4 Simultaneous Systems of Differential and Difference Equations

• It is common in economic models **for two or more variables to be determined simultaneously**. When the model is dynamic and involves two or more variables, a system of differential or difference equations arises.

4.1 Linear Differential Equation Systems

• We begin with the simplest case - a system of two linear differential equations - and solve it using the **substitution method**. We then proceed to a more general method, known as the **direct method**, that can be used to solve a system of linear differential equations with more than two equations.

4.1.1 The Substitution Method

• A linear system of two autonomous differential equations is expressed as

$$\frac{dy_1}{dt} = a_{11}y_1 + a_{12}y_2 + b_1$$
$$\frac{dy_2}{dt} = a_{21}y_1 + a_{22}y_2 + b_2$$

⇒ The system is linear because it contains only linear differential equations which, as usual, means that y_i and $\frac{dy_i}{dt}$ are not raised to any power other than one.

⇒ The system is autonomous because the coefficients, a_{ij}, and the terms, b_i, are constants.

⇒ The equations must be solved simultaneously because $\frac{dy_1}{dt}$ depends on the solution for y_2 and $\frac{dy_2}{dt}$ depends on the solution for y_1.

• As in previous sections on linear differential and difference equations, we separate the problem of finding the complete solutions into two parts. We first find the **homogeneous solutions** and then find **particular solutions**. The **complete solutions** are the sum of the homogeneous and particular solutions.

$\Rightarrow$ In symbols,

$$y_1 = y_1^h + y_1^p$$

$$y_2 = y_2^h + y_2^p$$

where y_i is the complete solution, y_i^h is the general homogeneous solution for y_i, and y_i^p is the particular solution for y_i.

Theorem 9. *The* ***complete solutions*** *to the system of two linear, first-order differential equations*

$$\frac{dy_1}{dt} = a_{11}y_1 + a_{12}y_2 + b_1$$
$$\frac{dy_2}{dt} = a_{21}y_1 + a_{22}y_2 + b_2$$

are the following:

1. ***Real and distinct roots***:

$$y_1(t) = C_1e^{r_1t} + C_2e^{r_2t} + \bar{y}_1$$

$$y_2(t) = \frac{r_1 - a_{11}}{a_{12}}C_1e^{r_1t} + \frac{r_2 - a_{11}}{a_{12}}C_2e^{r_2t} + \bar{y}_2$$

2. ***Real and equal roots***:

$$y_1(t) = C_1e^{rt} + C_2te^{rt} + \bar{y}_1$$

$$y_2(t) = \left[\frac{r - a_{11}}{a_{12}}(C_1 + C_2t) + \frac{C_2}{a_{12}}\right]e^{rt} + \bar{y}_2$$

where r_1, r_2 are the roots to the characteristic equation

$$r^2 - (a_{11} + a_{22})r + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\Longleftrightarrow r^2 - tr(A) + |A| = 0$$

Example 4. Solve the following system of differential equations, given the initial conditions $y_1(0) = 1$ and $y_2(0) = 3$:

$$\frac{dy_1}{dt} = y_1 - 3y_2 - 5$$
$$\frac{dy_2}{dt} = 0.25y_1 + 3y_2 - 5$$

Solutions:

Approach One: The Substitution Method

Step 1: The homogeneous form of the system is

$$\frac{dy_1}{dt} = y_1 - 3y_2$$
$$\frac{dy_2}{dt} = 0.25y_1 + 3y_2$$

$\Rightarrow$ Differentiate the first equation to get

$$\frac{d^2y_1}{dt^2} = \frac{dy_1}{dt} - 3\frac{dy_2}{dt}$$

$\Rightarrow$ Use the second equation to substitute for $\frac{dy_2}{dt}$. This gives

$$\Rightarrow \frac{d^2y_1}{dt^2} = \frac{dy_1}{dt} - 3\frac{dy_2}{dt} = \frac{dy_1}{dt} - 3(0.25y_1 + 3y_2)$$

$\Rightarrow$ Finally use the first equation again to obtain an expression for y_2:

$$y_2 = \frac{y_1 - \frac{dy_1}{dt}}{3} \Rightarrow 3y_2 = y_1 - \frac{dy_1}{dt}$$

