9.5 Transboundary Pollution Games

Practice Question 5. Consider the problem of a transboundary stock pollutant. There are $N = 2$ regions (e.g., provinces). The objective of the regulator in region i is to choose a pollution control strategy E_i that maximizes the discounted stream of net benefits:

$$\max_{E_i} \int_0^{\infty} e^{-t} \left(E_i - \frac{1}{2}E_i^2 - 3P^2 \right) dt$$

Emissions of pollution accumulate into a stock, $P(t)$, according to the following transition equation

$$\frac{dP}{dt} = z(E_1 + E_2) - P, \quad P(0) = P_0$$

and where $z \in [0, 1]$ is a technology parameter. It can for example represent carbon capture at the production stage.z can also be interpreted as the share of emissions that go to the pollution. The lower level of z, the cleaner is the technology. We assume that countries behave in a noncooperative way and wish to construct a Markov perfect Nash equilibrium of this pollution game.

(a) Find the OLNE and discuss the impact of drop of z from 1 to 0.8 on the steady stock of pollution P_{SS}^{OL}.

(b) Find the threshold z_O^* such that $P_{SS}^{OL}(z_O^*) = P_{SS}^{OL}(z = 1)$

(c) Find the MPNE with linear strategies and discuss the impact of drop of z from 1 to 0.8 on the steady stock of pollution.

(d) Find the threshold z_M^* such that $P_{SS}^{M}(z_M^*) = P_{SS}^{M}(z = 1)$

Solutions:

(a) **Maximum Principle Approach - OLNE**

Step 1: The current value Hamiltonian of Player i is

$$\mathcal{H}_i(P, \lambda_i, E_1, E_2) = \left(E_i - \frac{1}{2}E_i^2 - 3P^2 \right) + \lambda_i[z(E_1 + E_2) - P), \quad i = 1, 2$$

where $\lambda_i(t)$ are the current-valued costate variables.

Step 2: Maximizing the Hamiltonian with respect to the control variable, E_i, gives

$$\frac{\partial \mathcal{H}_i}{\partial E_i} = 1 - E_i + \lambda_i z = 0 \Rightarrow E_i = 1 + \lambda_i z, \quad i = 1, 2$$

Step 3: The differential equations for the costate variable are

$$\frac{d\lambda_i}{dt} - \lambda_i = -\frac{\partial \mathcal{H}_i}{\partial P} = -(-6P - \lambda_i)$$

$$\Rightarrow \frac{d\lambda_i}{dt} = 6P + 2\lambda_i$$

The above costate equations are independent of the control variables, so clearly

$$\lambda_1(t) = \lambda_2(t) = \lambda, \quad \forall t \in [0, +\infty)$$

Therefore, we need to solve only one adjoint equation:

$$\frac{d\lambda}{dt} = 6P + 2\lambda$$

Since $E_i = 1 + \lambda_i z$, then

$$E_1(t) = E_2(t) = 1 + \lambda z, \quad \forall t \in [0, +\infty)$$

Then the system of differential equations is

$$\frac{d\lambda}{dt} = 6P + 2\lambda$$

$$\frac{dP}{dt} = z(E_1 + E_2) - P = 2z(1 + \lambda z) - P = -P + 2z^2\lambda + 2z$$

Or:

$$\begin{pmatrix} \frac{dP}{dt} \\ \frac{d\lambda}{dt} \end{pmatrix} = \begin{pmatrix} -1 & 2z^2 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} P \\ \lambda \end{pmatrix} + \begin{pmatrix} 2z \\ 0 \end{pmatrix}$$

The steady states are P_{SS}^{OL} and λ_{SS}^{OL} where

$$\frac{d\lambda}{dt} = 6P + 2\lambda = 0 \Rightarrow \lambda = -3P$$

$$\frac{dP}{dt} = -P + 2z^2\lambda + 2z = -P - 6z^2p + 2z = 0$$

$$\Rightarrow P_{SS}^{OL} = \frac{2z}{1+6z^2}, \quad \lambda_{SS}^{OL} = \frac{-6z}{1+6z^2}$$

The coefficient matrix is

$$A = \begin{bmatrix} -1 & 2z^2 \\ \\ 6 & 2 \end{bmatrix}$$

The characteristic equation is

$$r^2 - (-1+2)r + (-2 - 12z^2) = 0$$

$$\Rightarrow r^2 - r - 2 - 12z^2 = 0$$

$$\Rightarrow r = \frac{1 \pm \sqrt{1 + 4(2+12z^2)}}{2} = \frac{1 \pm \sqrt{9+12z^2}}{2}$$

$$\Rightarrow r_1 = \frac{1 - \sqrt{9+12z^2}}{2} < 0, \quad r_2 = \frac{1 + \sqrt{9+12z^2}}{2} > 0$$

Then

$$\lambda(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \lambda_{SS}^{OL}$$

