) \times (n-k)$ identity matrix. The matrix $\boldsymbol{I} - \boldsymbol{D}$ is invertible, which allows us to solve for the aggregate profit functions:

$$\mathbf{\Pi} = (\boldsymbol{I} - \boldsymbol{D})^{-1}p\boldsymbol{q} = \begin{bmatrix} \boldsymbol{B}_{kk}^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{n-k} \end{bmatrix} p\boldsymbol{q},$$

where $\boldsymbol{B}_{kk}^{-1}$ is given by the following matrix

$$\Omega \equiv \frac{1}{f(v)} \begin{bmatrix} 1-(k-2)v & v & \cdots & v \\ v & 1-(k-2)v & \cdots & v \\ \vdots & \vdots & \ddots & \vdots \\ v & v & \cdots & 1-(k-2)v \end{bmatrix},$$

with $f(v) = (1+v)\big(1-(k-1)v\big) > 0$. Then the aggregate profit function of firm $i \in I$ at time t is

$$\Pi_i(t) = \frac{1}{f(v)}\Big(a - b\sum_{j\neq i} q_j(t) - bq_i(t)\Big)\Big(\big(1-(k-2)v\big)q_i(t) + v\sum_{m\in I\setminus i} q_m(t)\Big),$$

while for firm $o \in O$, the aggregate profit function at time t is

$$\Pi_o(t) = (a - b\sum_{j\neq o} q_j(t) - bq_o(t))q_o(t).$$

Taking the strategies of its $(J-1)$ rivals as given, each firm j chooses its own

[6] The weight given to rivals' profits is bounded from above by $\frac{1}{k-1}$, which guarantees that the aggregate stake of rivals in each cross-ownership participant, $(k-1)v$, is less than 1.

decision rule to maximize the discounted sum of the aggregate profits, which consists of its operating profits and the share of profits obtained through ownership interests in other firms, subject to the stock dynamics. For a typical firm $i \in I$,

$$\max_{q_i(t)\geq 0} \int_0^\infty e^{-rt}\left[\frac{1}{f(v)}\left(a-b\sum_{j\neq i}q_j(t)-bq_i(t)\right)\left((1-(k-2)v)q_i(t)+v\sum_{m\in I\setminus i}q_m(t)\right)\right]dt, \quad (1)$$

$$s.t. \quad \frac{dS}{dt}=F(S)-q_i-\sum_{j\neq i}q_j, \quad S(0)=S_0, \quad (2)$$

while for a typical firm $o \in O$,

$$\max_{q_o(t)\geq 0} \int_0^\infty e^{-rt}\left[(a-b\sum_{j\neq o}q_j-bq_o)q_o\right]dt, \quad (3)$$

$$s.t. \quad \frac{dS}{dt}=F(S)-q_o-\sum_{j\neq o}q_j, \quad S(0)=S_0, \quad (4)$$

where $r>0$ is the discount rate, the same for all firms.

We make the following assumption:

Assumption 1. *The intrinsic growth rate satisfies the following condition:*

$$\delta > \delta_0 \equiv max\left\{\frac{r\left[\left(k(k-1)v-n(1+v)\right)^2+(1+v)^2\right]}{2(1+v)^2}, \frac{a\left[\left(k(k-1)v-n(1+v)\right)^2+(1+v)^2\right]}{bS_y\left((k+n+1-k^2)v+n+1\right)^2}\right\}.$$

Assumption 1 implies that $\frac{\delta}{r}$ is strictly bounded from below, which guarantees that there exists a strictly interior stable steady-state stock. Similar imposition of such a lower bound can also be found in Dutta and Sundaram (1993a,b); Dockner and Sorger (1996); Benchekroun (2003, 2008); Colombo and Labrecciosa (2015, 2018).

Since each firm's problem is stationary, we restrict our attention to stationary Markov strategies, which are feedback decision rules whereby firms condition their exploitation rates of the resource on the current resource stock: $q_j=\phi_j(S(t))$. We characterize the Markov perfect Nash equilibrium (MPNE) for the above noncooperative differential game. More specifically,

Proposition 1. *Let ϕ_i^* and ϕ_o^* denote the production strategy for firm $i \in I=\{1,2,\cdots,k\}$ and firm $o \in O=\{k+1,\cdots,n\}$, respectively:*

$$\phi_i^*(S)=\begin{cases} 0 & \text{for } 0\leq S\leq S_1 \\ \frac{\left(1-(k-2)v\right)a-(1+v)\left(1-(k-1)v\right)(AS+B)}{\left((k+n+1-k^2)v+n+1\right)b} & \text{for } S_1<S\leq S_2 \\ q_i^v=\frac{\left(1-(k-2)v\right)a}{\left((k+n+1-k^2)v+n+1\right)b} & \text{for } S>S_2 \end{cases}, \quad (5)$$

$$\phi_o^*(S) = \begin{cases} 0 & \textit{for } 0 \le S \le S_1 \\ \frac{1+v}{1-(k-2)v} \frac{\big(1-(k-2)v\big)a-(1+v)\big(1-(k-1)v\big)(AS+B)}{\big((k+n+1-k^2)v+n+1\big)b} & \textit{for } S_1 < S \le S_2\,, \\ q_o^v = \frac{(1+v)a}{\big((k+n+1-k^2)v+n+1\big)b} & \textit{for } S > S_2 \end{cases} \tag{6}$$

where A, B, S_1, S_2 are constants (given in Appendix A) that depend on the vector of parameter values $(k, n, v, \delta, r, a, b)$. Then, the n-tuple vector of closed-loop strategies $(\phi_i^*, \cdots, \phi_i^*, \phi_o^*, \cdots, \phi_o^*)$ constitutes a MPNE in a common property renewable resource oligopoly with cross-ownership.

Proof. See Appendix A. □

Proposition 1 shows that firms' exploitation strategies depend crucially on the stock level of the productive asset. To visualize these results, we plot the production strategies of a typical insider (ϕ_i^*) and an outsider (ϕ_o^*) as a function of the asset stock (S) in Figure 1. When the asset's stock is too small (i.e., $S \le S_1$), both the insiders and

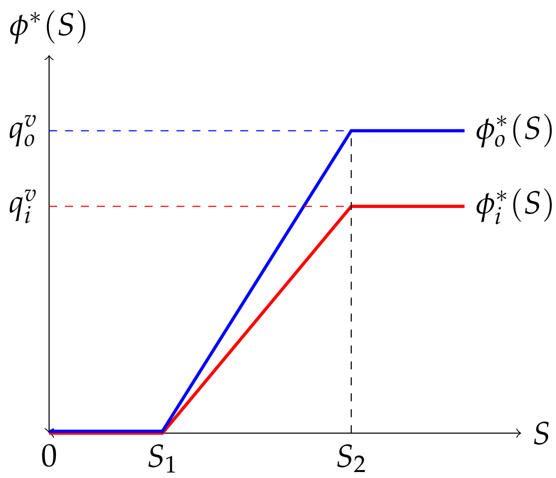


Figure 1: The equilibrium extraction strategies

outsiders will voluntarily cease their productions, despite the fact that they have free access to the asset and they compete in the oligopoly market. Similar results can be also found in Benhabib and Radner (1992), Benchekroun (2003, 2008) and Colombo and Labrecciosa (2018) where firms refrain from consumption/production and wait for the asset to grow to the maturity threshold. Therefore, depletion of the asset is avoided. When the level of the stock is very large or abundant (i.e., $S > S_2$), firms simply adopt the production strategy that coincides with the solution under a static Cournot game with a k-symmetric cross-ownership structure when the inverse demand is $p = a - bQ$ and the production cost is $c = 0$, i.e., q_i^v and q_o^v in Dai et al. (2022). Finally, when the stock level of the asset is intermediate (i.e., $S_1 < S \le S_2$), both the cross-ownership participants and non-participants will adopt a Markov strategy that is a monotonous non-decreasing function of S, strictly increasing over S_1 to S_2. [7]

Let $S^*(t)$ denote the equilibrium time path of the asset and $\Phi_v^*(S)$ denote the industry's production, i.e., $\Phi_v^*(S) = k\phi_i^*(S) + (n-k)\phi_o^*(S)$, or

$$\Phi_v^*(S) = \begin{cases} 0 & \text{for } 0 \leq S \leq S_1 \\ \frac{(-k^2+n+k)v+n}{1-(k-2)v} \frac{\left(1-(k-2)v\right)a-(1+v)\left(1-(k-1)v\right)(AS+B)}{\left((k+n+1-k^2)v+n+1\right)b} & \text{for } S_1 < S \leq S_2 \\ Q_v = \frac{\left((-k^2+n+k)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)b} & \text{for } S > S_2 \end{cases}. \tag{7}$$

Then, we have the following:

Corollary 1. *(i) For $\delta S_y < Q_v = \frac{\left((k+n-k^2)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)b}$, there exists a unique positive stationary asset stock given by*

$$S_1^\infty = \frac{a\left[(2\delta-r)(1+v)^2 - r\left(k(k-1)v-n(1+v)\right)^2\right]}{b\delta\left((k+n+1-k^2)v+n+1\right)\left[(2\delta-r)(1+v)+r\left(k(k-1)v-n(1+v)\right)\right]} \in (S_1, S_2)$$

that is globally asymptotically stable with

$$\lim_{t\to\infty} S^*(t) = S_1^\infty, \quad \forall S_0 > 0;$$

(ii) For $\delta S_y > Q_v = \frac{\left((k+n-k^2)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)b}$, there exist three positive stationary asset stocks, denoted by S_1^∞, S_2^∞, and S_3^∞ with $S_1 < S_1^\infty < S_2 < S_2^\infty < S_3^\infty$, where

$$S_2^\infty = \frac{a\left((k+n-k^2)v+n\right)}{b\delta\left((k+n+1-k^2)v+n+1\right)}, \quad S_3^\infty = 1 - \frac{a(1-S_y)\left((k+n-k^2)v+n\right)}{b\delta S_y\left((k+n+1-k^2)v+n+1\right)}.$$

For any initial stock $S_0 \in (0, S_2^\infty)$, the asset stock converges to S_1^∞, while for any $S_0 \in (S_2^\infty, \infty)$, the asset stock converges to S_3^∞.

Proof. See Appendix B. □

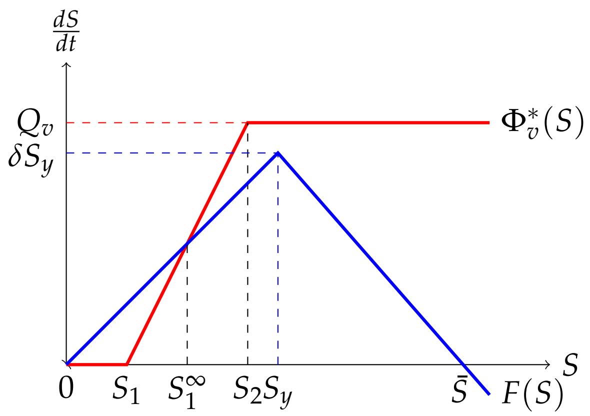
Corollary 1 demonstrates that when the static Cournot industry output with cross-ownership is larger than the maximum sustainable yield as in case (i), there is a unique positive steady-state stock to which the MPNE path of the asset's stock converges. However, if the maximum sustainable yield exceeds the static Cournot industry output with cross-ownership as in case (ii), there exist three positive steady-state stocks with

[7]Note that we would need the assumption made from Assumption 1 to ensure that

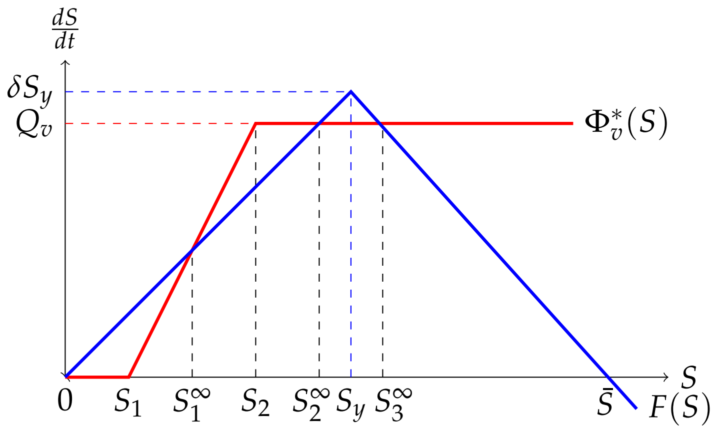
$$S_y > S_2 \iff \delta > \frac{a\left[\left(k(k-1)v-n(1+v)\right)^2+(1+v)^2\right]}{bS_y\left((k+n+1-k^2)v+n+1\right)^2}, \forall\, v \in (0, \frac{1}{k-1})$$

or equivalently, $\delta > \frac{a(n^2+1)}{bS_y(n+1)^2}$. When $S_y < S_2$, the N-tuple vector of closed-loop strategies $(\phi_i^*, \cdots, \phi_i^*, \phi_o^*, \cdots, \phi_o^*)$ is not a global MPNE.

the smallest and largest being stable and the middle one unstable. The above findings in cases (i) and (ii) of Corollary 1 are illustrated in Figure 2(a) and Figure 2(b), respectively.



(a) $\delta S_y < Q_v$



(b) $\delta S_y > Q_v$

Figure 2: The steady-state stocks

Note that when the initial resource stock and the implicit growth rate of the asset are large enough (i.e., $S_0 > S_2^\infty$ and $\delta > \frac{\left((k+n-k^2)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)bS_y}$), exploiting the asset at a rate corresponding to the static Cournot equilibrium with cross-ownership $(q_i^v, \cdots, q_i^v, q_o^v, \cdots, q_o^v)$ is sustainable as a MPNE. Firms can play this equilibrium endlessly. Interestingly, the steady state level of the asset stock S_3^∞ in this case does not depend on the discount rate r, which implies that the resource dynamics plays no role. However, when the implicit growth rate of the asset is not high enough (i.e., $\delta < \frac{\left((k+n-k^2)v+n\right)a}{\left((k+n+1-k^2)v+n+1\right)bS_y}$), playing the static Cournot equilibrium with cross-ownership is not sustainable. While firms can still adopt such rates of extraction if the initial asset stock is above S_2^∞, these production rates will only last for a finite period of time, after which firms will reduce their production as the stock level of the asset falls below S_2^∞ and eventually converges to $S_1^\infty \in (S_1, S_2)$.

3 The short-run effects of cross-ownership

We now use the MPNE characterized above to study the impact of cross-ownership on firms' production and the profitability of cross-ownership in the context of a common property renewable resource oligopoly. We define the profitability of cross-ownership as the difference between the equilibrium discounted sum of operating profits with and without cross-ownership. We then compare it with the static case and show that the cross-ownership paradox result may not hold in a renewable resource industry. Specifically, we show both analytically and numerically that there always exists an interval of the renewable resource stock for which a k-symmetric cross-ownership can be profitable, even though such rival cross-shareholdings are unprofitable in the corresponding static equilibrium framework.

It should be noted that throughout this section, the analysis is conducted in the short run, i.e., in the neighbourhood of a given initial resource stock S_0. Another crucial assumption we make in what follows is that rival cross-shareholdings occur only at time $t = 0$ with $S_0 > 0$ and these shareholdings are irreversible. This is an important consideration in our dynamic framework because the timing of cross-ownership can become an issue as opposed to the purely static case. A symmetric cross-ownership might be profitable for only a finite period, after which it would be dissolved. Hence, it suffices to confine our focus to such a situation, as our primary aim is to illustrate circumstances under which the profitability of cross-ownership is positive in our dynamic framework but would not be in the static counterpart.

3.1 Cross-ownership and firms' output

First, we compare the equilibrium strategies under cross-ownership with the ones without cross-ownership. The latter is obtained by setting $v = 0$ in Proposition 1, which is given by

$$\phi_c^*(S) = \begin{cases} 0 & \text{for } 0 \leq S \leq S_{1,N}, \\ \frac{a-(XS+Y)}{(n+1)b} & \text{for } S_{1,N} < S \leq S_{2,N}, \\ q_c = \frac{a}{(n+1)b} & \text{for } S > S_{2,N}, \end{cases}$$

where $X, Y, S_{1,N}$ and $S_{2,N}$ are constants that depend on the parameters of the model, obtained when setting $v = 0$ in A, B, S_1 and S_2, respectively. Let Ω_i and Ω_o denote the positive slopes of the equilibrium feedback strategies of a typical insider firm $\phi_i^*(S)$ and an outsider firm $\phi_o^*(S)$ under cross-ownership when $S \in [S_1, S_2]$, and denote by Ω_c the positive slope of the benchmark equilibrium strategy $\phi_c^*(S)$ when $S \in [S_{1,N}, S_{2,N}]$, respectively.

We distinguish between the case of $n = k$ in which all the firms in the industry engage in rival cross-shareholdings from the case of $n > k$ where only a subset of firms hold shares in each other.[8] We have the following results:

Lemma 1. *For any* $0 < v < \frac{1}{k-1}$ *and* $n = k \geq 2$, $S_1 > S_{1,N}, S_2 < S_{2,N}, q_c > q_i^v$, *and* $\Omega_i > \Omega_c$.

Lemma 2. *For any* $0 < v < \frac{1}{k-1}$ *and* $n > k \geq 2$, $S_1 > S_{1,N}$, $S_2 < S_{2,N}$, $q_i^v < q_c < q_o^v$, *and* $\Omega_i < \Omega_c < \Omega_o$.