Using this to substitute for y_2 gives

$$\Rightarrow \frac{d^2y_1}{dt^2} = \frac{dy_1}{dt} - 0.75y_1 - 3\left(y_1 - \frac{dy_1}{dt}\right) = 4\frac{dy_1}{dt} - 3.75y_1$$

$\Rightarrow$ Simplify and rearrange to get

$$\Rightarrow \frac{d^2y_1}{dt^2} - 4\frac{dy_1}{dt} + 3.75y_1 = 0$$

Step 2: The **characteristic equation** is

$$r^2 - 4r + 3.75 = 0$$

with roots

$$r_1 = \frac{4 - \sqrt{16 - 4 \cdot 3.75}}{2} = \frac{4-1}{2} = 1.5, \quad r_2 = \frac{4+1}{2} = 2.5$$

The roots are real and distinct.

$\Rightarrow$ The solution for y_1 then is

$$y_1(t) = C_1 e^{1.5t} + C_2 e^{2.5t}$$

$\Rightarrow$ The solution for y_2 can now be found using equation

$$y_2 = \frac{y_1 - \frac{dy_1}{dt}}{3}$$

First, differentiate the solution for y_1 to get

$$\frac{dy_1}{dt} = 1.5 C_1 e^{1.5t} + 2.5 C_2 e^{2.5t}$$

then substitute it and the solution for y_1 into equation

$$\Rightarrow y_2(t) = \frac{y_1 - \frac{dy_1}{dt}}{3} = \frac{C_1 e^{1.5t} + C_2 e^{2.5t} - 1.5 C_1 e^{1.5t} - 2.5 C_2 e^{2.5t}}{3} = -\frac{C_1 e^{1.5t}}{6} - \frac{C_2 e^{2.5t}}{2}$$

Step 3: Find the **particular solutions** for

$$\frac{dy_1}{dt} = y_1 - 3y_2 - 5$$
$$\frac{dy_2}{dt} = 0.25 y_1 + 3y_2 - 5$$

Set $\frac{dy_1}{dt} = 0, \frac{dy_2}{dt} = 0$ and solve for $\bar{y}_1, \bar{y}_2$:

$$\bar{y}_1 - 3\bar{y}_2 - 5 = 0$$
$$0.25\bar{y}_1 + 3\bar{y}_2 - 5 = 0$$

$$\Rightarrow \bar{y}_1 = 8, \quad \bar{y}_2 = 1$$

Step 4: The **complete solution** to the system of two linear equations is the sum of the homogeneous solutions and the particular solutions:

$$y_1(t) = C_1 e^{1.5t} + C_2 e^{2.5t} + 8$$

$$y_2(t) = -\frac{C_1 e^{1.5t}}{6} - \frac{C_2 e^{2.5t}}{2} + 1$$

Step 5: Use the initial conditions $y_1(0) = 1$ and $y_2(0) = 3$ to solve the constants of integration C_1 and C_2.

$$y_1(0) = C_1 + C_2 + 8 = 1$$

$$y_2(0) = -\frac{C_1}{6} - \frac{C_2}{2} + 1 = 3$$

$$\Rightarrow C_1 = -4.5, \quad C_2 = -2.5$$

The complete solutions then become

$$y_1(t) = -\frac{9}{2} e^{1.5t} - \frac{5}{2} e^{2.5t} + 8$$

$$y_2(t) = \frac{3}{4} e^{1.5t} + \frac{5}{4} e^{2.5t} + 1$$

Example 5. Solve the following system of differential equations, given the initial conditions $y_1(0) = 5$ and $y_2(0) = 0$:

$$\frac{dy_1}{dt} = -6y_1 - 8y_2 + 2$$
$$\frac{dy_2}{dt} = 2y_1 + 2y_2 + 2$$

Solutions: First find the **homogeneous solutions** and then **particular solutions**