Step 4: The boundary conditions are

$$P(0) = P_0$$

$$\lim_{t\to\infty} e^{-t}\lambda(t) = 0 \Rightarrow C_2 = 0$$

$$\Rightarrow \lambda(t) = C_1 e^{r_1 t} + \lambda_{SS}^{OL}$$

$$\Rightarrow \frac{d\lambda}{dt} = C_1 r_1 e^{r_1 t}$$

Then

$$\frac{d\lambda}{dt} = 6P + 2\lambda \Rightarrow P(t) = \frac{1}{6}\frac{d\lambda}{dt} - \frac{1}{3}\lambda = \frac{1}{6}(C_1 r_1 e^{r_1 t}) - \frac{1}{3}(C_1 e^{r_1 t} + \lambda_{SS}^{OL})$$

$$\Rightarrow P(t) = (\frac{1}{6}C_1 r_1 - \frac{1}{3}C_1)e^{r_1 t} - \frac{1}{3}\lambda_{SS}^{OL} = (\frac{1}{6}C_1 r_1 - \frac{1}{3}C_1)e^{r_1 t} + P_{SS}^{OL}$$

$$P(0) = \frac{1}{6}C_1 r_1 - \frac{1}{3}C_1 + P_{SS}^{OL} = P_0 \Rightarrow C_1 = \frac{6(P_0 - P_{SS}^{OL})}{r_1 - 2}$$

Step 5: The OLNE are

$$P(t) = (P_0 - P_{SS}^{OL})e^{r_1 t} + P_{SS}^{OL}$$

$$E_1(t) = E_2(t) = 1 + \lambda z = 1 + [C_1 e^{r_1 t} + \lambda_{SS}^{OL}]z$$

where

$$P_{SS}^{OL} = \frac{2z}{1+6z^2}, \quad \lambda_{SS}^{OL} = -3P_{SS}^{OL} = \frac{-6z}{1+6z^2}$$

$$r_1 = \frac{1-\sqrt{9+12z^2}}{2} < 0$$

To see the effect of a decrease of z in P_{SS}^{OL}, simply take the derivative

$$\frac{\partial P_{SS}^{OL}}{\partial z} = \frac{2(1+6z^2) - 2z(12z)}{(1+6z^2)^2} = \frac{2-12z^2}{(1+6z^2)^2}$$

$\Rightarrow$ If $2 - 12z^2 > 0$, or

$$0 \leq z < \sqrt{\frac{1}{6}} \approx 0.4$$

then

$$\frac{\partial P_{SS}^{OL}}{\partial z} > 0$$

An drop in z leads to a decrease in P_{SS}^{OL}.

$\Rightarrow$ If $2 - 12z^2 < 0$, or

$$\sqrt{\frac{1}{6}} \approx 0.4 < z \leq 1$$

then

$$\frac{\partial P_{SS}^{OL}}{\partial z} < 0$$

Thus an drop in z from 1 to 0.8 leads to an increase in P_{SS}^{OL}.

- **Alternatively**, to see the effect of a decrease of z from 1 to 0.8 in P_{SS}^{M}, we can simply evaluate these values at SS, then

$$P_{SS}^{OL}(z=1) = \frac{2}{1+6} = \frac{2}{7} = 0.29$$

$$P_{SS}^{OL}(z=0.8) = \frac{2(0.8)}{1+6(0.64)} = 0.33 > P_{SS}^{OL}(z=1) = 0.29$$

Comments: z is the share of emissions that go to the pollution. So the lower z, the cleaner is the technology.

$\Rightarrow$ In one person situation or joint welfare maximization, a decrease in z or an improvement in technology is welfare enhancing, as now he can emit more.

$$W \uparrow = \int_0^\infty e^{-t} \left(E - \frac{1}{2}E^2 - 3P^2 \right) dt$$

$\Rightarrow$ But with two players, whether a drop in z is welfare enhancing is a research question?

- In an open loop equilibrium with two players, each player is committing to a fixed time path of the emission $E_i(t)$, taking the other player's emission decision $E_{-i}(t)$ as given. Thus each player is trying to be free riders, as a result more of emissions will be added to the atmosphere, and the pollution level will be higher in the open-loop equilibrium than in the optimal joint welfare maximization situation.
- With an improvement of technology (a decrease in z), each player can emit more, and thus pollution level goes up.

(b) We have

$$P_{SS}^{OL}(z=1) = \frac{2}{1+6} = \frac{2}{7}$$

Set

$$P_{SS}^{OL} = \frac{2z}{1+6z^2} = \frac{2}{7}$$
$$\Rightarrow 12z^2 - 14z + 2 = 0$$
$$\Rightarrow z_O^* = \frac{1}{6}$$

(c) **Dynamic Programming Approach - MPNE**

- The Hamilton-Jacobi-Bellman equation for player i is

$$V_i(P) = \max_{E_i \geq 0} \left[\left(E_i - \frac{1}{2}E_i^2 - 3P^2 \right) + V_i'(P)\Big(z(E_1+E_2) - P \Big) \right]$$

- Differentiating the right-hand side with respect to E_i and equating to 0 yields

$$1 - E_i + zV_i'(P) = 0$$

$$\Rightarrow E_i(P) = 1 + zV_i'(P)$$

- Since the game is symmetric, we focus on the symmetric equilibrium strategies. Also game is linear-quadratic, we can make the informed guess that the value function is the same for both players and takes the following form:

$$V_i(P) = \frac{A}{2}P^2 + BP + C, \quad i = 1,2$$

$$\Rightarrow V_i'(P) = AP + B, \quad i = 1,2$$

$$\Rightarrow E_i(P) = 1 + zV_i'(P) = 1 + z(AP + B) = E_1(P) = E_2(P)$$

- Then we have

$$\begin{aligned}\frac{A}{2}P^2 + BP + C =&(1 + z(AP + B))\left(1 - \frac{1}{2}(1 + z(AP + B))\right) - 3P^2 \\ &+ (AP + B)\left(2z(1 + z(AP + B)) - P\right) \\ =&\frac{1}{2}\left(1 - z^2(AP + B)^2\right) - 3P^2 + (AP + B)\left((2Az^2 - 1)P + 2z + 2Bz^2\right)\end{aligned}$$