Proof. See Appendix C. □

Lemma 1 and 2 show that when competing firms in the same industry hold shares in each other, it has the following impact: (i) the range of asset stocks for which firms adopt

[8] When $n = k$, the set O is empty. Thus by definition, ϕ_o^* does not exist as there are no outsiders. So we only have ϕ_i^* in the case of n-symmetric cross-ownership, as if firms were merging.

the positive and increasing Markov strategies ($[S_1, S_2]$) shrinks due to the fact that the maturity threshold (S_1) increases while the threshold beyond which firms commit to the static Cournot strategies (S_2) decreases; (ii) for $S \in [S_1, S_2]$ and any $n = k$, i.e., when all firms engage in cross-ownership, the linear curve of each firm becomes steeper; (iii) for $S \in [S_1, S_2]$ and any $n > k$, the outsiders become more responsive to changes in the asset's stock while the insiders become less responsive, i.e., the linear curve of a typical outsider becomes steeper while that of an insider becomes flatter; (iv) the result that q_i^v decreases and q_o^v increases directly follows from the static theory: when firms hold an ownership stake in their competitors, they will compete less aggressively and thus reduce their outputs, because their profit gains may come at the loss of other firms in which they have shareholdings. But in terms of strategic substitutes in Cournot competition, outsiders will respond by increasing their production.

3.1.1 The case of $n = k$

Proposition 2. *For any $n = k \geq 3$ and $v \in (0, \hat{v})$, there exists a $\hat{S}_1 \in (S_1, S_2)$ and a $\hat{S}_2 \in (S_2, S_{2,N})$ such that $\phi_i^*(S) > \phi_c^*(S)$ if and only if $\hat{S}_1 < S < \hat{S}_2$, where $\hat{v} = \frac{(n+1)\left(r-2\delta+n(\delta-r)\right)}{(\delta-r)\left(n^2(n-2)\right)+2\delta-r+n\delta}$.*

Proof. See Appendix D. □

Proposition 2 states that under some conditions, there exists an interval of initial resource stocks for which the cross-owner participants may increase their production following the symmetric cross-ownership. This is illustrated in panel (a) of Figure 3, which plots all the possible scenarios for the individual production strategies in the cases with and without cross-ownership when all the firms in the industry engage in rival cross-shareholdings. This finding is quite surprising and goes against the static oligopoly theory and cross-ownership theory. Indeed, when all firms participate in cross-ownership, they have an incentive to reduce competition as one firm's gain may come at the loss of the other firms it has shareholdings. As such, every cross-ownership participant unilaterally reduces its output in terms of Cournot competition. However, in our context where an oligopoly exploits a common property productive asset, there is another channel through which cross-ownership affects firms' extraction rate, in addition to the reduced competition at the level of the output market. That is, cross-ownership also affects how firms interact with each other at the resource level.

To explain this further, note that the value function of a typical insider firm $i \in I$, which corresponds to the sum of the aggregate profits along the equilibrium path discounted to infinity at rate r when the asset's stock is S, is denoted by (See Proposition

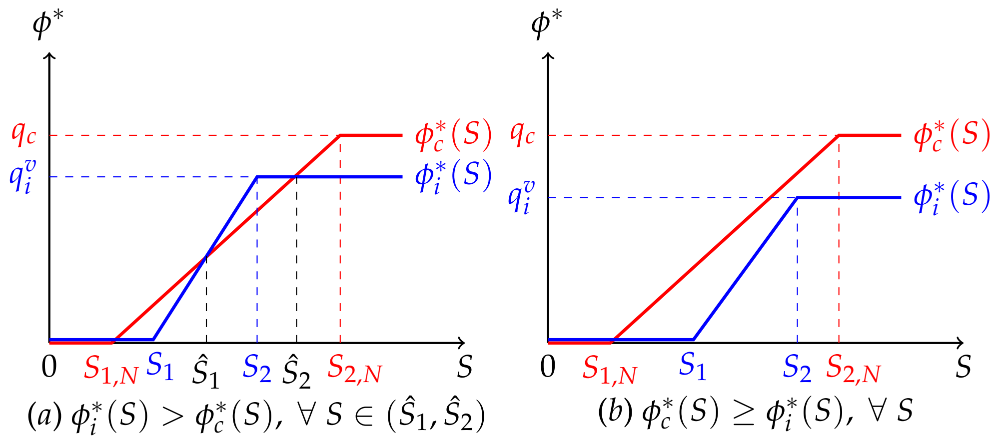


Figure 3: Individual MPNE with and without cross-ownership when $n = k$

1 and Appendix A)

$$V_i(S) = \begin{cases} \left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) & \text{for } 0 \leq S \leq S_1 \\ W(S) = \frac{A}{2}S^2 + BS + C & \text{for } S_1 < S \leq S_2 \\ \frac{\Pi_i}{r} & \text{for } S > S_2 \end{cases} \cdot$$

For $S > S_2$, the value function of a typical insider firm $i \in I$ is equal to the aggregate profit in the static Cournot equilibrium with cross-ownership discounted to infinity at r, independent of the asset's stock. This means that for any $S > S_2$, resource scarcity plays no role and the value of an insider firm is simply the discounted market rent due to the induced market power by cross-ownership. On the other hand for any $S < S_2$, the value function accounts for both the market rent and the resource rent, which depends on the stock of the asset. More specifically, the value to an insider firm $i \in I$ of an additional until of common stock S, or the resource rent, can be determined as

$$V_i'(S) = \begin{cases} \frac{r}{\delta S_1}\left(\frac{S}{S_1}\right)^{\frac{r}{\delta}-1} W(S_1) & \text{for } 0 \leq S \leq S_1 \\ W'(S) = AS + B & \text{for } S_1 < S \leq S_2 \\ 0 & \text{for } S > S_2 \end{cases} \cdot$$

It can be seen from the above that for $S < S_2$, the rent of an additional unit of stock to the insider is decreasing with the stock of the asset, and it tends to infinity as the stock approaches zero.

In the absence of resource rents, reduced competition in the output market due to cross-ownership leads to a decrease in the production of each firm. However, in the presence of resource scarcity, i.e., when the stock of the asset is below the threshold S_2 beyond which firms commit to the static Cournot strategies, the resource rent is positive and affects firms' production. The impact of rival cross-shareholdings on the resource rent manifests as a negative effect for relatively large stocks and a positive effect for relatively small stocks.

Notice that this specific outcome – the insiders may increase their production as a result of cross-ownership – occurs only when the number of firms is larger than two (i.e., $n = k \geq 3$) and the ownership stake is not large enough (i.e., $v \in (0, \hat{v})$. That is, cross-owners will only slightly reduce their production due to a relatively low shareholding in the output market, but this results in a sufficiently abundant stock of the asset, which in turn reduces the marginal valuation of the resource stock by each cross-ownership participant. A smaller resource rent thus incentivizes each cross-owner to expand its production. This output expansion is the resource rent effect of cross-ownership. Proposition 2 and Figure 3(a) show that the effect on individual firm's production of the change in the resource rent due to cross-shareholdings outweighs the effect of reduced competition at the output market for $n = k \geq 3, v \in (0, \hat{v})$ and $S \in (\hat{S}_1, \hat{S}_2)$.

Otherwise, for any $n = k \geq 3, v \in (0, \hat{v})$ and $S \notin (\hat{S}_1, \hat{S}_2)$, the result of $\phi_c^*(S) \geq \phi_i^*(S)$ holds. More specifically, for any given resource stock such that exploitation rates are strictly positive (i.e., $S > S_{1,N}$), $\phi_c^* > \phi_i^*$, i.e., interlock cross-ownership results in a lower production for each individual firm. This result is consistent with the traditional oligopoly theory, as when firms cooperate due to their ownership stakes, each firm will unilaterally reduce its output to lessen competition. The same also holds either when a duopoly controls the industry ($n = k = 2$) or when the number of firms in the industry is large but the ownership stake is high enough ($n = k \geq 3, v \in [\hat{v}, \frac{1}{n-1})$), as shown in Figure 3(b).[9] Colombo and Labrecciosa (2018) also find similar results in a duopoly setting. However, our result extends beyond their 2-firm case and holds in a more general oligopoly setting for a large enough ownership stake.

3.1.2 The case of $n > k$

Proposition 3. *For any $n > k$ and $0 < v < \frac{1}{k-1}$, there exists a $\tilde{S} \in (S_1, S_2)$ such that*

$$\phi_c^*(S) - \phi_o^*(S) \begin{cases} > 0 & if \;\; S_{1,N} < S < \tilde{S} \\ \leq 0 & if \;\; S \geq \tilde{S} \end{cases},$$

while for any $S > S_{1,N}$, $\phi_c^*(S) > \phi_i^*(S)$, and for any $S \leq S_{1,N}$, $\phi_i^*(S) = \phi_c^*(S) = \phi_o^*(S) = 0$.

Proof. Directly follow from Lemma 2. □

Figure 4 illustrates the findings of Proposition 3. Except for the region $S \leq S_{1,N}$ where both the insiders and outsiders keep their production at zero, the production of a typical insider is strictly lower than the one without cross-ownership, while the production of a typical outsider is smaller for low stocks ($S_{1,N} < S < \tilde{S}$) and larger for high stocks ($S > \tilde{S}$) than the case without cross-ownership. That is, there exists a range of initial resource stocks such that the outsiders may also lower their production as a result of cross-ownership. This result is quite counterintuitive, as one would expect that

[9] See Appendix D for more details.

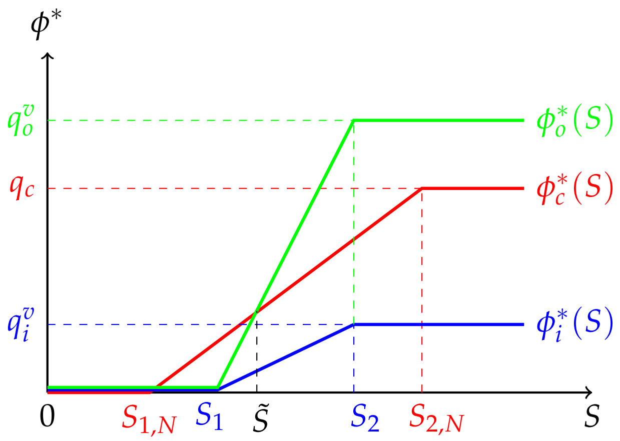


Figure 4: Individual MPNE with and without cross-ownership when $n > k$

in terms of strategic substitutes in Cournot competition, outsiders will always expand their production in response to the output reduction brought by cross-owners.

However, in our dynamic case where firms exploit a common stock while cross-ownership non-participants exhibit a more pronounced response to the asset's stock compared to the participants, decreasing production is also in the best interest of these outsiders for the relatively low levels of stocks ($S_{1,N} < S < \tilde{S}$). In particular, the outsiders will voluntarily cease their production for $S \leq S_1$ and leave the asset to grow. As the stock gradually increases within the range $(S_1, \tilde{S})$, the asset remains sufficiently scarce, leading to an increase in the marginal valuation of the resource stock for each cross-ownership non-participant. A higher resource rent thus provides an incentive for the outsiders to reduce their production. Proposition 3 and Figure 4 show that the output reduction resulting from the resource rent effect of cross-ownership dominates the static effect of production expansion for $S_{1,N} < S < \tilde{S}$.

3.2 Cross-ownership and industry output

Next, we compare the equilibrium industry outputs with and without cross-ownership. Recall that the total industry production with cross-ownership is denoted by (7) while the one without cross-ownership is given by

$$\Phi_c^*(S) = n\phi_c^*(S) = \begin{cases} 0 & \text{for } 0 \leq S \leq S_{1,N} \\ \frac{n\left(a-(XS+Y)\right)}{(n+1)b} & \text{for } S_{1,N} < S \leq S_{2,N} \\ Q_c = \frac{na}{(n+1)b} & \text{for } S > S_{2,N} \end{cases}.$$

Following Proposition 2, an immediate result can be established regarding comparing the industry equilibrium production strategies with and without cross-ownership when $n = k$, summarized in Corollary 2.

Corollary 2. *For any $n = k \geq 3$ and $v \in (0, \hat{v})$, there exists a $\hat{S}_1 \in (S_1, S_2)$ and a $\hat{S}_2 \in (S_2, S_{2,N})$ such that $\Phi_v^*(S) > \Phi_c^*(S)$ if and only if $\hat{S}_1 < S < \hat{S}_2$, where $\hat{v} =$*

$\frac{(n+1)\big(r-2\delta+n(\delta-r)\big)}{(\delta-r)\big(n^2(n-2)\big)+2\delta-r+n\delta}$.

Now, let us turn to the case of $n > k$. Denote by ζ_v and ζ_c the positive slopes of the industry production for the cases with and without cross-ownership, then we can easily observe:

Lemma 3. *For any $n \geq k \geq 2$ and $0 < v < \frac{1}{k-1}$, $Q_v < Q_c$ and $\zeta_v > \zeta_c$.*

Proof. See Appendix E. □

Further, we can establish the following proposition:

Proposition 4. *For any $n > k \geq 2$ and $v \in (0, \frac{1}{k-1})$, there exists a $\tilde{S}_1 \in (S_1, S_2)$ and a $\tilde{S}_2 \in (S_2, S_{2,N})$ such that $\Phi_v^*(S) > \Phi_c^*(S)$ if and only if $S \in (\tilde{S}_1, \tilde{S}_2)$.*

Proof. See Appendix F. □

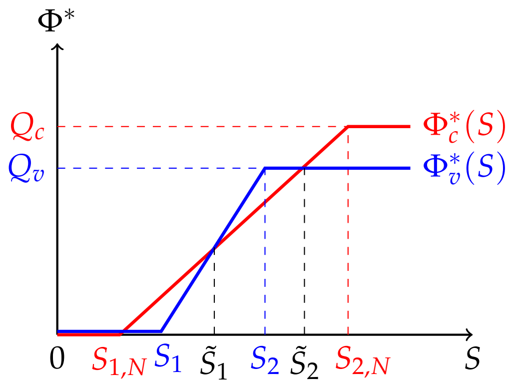


Figure 5: Industry production with and without cross-ownership when $n > k$

Similar to Corollary 2, Proposition 4 also demonstrates that there exists some range of stocks such that the total industry production can increase as a result of cross-ownership. This surprising result, illustrated in Figure 5, contradicts the traditional oligopoly and cross-ownership theory. Indeed, when firms acquire an ownership stake in their rivals, they compete less aggressively and thus unilaterally reduce their output. But in terms of strategic substitutes in Cournot competition, firms that do not participate in cross-ownership will respond by expanding their production. However, the output reduction brought by cross-owners will outweigh the outsiders' production expansion, leading to an overall decrease in industry production. This is the standard static result of output reduction brought by cross-ownership. But as explained earlier, in our dynamic context where an oligopoly exploits a common property renewable resource, the output expansion resulting from the resource rent effect due to the change in the marginal valuation of the resource stock by firms as a result of cross-ownership will dominate the static effect for any $S \in (\tilde{S}_1, \tilde{S}_2)$. Outside this range, the industry output is strictly lower following cross-ownership, except for $S \leq S_{1,N}$ where the production remains at zero.

3.3 The profitability of cross-ownership

We now move to conduct the profitability analysis of cross-ownership in a renewable resource industry and highlight the contrast with results from the static oligopoly theory. While in the case of $k = n$, one expects cross-ownership to be profitable; in the scenario where only a subset of firms are involved in partial cross-ownership ($k < n$), the profitability of cross-ownership is no longer clear.

In a static framework, three countervailing effects are in operation when firms decide to participate in cross-ownership in an oligopolistic market. One is the positive effect on cross-owners' profits due to the partial elimination of previous rivalry, the second is the negative effect of non-participants' production expansion in terms of strategic substitutability, and the last one is how aggressively outsiders will respond depending on the levels of shareholdings. The relative size of these three effects drives the final result concerning the profitability of cross-ownership. More specifically, firms can never profit from cross-shareholdings if $\frac{k}{n} \leq \frac{k}{2k-1}$, but will also have an incentive to do so if $\frac{k}{n} > \frac{k}{k+\sqrt{k}-1}$; for participation ratios in between the lower threshold $(\frac{k}{2k-1})$ and upper threshold $(\frac{k}{k+\sqrt{k}-1})$, there exists a large range of shareholdings for which a k-symmetric cross-ownership can be profitable (See Proposition 2 in Dai et al. (2022)).

However, as we shall show below, this static result may not necessarily carry over to the case of a renewable resource industry where the resource stock, if left unexploited, reproduces itself naturally at a rate that depends on the size of the stock. To see this, note that the equilibrium discounted sum of operating profits for a typical insider firm $i \in I$ under the $k-$symmetric cross-ownership structure is given by[10]

$$V_S = (1 - (k-1)v)V_i(S),$$

while that for an individual firm without cross-ownership is obtained by setting $v = 0$, denoted by

$$V_c = \begin{cases} \left(\frac{S}{S_{1,N}}\right)^{\frac{r}{\delta}} W_c(S_{1,N}) & \text{for } 0 \leq S \leq S_{1,N} \\ W_c(S) & \text{for } S_{1,N} < S \leq S_{2,N} \\ \frac{\pi_c}{r} & \text{for } S > S_{2,N} \end{cases},$$

where $W_c(S) = \frac{X}{2}S^2 + YS + Z, \pi_c = \frac{a^2}{b(n+1)^2}$, and X, Y, Z are obtained when setting $v = 0$ in A, B, C, respectively. Then, a $k-$symmetric cross-ownership is profitable when

$$G(k, n, v, S) = V_S - V_c > 0.$$

Clearly, the profitability of cross-ownership in a common pool productive asset depends on k, n, v, but also crucially depends on the asset stock S.

[10]Recall that $V_i(S)$ corresponds to the equilibrium discounted sum of aggregate profits or accounting profits for a typical insider firm, which includes not only the profits from its own operations but also the share of profits in other firms. We use operating or economic profits rather than accounting profits to define profitability.