- Solve by identification:

$$\frac{A}{2} = -\frac{z^2}{2}A^2 - 3 + A(2Az^2 - 1) \Rightarrow 3z^2A^2 - 3A - 6 = 0$$

$$\Rightarrow A = \frac{3 \pm \sqrt{9 + 72z^2}}{6z^2} = \frac{1 \pm \sqrt{1 + 8z^2}}{2z^2}$$

and

$$B = -ABz^2 + B(2Az^2 - 1) + A(2z + 2Bz^2)$$

$$\Rightarrow B = \frac{2Az}{2 - 3Az^2}$$

and

$$C = \frac{1}{2} - \frac{z^2B^2}{2} + B(2z + 2Bz^2) = \frac{3}{2}B^2z^2 + 2Bz + \frac{1}{2}$$

- Selection of the negative root

$$A = \frac{1 - \sqrt{1 + 8z^2}}{2z^2}$$

guarantees the global stability of the state trajectory $x(t)$, which can be obtained by solving the differential equation :

$$\frac{dP}{dt} = z(E_1 + E_2) - P = 2z(1 + z(AP + B)) - P = (2z^2A - 1)P + 2z + 2z^2B$$

$$\Rightarrow P(t) = Ke^{(2z^2A-1)t} + P_{SS}^M$$

where K is a constant of integration and

$$P_{SS}^M = \frac{2z + 2z^2B}{1 - 2z^2A}$$

Aside:

$$2z^2A - 1 = 2z^2\frac{1 - \sqrt{1+8z^2}}{2z^2} = -\sqrt{1+8z^2} < 0$$

Indeed

$$\text{As} \quad t \to \infty, \quad P(t) = Ke^{(2z^2A-1)t} + P_{SS}^M \to P_{SS}^M$$

we have

$$2z^2B = 2z^2\frac{2Az}{2 - 3Az^2} = 2z^2\frac{2z\frac{1-\sqrt{1+8z^2}}{2z^2}}{2 - 3z^2\frac{1-\sqrt{1+8z^2}}{2z^2}} = \frac{4z(1 - \sqrt{1+8z^2})}{1 + 3\sqrt{1+8z^2}}$$

then

$$P_{SS}^M = \frac{2z + 2z^2B}{1 - 2z^2A} = \frac{2z + \frac{4z(1-\sqrt{1+8z^2})}{1+3\sqrt{1+8z^2}}}{\sqrt{1+8z^2}} = \frac{2z(\sqrt{1+8z^2} + 3)}{(3\sqrt{1+8z^2} + 1)(\sqrt{1+8z^2})}$$

- Using boundary condition

$$P(0) = K + P_{SS}^M = P_0 \Rightarrow K = P_0 - P_{SS}^M$$

$$\Rightarrow P(t) = (P_0 - P_{SS}^M)e^{(2z^2A-1)t} + P_{SS}^M$$

To see the effect of a decrease of z from 1 to 0.8 in P_{SS}^M, simply evaluate these values at SS, then

$$P_{SS}^M(z=1) = \frac{2(3+3)}{(9+1)3} = \frac{12}{30} = \frac{2}{5} = 0.4$$

$$P_{SS}^M(z=0.8) = \frac{1.6(\sqrt{1+8\cdot 0.64} + 3)}{(3\sqrt{1+8\cdot 0.64} + 1)(\sqrt{1+8\cdot 0.64})} = \frac{8.752}{20.77} = 0.42 > P_{SS}^M(z=1) = 0.4$$

Thus an drop in z from 1 to 0.8 leads to an increase in P_{SS}^M.

- **Comments:** With Markovian strategies, each player is choosing his strategies based on what the other player is doing such that he is re-optimizing each period. And actually for each player, the value function or the player's welfare is

$$V_i(P) = \frac{A}{2}P^2 + BP + C, \quad i = 1, 2$$

with

$$A = \frac{1 - \sqrt{1 + 8z^2}}{2z^2}, \quad B = \frac{2Az}{2 - 3Az^2}, \quad C = \frac{3}{2}B^2z^2 + 2Bz + \frac{1}{2}$$

To see the effect of a decrease in z on the welfare, simply take the derivative:

$$\frac{dV_i}{dz} = \cdots$$

(d) We have

$$P_{SS}^M(z = 1) = \frac{2(3 + 3)}{(9 + 1)3} = \frac{12}{30} = \frac{2}{5} = 0.4$$

Set

$$P_{SS}^M = \frac{2z(\sqrt{1 + 8z^2} + 3)}{(3\sqrt{1 + 8z^2} + 1)(\sqrt{1 + 8z^2})} = 0.4$$

$$\Rightarrow z_M^* = 0.3$$

Practice Question 6. Consider a stock public good G that benefits two cities 1 and 2. The utility from the public good for each city is given

$$U(G) = \begin{cases} G\left(1 - \frac{1}{2}G\right) & \text{for } G \leq 1 \\ \frac{1}{2} & \text{for } G \geq 1 \end{cases}$$

Throughout this question we focus on $G \leq 1$. The public good depreciates at a rate δ. Denote by g_i the contribution of city i to the build up of the public good. The cost of contributing g_i is

$$c(g_i) = \frac{1}{2}g_i^2$$

The objective of city i is thus

$$\max_{g_i \geq 0} \int_0^\infty (U(G) - c(g_i))e^{-rt}dt = \max_{g_i \geq 0} \int_0^\infty \left(G\left(1 - \frac{1}{2}G\right) - \frac{1}{2}g_i^2\right)e^{-rt}dt$$

$$s.t. \quad \frac{dG}{dt} = g_1 + g_2 - \delta G, \quad G(0) = G_0$$

For simplicity we set $\delta = 1$ and $r = 1$.

(a) Find the Open-Loop Nash Equilibrium (OLNE)

(b) Find a Markov Perfect Nash Equilibrium (MPNE)

(c) Find the Social Optimum (you can either find the socially optimal contribution paths or the policies)

(d) Find the steady states level of public good under (a), (b) and (c); denote them respectively G_{SS}^{OL}, G_{SS}^{M} and G_{SS}^{SO}. Compare them and give an economic interpretation of your result.

(e) Compare the contribution paths under (a), (b) and (c). Does this corroborate your answer in (d)?

Solutions:

(a) **Maximum Principle Approach - OLNE**

Step 1: The current value Hamiltonian of city i is

$$\mathcal{H}_i(G, \lambda_i, g_1, g_2) = G\left(1 - \frac{1}{2}G\right) - \frac{1}{2}g_i^2 + \lambda_i(g_1 + g_2 - \delta G), \quad i = 1, 2$$

where $\lambda_i(t)$ are the current-valued costate variables.

Step 2: Maximizing the Hamiltonian with respect to the control variable, g_i, gives

$$\frac{\partial \mathcal{H}_i}{\partial g_i} = -g_i + \lambda_i = 0 \Rightarrow g_i = \lambda_i, \quad i = 1, 2$$

Step 3: The differential equations for the costate variable are

$$\frac{d\lambda_i}{dt} - r\lambda_i = -\frac{\partial \mathcal{H}_i}{\partial G} = -(1 - G - \lambda_i\delta)$$

$$\Rightarrow \frac{d\lambda_i}{dt} = G - 1 + (r+\delta)\lambda_i$$

The above costate equations are independent of the control variables, so clearly

$$\lambda_1(t) = \lambda_2(t) = \lambda, \quad \forall t \in [0, +\infty)$$

Therefore, we need to solve only one adjoint equation:

$$\frac{d\lambda}{dt} = G - 1 + (r+\delta)\lambda$$

Since $g_i = \lambda_i, \quad i = 1, 2$, then

$$g_1(t) = g_2(t) = \lambda, \quad \forall t \in [0, +\infty)$$

Then given $\delta = r = 1$, the system of differential equations is

$$\frac{d\lambda}{dt} = G - 1 + (r+\delta)\lambda = G - 1 + 2\lambda$$

$$\frac{dG}{dt} = g_1 + g_2 - \delta G = 2\lambda - G$$

Or:

$$\begin{pmatrix} \frac{d\lambda}{dt} \\ \\ \frac{dG}{dt} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \lambda \\ \\ G \end{pmatrix} + \begin{pmatrix} -1 \\ \\ 0 \end{pmatrix}$$

The steady states are G_{SS}^{OL} and λ_{SS}^{OL} where

$$\frac{dG}{dt} = 2\lambda - G = 0 \Rightarrow G = 2\lambda$$

$$\frac{d\lambda}{dt} = G - 1 + 2\lambda = 0 \Rightarrow 4\lambda - 1 = 0 \Rightarrow \lambda_{SS}^{OL} = \frac{1}{4}$$

$$\Rightarrow G_{SS}^{OL} = \frac{1}{2} = 0.5$$

The coefficient matrix is

$$A = \begin{bmatrix} 2 & 1 \\ \\ 2 & -1 \end{bmatrix}$$

So the characteristic equation is

$$k^2 - (2-1)k + (-2-2) = 0$$

$$\Rightarrow k^2 - k - 4 = 0$$

$$\Rightarrow k = \frac{1 \pm \sqrt{1+16}}{2} = \frac{1 \pm \sqrt{17}}{2}$$

$$\Rightarrow k_1 = \frac{1-\sqrt{17}}{2} = -1.56 < 0, \quad k_2 = \frac{1+\sqrt{17}}{2} > 0$$