It is useful to distinguish five regions for the resource stocks, with Region I: $S \in [0, S_{1,N})$, Region II: $S \in [S_{1,N}, S_1)$, Region III: $S \in [S_1, S_2)$, Region IV: $S \in [S_2, S_{2,N})$, and Region V: $S \in [S_{2,N}, \infty)$. The profitability function $G(k, n, v, S)$ can then be expressed as

$$G(k,n,v,S) = \begin{cases} (1-(k-1)v)\left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) - \left(\frac{S}{S_{1,N}}\right)^{\frac{r}{\delta}} W_c(S_{1,N}) & \text{for } 0 \leq S < S_{1,N} \\ (1-(k-1)v)\left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) - W_c(S) & \text{for } S_{1,N} \leq S < S_1 \\ (1-(k-1)v)W(S) - W_c(S) & \text{for } S_1 \leq S < S_2 \\ \frac{\pi_i^v}{r} - W_c(S) & \text{for } S_2 \leq S < S_{2,N} \\ \frac{\pi_i^v}{r} - \frac{\pi_c}{r} & \text{for } S \geq S_{2,N} \end{cases} .$$

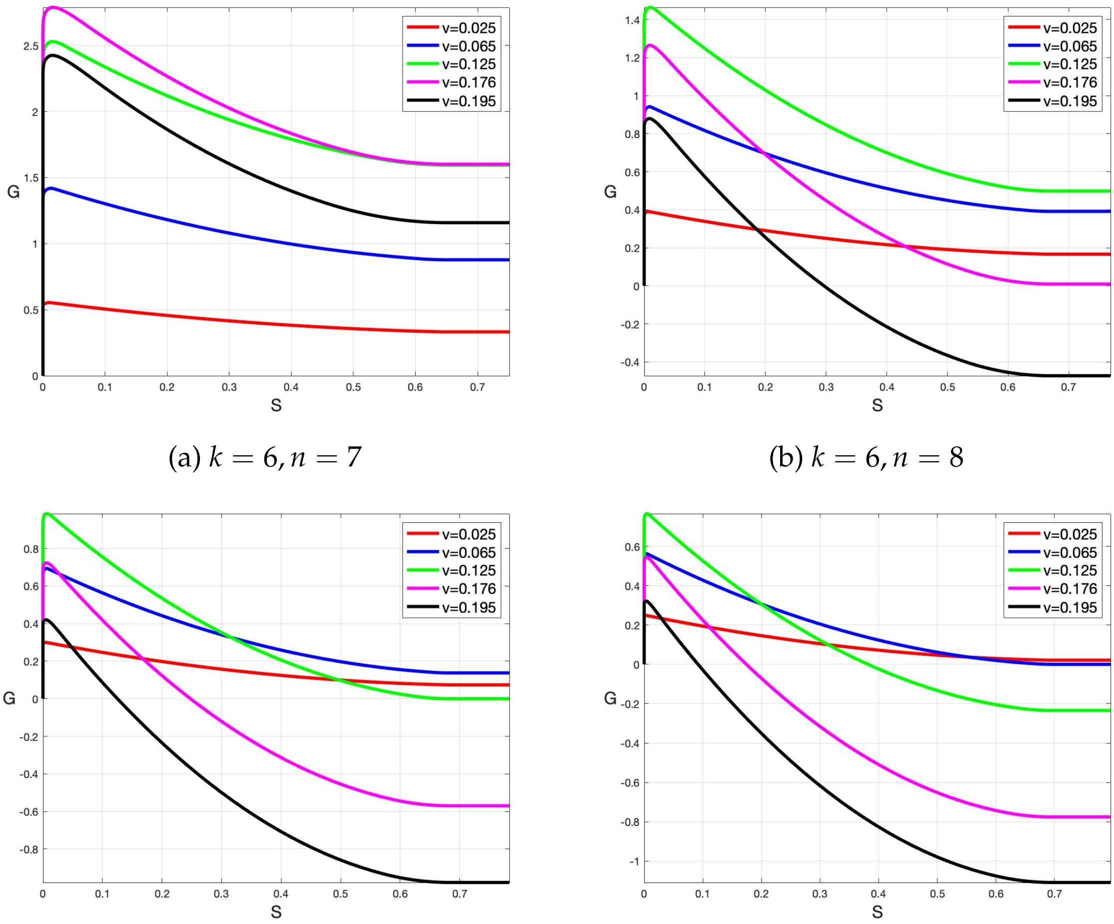
It can be easily observed that for any $S \geq S_{2,N}$, the dynamic profitability is simply equal to the static profitability discounted to infinity at rate r. Therefore, the cross-ownership paradox also holds in the dynamic equilibrium for any S that is large enough. However, our dynamic profitability holds beyond this static result. More specifically, we have

Result 1. *If a k-symmetric cross-ownership is profitable in the static Cournot equilibrium, it will also be profitable in the dynamic equilibrium for all S.*

To illustrate this result, we plot in Figure 6 the dynamic profitability as a function of the resource stock when the participation ratio satisfies $\frac{k}{n} > \frac{k}{2k-1}$. Using parameter values of $a = 5, b = 0.5, r = 0.15, S_y = 0.75, \delta = 12$, we fix $k = 6$ and vary $n = 7, 8, 9, 10$ and plot G as a function of S for different levels of ownership v. When k is fixed at 6, the static profitability of cross-ownership is always positive for any admissible $v \in (0, \frac{1}{k-1})$ when $n = 7$, while for $n = 8, 9, 10$, it is positive if $v < 17.6\%, 12.5\%, 6.5\%$, respectively (Dai et al., 2022). Further, we can observe that this static result also carries over to the renewable resource industry for all S as long as v satisfies the corresponding conditions throughout Figures 6(a), 6(b), 6(c) and 6(d). Simulations using many other combinations of k and n satisfying $\frac{k}{n} > \frac{k}{2k-1}$ also find such findings, suggesting that Result 1 is quite robust.

Moreover, a closer look at Figure 6 seems to indicate that there exists a range of initial resource stocks such that the symmetric cross-ownership can be profitable, even though it is unprofitable in the corresponding static framework.[11] For instance, in the case of $k = 6$ and $n = 8$, the static profitability is negative if each of the 6 firms holds more than 17.6%, but as shown in Figure 6(b), the symmetric cross-ownership can be profitable even for $v > 17.6\%$ over some range of resource stocks. Similarly as illustrated in Figure 6(c), the dynamic profitability for $v > 12.5\%$ when $k = 6$ and $n = 9$ is positive over some interval of the initial resource stocks, while such cross-shareholdings are not profitable in the static model. Moreover, compared to the static case where the symmetric cross-ownership is not profitable for $v > 6.5\%$ when $k = 6$

[11] It should be noted that the segment for G when S is small is not a vertical line. This is because the values of $S_{1,N}$ and S_1 are relatively too small compared to S_2 and $S_{2,N}$. Trying to plot the whole region will result in such a display.



(a) $k = 6, n = 7$

(b) $k = 6, n = 8$

(c) $k = 6, n = 9$

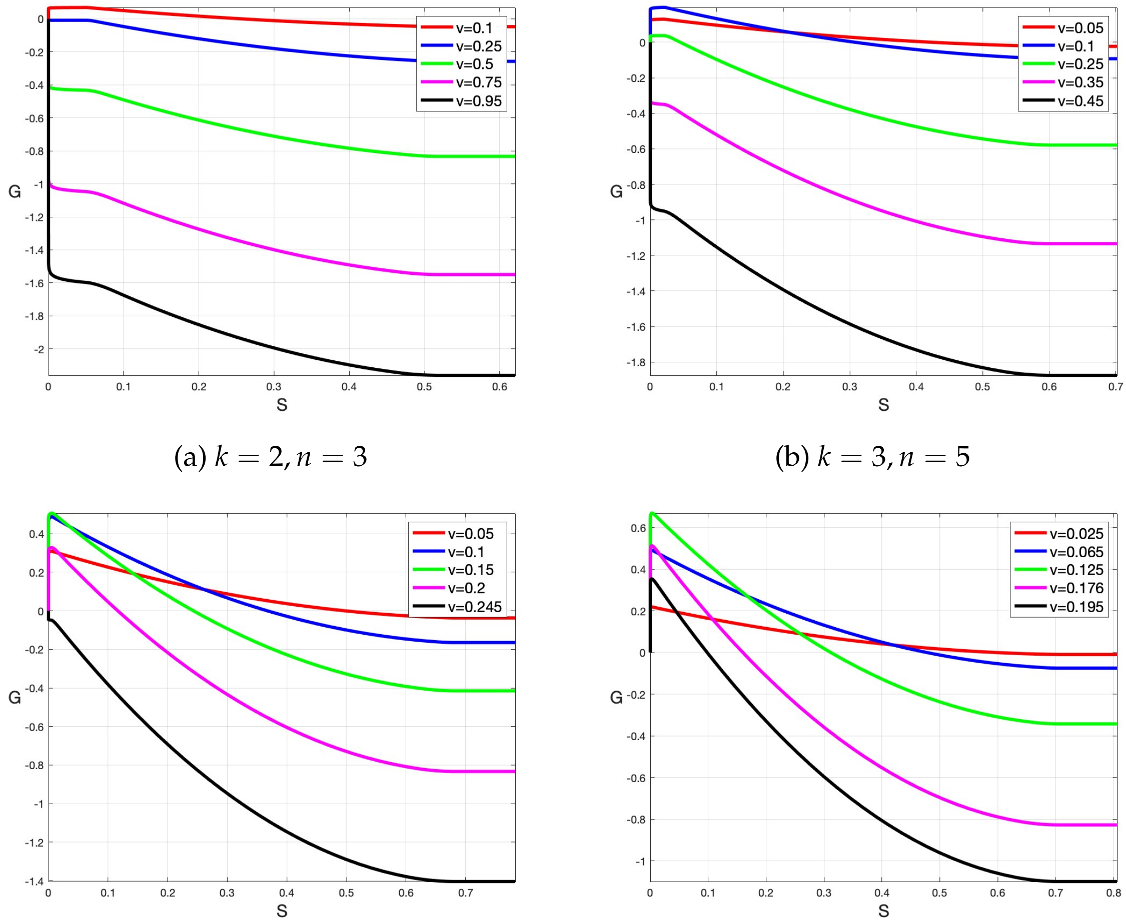
(d) $k = 6, n = 10$

Figure 6: Dynamic profitability as a function of S when $\frac{k}{n} > \frac{k}{2k-1}$

and $n = 10$, it can be profitable in the dynamic model for some initial resource stocks, as demonstrated in Figure 6(d). In addition, the larger the shareholding, the smaller the range of initial resource stocks for which the dynamic profitability is positive.

We now move to check whether these findings can also hold when the participation ratio satisfies $\frac{k}{n} \leq \frac{k}{2k-1}$ in which the static profitability is strictly negative for any admissible $v \in (0, \frac{1}{k-1})$. Using the same parameter values as in Figure 6, Figure 7 illustrates the dynamic profitability G as a function of the initial resource stock S for different levels of shareholdings v when $\frac{k}{n} \leq \frac{k}{2k-1}$. While Figures 7(a), 7(b), 7(c) show that for some large level of ownership v, firms can never profit from rival cross-shareholdings for all S, we also observe that there always exists some range of initial stocks such that the profitability of cross-ownership is positive, even in the least possible case of $k = 2$ and $n = 3$. In addition, the range of resource stocks for which a k-symmetric cross-ownership is profitable shrinks as v increases. Simulations using a wide range of k and n that satisfy $\frac{k}{n} \leq \frac{k}{2k-1}$ also support these findings. We now formally summarize these results in below:

Result 2. *There exists an interval of stocks such that a k-symmetric cross-ownership can be profitable, even though such rival cross-shareholdings are strictly unprofitable in the corresponding*



(a) $k = 2, n = 3$

(b) $k = 3, n = 5$

(c) $k = 5, n = 9$

(d) $k = 6, n = 11$

Figure 7: Dynamic profitability as a function of S when $\frac{k}{n} \leq \frac{k}{2k-1}$

static equilibrium framework. Moreover, this interval decreases in the level of shareholdings.

Up until now, we have not yet explained the difference in the profitability of cross-ownership between the static and dynamic frameworks. However, as already mentioned earlier, a key factor that drives the significantly different results is the presence of a scarcity rent on the renewable resource, which is absent in the static model. Note that from the first-order conditions of the Hamilton-Jacobi-Bellman (HJB) equations to the problems of (1)–(4), the best response functions for an insider firm $i \in I$ and an outsider firm $o \in O$ are respectively given by[12]

$$\phi_i = \frac{\big(1-(k-2)v\big)a - \big(1-(k-2)v\big)(n-k)b\phi_o - (1+v)\big(1-(k-1)v\big)V_i'(S)}{\big[1+k+\big(1-k(k-2)\big)v\big]b}, \quad (8)$$

and

$$\phi_o = \frac{a - bk\phi_i - V_o'(S)}{b(n-k+1)}, \quad (9)$$

where $V_i'(S) = \frac{\partial V_i(S)}{\partial S}$ and $V_o'(S) = \frac{\partial V_o(S)}{\partial S}$ denote the resource rent or the marginal valuation of an additional unit of the resource stock by cross-ownership participants

[12]See Appendix A for more details.

and non-participants, respectively. In the purely static Cournot model with cross-ownership, both the terms $V_i'(S)$ and $V_o'(S)$ are 0, as by definition firms' production decisions are independent of S. The corresponding pair of reaction functions thus become

$$\phi_i = \frac{(1-(k-2)v)a - (1-(k-2)v)(n-k)b\phi_o}{\left[1+k+\left(1-k(k-2)\right)v\right]b}, \quad \phi_o = \frac{a - bk\phi_i}{b(n-k+1)},$$

in which the best response of an outsider firm to the change in the production of an insider firm due to cross-ownership, or vice versa, is simply the movement along the reaction function. However, in our dynamic framework with the presence of resource rents ($V_i'(S) > 0, V_o'(S) > 0$) as shown in (8) and (9), attempting to move along the reaction functions due to cross-shareholdings will unexpectedly lead to a shift in those reaction functions, because there exists a dynamic link between the level of stock to the rate of production through the growth function in (2) and (4). This additional feature is highly relevant to the profitability analysis of cross-ownership in a renewable resource industry.

To see how these reaction functions are shifting, consider the impact of cross-ownership on the resource rents over the interval (S_1, S_2), where

$$V_i(S) = \frac{1}{2}AS^2 + BS + C, \quad V_o(S) = \frac{1}{2}DS^2 + ES + F,$$

and the resource rents for a typical insider firm $i \in I$ and a typical outsider firm $o \in O$ are respectively given by

$$V_i'(S) = \frac{\partial V_i(S)}{\partial S} = AS + B, \quad V_o'(S) = \frac{\partial V_o(S)}{\partial S} = DS + E.$$

It would be ideal to explicitly demonstrate how the resource rents are changing with v, but the equations of $\frac{\partial V_i'(S)}{\partial v}$ and $\frac{\partial V_o'(S)}{\partial v}$ are rather cumbersome. To exemplify the concept, we consider without loss of generality the case of $k = 2$ and $n = 3$, which is supposedly the least likely to be profitable. Then

$$\frac{\partial V_i'(S)}{\partial v} = \frac{-4b(2\delta - r)(v+2)(v^3+4v^2+6v+1)}{(1-v)^2(1+v)^2(v+3)^3}\left(S - S_{iR}\right),$$

$$\frac{\partial V_o'(S)}{\partial v} = \frac{-4b(2\delta - r)(v+2)}{(v+3)^3}\left(S - S_{oR}\right),$$

where

$$S_{iR} = \frac{a(v^4+6v^3+16v^2+16v+1)}{2b\delta(v+2)(v^3+4v^2+6v+1)}, \quad S_{oR} = \frac{a(1+v)}{2b\delta(v+2)}.$$

It can be easily checked that for any $v \in (0, \frac{1}{k-1})$, $S_1 < S_{oR} < S_{iR} < S_2$. Therefore, we have

$$\frac{\partial V_i'(S)}{\partial v} \begin{cases} > 0 & \text{for } S_1 < S < S_{iR} \\ = 0 & \text{for } S = S_{iR} \\ < 0 & \text{for } S_{iR} < S < S_2 \end{cases}, \tag{10}$$

and

$$\frac{\partial V_o'(S)}{\partial v} \begin{cases} > 0 & \text{for } S_1 < S < S_{oR} \\ = 0 & \text{for } S = S_{oR} \\ < 0 & \text{for } S_{oR} < S < S_2 \end{cases}. \tag{11}$$

Conditions (10) and (11) indicate that at a relatively large stock, i.e., $S \in (S_{iR}, S_2)$, engaging in cross-shareholdings will result in a decrease in both the insiders and outsiders' marginal valuation of the resource stock. At the same time, the reverse is true at a relatively small level of stock for $S \in (S_1, S_{oR})$. In addition, for some intermediate levels of stock, i.e., $S \in (S_{oR}, S_{iR})$, cross-ownership between rival firms leads to an increase in the cross-owners' marginal valuation of the resource stock but a reduction in the resource rent of the outsiders.