Then

$$\lambda(t) = C_1 e^{k_1 t} + C_2 e^{r_2 t} + \lambda_{SS}^{OL}$$

Step 4: The boundary conditions are

$$G(0) = G_0$$

$$\lim_{t\to\infty} e^{-t}\lambda(t) = 0 \Rightarrow C_2 = 0$$

$$\Rightarrow \lambda(t) = C_1 e^{k_1 t} + \lambda_{SS}^{OL}$$

$$\Rightarrow \frac{d\lambda}{dt} = C_1 k_1 e^{k_1 t}$$

Then

$$\frac{d\lambda}{dt} = G - 1 + 2\lambda \Rightarrow G(t) = \frac{d\lambda}{dt} - 2\lambda + 1 = C_1 k_1 e^{k_1 t} - 2(C_1 e^{k_1 t} + \lambda_{SS}^{OL}) + 1$$

$$\Rightarrow G(t) = (C_1 k_1 - 2C_1)e^{k_1 t} - 2\lambda_{SS}^{OL} + 1 = (C_1 k_1 - 2C_1)e^{k_1 t} + G_{SS}^{OL}$$

$$G(0) = C_1 k_1 - 2C_1 + G_{SS}^{OL} = G_0 \Rightarrow C_1 = \frac{G_0 - G_{SS}^{OL}}{k_1 - 2}$$

Step 5: The OLNE are

$$G(t) = C_1(k_1 - 2)e^{k_1 t} + G_{SS}^{OL} = (G_0 - G_{SS}^{OL})e^{k_1 t} + G_{SS}^{OL}$$

$$g_1(t) = g_2(t) = \lambda(t) = \frac{G_0 - G_{SS}^{OL}}{k_1 - 2} e^{k_1 t} + \lambda_{SS}^{OL}$$

where

$$G_{SS}^{OL} = \frac{1}{2} = 0.5, \quad \lambda_{SS}^{OL} = \frac{1}{4}, \quad k_1 = \frac{1-\sqrt{17}}{2} = -1.56 < 0$$

(b) **Dynamic Programming Approach - MNE**

- The Hamilton-Jacobi-Bellman equation for city i is

$$rV_i(G) = \max_{g_i \geq 0} \left[\left(G - \frac{1}{2}G^2 - \frac{1}{2}g_i^2 \right) + V_i'(G)\left(g_1 + g_2 - \delta G \right) \right]$$

- Differentiating the right-hand side with respect to g_i and equating to 0 yields

$$-g_i + V_i'(G) = 0$$

$$\Rightarrow g_i(G) = V_i'(G)$$

- Since the game is symmetric, we focus on the symmetric equilibrium strategies. Also game is linear-quadratic, we can make the informed guess that the value function is the same for both players and takes the following form:

$$V_i(G) = \frac{A}{2}G^2 + BG + C, \quad i = 1, 2$$

$$\Rightarrow V_i'(G) = AG + B, \quad i = 1, 2$$

$$\Rightarrow g_i(G) = V_i'(G) = AG + B, \quad i = 1, 2$$

- Then given $r = \delta = 1$ we have

$$\begin{aligned}\frac{A}{2}G^2 + BG + C =& \left(G - \frac{1}{2}G^2 - \frac{1}{2}(AG + B)^2 \right) + (AG + B)\left(2(AG + B) - G \right) \\ =& G - \frac{1}{2}G^2 + \frac{3}{2}(AG + B)^2 - AG^2 - BG\end{aligned}$$

- Solve by identification:

$$\frac{A}{2} = -\frac{1}{2} + \frac{3}{2}A^2 - A \Rightarrow \frac{3}{2}A^2 - \frac{3}{2}A - \frac{1}{2} = 0 \Rightarrow 3A^2 - 3A - 1 = 0$$

$$\Rightarrow A = \frac{3 \pm \sqrt{9 + 12}}{6} = \frac{3 \pm \sqrt{21}}{6}$$

and

$$B = 1 + 3AB - B \Rightarrow B = \frac{1}{2 - 3A}$$

and

$$C = \frac{3}{2}B^2$$

- To guarantees the global stability of the state trajectory $G(t)$,

$$\frac{dG}{dt} = g_1 + g_2 - G = 2(AG + B) - G = (2A - 1)G + 2B$$

we would need to satisfy the following condition

$$\frac{\frac{dG}{dt}}{dt} = 2A - 1 < 0$$

Thus we select the negative root

$$\Rightarrow A = \frac{3 - \sqrt{21}}{6} = -0.26$$

$$\Rightarrow B = \frac{1}{2 - 3A} = 0.36$$

- Then we have

$$g_1(G) = g_2(G) = AG + B = -0.26G + 0.36$$

$$\frac{dG}{dt} = (2A - 1)G + 2B = -1.52G + 0.72$$

$$\Rightarrow G(t) = ke^{-1.52t} + G_{SS}^{M}$$

Since

$$G(0) = k + G_{SS}^{M} = G_0$$

$$\Rightarrow k = G_0 - G_{SS}^{M}$$

$$\Rightarrow G(t) = (G_0 - G_{SS}^{M})e^{-1.52t} + G_{SS}^{M}$$

where

$$G_{SS}^{M} = \frac{2B}{1 - 2A} = 0.47$$

(c) **Method one: Maximum Principle Approach**

Step 1: The current value Hamiltonian for the social planner is

$$\mathcal{H}(G, \lambda, g_1, g_2) = G\left(1 - \frac{1}{2}G\right) - \frac{1}{2}g_1^2 + G\left(1 - \frac{1}{2}G\right) - \frac{1}{2}g_2^2 + \lambda(g_1 + g_2 - \delta G)$$

$$\Rightarrow \mathcal{H}(G, \lambda, g_1, g_2) = 2G - G^2 - \frac{1}{2}g_1^2 - \frac{1}{2}g_2^2 + \lambda(g_1 + g_2 - \delta G)$$

where $\lambda(t)$ are the current-valued costate variables.