When a subset of competitors engage in rival cross-shareholdings, this results in an increase of the resource rents for these participants but also for those outsiders if the initial resource stock is relatively small, i.e., $S \in (S_1, S_{oR})$. Unlike in the static case where outsiders respond aggressively by increasing their production to mitigate any profit gains of insiders through output reduction due to ownership links, the existence of resource rent could attenuate such an increase and might even lead to a reduction in outsiders' production. This latter case can well occur when $S \in (S_1, \tilde{S})$, as discussed earlier in Proposition 3. Consequently, this engenders a more cautious response from outsiders, in which the optimal production of an outsider firm $o \in O$, given its rivals' production, tends to be lower compared to the static scenario where resource rents are absent. The 'moderation' of the outsiders' response to a reduction in production by the insiders, influenced by the existence of the resource rent, elucidates the fact that there exists a stock range within which a symmetric cross-ownership can be profitable, despite such rival cross-shareholdings being unprofitable in the corresponding static equilibrium framework. A similar result can be found in Benchekroun and Gaudet (2015), who find that there always exists an interval of the asset's stock such that any merger is profitable. It should be noted that the above analysis is conducted for the least possible profitable case: $k = 2$ and $n = 3$. Similar findings can be also obtained using other combinations k and n satisfying $\frac{k}{n} \leq \frac{k}{2k-1}$ that is strictly unprofitable for any admissible $v \in (0, \frac{1}{k-1})$, or any $\frac{k}{n} > \frac{k}{2k-1}$ that is unprofitable for $v \in (\bar{v}, \frac{1}{k-1})$ in the static framework.

Result 1 and Result 2 also help us clarify the question of whether the output expansion resulting from cross-ownership can actually occur, as firms may never find it profitable to engage in rival cross-shareholdings in the first place. For the case of $n = k$

in Corollary 2, the answer is obvious, since a symmetric industry-wide cross-ownership is always profitable for all S. Consequently, firms will always find it profitable to engage in cross-shareholdings for any S, and thus there will always exist a range of $(\hat{S}_1, \hat{S}_2)$ such that the industry production increases as a result of profitable cross-ownership for any $n = k \geq 3$ and $v \in (0, \hat{v})$. As for the case of $n > k$ in Proposition 4, it is less straightforward. To illustrate this, we plot in Figure 8 both the industry outputs with and without cross-ownership and the dynamic profitability as a function of S, using the same parameter values as in Figure 6 but fixing $v = 0.15$ for $k = 6, n = 10$ in Figure 8. It can be easily observed from Figure 8 that the range of resource stocks $(\tilde{S}_1, \tilde{S}_2)$

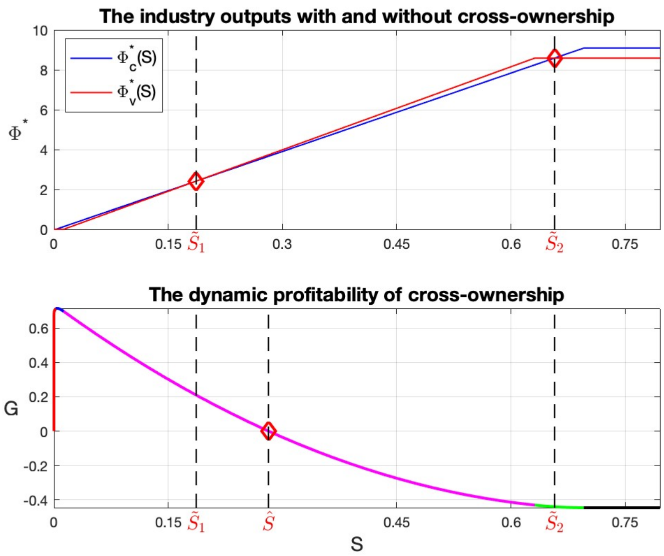


Figure 8: Industry output and dynamic profitability for $k = 6, n = 10$

for which the industry output increases following rival cross-shareholdings intersect with the interval of resource stocks $(0, \hat{S})$ for which a k-symmetric cross-ownership is profitable, where $G(k, n, v, \hat{S}) = (1 - (k-1)v)W(\hat{S}) - W_c(\hat{S}) = 0$. That is, for any $S \in (\tilde{S}_1, \hat{S})$, not only is it profitable for firms to engage in rival cross-shareholdings, but also the industry production will increase as a result of cross-ownership for any $n > k$ and $v \in (0, \frac{1}{k-1})$.

3.4 The short-run welfare implications

In the preceding analysis, we have illustrated the private incentives that motivate rival firms to engage in cross-shareholding. Additionally, we have demonstrated that industry output can increase following profitable rival cross-shareholdings. One direct implication of these results is that consumer surplus might also increase as a result of cross-ownership. This aspect holds significant relevance for discussions surrounding competition policies, as there is a growing call for more stringent regulations of these non-controlling minority shareholdings that are currently subject to a very lenient approach by antitrust authorities. In this subsection, we examine the welfare implications of cross-ownership in the context of a renewable resource industry, where welfare is defined as the sum of consumer surplus (CS) and producer surplus (PS) or industry profits. The latter is defined as the sum of the operating profits of the cross-ownership participants that belong to the subset I of insiders and of the non-participants that belong to the subset O of outsiders.

From our earlier analysis, we know that starting from any initial resource stock $S \in (\hat{S}_1, \hat{S}_2)$ when $n = k \geq 3$ and $v \in (0, \hat{v})$, or any $S \in (\tilde{S}_1, \hat{S})$ when $n > k$, the industry production expands and thus CS increases following a profitable cross-ownership. We now show that industry profits can also go up for these cases. The PS generated by the exploitation of the common property renewable resource under the k-symmetric cross-ownership structure is given by

$$PS_v = kV(S) + (n-k)V_o(S) = \begin{cases} \frac{\left(1-(k-1)v\right)\left((-k^2+n+k)v+n\right)}{1-(k-2)v} \left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) & \text{for } 0 \leq S \leq S_1 \\ \frac{\left(1-(k-1)v\right)\left((-k^2+n+k)v+n\right)}{1-(k-2)v} W(S) & \text{for } S_1 < S \leq S_2 \\ \frac{k\pi_i^v + (n-k)\pi_o^v}{r} & \text{for } S > S_2 \end{cases},$$

while the one without cross-ownership is denoted by

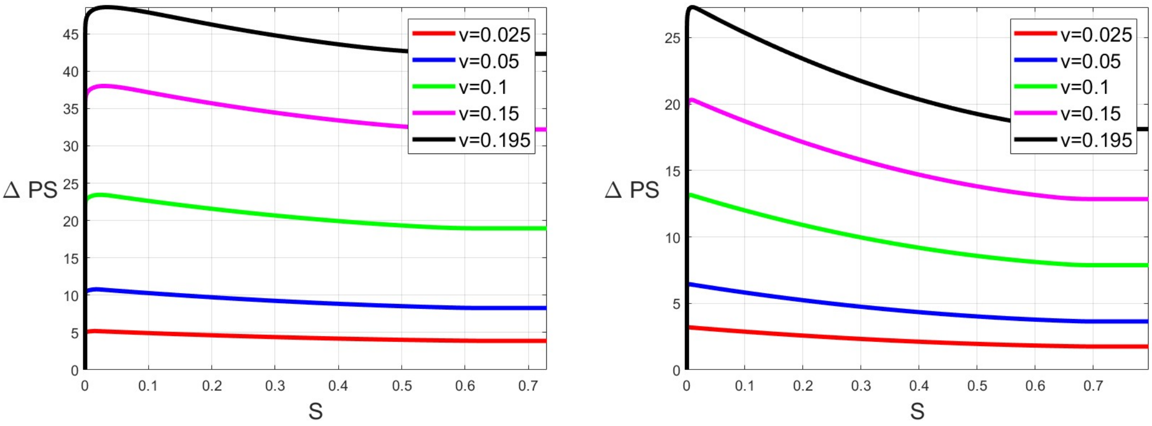
$$PS_c = nV_c = \begin{cases} n\left(\frac{S}{S_{1,N}}\right)^{\frac{r}{\delta}} W_c(S_{1,N}) & \text{for } 0 \leq S \leq S_{1,N} \\ nW_c(S) & \text{for } S_{1,N} < S \leq S_{2,N} \\ \frac{n\pi_c}{r} & \text{for } S > S_{2,N} \end{cases}.$$

Thus, the change in PS can be defined as

$$\begin{aligned} \Delta PS &= PS_v - PS_c \\ &= \begin{cases} \frac{\left(1-(k-1)v\right)\left((-k^2+n+k)v+n\right)}{1-(k-2)v} \left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) - n\left(\frac{S}{S_{1,N}}\right)^{\frac{r}{\delta}} W_c(S_{1,N}) & \text{for } 0 \leq S < S_{1,N} \\ \frac{\left(1-(k-1)v\right)\left((-k^2+n+k)v+n\right)}{1-(k-2)v} \left(\frac{S}{S_1}\right)^{\frac{r}{\delta}} W(S_1) - nW_c(S) & \text{for } S_{1,N} \leq S < S_1 \\ \frac{\left(1-(k-1)v\right)\left((-k^2+n+k)v+n\right)}{1-(k-2)v} W(S) - nW_c(S) & \text{for } S_1 \leq S < S_2 \\ \frac{k\pi_i^v + (n-k)\pi_o^v}{r} - nW_c(S) & \text{for } S_2 \leq S < S_{2,N} \\ \frac{k\pi_i^v + (n-k)\pi_o^v}{r} - \frac{n\pi_c}{r} & \text{for } S \geq S_{2,N} \end{cases}. \end{aligned}$$

Using the same parameter values as in Figure 6, we plot the PS change as a function of the initial stock S for different levels of v when $k = 6, n = 6$ in Figure 9(a) and when $k = 6, n = 10$ in Figure 9(b). It can be easily observed that for all S, the change in PS is positive. Simulations using many other combinations of k, n and v also show the same result. This indicates that for the above-mentioned two scenarios, a profitable cross-ownership can not only increase industry production and CS, but also boost industry profits, leading to a higher overall welfare.

Result 3. *Profitable rival cross-shareholdings can lead to a higher CS, PS and welfare in the short run if any of the following scenarios occurs:*



(a) $k = 6, n = 6$ (b) $k = 6, n = 10$

Figure 9: The change in PS as a function of S

(i) $n = k \geq 3$, $v \in (0, \hat{v})$ *and* $S \in (\hat{S}_1, \hat{S}_2)$, *where* $\hat{S}_1 \in (S_1, S_2)$,$\hat{S}_2 \in (S_2, S_{2,N})$ *and* $\hat{v} = \frac{(n+1)\left(r-2\delta+n(\delta-r)\right)}{(\delta-r)\left(n^2(n-2)\right)+2\delta-r+n\delta}$;

(ii) $n > k \geq 2$, *and* $S \in (\tilde{S}_1, \hat{S})$, *where* $\tilde{S}_1 \in (S_1, S_2)$ *and* $G(k, n, v, \hat{S}) = 0$ *with* $\hat{S} \in (\tilde{S}_1, S_2)$.

This result is in sharp contrast with the static oligopoly and cross-ownership theory, according to which cross-ownership always leads to a welfare loss in the absence of any efficiency gains. Indeed, when firms engage in rival cross-shareholdings, they tend to compete less aggressively with each other and thus unilaterally reduce their production. This happens because any increase in the acquiring firm's activities could diminish the returns from its shareholdings in the target firm. Although the outsider firms that are not involved in cross-shareholdings respond by increasing their production, the reduction in outputs from the cross-owners more than offsets this increase. As a result, the total industry output falls and the market price increases. While this benefits the industry by boosting profits, it reduces consumer surplus. However, the loss in consumer surplus dominates the gains in industry profits, resulting in an overall welfare loss. But in our dynamic framework where an oligopoly exploits a common pool productive asset, the presence of the resource rent effect dominates this standard static market power effect conferred by cross-ownership for the above-stated scenarios in Result 3. The former increases production, which outweighs the output reduction induced by the latter, leading to a higher industry output and thus CS. The industry profits also increase, because both insiders and outsiders expand their production at a slightly decreased price due to a relatively moderate response. Consequently, the social welfare is higher in the short run following the profitable rival cross-shareholding activities. This result thus suggests that competition authorities should be cautious when ruling in the renewable resource sector, as cross-ownership may turn out to be welfare-improving.

4 The long-run impact of cross-ownership

In this section, we explore the effects of cross-ownership on the long-run steady-state resource stocks, industry outputs, profitability, and social welfare. More specifically, we compare the outcomes under the k- symmetric cross-ownership structure ($v > 0$) and the one without cross-ownership ($v = 0$), and then we characterize conditions under which cross-ownership may lead to an increase in the industry output and social welfare at the steady state.

4.1 The effects on steady-state resource stocks and industry outputs

We start by analyzing the impact of cross-ownership on the productive asset's stock at the steady state and the industry's production. First, note that

Lemma 4. *For any* $2 \leq k \leq n$ *and* $0 < v < \frac{1}{k-1}$, $S_1^\infty > S_{1,N}^\infty$, $S_2^\infty < S_{2,N}^\infty$ *and* $S_3^\infty < S_{3,N}^\infty$.

Proof. See Appendix G. □

As discussed earlier in Corollary 1, the steady state level of the asset depends crucially on the initial resource stock. We can thus distinguish the following three cases. First, let us consider

$$\delta S_y < Q_v = \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b} < Q_c = \frac{an}{b(n+1)}, \tag{LC1}$$

in which there is only one positive stationary asset stock in both the cases with and without cross-ownership given by S_1^∞ and $S_{1,N}^\infty$, respectively, as shown in Figure 10(a). From Lemma 4, we know that $S_1^\infty > S_{1,N}^\infty$ for all $2 \leq k \leq n$ and $0 < v < \frac{1}{k-1}$. Thus, the long-run industry outputs with and without cross-ownership are respectively given by

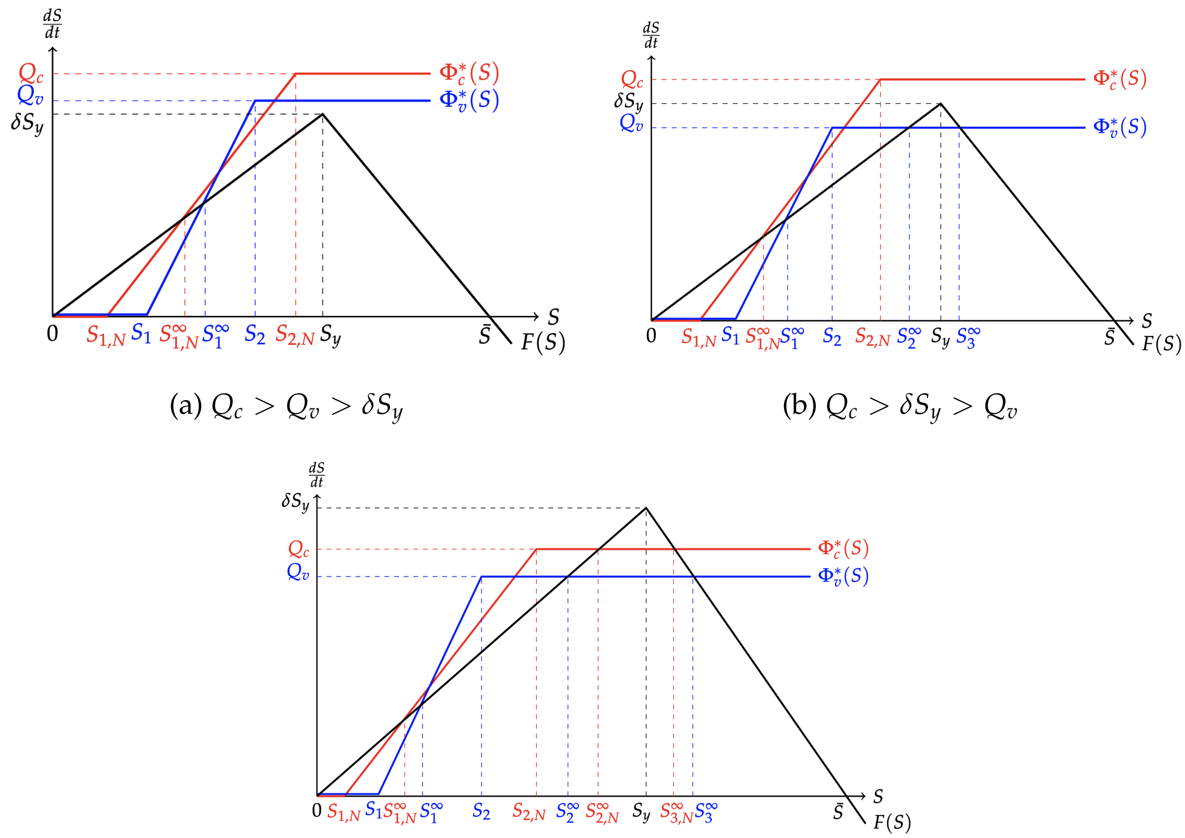
$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) = \delta S_1^\infty, \quad \lim_{t\to\infty} \Phi_c^*(S_c^*(t)) = \delta S_{1,N}^\infty, \quad \forall\, S_0 > 0,$$

with $\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) > \lim_{t\to\infty} \Phi_c^*(S_c^*(t))$. That is, following the rival cross-shareholding activities, both the productive asset's stock at the steady state and the industry's production has gone up.

Next, we consider the situation

$$Q_c = \frac{an}{b(n+1)} > \delta S_y > Q_v = \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b}, \tag{LC2}$$

in which there are three positive stationary stocks in the case of cross-ownership given by S_1^∞, S_2^∞ and S_3^∞ respectively, while there is only one positive stationary stock in the case of no cross-ownership given by $S_{1,N}^\infty$, as shown in Figure 10(b). Clearly, we have $S_3^\infty > S_2^\infty > S_1^\infty > S_{1,N}^\infty$ for all $2 \leq k \leq n,\ 0 < v < \frac{1}{k-1}$. That is, regardless of the initial resource stock, the asset's stock will converge to a larger steady-state level following



(a) $Q_c > Q_v > \delta S_y$

(b) $Q_c > \delta S_y > Q_v$

(c) $\delta S_y > Q_c > Q_v$

Figure 10: The steady-state stocks under different scenarios

the cross-ownership activities. For any $S_0 \in (0, S_2^\infty)$, we have

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) = \delta S_1^\infty > \lim_{t\to\infty} \Phi_c^*(S_c^*(t)) = \delta S_{1,N}^\infty,$$

and for any $S_0 > S_2^\infty$, we have

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) = \delta S_y \left(\frac{1-S_3^\infty}{1-S_y}\right) = Q_v > \lim_{t\to\infty} \Phi_c^*(S_c^*(t)) = \delta S_{1,N}^\infty.$$

Therefore, we can also claim that cross-ownership will result in a higher steady-state resource stock and industry output in this scenario.

Finally, let us consider

$$\delta S_y > Q_c = \frac{an}{b(n+1)} > Q_v = \frac{((k+n-k^2)v+n)a}{((k+n+1-k^2)v+n+1)b}, \tag{LC3}$$

in which there are three positive steady-state stocks in both the case with cross-ownership and without cross-ownership as shown in Figure 10(c). The steady-state stocks with cross-ownership are given by S_1^∞, S_2^∞ and S_3^∞, while those without cross-ownership are denoted by $S_{1,N}^\infty$, $S_{2,N}^\infty$ and $S_{3,N}^\infty$, respectively. From Lemma 4 and Figure

10(c), we know that

$$S_{1,N}^{\infty} < S_1^{\infty} < S_2^{\infty} < S_{2,N}^{\infty} < S_{3,N}^{\infty} < S_3^{\infty}, \quad \forall\, 2 \le k \le n,\ 0 < v < \frac{1}{k-1}.$$

For any $S_0 \in (0, S_2^{\infty})$,

$$\lim_{t\to\infty} S_v^*(t) = S_1^{\infty} > \lim_{t\to\infty} S_c^*(t) = S_{1,N}^{\infty},$$

and thus

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) = \delta S_1^{\infty} > \lim_{t\to\infty} \Phi_c^*(S_c^*(t)) = \delta S_{1,N}^{\infty}.$$

In addition, for any $S_0 \in (S_2^{\infty}, S_{2,N}^{\infty})$,

$$\lim_{t\to\infty} S_v^*(t) = S_3^{\infty} > \lim_{t\to\infty} S_c^*(t) = S_{1,N}^{\infty},$$

and thus we have

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) = \delta S_y \left(\frac{1-S_3^{\infty}}{1-S_y}\right) = Q_v > \lim_{t\to\infty} \Phi_c^*(S_c^*(t)) = \delta S_{1,N}^{\infty}.$$

Furthermore, for any $S_0 > S_{2,N}^{\infty}$,

$$\lim_{t\to\infty} S_v^*(t) = S_3^{\infty} > \lim_{t\to\infty} S_c^*(t) = S_{3,N}^{\infty},$$

and thus

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) = \delta S_y \left(\frac{1-S_3^{\infty}}{1-S_y}\right) = Q_v < \lim_{t\to\infty} \Phi_c^*(S_c^*(t)) = \delta S_y \left(\frac{1-S_{3,N}^{\infty}}{1-S_y}\right) = Q_c.$$

To conclude in this scenario, regardless of the initial resource stock, the stationary asset stock is always higher following cross-ownership, and there exist cases where the long-run industry output increases as a result of cross-ownership.

Based on the findings from all these three possible cases, we can therefore summarize in the following propositions the impact of cross-ownership on the long-run resource stocks and the industry's production.

Proposition 5. *Regardless of the initial resource stock, cross-ownership results in a larger steady-state level of the productive asset's stock.*

Proposition 6. *For any $2 \le k \le n$ and $0 < v < \frac{1}{k-1}$,*

$$\lim_{t\to\infty} \Phi_v^*(S_v^*(t)) > \lim_{t\to\infty} \Phi_c^*(S_c^*(t))$$

if one of the following conditions holds: (i) $\delta S_y < Q_c = \frac{an}{b(n+1)}$, or (ii) $\delta S_y > Q_c = \frac{an}{b(n+1)}$, and $S_0 \in (0, S_{2,N}^{\infty})$.

Proposition 6 demonstrates that at the stationary equilibrium, there exist conditions under which cross-shareholdings between rival firms can lead to an increase in the industry output. The result is quite surprising and sharply contrasts with the predictions of static oligopoly and cross-ownership theory. Indeed, when a subset of firms partially internalize their previous rivalry due to their ownership links, they unilaterally reduce their production. But in terms of strategic substitutes in Cournot competition, other non-participating firms will respond by expanding their production, aiming to capture a larger market share. Nonetheless, the reduction in output by cross-owners outweighs the production expansion by outsiders, resulting in an overall decrease in industry production. In our context where an oligopoly exploits a productive asset, this static result would hold if the initial resource stock is large enough and the implicit growth rate exceeds a certain threshold (i.e., $S_0 > S^{\infty}_{2,N}$ and $\delta > \frac{an}{bS_y(n+1)}$). However, when the implicit growth rate falls below a certain threshold (i.e., $\delta < \frac{an}{bS_y(n+1)}$) or the initial resource stock is small enough (i.e.,$S_0 < S^{\infty}_{2,N}$), rival cross-shareholdings can lead to an increase in the industry's production. This is because in our dynamic framework with a productive common asset, cross-ownership between rival firms influences the industry's exploitation rate through two channels: the output market and the interaction at the resource level. The former is the traditional channel through which reduced competition in the output market due to ownership links makes the industry output fall, while the latter is specific to the renewable resource industry whereas cross-ownership between rival firms results in a larger long-run stock of the asset and consequently allows for greater extraction by the industry. Proposition 6 shows that the latter impact of cross-ownership dominates the former one.

4.2 Cross-ownership and long-run profitability

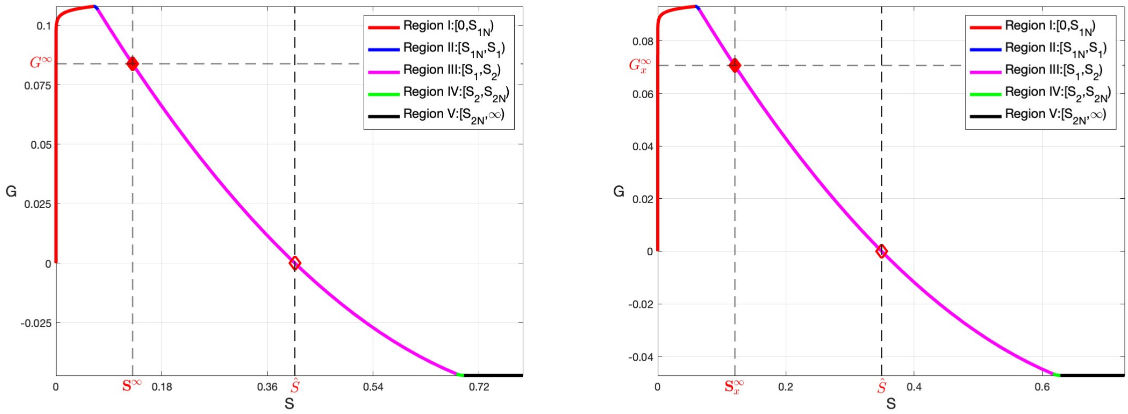
Despite our solid explanations on why cross-ownership could lead to an increased industry output in the long run, one might still question whether such a scenario would ever materialize, as firms may not find it profitable to engage in cross-shareholdings in the transition towards the steady-state level of the stock. To illustrate this, it is sufficient to give some examples. For simplicity, we limit our analysis to the first situation $\delta S_y < Q_c = \frac{an}{b(n+1)}$ where there is only one steady state before cross-ownership.

First, consider $\delta S_y < Q_v < Q_c$ in which there is only one steady state before and after cross-ownership as shown in Figure 10(a). The stationary asset stocks are given by $S^{\infty}_{1,N}$ and S^{∞}_{1}, both falling into region III: $S \in [S_1, S_2)$ with $S_1 < S^{\infty}_{1,N} < S^{\infty}_{1} < S_2$, and the associated long-run industry outputs are $\delta S^{\infty}_{1,N}$ and δS^{∞}_{1}, respectively. Therefore, the profitability function is given by

$$G^{\infty}(k, n, v, \boldsymbol{S}^{\infty}) = (1 - (k-1)v)W(S^{\infty}_{1}) - W_c(S^{\infty}_{1,N}),$$

where $\boldsymbol{S}^{\infty} = \{S^{\infty}_{1,N}, S^{\infty}_{1}\}$. To show that $G^{\infty}(k, n, v, \boldsymbol{S}^{\infty})$ can be positive, we refer back to the least possibly profitable case of $k = 2, n = 3$. Using parameter values of $a = 5, b = 0.5, r = 0.15, S_y = 0.75$, we consider $v = 0.1$ that is strictly unprofitable in the static framework and choose $\delta = 9$ that satisfies both Assumption 1 and condition

([LC1](#)). Figure 11(a) reproduces the dynamic profitability as a function of S in the short run and adds the long-run profitability when $\boldsymbol{S}^{\infty} = \{S_{1,N}^{\infty}, S_1^{\infty}\} = \{0.2634, 0.2782\}$ and $G^{\infty} = 0.0839$. In the short run, for any $S_0 < \hat{S} = 0.4065$, the profitability of cross-ownership is positive, while for any $S_0 > \hat{S}$, each of the two firms will find it unprofitable to hold a 10% share of the other in a triopoly industry. That is, a symmetric cross-ownership between two firms that is profitable at $t = 0$ if $S_0 < \hat{S}$ will remain profitable throughout the transition to the steady-state level of the stock. Moreover, the unprofitable cross-ownership at $t = 0$ for $S > \hat{S}$ can become profitable as the stock evolves to the steady-state value.



(a) $Q_c > Q_v > \delta S_y$

(b) $Q_c > \delta S_y > Q_v$

Figure 11: Transition from short-run to long-run profitability

Now, let us look at the case of $Q_c > \delta S_y > Q_v$ in which there is only one positive stationary stock before cross-ownership given by $S_{1,N}^{\infty}$ and there are three positive stationary stocks after cross-ownership given by S_1^{∞}, S_2^{∞} and S_3^{∞} respectively, as shown in Figure 10(b). We continue to use the example of $k = 2, n = 3$ and $v = 0.1$ but set $\delta = 9.95$ that satisfies both Assumption 1 and condition (LC2), while keeping the other parameter values unchanged. In this case, the pair of long-run steady-state stocks can be either $\boldsymbol{S}_x^{\infty} = \{S_{1,N}^{\infty}, S_1^{\infty}\} = \{0.2395, 0.2528\}$ if $S_0 < S_2^{\infty} = 0.7418$ or $\boldsymbol{S}_y^{\infty} = \{S_{1,N}^{\infty}, S_3^{\infty}\} = \{0.2395, 0.7527\}$ when $S_0 > S_2^{\infty} = 0.7418$ and the corresponding profitability is given by

$$G_x^{\infty}(k, n, v, \boldsymbol{S}_x^{\infty}) = (1 - (k-1)v)W(S_1^{\infty}) - W_c(S_{1,N}^{\infty}) = 0.0706 > 0,$$

$$G_y^{\infty}(k, n, v, \boldsymbol{S}_y^{\infty}) = \frac{\pi_i^v}{r} - W_c(S_{1,N}^{\infty}) = 0.6155 > 0.$$

A similar graph is produced in Figure 11(b) for the short-run profitability as a function of S but only adds the long-run profitability G_x^{∞} at $\boldsymbol{S}_x^{\infty} = \{S_{1,N}^{\infty}, S_1^{\infty}\}$. At $t = 0$, a symmetric cross-ownership between 2 firms in a 3-firm industry will profit from this mutual shareholding of 10% if $S_0 < \hat{S} = 0.3492$, and they will continue to find it profitable in the transition to the new steady state at $\boldsymbol{S}_x^{\infty} = \{S_{1,N}^{\infty}, S_1^{\infty}\}$. However, if the initial stock is $S_0 \in (\hat{S}, S_2^{\infty})$ or $S_0 > S_2^{\infty}$, the profitability of cross-ownership is strictly negative in the short run, but in the transition towards the new steady state, this original unprofitability will turn into positive with the former converging to $\mathbf{S}_x^\infty$ and the latter to $\mathbf{S}_y^\infty$.

Simulations using many other combinations of k, n, v can also find such results. These results thus confirm that firms will find it profitable to engage in cross-shareholdings in the transition to the steady state of the stocks. Consequently, the long-run expansion of industry production becomes a viable prospect, suggesting the potential for an increase in consumer surplus in the long run as a result of cross-ownership.

4.3 The long-run welfare implications

Now let us turn to the comparison of CS, PS and welfare at the stationary equilibrium. By Proposition 6, we know that the stationary CS is higher when firms engage in rival cross-shareholdings than in the case without cross-ownership for $\delta S_y < Q_c$, or $\delta S_y > Q_c$ and $S_0 \in (0, S_{2,N}^\infty)$. Thus, it remains to show that stationary industry profits can also be higher in these scenarios as a result of cross-ownership. We discuss the three respective cases below.

(i) If $\delta S_y < Q_v < Q_c$ in which there is only one stable steady state before and after cross-ownership, then the pair of stationary asset stocks is given by $\mathbf{S}^\infty = \{S_{1,N}^\infty, S_1^\infty\}$. Thus, the change in stationary PS is given by

$$\Delta PS(\mathbf{S}^\infty) = PS_v(S_1^\infty) - PS_c(S_{1,N}^\infty) = \frac{(a - b\delta S_1^\infty)\delta S_1^\infty}{r} - \frac{(a - b\delta S_{1N}^\infty)\delta S_{1N}^\infty}{r}.$$

(ii) If $Q_v < \delta S_y < Q_c$ in which there is only one positive stable stationary stock before cross-ownership and there are two positive stable stationary stocks after cross-ownership, then the pair of steady-state stocks can be either $\mathbf{S}_x^\infty = \{S_{1,N}^\infty, S_1^\infty\}$ or $\mathbf{S}_y^\infty = \{S_{1,N}^\infty, S_3^\infty\}$. The stationary PS changes are

$$\Delta PS(\mathbf{S}_x^\infty) = PS_v(S_1^\infty) - PS_c(S_{1,N}^\infty) = \frac{(a - b\delta S_1^\infty)\delta S_1^\infty}{r} - \frac{(a - b\delta S_{1N}^\infty)\delta S_{1N}^\infty}{r},$$
$$\Delta PS(\mathbf{S}_y^\infty) = PS_v(S_3^\infty) - PS_c(S_{1,N}^\infty) = \frac{k\pi_i^v + (n-k)\pi_o^v}{r} - \frac{(a - b\delta S_{1N}^\infty)\delta S_{1N}^\infty}{r}.$$

(iii) Finally, if $\delta S_y > Q_c$ and $S_0 \in (0, S_{2,N}^\infty)$, the pair of stationary stocks is either $\mathbf{S}_z^\infty = \{S_{1,N}^\infty, S_1^\infty\}$ or $\mathbf{S}_w^\infty = \{S_{1,N}^\infty, S_3^\infty\}$, with the PS changes at the stationary equilibrium given by

$$\Delta PS(\mathbf{S}_z^\infty) = PS_v(S_1^\infty) - PS_c(S_{1,N}^\infty) = \frac{(a - b\delta S_1^\infty)\delta S_1^\infty}{r} - \frac{(a - b\delta S_{1N}^\infty)\delta S_{1N}^\infty}{r},$$
$$\Delta PS(\mathbf{S}_w^\infty) = PS_v(S_3^\infty) - PS_c(S_{1,N}^\infty) = \frac{k\pi_i^v + (n-k)\pi_o^v}{r} - \frac{(a - b\delta S_{1N}^\infty)\delta S_{1N}^\infty}{r}.$$

It should be noted that $\mathbf{S}^\infty = \{S_{1,N}^\infty, S_1^\infty\}$ is not the same as $\mathbf{S}_x^\infty = \{S_{1,N}^\infty, S_1^\infty\}$ or

$\mathbf{S}_z^\infty = \{S_{1,N}^\infty, S_1^\infty\}$, as these cases correspond to different initial conditions when $\delta S_y < Q_v < Q_c$, $Q_v < \delta S_y < Q_c$ and $Q_c < \delta S_y$, respectively. The same applies to $\mathbf{S}_y^\infty$ and $\mathbf{S}_w^\infty$.

We now show that the PS change at the stationary equilibrium is always positive, irrespective of the initial conditions, and thus we can establish the following:

Proposition 7. *Profitable rival cross-shareholdings can lead to a higher CS, PS and welfare at the steady state for* $\delta S_y < Q_c = \frac{an}{b(n+1)}$, *or* $\delta S_y > Q_c = \frac{an}{b(n+1)}$ *and* $S_0 \in (0, S_{2,N}^\infty)$.

Proof. See Appendix H. □

Proposition 7 indicates that cross-ownership can turn out to be welfare-improving in the long run. This result, combined with our findings in Result 3, suggests that welfare can increase as a result of cross-shareholdings both in the short run and long run. Therefore, antitrust authorities should exercise caution when regulating renewable resource industries, as strict policies that restrict cooperation among users of common-pool renewable resources could ultimately harm consumers and society. Unintentionally, these measures might produce the exact opposite effect of what is intended.

5 Conclusion

We use a Markov Perfect Nash Equilibrium of a dynamic oligopoly game where firms exploit a productive asset to analyze the impact of cross-ownership on the equilibrium production strategies, steady-state resource stocks, profitability and social welfare. We show that cross-ownership may not only lead to a higher market output and social welfare in the short run, but also a higher steady-state stock, industry production, and greater social welfare at the steady state. Moreover, there exists an interval of stocks for which a symmetric cross-ownership can be profitable, even though such rival cross-shareholdings are strictly unprofitable in the corresponding static equilibrium framework. These findings have important implications regarding antitrust regulation of dynamic oligopolies. The lessons from the static oligopoly theory may lead to misleading antitrust decisions and a specific examination is required when dealing with dynamic oligopolies, such as those exploiting a productive asset.