Step 2: Maximizing the Hamiltonian with respect to the control variable, g_i, gives

$$\frac{\partial \mathcal{H}}{\partial g_i} = -g_i + \lambda = 0 \Rightarrow g_i = \lambda, \quad i = 1, 2$$

Step 3: The differential equation for the costate variable is

$$\frac{d\lambda}{dt} - r\lambda = -\frac{\partial \mathcal{H}}{\partial G} = -(2 - 2G - \lambda\delta)$$

$$\Rightarrow \frac{d\lambda}{dt} = 2G - 2 + (r + \delta)\lambda$$

Then given $\delta = r = 1$, the system of differential equations is

$$\frac{d\lambda}{dt} = 2G - 2 + 2\lambda$$

$$\frac{dG}{dt} = g_1 + g_2 - \delta G = 2\lambda - G$$

Or:

$$\begin{pmatrix} \frac{d\lambda}{dt} \\ \frac{dG}{dt} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \lambda \\ G \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

The steady states are G_{SS}^{SO} and λ_{SS}^{SO} where

$$\frac{dG}{dt} = 2\lambda - G = 0 \Rightarrow G = 2\lambda$$

$$\frac{d\lambda}{dt} = 2G - 2 + 2\lambda = 0 \Rightarrow 4\lambda - 2 + 2\lambda = 0 \Rightarrow \lambda_{SS}^{SO} = \frac{1}{3}$$

$$\Rightarrow G_{SS}^{SO} = \frac{2}{3} = 0.67$$

The coefficient matrix is

$$A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

So the characteristic equation is

$$k^2 - (2-1)k + (-2-4) = 0$$

$$\Rightarrow k^2 - k - 6 = 0 \Rightarrow (k+2)(k-3) = 0$$

$$\Rightarrow k = -2 \text{ or } k = 3$$

Then

$$\lambda(t) = C_1 e^{-2t} + C_2 e^{3t} + \lambda_{SS}^{SO}$$

Step 4: The boundary conditions are

$$G(0) = G_0$$

$$\lim_{t\to\infty} e^{-t}\lambda(t) = 0 \Rightarrow C_2 = 0$$

$$\Rightarrow \lambda(t) = C_1 e^{-2t} + \lambda_{SS}^{SO}$$

$$\Rightarrow \frac{d\lambda}{dt} = -2C_1 e^{-2t}$$

Then

$$\frac{d\lambda}{dt} = 2G - 2 + 2\lambda \Rightarrow G(t) = \frac{1}{2}\left(\frac{d\lambda}{dt} - 2\lambda + 2\right) = -C_1 e^{-2t} - (C_1 e^{-2t} + \lambda_{SS}^{SO}) + 1$$

$$\Rightarrow G(t) = -2C_1 e^{-2t} - \lambda_{SS}^{SO} + 1 = -2C_1 e^{-2t} + G_{SS}^{SO}$$

$$G(0) = -2C_1 + G_{SS}^{SO} = G_0 \Rightarrow C_1 = \frac{G_0 - G_{SS}^{SO}}{-2}$$

Step 5: The social optimum are

$$G(t) = (G_0 - G_{SS}^{SO})e^{-2t} + G_{SS}^{SO}$$

$$g_1(t) = g_2(t) = \lambda(t) = -\frac{G_0 - G_{SS}^{OL}}{2} e^{-2t} + \lambda_{SS}^{SO}$$

where

$$G_{SS}^{SO} = \frac{2}{3} = 0.67, \quad \lambda_{SS}^{SO} = \frac{1}{3}$$

Method 2: Dynamic Programming Approach

- The Hamilton-Jacobi-Bellman equation for the society is

$$rV(G) = \underset{g_i \geq 0}{max} \left[\left(G - \frac{1}{2}G^2 - \frac{1}{2}g_1^2\right) + \left(G - \frac{1}{2}G^2 - \frac{1}{2}g_2^2\right) + V'(G)\left(g_1 + g_2 - \delta G\right)\right]$$

$$\Rightarrow rV(G) = \underset{g_i \geq 0}{max} \left[2G - G^2 - \frac{1}{2}g_1^2 - \frac{1}{2}g_2^2 + V'(G)\left(g_1 + g_2 - \delta G\right)\right]$$

- Differentiating the right-hand side with respect to g_i and equating to 0 yields

$$-g_i + V'(G) = 0$$

$$\Rightarrow g_i(G) = V'(G), \quad i = 1, 2$$

- Since the game is linear-quadratic, we can make the informed guess that the value function takes the following form:

$$V(G) = \frac{A}{2}G^2 + BG + C$$

$$\Rightarrow V'(G) = AG + B$$

$$\Rightarrow g_1(G) = g_2(G) = V'(G) = AG + B$$

- Then given $r = \delta = 1$ we have

$$\begin{aligned}\frac{A}{2}G^2 + BG + C =& 2G - G^2 - (AG + B)^2 + (AG + B)\left(2(AG + B) - G\right)\\ =& 2G - G^2 + (AG + B)^2 - AG^2 - BG\end{aligned}$$

- Solve by identification:

$$\frac{A}{2} = -1 + A^2 - A \Rightarrow A^2 - \frac{3}{2}A - 1 = 0 \Rightarrow 2A^2 - 3A - 2 = 0$$

$$\Rightarrow (2A + 1)(A - 2) = 0$$

$$\Rightarrow A = -\frac{1}{2}, \quad A = 2$$

and

$$B = 2 + 2AB - B \Rightarrow B = \frac{1}{1 - A}$$

and

$$C = B^2$$

- To guarantees the global stability of the state trajectory $G(t)$,

$$\frac{dG}{dt} = g_1 + g_2 - G = 2(AG + B) - G = (2A - 1)G + 2B$$

we would need to satisfy the following condition

$$\frac{\frac{dG}{dt}}{dt} = 2A - 1 < 0$$

Thus we select the negative root

$$\Rightarrow A = -\frac{1}{2}$$

$$\Rightarrow B = \frac{1}{1 - A} = \frac{2}{3}$$

- Then we have

$$g_1(G) = g_2(G) = AG + B = -\frac{1}{2}G + \frac{2}{3}$$

$$\frac{dG}{dt} = (2A - 1)G + 2B = -2G + \frac{4}{3}$$

$$\Rightarrow G(t) = ke^{-2t} + G_{SS}^{SO}$$

Since

$$G(0) = k + G_{SS}^{SO} = G_0$$

$$\Rightarrow k = G_0 - G_{SS}^{SO}$$

$$\Rightarrow G(t) = (G_0 - G_{SS}^{SO})e^{-2t} + G_{SS}^{SO}$$

where

$$G_{SS}^{SO} = \frac{2B}{1 - 2A} = \frac{2}{3} = 0.67$$

(d) **The steady state level of public good** from open-loop, Markov, and socially optimal are respectively

$$G_{SS}^{SO} = \frac{2}{3} = 0.67 > G_{SS}^{OL} = \frac{1}{2} = 0.5 > G_{SS}^{M} = 0.47$$

The socially optimal steady state stock of public good is the highest, followed by the open-loop Nash equilibrium and MNE is the smallest. This makes total sense. As in an OLNE, each city is committing to a fixed time path of contribution of the public good, taking the other one's contribution as given, thus each city is trying to free-ride on the other's contribution and this leads to a lower steady state stock of the public good. For MNE, each city will re-optimize each period based on the stock of the public good and thus the stock level is the lowest.

(e) **The contribution path in each case is**
OLNE:

$$G(t) = (G_0 - 0.5)e^{-1.56t} + 0.5$$

MNE:

$$G(t) = (G_0 - 0.47)e^{-1.52t} + 0.47$$

Socially Optimum:

$$G(t) = (G_0 - 0.67)e^{-2t} + 0.67$$

This corroborate the results in part (d). As we can see, the exponential parameter is highest in socially optimum, followed by OLNE and the lowest is MNE. So starting form any initial value, the contribution in the socially optimum case is the fastest and converges to a higher steady state stock level of the public good.

Practice Question 7. **Part A:** A firm is licensed to produce a polluting product, the quantity of which is denoted by C, from $t = 0$ until T. The horizon T is fixed. The firm is able to extract all consumer surplus which is given by $\ln C$, and for simplicity we assume the firm's cost of production is zero. Production generates a pollution stock P, which follows the dynamics of

$$\frac{dP}{dt} = C - kP, \quad P(0) = P_0$$

where k is the decay rate. The damage caused by pollution is

$$D(P) = \gamma P$$

The regulator imposes at each moment a tax on the firm equal to $D(P)$ (This is like a lump-sum one-time tax). Therefore the objective of the firm is

$$\max_{C>0} \int_0^T (\ln C - \gamma P)e^{-rt}dt$$

where $r > 0$ denotes the discount rate. The level of $P(T)$ is free, i.e., to be chosen freely by the firm.

(a) Solve for the firm's optimal path of C_f using the maximum principle.

(b) What is the approximate maximum willingness to pay of the firm for a technology that would reduce the initial stock of pollution from P_0 to $P_0 - \Delta P_0$?

Part B: Consider now the problem of social planner whose time horizon is infinite

$$\max_{C>0} \int_0^T (\ln C - \gamma P)e^{-rt}dt$$

(c) Find the optimal consumption strategy and path using a dynamic programming approach.

(d) The planner is considering levying a tax at time T from the polluting firrm (that would be announced at time 0). Can the planner induce the firm, through this tax, to choose the socially optimal path? If so how? and if not why not? Prove your answer.

Solutions:

(a) **Step 1:** The current value Hamiltonian for the firm is

$$\mathcal{H}(P, \lambda, C) = \ln C - \gamma P + \lambda(C - kP)$$

where $\lambda(t)$ is the current-valued costate variable.

Step 2: Maximizing the Hamiltonian with respect to the control variable, C, gives

$$\frac{\partial \mathcal{H}}{\partial C} = \frac{1}{C} + \lambda = 0 \Rightarrow C = -\frac{1}{\lambda}$$

Step 3: The differential equations for the costate variable is

$$\frac{d\lambda}{dt} - r\lambda = -\frac{\partial \mathcal{H}}{\partial P} = -(-\gamma - \lambda k)$$

$$\Rightarrow \frac{d\lambda}{dt} = (r+k)\lambda + \gamma$$

$$\Rightarrow \lambda(t) = ae^{(r+k)t} - \frac{\gamma}{r+k}$$

where a is the coefficient of integration.