We have focused on cross-ownership arrangements that are fixed at the beginning of the game, which made it easier to compare them with the corresponding static framework. However, if the decision to participate in cross-shareholdings were endogenous, we might expect more firms to join these "partial mergers". Yet, the endogenization of cross-ownership formation in this context remains complex and challenging, and we leave it for future research for which our model could serve as a useful starting point. Moreover, while we have focused on the case of a single common property resource, i.e. single species, it would be interesting to examine the case of multiple interdependent resources, e.g., multiple species allowing for predatory-prey interactions and aquaculture (see e.g., Regnier and Schubert (2017)). Analyzing the incentives of different market actors in the fisheries to engage in (horizontal and possibly vertical) cross-ownership would be a valuable and promising avenue for future research.

Appendices

A Proof of Proposition 1

Proof. The vector $(\phi_i^*, \cdots, \phi_i^*, \phi_o^*, \cdots, \phi_o^*)$ constitutes a MPNE if there exist n continuously differentiable value functions $(V_i, \cdots, V_i, V_o, \cdots, V_o)$ such that the functions $\phi_i^*(S)$ and $\phi_o^*(S)$ are solutions to the problems

$$rV_i(S) = max_{\phi_i}\left\{ \frac{a - b\phi_{-i} - b\phi_i}{(1+v)\big(1-(k-1)v\big)}\big((1-(k-2)v)\phi_i + v\sum_{m\in I\setminus i}\phi_m\big) + V_i'(S)(F(S) - \phi_{-i} - \phi_i)\right\}, \tag{12}$$

for $i \in I = \{1, 2, \cdots, k\}$ and

$$rV_o(S) = max_{\phi_o}\left\{(a - b\phi_{-o} - b\phi_o)\phi_o + V_o'(S)(F(S) - \phi_{-o} - \phi_o)\right\}, \tag{13}$$

for $o \in O = \{k+1, \cdots, n\}$. Consider the following value functions

$$V_i(S) = \begin{cases} \left(\frac{S}{S_i^1}\right)^{\frac{r}{\delta}} W(S_i^1) & \text{for } 0 \le S \le S_i^1, \\ W(S) & \text{for } S_i^1 < S \le S_i^2, \\ \frac{\Pi_i}{r} & \text{for } S > S_i^2, \end{cases}$$

$$V_o(S) = \begin{cases} \frac{(1+v)\left(1-(k-1)v\right)}{1-(k-2)v}\left(\frac{S}{S_o^1}\right)^{\frac{r}{\delta}} W(S_o^1) & \text{for } 0 \le S \le S_o^1, \\ \frac{(1+v)\left(1-(k-1)v\right)}{1-(k-2)v} W(S) & \text{for } S_o^1 < S \le S_o^2, \\ \frac{\Pi_o}{r} & \text{for } S > S_o^2, \end{cases}$$

where $W(S) = \frac{A}{2}S^2 + BS + C$, and

$$\Pi_i = \frac{\pi_i^v}{1-(k-1)v} = \frac{1}{1-(k-1)v}\frac{(1+v)\big(1-(k-2)v\big)a^2}{\big((k+n+1-k^2)v+n+1\big)^2 b},$$

$$\Pi_o = \pi_o^v = \frac{(1+v)^2a^2}{\big((k+n+1-k^2)v+n+1\big)^2 b} \equiv \frac{(1+v)\big(1-(k-1)v\big)}{1-(k-2)v}\Pi_i,$$

$$A = \frac{b(r-2\delta)\big(1-(k-2)v\big)\big((k+n+1-k^2)v+n+1\big)^2}{2(1+v)\big(1-(k-1)v\big)\big((k+n-k^2)v+n\big)^2}, \tag{14}$$

$$B = \frac{a(2\delta-r)\big(1-(k-2)v\big)\left[\big(k(k-1)v-n(1+v)\big)^2+(1+v)^2\right]}{2\delta(1+v)\big(1-(k-1)v\big)\big((k+n-k^2)v+n\big)^2}, \tag{15}$$

$$C = \frac{a^2\big(1-(k-2)v\big)C_1C_2}{4br\delta^2(1+v)\big(1-(k-1)v\big)\big((k+n-k^2)v+n\big)^2\big((k+n+1-k^2)v+n+1\big)^2}, \tag{16}$$

$$C_1 = \big(2\delta + rk(k-1)\big(2n-k(k-1)\big) - r(n^2+1)\big)v^2 + \big(4\delta + 2rkn(k-1) - 2r(n^2+1)\big)v + 2\delta - r(n^2+1),$$

$$C_2 = \big((2\delta-r)\big(k(k-1)-n\big)^2 - r\big)v^2 + \big(4\delta n^2 + 2kn(k-1)(r-2\delta) - 2r(n^2+1)\big)v + 2\delta n^2 - r(n^2+1),$$

$$S_i^1 = \frac{(1-(k-2)v)a - (1+v)(1-(k-1)v)B}{(1+v)(1-(k-1)v)A} = S_o^1, \quad S_i^2 = -\frac{B}{A} = S_o^2.$$

In the following, we show that (i) the value functions $V_i(S)$ and $V_o(S)$ are continuously differentiable; (ii) the functions $\phi_i^*(S)$ and $\phi_o^*(S)$ given by (5) and (6) are solutions to the problems (12) and (13).

First, note that there exists a unique relationship between $V_i(S)$ and $V_o(S)$ such that $V_o(S) = \frac{(1+v)\left(1-(k-1)v\right)}{1-(k-2)v} V_i(S)$. Therefore, we only need to prove that $V_i(S)$ is continuously differentiable. Clearly, the value function $V_i(S)$ is continuously differentiable over $(0, S_i^1)$, (S_i^1, S_i^2) and (S_i^2, ∞), respectively, with

$$V_i'(S) = \begin{cases} \frac{r}{\delta S_i^1}\left(\frac{S}{S_i^1}\right)^{\frac{r}{\delta}-1} W(S_i^1) & \text{for } 0 \le S \le S_i^1, \\ W'(S) & \text{for } S_i^1 < S \le S_i^2, \\ 0 & \text{for } S > S_i^2, \end{cases}$$

We then need to check that the function $V_i(S)$ is continuously differentiable at S_i^1 and S_i^2. We have

$$\lim_{S \to S_i^1, S < S_i^1} V_i(S) = W(S_i^1) = \lim_{S \to S_i^1, S > S_i^1} V_i(S),$$

$$\lim_{S \to S_i^2, S < S_i^2} V_i(S) = W(S_i^2) = \Pi_i = \lim_{S \to S_i^2, S > S_i^2} V_i(S).$$

Thus, $V_i(S)$ is continuous at both S_i^1 and S_i^2. Also, note that $\lim_{S \to S_i^1, S < S_i^1} V_i'(S) = \frac{r}{\delta S_i^1} W(S_i^1)$. It can be easily checked that $\frac{r}{\delta S_i^1} W(S_i^1) = W'(S_i^1) = \frac{1-(k-2)v}{(1+v)\left(1-(k-1)v\right)} a$. Thus, we must have

$$\lim_{S \to S_i^1, S < S_i^1} V_i'(S) = W'(S_i^1) = \lim_{S \to S_i^1, S > S_i^1} V_i'(S),$$

i.e., $V_i'(S)$ is continuous at S_i^1. Similarly, we have

$$\lim_{S \to S_i^2, S < S_i^2} V_i'(S) = 0 = \lim_{S \to S_i^2, S > S_i^2} V_i'(S).$$

So $V_i'(S)$ is continuous at both S_i^1 and S_i^2. Therefore, we can conclude that both the functions $V_i(S)$ and $V_o(S)$ are continuously differentiable over $[0, \infty)$.

Next, we show that $\phi_i^*(S)$ and $\phi_o^*(S)$ given by (5) and (6) are solutions to the problems (12) and (13), where $V_1(S) = V_2(S) = \cdots = V_k(S)$, and $V_{k+1}(S) = V_{k+2}(S) = \cdots = V_n(S)$. First, for $S \ge S_i^1$ and $S \ge S_o^1$, the system of problems (12) and (13) admit a pair of interior solutions. The HJB equation for firm $i \in I$ is

$$rV_i(S) = max_{q_i}\left\{\frac{a - b\sum_{j \neq i} q_j - bq_i}{(1+v)(1-(k-1)v)}\left((1-(k-2)v)q_i + v\sum_{m \in I \setminus i} q_m\right) + V_i'(S)(F(S) - \sum_{j \neq i} q_j - q_i)\right\},$$

and that for firm $o \in O$ is given by

$$rV_o(S) = max_{q_o}\left\{(a - b\sum_{j \neq o} q_j - bq_o)q_o + V_o'(S)(F(S) - \sum_{j \neq o} q_j - q_o)\right\}.$$

FOCs of the right-hand side yield

$$\frac{(a - b\sum_{j\neq i} q_j - bq_i)(1-(k-2)v) - b\big((1-(k-2)v)q_i + v\sum_{m\in I\setminus i} q_m\big)}{(1+v)\big(1-(k-1)v\big)} - V_i'(S) = 0,$$

$$a - b\sum_{j\neq o} q_j - 2bq_o - V_o'(S) = 0.$$

Symmetry yields

$$\frac{(a - bkq_i - b(n-k)q_o)(1-(k-2)v) - b(1+v)q_i}{(1+v)\big(1-(k-1)v\big)} - V_i'(S) = 0,$$

$$a - bkq_i - b(n-k+1)q_o - V_o'(S) = 0.$$

So the best response functions can be expressed as

$$q_i = \frac{(1-(k-2)v)a - (1-(k-2)v)(n-k)bq_o - (1+v)\big(1-(k-1)v\big)V_i'(S)}{\big[1+k+\big(1-k(k-2)\big)v\big]b},$$

$$q_o = \frac{a - bkq_i - V_o'(S)}{b(n-k+1)}.$$

Guess the value function as $V_i(S) = \frac{A}{2}S^2 + BS + C$ and $V_o(S) = \frac{D}{2}S^2 + ES + F$, then $V_i'(S) = AS + B$ and $V_o'(S) = DS + E$. So the FOCs become

$$(a - bkq_i - b(n-k)q_o)(1-(k-2)v) - b(1+v)q_i = (AS+B)(1+v)\big(1-(k-1)v\big),$$

$$a - bkq_i - b(n-k+1)q_o = DS + E.$$

Solving for (q_i, q_o) yields

$$q_i^* = \frac{(1-(k-2)v)a - (1+v)\big(1-(k-1)v\big)(n-k+1)(AS+B) + \big(1-(k-2)v\big)(n-k)(DS+E)}{\big((k+n+1-k^2)v+n+1\big)b} \tag{17}$$

$$q_o^* = \frac{(1+v)a + (1+v)\big(1-(k-1)v\big)k(AS+B) - \big[1+k+\big(1-k(k-2)\big)v\big](DS+E)}{\big((k+n+1-k^2)v+n+1\big)b} \tag{18}$$

Substitute q_i^* and q_o^* into the HJB equations, then we have

$$r\big(\frac{A}{2}S^2 + BS + C\big) = \frac{a - bkq_i^* - b(n-k)q_o^*}{(1-(k-1)v)}q_i^* + (AS+B)(\delta S - kq_i^* - (n-k)q_o^*),$$

$$r\big(\frac{D}{2}S^2 + ES + F\big) = (a - bkq_i^* - b(n-k)q_o^*)q_o^* + (DS+E)(\delta S - kq_i^* - (n-k)q_o^*).$$

Using the undetermined coefficients technique (Dockner et al., 2000), we can solve for A, B, C as those defined in (14), (15) and (16), respectively, and we have

$$D = \frac{(1+v)\big(1-(k-1)v\big)}{1-(k-2)v}A, \quad E = \frac{(1+v)\big(1-(k-1)v\big)}{1-(k-2)v}B, \quad F = \frac{(1+v)\big(1-(k-1)v\big)}{1-(k-2)v}C.$$

Therefore, we must have

$$V_o'(S) = DS + E = \frac{(1+v)\big(1-(k-1)v\big)}{1-(k-2)v}(AS+B) = \frac{(1+v)\big(1-(k-1)v\big)}{1-(k-2)v}V_i'(S).$$

Substituting these coefficients back to (17) and (18) yields

$$\phi_i^*(S) = q_i^* = \frac{\big(1-(k-2)v\big)a - (1+v)\big(1-(k-1)v\big)(AS+B)}{\big((k+n+1-k^2)v+n+1\big)b},$$

$$\phi_o^*(S) = q_o^* = \frac{1+v}{1-(k-2)v}\frac{\big(1-(k-2)v\big)a - (1+v)\big(1-(k-1)v\big)(AS+B)}{\big((k+n+1-k^2)v+n+1\big)b}.$$

Moreover, it can be easily observed that there exists a unique relationship between ϕ_i^* and ϕ_o^* such that $\phi_o^*(S) = \frac{1+v}{1-(k-2)v}\phi_i^*(S)$.

The level of stocks S_i^2 and S_o^2 are determined such that $V_i(S)$ and $V_o(S)$ are continuously differentiable in the neighborhood of S_i^2 and S_o^2, respectively. Thus, we have $S_i^2 = -\frac{B}{A} = S_o^2$. Finally, for $S < S_i^1$ and $S < S_o^1$, the system of problems (12) and (13) admit a pair of corner solutions such that $\phi_i^*(S) = 0 = \phi_o^*(S)$. Therefore, we must have

$$S_i^1 = \frac{\big(1-(k-2)v\big)a - (1+v)\big(1-(k-1)v\big)B}{(1+v)\big(1-(k-1)v\big)A} = S_o^1.$$

Since the thresholds of the stocks are the same for the insiders and outsiders, let $S_1 = S_i^1 = S_o^1$ and $S_2 = S_i^2 = S_o^2$ thereafter for ease of exposition. After simplification, we have

$$S_1 = \frac{a\Big[(2\delta-r)(1+v)^2 - r\big(k(k-1)v-n(1+v)\big)^2\Big]}{b\delta(2\delta-r)\big((k+n+1-k^2)v+n+1\big)^2},\ S_2 = \frac{a\Big[\big(k(k-1)v-n(1+v)\big)^2+(1+v)^2\Big]}{b\delta\big((k+n+1-k^2)v+n+1\big)^2}.$$

□

B Proof of Corollary 1

Proof. The stationary asset stocks are characterized by

$$\frac{dS}{dt} = F(S) - k\phi_i^*(S) - (n-k)\phi_o^*(S) = F(S) - \Phi_v^*(S).$$

(i) If $F(S)_{max} = \delta S_y < \Phi_v^*(S)_{max} = kq_i^v + (n-k)q_o^v = \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b}$, there is one and only one positive root given by solving

$$\frac{dS}{dt} = \delta S - \left(k+(n-k)\frac{1+v}{1-(k-2)v}\right)\frac{\big(1-(k-2)v\big)a - \big(1-(k-2)v-(k-1)v^2\big)(AS+B)}{\big((k+n+1-k^2)v+n+1\big)b} = 0,$$

which yields

$$S_1^\infty = \frac{a\left[(2\delta - r)(1+v)^2 - r\big(k(k-1)v - n(1+v)\big)^2\right]}{b\delta\big((k+n+1-k^2)v+n+1\big)\left[(2\delta-r)(1+v)+r\big(k(k-1)v-n(1+v)\big)\right]}.$$

To ensure $S_1^\infty > 0$, we need $(2\delta - r)(1+v)^2 - r\big(k(k-1)v - n(1+v)\big)^2 > 0$, or

$$\delta > \frac{r\left[\big(k(k-1)v - n(1+v)\big)^2 + (1+v)^2\right]}{2(1+v)^2},$$

which also guarantees: $(2\delta - r)(1+v) + r\big(k(k-1)v - n(1+v)\big) > 0$. To see this, note that from $(2\delta - r)(1+v)^2 > r\big(k(k-1)v - n(1+v)\big)^2$, we have

$$(2\delta - r)(1+v) > r\frac{\big(k(k-1)v - n(1+v)\big)^2}{1+v}.$$

We now show that

$$r\frac{\big(k(k-1)v - n(1+v)\big)^2}{1+v} > -r\big(k(k-1)v - n(1+v)\big),$$

$$\iff \big(k(k-1)v - n(1+v)\big)^2 > n(1+v)^2 - kv(k-1)(1+v),$$

$$\iff k^2(k-1)^2v^2 - 2knv(k-1)(1+v) + n^2(1+v)^2 - n(1+v)^2 + kv(k-1)(1+v) > 0,$$

$$\iff \big(k(k-1)v - n(1+v)\big)\big(k(k-1)v - (n-1)(1+v)\big) > 0,$$

which is true, since $k \leq n$, we have

$$k(k-1)v - n(1+v) \leq n(n-1)v - n(1+v) = n\big((n-2)v - 1\big) < 0,$$

$$k(k-1)v - (n-1)(1+v) \leq n(n-1)v - (n-1)(1+v) = (n-1)\big((n-1)v - 1\big) < 0.$$

Moreover,

$$\frac{dS}{dt} = F(S) - \Phi_v^*(S) < 0, \quad \forall\, S > S_1^\infty,$$
$$\frac{dS}{dt} = F(S) - \Phi_v^*(S) > 0, \quad \forall\, S < S_1^\infty.$$

Therefore, for any initial asset's stock S_0, the asset's stock equilibrium path converges asymptotically to S_1^∞, i.e., S_1^∞ is globally stable.