Step 4: The transversality condition is

$$\lambda(T) = 0$$

$$\Rightarrow ae^{(r+k)T} - \frac{\gamma}{r+k} = 0$$

$$\Rightarrow a = \frac{\frac{\gamma}{r+k}}{e^{(r+k)T}}$$

$$\Rightarrow \lambda(t) = ae^{(r+k)t} - \frac{\gamma}{r+k} = \frac{\frac{\gamma}{r+k}}{e^{(r+k)T}}e^{(r+k)t} - \frac{\gamma}{r+k} = \frac{\gamma}{r+k}\left[e^{(r+k)(t-T)} - 1\right]$$

Then

$$C(t) = -\frac{1}{\lambda(t)} = \frac{r+k}{\gamma}\frac{1}{1-e^{(r+k)(t-T)}}$$

(b) Since $\lambda(t)$ is the shadow price or marginal value of the pollution $P(t)$, then it should be negative. For the firm reduce the initial stock of pollution from P_0 to $P_0 - \Delta P_0$, the approximate maximum willingness to pay for a technology will be

$$-\lambda(0)\Delta P_0 = -\frac{\gamma}{r+k}\left[e^{-(r+k)T} - 1\right]\Delta P_0$$

(c) The Hamilton-Jacobi-Bellman equation for the social planner is

$$rV(P) = \max_{C>0}\left[\ln C - \gamma P + V'(P)\Big(C - kP\Big)\right]$$

Differentiating the right-hand side with respect to C and equating to 0 yields

$$\frac{1}{C} + V'(P) = 0 \Rightarrow C = -\frac{1}{V'(P)}$$

Since the game is linear state, we can make the informed guess that the value function takes the following form:

$$V(P) = AP + B \Rightarrow V'(P) = A \Rightarrow C = -\frac{1}{V'(P)} = -\frac{1}{A}$$

Then the HJB becomes

$$r(AP+B) = \ln(-\frac{1}{A}) - \gamma P + A\Big(-\frac{1}{A} - kP\Big)$$

Solve by identification:

$$rA = -\gamma - Ak \Rightarrow (r+k)A = -\gamma \Rightarrow A = -\frac{\gamma}{r+k}$$

and

$$rB = \ln(-\frac{1}{A}) - 1 = \ln\left(\frac{r+k}{\gamma}\right) - 1$$
$$\Rightarrow B = \frac{1}{r}\left[\ln\left(\frac{r+k}{\gamma}\right) - 1\right]$$

So the policy function is a constant

$$\Rightarrow C(P) = -\frac{1}{A} = \frac{r+k}{\gamma}$$

(d) The main difference between the firm and the social planner lies in the time horizon. To induce the firm to choose the socially optimal path, the social planner introduces a per unit tax

$$\tau = \frac{\gamma}{r+k}$$

on $P(T)$ at time T for the polluting firm such that

$$S(P(T)) = \tau P(T)$$

where $S(P(T))$ is the tax payment at time T, or $-S(T)$ is the Salvage value or Scarp value at time T. Then the objective of the firm becomes

$$\max_{C>0} \int_0^T (\ln C - \gamma P)e^{-rt}dt - S(P(T))$$

Using maximum principle, we would have the same set of optimal conditions except the transversality condition.

Step 1: The current value Hamiltonian for the firm is

$$\mathcal{H}(P, \lambda, C) = \ln C - \gamma P + \lambda(C - kP)$$

where $\lambda(t)$ is the current-valued costate variable.

Step 2: Maximizing the Hamiltonian with respect to the control variable, C, gives

$$\frac{\partial \mathcal{H}}{\partial C} = \frac{1}{C} + \lambda = 0 \Rightarrow C = -\frac{1}{\lambda}$$

Step 3: The differential equations for the costate variable is

$$\frac{d\lambda}{dt} - r\lambda = -\frac{\partial \mathcal{H}}{\partial P} = -(-\gamma - \lambda k)$$

$$\Rightarrow \frac{d\lambda}{dt} = (r + k)\lambda + \gamma$$

$$\Rightarrow \lambda(t) = ae^{(r+k)t} - \frac{\gamma}{r + k}$$

where a is the coefficient of integration.

Step 4: The transversality condition now becomes

$$\lambda(T) = -\frac{\partial S(P(T))}{\partial P}$$

If the social planner introduces a per unit tax on $P(T)$ at time T, then

$$S(P(T)) = \tau P(T)$$

$$\Rightarrow \lambda(T) = -\frac{\partial S(P(T))}{\partial P} = -\tau = -\frac{\gamma}{r + k}$$

Then

$$\Rightarrow \lambda(T) = ae^{(r+k)T} - \frac{\gamma}{r + k} == -\frac{\gamma}{r + k}$$

So we must have

$$a = 0$$

$$\Rightarrow C(t) = -\frac{1}{\lambda(t)} = -\frac{1}{ae^{(r+k)t} - \frac{\gamma}{r+k}} = \frac{r + k}{\gamma}$$

Thus the firm's optimal path coincides with the socially optimal one. Then we have finished our proof.