(ii) If $F(S)_{max} = \delta S_y > \Phi_v^*(S)_{max} = kq_i^v + (n-k)q_o^v = \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b}$, there are three positive roots. One is given by S_1^∞, and the other two are given by solving $\frac{dS}{dt} = \delta S - \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b} = 0$, and $\frac{dS}{dt} = \delta S_y\big(\frac{1-S}{1-S_y}\big) - \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b} = 0$, respectively. Solving for S yields

$$S_2^\infty = \frac{\big((k+n-k^2)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b\delta}, \quad S_3^\infty = 1 - \frac{\big((k+n-k^2)v+n\big)a(1-S_y)}{\big((k+n+1-k^2)v+n+1\big)b\delta S_y}.$$

Moreover, we have

$$\frac{dS}{dt} = F(S) - \Phi_v^*(S) > 0, \quad \forall\, 0 < S < S_1^\infty,$$

$$\frac{dS}{dt} = F(S) - \Phi_v^*(S) < 0, \quad \forall\, S_1^\infty < S < S_2^\infty.$$

Therefore, for any initial stock $S_0 \in (0, S_2^\infty)$, the asset's stock equilibrium path converges monotonically to S_1^∞, i.e., S_1^∞ is stable. In addition,

$$\frac{dS}{dt} = F(S) - \Phi_v^*(S) > 0, \quad \forall\, S_2^\infty < S < S_3^\infty,$$

$$\frac{dS}{dt} = F(S) - \Phi_v^*(S) < 0, \quad \forall\, S > S_3^\infty.$$

Therefore, for any initial stock $S_0 \in (S_2^\infty, \infty)$, the asset's stock equilibrium path converges monotonically to S_3^∞, i.e., S_3^∞ is stable. The stationary asset stock S_2^∞ is thus unstable.

□

C Proof of Lemma 1 and Lemma 2

Proof. Note that

$$S_2(v) = \frac{a\big[\big(k(k-1)v - n(1+v)\big)^2 + (1+v)^2\big]}{b\delta\big((k+n+1-k^2)v+n+1\big)^2},\ S_1(v) = \frac{a\big[(2\delta-r)(1+v)^2 - r\big(k(k-1)v - n(1+v)\big)^2\big]}{b\delta(2\delta-r)\big((k+n+1-k^2)v+n+1\big)^2},$$

and thus

$$\frac{\partial S_2(v)}{\partial v} = \frac{2ak(k-1)\big(k(k-1)v - (n-1)(1+v)\big)}{b\delta\big((k+n+1-k^2)v+n+1\big)^3},$$

$$\frac{\partial S_1(v)}{\partial v} = \frac{2ak(k-1)\big[(2\delta-r)(1+v) - r\big(k(k-1)v - n(1+v)\big)\big]}{b\delta(2\delta-r)\big((k+n+1-k^2)v+n+1\big)^3}.$$

By Assumption 1, it directly follows that

$$\delta > \frac{r\big[\big(k(k-1)v - n(1+v)\big)^2 + (1+v)^2\big]}{2(1+v)^2} = \frac{r\big(k(k-1)v - n(1+v)\big)^2}{2(1+v)^2} + \frac{r}{2} > \frac{r}{2},$$

or $2\delta - r > 0$. Since $k \le n$, we have

$$k(k-1)v - (n-1)(1+v) \le n(n-1)v - (n-1)(1+v) = (n-1)\big((n-1)v - 1\big) < 0,$$

and

$$k(k-1)v - n(1+v) \le n(n-1)v - n(1+v) = n\big((n-2)v - 1\big) < 0,$$

$$\Rightarrow (2\delta - r)(1+v) - r\big(k(k-1)v - n(1+v)\big) > 0.$$

Therefore, $\frac{\partial S_1(v)}{\partial v} > 0, \frac{\partial S_2(v)}{\partial v} < 0$. That is, for any $v > 0$, $S_1(v) > S_1(0) = S_{1,N}$ and $S_2(v) < S_2(0) = S_{2,N}$.

The positive slopes of the MPNE are given by

$$\Omega_i = \frac{(2\delta - r)\big(1-(k-2)v\big)\big((k+n+1-k^2)v+n+1\big)}{2\big((k+n-k^2)v+n\big)^2},$$

$$\Omega_o = \frac{(2\delta - r)(1+v)\big((k+n+1-k^2)v+n+1\big)}{2\big((k+n-k^2)v+n\big)^2}, \quad \Omega_c = \frac{(2\delta-r)(1+n)}{2n^2}.$$

We have

$$\frac{\partial \Omega_o(v)}{\partial v} = \frac{k(k-1)(2\delta-r)\big((n+2)(1+v)-kv(k-1)\big)}{2\big((k+n-k^2)v+n\big)^3} > 0,$$

since $2\delta - r > 0$,

$$(n+2)(1+v)-kv(k-1) \geq (n+2)(1+v)-nv(n-1) = 2(1+v)+n\big(1-(n-2)v\big) > 0,$$

$$(k+n-k^2)v+n = n+nv-k(k-1)v \geq n-k+nv > 0.$$

Moreover,

$$\frac{\partial \Omega_i(v)}{\partial v} = \frac{(k-1)(r-2\delta)\big[n(n+1)-k(n+2)+\big(n(n+1)-k^2(n-k)-3k\big)v\big]}{2\big((k+n-k^2)v+n\big)^3}.$$

In the case of $n = k$,

$$\frac{\partial \Omega_i}{\partial v} = \frac{k(k-1)(2\delta-r)\big(1-(k-2)v\big)}{2\big((2k-k^2)v+k\big)^3} = \frac{(k-1)(2\delta-r)}{2k^2\big(1-(k-2)v\big)^2} > 0.$$

That is, for any $v > 0$ and $n = k$, $\Omega_i(v) > \Omega_i(0) = \Omega_c$. However, in the case of $n > k$, since $r - 2\delta < 0$, $n(n+1) - k(n+2) = n(n-k+1) - 2k \geq 2n - 2k > 0$, and

$$F(k,n) \equiv n(n+1) - k^2(n-k) - 3k > 0, \tag{C1}$$

we must have $\frac{\partial \Omega_i}{\partial v} < 0$. We now prove that condition (C1) is always true for all $n > k$. Note that $F(k,n)$ is a quadratic U-shaped function of n:

$$F(k,n) = n^2 + (1-k^2)n + k^3 - 3k,$$

with $\Delta(k) = (1-k^2)^2 - 4(k^3-3k) = k^4 - 2k^2 + 1 - 4k^3 + 12k$. Since

$$\Delta'(k) = 4k^3 - 12k^2 - 4k + 12 = 4(k+1)(k-1)(k-3) \begin{cases} < 0 & \text{if } k = 2 \\ = 0 & \text{if } k = 3 \\ > 0 & \text{if } k \geq 4 \end{cases},$$

we have the following cases:

(i) If $k = 2$, $\Delta'(k) > 0$ and $\Delta(2) = 9 - 4 \times 2 = 1 > 0$;

(ii) If $k = 3$, $\Delta'(k) = 0$, and $\Delta(3) = 64 - 4 \times 18 = -8 < 0$;

(iii) If $k \geq 4$, $\Delta'(k) > 0$ and since $\Delta(4) = 225 - 4 \times 52 = 17 > 0$, we must have $\Delta(k) > 0$ for all $k > 4$.

That is,

(i) when $k = 2$, $F(2, n) = n^2 - 3n + 2 = (n-1)(n-2)$ has two roots: $n_1 = 1, n_2 = 2$. Therefore, for any $n > n_2 = k = 2$, we have $F(2, n) > 0$.

(ii) When $k = 3$, $F(k, n)$ has no real roots of n, and thus it is always positive, i.e.,

$$F(3, n) = n^2 - 8n + 18 = (n-4)^2 + 2 > 0 \quad \forall\, n > k = 3.$$

(iii) However, when $k \geq 4$, $F(k, n)$ has two real roots: $n_2 = \frac{k^2-1+\sqrt{\Delta(k)}}{2} > n_1 = \frac{k^2-1-\sqrt{\Delta(k)}}{2}$. Thus, $F(k, n)$ is strictly positive for any $n > n_2$ or $n < n_1$, and negative for any $n \in (n_1, n_2)$. We now show that for any $n > k \geq 4$, $n > n_2$ always holds.

$$\begin{aligned} k > n_2 &= \frac{k^2 - 1 + \sqrt{\Delta(k)}}{2}, \\ &\iff (2k - k^2 + 1)^2 > (1 - k^2)^2 - 4(k^3 - 3k), \\ &\iff 4k^2 + 4k(1 - k^2) + 4(k^3 - 3k) > 0 \iff 4k(k-2) > 0. \end{aligned}$$

Therefore, for any $v > 0$ and $n > k$, we must have

$$\Omega_i(v) < \Omega_i(0) = \Omega_c = \Omega_o(0) < \Omega_o(v).$$

Finally, we show that

$$\frac{\partial q_i^v}{\partial v} = \frac{a(k-1)(k-n-1)}{b\big((k+n+1-k^2)v+n+1\big)^2} < 0, \quad \frac{\partial q_o^v}{\partial v} = \frac{ak(k-1)}{b\big((k+n+1-k^2)v+n+1\big)^2} > 0.$$

Thus, for any $v > 0$, $q_i^v(v) < q_i^v(0) = q_c = q_o^v(0) < q_o^v(v)$. □

D Proof of Proposition 2

Proof. Note that when $n = k$, $S_2 = \frac{a\left[n^2\left(1-(n-2)v\right)^2+(1+v)^2\right]}{b\delta\left((2n+1-n^2)v+n+1\right)^2}$, we thus have

$$\begin{aligned} \phi_i^*(S_2) - \phi_c^*(S_2) &= \frac{(1-(n-2)v)a}{((2n+1-n^2)v+n+1)b} - \frac{a-(XS_2+Y)}{(n+1)b} \\ &= \frac{av(n-1)\Gamma}{b\delta n(n+1)\big((2n+1-n^2)v+n+1\big)^2}, \end{aligned}$$

where $\Gamma = n^2(\delta - r)(1-(n-2)v) - (2\delta - r + \delta n)(1+v)$. Therefore, $\phi_i^*(S_2) - \phi_c^*(S_2)$ has the same sign as Γ. We can express Γ as

$$\Gamma(n,v) \equiv \big[(r-\delta)\big(n^2(n-2)\big) + r - 2\delta - n\delta\big]v + (n+1)\big(r - 2\delta + n(\delta - r)\big),$$

where $\Gamma(n,v)$ is a linear function in v with

$$\frac{\partial\Gamma(n,v)}{\partial v} = (r-\delta)\big(n^2(n-2)\big) + r - 2\delta - n\delta.$$

By Assumption 1, it follows that $\delta - r > 0$. To see this, note that

$$\delta - r > \frac{r\big[\big(k(k-1)v - n(1+v)\big)^2 - (1+v)^2\big]}{2(1+v)^2}.$$

We now show that

$$\big(k(k-1)v - n(1+v)\big)^2 - (1+v)^2 = \big(k(k-1)v - (n-1)(1+v)\big)\big(k(k-1)v - (n+1)(1+v)\big) > 0.$$

This is true since $k \leq n \Rightarrow k(k-1)v \leq n(n-1)v$, which means that

$$\big(k(k-1)v - (n-1)(1+v)\big) \leq n(n-1)v - (n-1)(1+v) = (n-1)\big((n-1)v - 1\big) < 0,$$

$$\big(k(k-1)v - (n+1)(1+v)\big) \leq n(n-1)v - (n+1)(1+v) = n\big((n-2)v - 1\big) - (1+v) < 0.$$

Since both $2\delta - r > 0$ and $\delta - r > 0$, we must have $\frac{\partial\Gamma(n,v)}{\partial v} < 0$.

Assumption 1 also implies that $\delta > \frac{(n^2+1)}{2}r$. To see this, note that both $\mathcal{H}_1(k,n,v) \equiv \frac{r\left[\left(k(k-1)v-n(1+v)\right)^2+(1+v)^2\right]}{2(1+v)^2}$ and $\mathcal{H}_2(k,n,v) \equiv \frac{a\left[\left(k(k-1)v-n(1+v)\right)^2+(1+v)^2\right]}{bS_y\left((k+n+1-k^2)v+n+1\right)^2}$ are strictly decreasing functions in v:

$$\frac{\partial\mathcal{H}_1(k,n,v)}{\partial v} = \frac{k(k-1)r\big(k(k-1)v - n(1+v)\big)}{(1+v)^3} < 0,$$

$$\frac{\partial\mathcal{H}_2(k,n,v)}{\partial v} = \frac{2k(k-1)a\big(k(k-1)v - (n-1)(1+v)\big)}{bS_y\big((k+n+1-k^2)v+n+1\big)^3} < 0.$$

Therefore, for any $v \in (0, \frac{1}{k-1})$, Assumption 1 is equivalent to

$$\delta > \delta_0 \equiv max\left\{\mathcal{H}_1(k,n,0), \mathcal{H}_2(k,n,0)\right\} = max\left\{\frac{r(n^2+1)}{2}, \frac{a(n^2+1)}{bS_y(n+1)^2}\right\}.$$

Notice that

$$\Gamma(n,0) = (n+1)\big(r - 2\delta + n(\delta - r)\big) = (n+1)\big((n-2)\delta - (n-1)r\big),$$

so when $n = 2$, $\Gamma(n,0) < 0$, and for $n \geq 3$, we have $\Gamma(n,0) > 0$ if $\delta > \frac{n-1}{n-2}r$, and

$\Gamma(n,0) < 0$ if $\delta < \frac{n-1}{n-2}r$. It can be easily observed that for all $n \geq 3$,

$$\frac{(n^2+1)}{2} - \frac{n-1}{n-2} = \frac{n(n^2-2n-1)}{2(n-2)} > 0,$$

and thus $\delta > \frac{(n^2+1)}{2}r > \frac{n-1}{n-2}r$. Therefore, we have $\Gamma(n,0) < 0$ for $n = 2$ and $\Gamma(n,0) > 0$ for $n \geq 3$.

Also, note that

$$\begin{aligned}\Gamma(n, \frac{1}{n-1}) &= \big[(r-\delta)\big(n^2(n-2)\big) + r - 2\delta - n\delta\big]\frac{1}{n-1} + (n+1)\big(r - 2\delta + n(\delta - r)\big)\\ &= -\frac{n(2\delta - r + nr)}{n-1} < 0.\end{aligned}$$

Given the linearity of the function $\Gamma(n,v)$, we must have $\Gamma(n,v) < 0$ for all $n = 2$ and $v \in (0, \frac{1}{n-1})$, and when $n \geq 3$, there must exist some threshold shareholding $\hat{v}$ such that $\Gamma(n,v) > 0$ for any $v \in (0,\hat{v})$ and $\Gamma(n,v) < 0$ for any $v \in (\hat{v}, \frac{1}{n-1})$, where $\Gamma(n,\hat{v}) = 0$, i.e., $\hat{v} = \frac{(n+1)\big(r-2\delta+n(\delta-r)\big)}{(\delta-r)\big(n^2(n-2)\big)+2\delta-r+n\delta} > 0$. We now show that $\hat{v}$ is less than the upper bound of shareholdings $\frac{1}{n-1}$ by direct comparison:

$$\hat{v} = \frac{(n+1)\big(r-2\delta+n(\delta-r)\big)}{(\delta-r)\big(n^2(n-2)\big)+2\delta-r+n\delta} < \frac{1}{n-1},$$

$$\Longleftrightarrow (n^2-1)\big(r-2\delta+n(\delta-r)\big) < (\delta-r)\big(n^2(n-2)\big)+2\delta-r+n\delta,$$

$$\Longleftrightarrow -n(2\delta-r+nr) < 0.$$

To conclude, we have the following cases: If $n = 2$, then $\phi_i^*(S_2) < \phi_c^*(S_2)$ for all $v \in (0, \frac{1}{n-1})$; If $n \geq 3$, then $\phi_i^*(S_2) < \phi_c^*(S_2)$ for $v \in (\hat{v}, \frac{1}{n-1})$, $\phi_i^*(S_2) = \phi_c^*(S_2)$ when $v = \hat{v}$, and $\phi_i^*(S_2) > \phi_c^*(S_2)$ for $v \in (0,\hat{v})$. Given that $S_1 > S_{1,N}$, $S_2 < S_{2,N}$, $q_c > q_i^v$ and $\Omega_i > \Omega_c$ (from Lemma 1), we must have the following scenarios:

(i) For any $n = k = 2$ and $v \in (0, \frac{1}{n-1})$, or $n = k \geq 3$ and $v \in [\hat{v}, \frac{1}{n-1})$, $\phi_c^*(S) \geq \phi_i^*(S)$;

(ii) For any $n = k \geq 3$ and $v \in (0,\hat{v})$, there exists a $\hat{S}_1 \in (S_1, S_2)$ and a $\hat{S}_2 \in (S_2, S_{2,N})$ such that $\phi_i^*(S) > \phi_c^*(S)$ if and only if $\hat{S}_1 < S < \hat{S}_2$.

□

E Proof of Lemma 3

Proof. Note that the positive slopes of the industry production with and without cross-ownership are given by

$$\zeta_v = \frac{(2\delta-r)\big((k+n+1-k^2)v+n+1\big)}{2\big((k+n-k^2)v+n\big)}, \quad \zeta_c = \frac{(2\delta-r)(1+n)}{2n}.$$

We have

$$\frac{dQ_v}{dv} = -\frac{ak(k-1)}{b\big((k+n+1-k^2)v+n+1\big)^2} < 0, \quad \frac{d\zeta_v}{dv} = \frac{k(k-1)(2\delta-r)}{2\big((k+n-k^2)v+n\big)^2} > 0.$$

Therefore, for any $n \geq k \geq 2$ and $v > 0$, we have

$$Q_v(v) < Q_v(0) = Q_c, \quad \zeta_v(v) > \zeta_v(0) = \zeta_c.$$

□

F Proof of Proposition 4

Proof. Note that

$$\begin{aligned}
\Phi_v^*(S_2) - \Phi_c^*(S_2) &= \frac{\big((-k^2+n+k)v+n\big)a}{\big((k+n+1-k^2)v+n+1\big)b} - \frac{n(a-(XS_2+Y)}{(n+1)b} \\
&= \frac{akv(k-1)\big[n(\delta-r)\big(n(1+v)-k(k-1)v\big)-(2\delta-r+\delta n)(1+v)\big]}{b\delta n(n+1)\big((k+n+1-k^2)v+n+1\big)^2}.
\end{aligned}$$

Thus, for any $v \in (0, \frac{1}{k-1})$ and $n > k \geq 2$, $\Phi_v^*(S_2) - \Phi_c^*(S_2)$ has the same sign as

$$\begin{aligned}
\Theta(k,n,v) &\equiv n(\delta-r)\big(n(1+v)-k(k-1)v\big)-(2\delta-r+\delta n)(1+v) \\
&= \big[(n+1)\big((n-2)\delta-(n-1)r\big)-k(k-1)n(\delta-r)\big]v+(n+1)\big((n-2)\delta-(n-1)r\big),
\end{aligned}$$

which is linear in v. Notice that $\Theta(k,n,0) = (n+1)\big((n-2)\delta-(n-1)r\big)$, therefore, we can rewrite $\Theta(k,n,v)$ as

$$\Theta(k,n,v) = \big(\Theta(k,n,0)-k(k-1)n(\delta-r)\big)v+\Theta(k,n,0),$$

and we have

$$\begin{aligned}
\Theta(k,n,\frac{1}{k-1}) &= \big(\Theta(k,n,0)-k(k-1)n(\delta-r)\big)\frac{1}{k-1}+\Theta(k,n,0) \\
&= \frac{k}{k-1}\big[\big(n(n-k)-2\big)\delta-\big(n(n-k)+n-1\big)r\big].
\end{aligned}$$

Following Assumption 1, we have $\delta > \frac{(n^2+1)}{2}r$. It can be easily shown that for all $n > k \geq 2$,

$$\frac{(n^2+1)}{2} - \frac{n-1}{n-2} = \frac{n(n^2-2n-1)}{2(n-2)} > 0,$$

and

$$\frac{(n^2+1)}{2} - \frac{n(n-k)+n-1}{n(n-k)-2} = \frac{n(n+1)\big((n-1)(n-k)-2\big)}{2\big(n(n-k)-2\big)} \geq 0.$$

Therefore, we have

$$\delta > \frac{(n^2+1)}{2}r > \frac{n-1}{n-2}r > r, \quad \delta > \frac{(n^2+1)}{2}r \geq \frac{n(n-k)+n-1}{n(n-k)-2}r > r,$$

which means that for any $n > k \geq 2$, $\Theta(k,n,0) > 0$, $\Theta(k,n,\frac{1}{k-1}) \geq 0$. This combined with the fact that $\Theta(k,n,v)$ is linear in v completes the proof that $\Theta(k,n,v) > 0$ and thus $\Phi_v^*(S_2) > \Phi_c^*(S_2)$ for all $n > k \geq 2$ and $v \in (0, \frac{1}{k-1})$. Given that $S_1 > S_{1,N}$ and $S_2 < S_{2,N}$ (from Lemma 2), and $Q_v < Q_c$ and $\zeta_v > \zeta_c$ (from Lemma 3), there must exist a $\tilde{S}_1 \in (S_1, S_2)$ and a $\tilde{S}_2 \in (S_2, S_{2,N})$ such that $\Phi_v^*(S) > \Phi_c^*(S)$ for any $S \in (\tilde{S}_1, \tilde{S}_2)$. □

G Proof of Lemma 4

Proof. The stationary resource stocks with cross-ownership are given by

$$S_1^\infty(v) = \frac{a\left[(2\delta-r)(1+v)^2 - r\big(k(k-1)v - n(1+v)\big)^2\right]}{b\delta\big((k+n+1-k^2)v+n+1\big)\left[(2\delta-r)(1+v)+r\big(k(k-1)v-n(1+v)\big)\right]},$$

$$S_2^\infty(v) = \frac{a\big((k+n-k^2)v+n\big)}{b\delta\big((k+n+1-k^2)v+n+1\big)}, \quad S_3^\infty(v) = 1 - \frac{a(1-S_y)\big((k+n-k^2)v+n\big)}{b\delta S_y\big((k+n+1-k^2)v+n+1\big)},$$

while those without cross-ownership are denoted by

$$S_{1,N}^\infty = S_1^\infty(0) = \frac{a(2\delta - r(1+n^2))}{b\delta(1+n)(2\delta - r(1+n))},$$

$$S_{2,N}^\infty = S_2^\infty(0) = \frac{an}{b\delta(1+n)}, \quad S_{3,N}^\infty = S_3^\infty(0) = 1 - \frac{an(1-S_y)}{b\delta S_y(1+n)}.$$

Then, we have

$$\frac{\partial S_1^\infty}{\partial v} = \frac{2ak(k-1)(\delta-r)\left[(2\delta-r)(1+v)^2 + r\big(k(k-1)v-n(1+v)\big)^2\right]}{b\delta\big((k+n+1-k^2)v+n+1\big)^2\left[(2\delta-r)(1+v)+r\big(k(k-1)v-n(1+v)\big)\right]^2} > 0$$

$$\frac{\partial S_2^\infty}{\partial v} = -\frac{ak(k-1)}{b\delta\big((k+n+1-k^2)v+n+1\big)^2} < 0,$$

$$\frac{\partial S_3^\infty}{\partial v} = \frac{ak(k-1)(1-S_y)}{b\delta S_y\big((k+n+1-k^2)v+n+1\big)^2} > 0.$$

Therefore, for any $v > 0$, $S_1^\infty(v) > S_1^\infty(0) = S_{1,N}^\infty$, $S_2^\infty(v) < S_2^\infty(0) = S_{2,N}^\infty$ and $S_3^\infty(v) < S_3^\infty(0) = S_{3,N}^\infty$. □

H Proof of Proposition 7

Proof. We need to show that for all δ that satisfies Assumption 1, $\Delta PS(S^{\infty}_{1,N}, S^{\infty}_1) > 0$ and $\Delta PS(S^{\infty}_{1,N}, S^{\infty}_3) > 0$. First, we have

$$\Delta PS(S^{\infty}_{1,N}, S^{\infty}_1) = \frac{(a - b\delta S^{\infty}_1)\delta S^{\infty}_1}{r} - \frac{(a - b\delta S^{\infty}_{1N})\delta S^{\infty}_{1N}}{r}.$$

Since the instantaneous industry profit function $p(Q)Q$ is an inverted U-shape function and both the steady-state industry outputs δS^{∞}_1 and δS^{∞}_{1N} are on the increasing part, with $S^{\infty}_1 > S^{\infty}_{1,N}$ by Proposition 5, we must have

$$(a - b\delta S^{\infty}_1)\delta S^{\infty}_1 > (a - b\delta S^{\infty}_{1N})\delta S^{\infty}_{1N},$$

and thus $\Delta PS(S^{\infty}_{1,N}, S^{\infty}_1) > 0$.

Next, notice that

$$\begin{aligned}
\Delta PS(S^{\infty}_{1,N}, S^{\infty}_3) &= \frac{k\pi^v_i + (n-k)\pi^v_o}{r} - \frac{(a - b\delta S^{\infty}_{1N})\delta S^{\infty}_{1N}}{r} \\
&= \frac{a^2\Lambda_1\Lambda_2}{br(n+1)^2(2\delta - r - nr)^2\big((k+n+1-k^2)v + n + 1\big)^2},
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1 &= k(k-1)v\big(2\delta - r(1+n^2)\big) + nr(n^2-1)(1+v) > 0, \\
\Lambda_2(v) &= \big((n^2-1)(2\delta - r) - 2kn(\delta - r)(k-1)\big)v + (n^2-1)(2\delta - r).
\end{aligned}$$

Since $\Lambda_2(v)$ is a linear function in v with

$$\Lambda_2(0) = (n^2-1)(2\delta - r) > 0,$$

and

$$\begin{aligned}
\Lambda_2\Big(\frac{1}{k-1}\Big) &= \frac{k}{k-1}\Big[\big(2n(n-k) + 2(n-1)\big)\delta - \big((n^2-1) - 2n(k-1)\big)r\Big] \\
&> \frac{k}{k-1}\Big[\big(2n(n-k) + 2(n-1)\big)\frac{(n^2+1)}{2}r - \big((n^2-1) - 2n(k-1)\big)r\Big] \\
&= \frac{knr}{k-1}(n^2-1)(n-k+1) > 0.
\end{aligned}$$

We must have $\Lambda_2(v) > 0$ for all $v \in (0, \frac{1}{k-1})$ and thus $\Delta PS(S^{\infty}_{1,N}, S^{\infty}_3) > 0$.

For both cases, we have shown that the stationary PS is higher when firms engage in rival cross-shareholdings than in the case without cross-ownership, irrespective of the initial conditions. This result, together with Proposition 6, concludes that social welfare can be higher at the steady state when rival firms participate in cross-ownership. □

References

Adler, Jonathan H (2004), "Conservation cartels." *Regulation*, 27, 38.

Amir, Rabah (1996), "Cournot oligopoly and the theory of supermodular games." *Games and Economic Behavior*, 15(2), 132–148.

Amir, Rabah and David Rietzke (2023), "Monotone comparative statics and bounds on strategic influence." *Available at SSRN 4542103*.

Başar, Tamer and Georges Zaccour (2018), *Handbook of dynamic game theory*. Springer.

Benchekroun, Hassan (2003), "Unilateral production restrictions in a dynamic duopoly." *Journal of Economic Theory*, 111(2), 214–239.

Benchekroun, Hassan (2008), "Comparative dynamics in a productive asset oligopoly." *Journal of Economic Theory*, 138(1), 237–261.

Benchekroun, Hassan, Miao Dai, and Ngo Van Long (2022), "The impact of cross-ownership on the value of a clean technology in the energy market." *Environmental Modeling & Assessment*, 27(6), 1007–1019.

Benchekroun, Hassan and Gérard Gaudet (2015), "On the effects of mergers on equilibrium outcomes in a common property renewable asset oligopoly." *Journal of Economic Dynamics and Control*, 52, 209–223.

Benhabib, Jess and Roy Radner (1992), "The joint exploitation of a productive asset: a game-theoretic approach." *Economic Theory*, 2, 155–190.

Benndorf, Volker and Johannes Odenkirchen (2021), "An experiment on partial cross-ownership in oligopolistic markets." *International Journal of Industrial Organization*, 78, 102773.

Bresnahan, Timothy F and Steven C Salop (1986), "Quantifying the competitive effects of production joint ventures." *International Journal of Industrial Organization*, 4(2), 155–175.

Brito, Duarte, Luís Cabral, and Helder Vasconcelos (2014a), "Divesting ownership in a rival." *International Journal of Industrial Organization*, 34, 9–24.

Brito, Duarte, António Osório, Ricardo Ribeiro, and Helder Vasconcelos (2018a), "Unilateral effects screens for partial horizontal acquisitions: The generalized hhi and guppi." *International Journal of Industrial Organization*, 59, 127–189.

Brito, Duarte, Ricardo Ribeiro, and Helder Vasconcelos (2014b), "Measuring unilateral effects in partial horizontal acquisitions." *International Journal of Industrial Organization*, 33, 22–36.

Brito, Duarte, Ricardo Ribeiro, and Helder Vasconcelos (2018b), "Quantifying the coordinated effects of partial horizontal acquisitions." *European Economic Review*, 110, 108–149.

Clark, Colin W (2010), *Mathematical bioeconomics: the mathematics of conservation*, volume 91. John Wiley & Sons.

Colombo, Luca and Paola Labrecciosa (2013), "On the convergence to the cournot equilibrium in a productive asset oligopoly." *Journal of Mathematical Economics*, 49(6), 441–445.

Colombo, Luca and Paola Labrecciosa (2015), "On the markovian efficiency of bertrand and cournot equilibria." *Journal of Economic Theory*, 155, 332–358.

Colombo, Luca and Paola Labrecciosa (2018), "Consumer surplus-enhancing cooperation in a natural resource oligopoly." *Journal of Environmental Economics and Management*, 92, 185–193.

Commission, European, Directorate-General for Competition, L Boucon, P Ypma, I Kokkoris, and P McNally (2016), *Support study for impact assessment concerning the review of Merger Regulation regarding minority shareholdings – Final report*. Publications Office.

Conrad, Jon M (2010), *Resource economics*. Cambridge University Press.

Dai, Miao, Hassan Benchekroun, and Ngo Van Long (2022), "On the profitability of cross-ownership in cournot nonrenewable resource oligopolies: Stock size matters." *Journal of Environmental Economics and Management*, 111, 102597.

Dawid, Herbert and Michael Kopel (1997), "On the economically optimal exploitation of a renewable resource: the case of a convex environment and a convex return function." *Journal of Economic Theory*, 76(2), 272–297.

Deacon, Robert T (2012), "Fishery management by harvester cooperatives."

Dietzenbacher, Erik, Bert Smid, and Bjørn Volkerink (2000), "Horizontal integration in the dutch financial sector." *International Journal of Industrial Organization*, 18(8), 1223–1242.

Dockner, Engelbert J, Steffen Jorgensen, Ngo Van Long, and Gerhard Sorger (2000), *Differential games in economics and management science*. Cambridge University Press.

Dockner, Engelbert J and Gerhard Sorger (1996), "Existence and properties of equilibria for a dynamic game on productive assets." *Journal of Economic Theory*, 71(1), 209–227.

Dutta, Prajit K and Rangarajan K Sundaram (1993a), "How different can strategic models be?" *Journal of Economic Theory*, 60(1), 42–61.

Dutta, Prajit K and Rangarajan K Sundaram (1993b), "The tragedy of the commons?" *Economic Theory*, 3, 413–426.

Farrell, Joseph and Carl Shapiro (1990), "Asset ownership and market structure in oligopoly." *The RAND Journal of Economics*, 275–292.

Flath, David (1991), "When is it rational for firms to acquire silent interests in rivals?" *International Journal of Industrial Organization*, 9(4), 573–583.

Flath, David (1992), "Horizontal shareholding interlocks." *Managerial and Decision Economics*, 13(1), 75–77.

Fotis, Panagiotis and Nikolaos Zevgolis (2016), *The competitive effects of minority shareholdings: legal and economic issues*. Bloomsbury Publishing.

Gaudet, Gérard and Stephen W Salant (1991), "Increasing the profits of a subset of firms in oligopoly models with strategic substitutes." *The American Economic Review*, 81(3), 658–665.

Gilo, David (2000), "The anticompetitive effect of passive investment." *Michigan Law Review*, 99(1), 1–47.

Gilo, David, Yossi Moshe, and Yossi Spiegel (2006), "Partial cross ownership and tacit collusion." *The Rand Journal of Economics*, 37(1), 81–99.

Huse, Cristian, Ricardo Ribeiro, and Frank Verboven (2024), "Common-ownership versus cross-ownership: Evidence from the automobile industry*." *The Journal of Industrial Economics*.

Jovanovic, Dragan and Christian Wey (2014), "Passive partial ownership, sneaky takeovers, and merger control." *Economics Letters*, 125(1), 32–35.

Karp, Larry (1992), "Social welfare in a common property oligopoly." *International Economic Review*, 353–372.

Kopel, Michael and Ferenc Szidarovszky (2006), "Resource dynamics under partial cooperation in an oligopoly." *Journal of Optimization Theory and Applications*, 128, 393–410.

Levhari, David and Leonard J Mirman (1980), "The great fish war: an example using a dynamic cournot-nash solution." *The Bell Journal of Economics*, 322–334.

Long, Ngo Van (2010), *A survey of dynamic games in economics*, volume 1. World Scientific.

López, Ángel L and Xavier Vives (2019), "Overlapping ownership, r&d spillovers, and antitrust policy." *Journal of Political Economy*, 127(5), 2394–2437.

Malueg, David A (1992), “Collusive behavior and partial ownership of rivals.” *International Journal of Industrial Organization*, 10(1), 27–34.

Nain, Amrita and Yan Wang (2018), “The product market impact of minority stake acquisitions.” *Management Science*, 64(2), 825–844.

O’Brien, Daniel P and Steven C Salop (2000), “Competitive effects of partial ownership: financial interest and corporate control.” *Antitrust Law Journal*, 67(3), 559–614.

Posner, Eric A, Fiona M Scott Morgan, and E Glen Weyl (2016), “A proposal to limit the anticompetitive power of institutional investors.” *Antitrust LJ*, 81, 669.

Regnier, Esther and Katheline Schubert (2017), “To what extent is aquaculture socially beneficial? a theoretical analysis.” *American Journal of Agricultural Economics*, 99(1), 186–206.

Reinganum, Jennifer F and Nancy L Stokey (1985), “Oligopoly extraction of a common property natural resource: The importance of the period of commitment in dynamic games.” *International Economic Review*, 161–173.

Reynolds, Robert J and Bruce R Snapp (1986), “The competitive effects of partial equity interests and joint ventures.” *International Journal of Industrial Organization*, 4(2), 141–153.

Salant, Stephen W, Sheldon Switzer, and Robert J Reynolds (1983), “Losses from horizontal merger: the effects of an exogenous change in industry structure on cournot-nash equilibrium.” *The Quarterly Journal of Economics*, 98(2), 185–199.

Sandal, Leif K and Stein I Steinshamn (2004), “Dynamic cournot-competitive harvesting of a common pool resource.” *Journal of Economic Dynamics and Control*, 28(9), 1781–1799.

Sorger, Gerhard (1998), “Markov-perfect nash equilibria in a class of resource games.” *Economic Theory*, 11, 79–100.