Step 1: The homogeneous form of the system is

$$\frac{dy_1}{dt} = -6y_1 - 8y_2$$
$$\frac{dy_2}{dt} = 2y_1 + 2y_2$$

$\Rightarrow$ Differentiate the first equation to get

$$\frac{d^2y_1}{dt^2} = -6\frac{dy_1}{dt} - 8\frac{dy_2}{dt}$$

$\Rightarrow$ Use the second equation to substitute for $\frac{dy_2}{dt}$. This gives

$$\Rightarrow \frac{d^2y_1}{dt^2} = -6\frac{dy_1}{dt} - 8\frac{dy_2}{dt} = -6\frac{dy_1}{dt} - 8(2y_1 + 2y_2)$$

$\Rightarrow$ Finally use the first equation again to obtain an expression for y_2:

$$y_2 = -\frac{6y_1 + \frac{dy_1}{dt}}{8}$$

Using this to substitute for y_2 gives

$$\Rightarrow \frac{d^2y_1}{dt^2} = -6\frac{dy_1}{dt} - 16y_1 + 16\left(\frac{6y_1 + \frac{dy_1}{dt}}{8}\right) = -4\frac{dy_1}{dt} - 4y_1$$

$\Rightarrow$ Simplify and rearrange to get

$$\Rightarrow \frac{d^2y_1}{dt^2} + 4\frac{dy_1}{dt} + 4y_1 = 0$$

Step 2: The **characteristic equation** is

$$r^2 + 4r + 4 = 0$$

with roots

$$r_1 = r_2 = -2$$

The roots are real and equal.

$\Rightarrow$ The solution for y_1 then is

$$y_1(t) = C_1e^{-2t} + C_2te^{-2t}$$

$\Rightarrow$ The solution for y_2 can now be found using equation

$$y_2 = -\frac{6y_1 + \frac{dy_1}{dt}}{8}$$

First, differentiate the solution for y_1 to get

$$\frac{dy_1}{dt} = -2C_1e^{-2t} - 2C_2te^{-2t} + C_2e^{-2t}$$

then substitute it and the solution for y_1 into equation

$$\begin{aligned}\Rightarrow y_2(t) &= -\frac{6y_1 + \frac{dy_1}{dt}}{8} \\ &= -\frac{6(C_1e^{-2t} + C_2te^{-2t}) - 2C_1e^{-2t} - 2C_2te^{-2t} + C_2e^{-2t}}{8} \\ &= \left(-\frac{1}{2}C_1 - \frac{1}{2}C_2t - \frac{1}{8}C_2\right)e^{-2t}\end{aligned}$$

Step 3: Find the **particular solutions** for

$$\begin{aligned}\frac{dy_1}{dt} &= -6y_1 - 8y_2 + 2 \\ \frac{dy_2}{dt} &= 2y_1 + 2y_2 + 2\end{aligned}$$

Set $\frac{dy_1}{dt} = 0, \frac{dy_2}{dt} = 0$ and solve for $\bar{y}_1, \bar{y}_2$:

$$\begin{aligned}-6\bar{y}_1 - 8\bar{y}_2 + 2 &= 0 \\ 2\bar{y}_1 + 2\bar{y}_2 + 2 &= 0\end{aligned}$$

$$\Rightarrow \bar{y}_1 = -5, \quad \bar{y}_2 = 4$$

Step 4: The **complete solution** to the system of two linear equations is the sum of the homogeneous solutions and the particular solutions:

$$y_1(t) = C_1e^{-2t} + C_2te^{-2t} - 5$$

$$y_2(t) = \left(-\frac{1}{2}C_1 - \frac{1}{2}C_2t - \frac{1}{8}C_2\right)e^{-2t} + 4$$

Step 5: Use the initial conditions $y_1(0) = 5$ and $y_2(0) = 0$ to solve the constants of integration C_1 and C_2.

$$y_1(0) = C_1 - 5 = 5 \Rightarrow C_1 = 10$$

$$y_2(0) = -\frac{1}{2}C_1 - \frac{1}{8}C_2 + 4 = -5 - \frac{1}{8}C_2 + 4 = 0 \Rightarrow C_2 = -8$$

The complete solutions then become

$$y_1(t) = 10e^{-2t} - 8te^{-2t} - 5$$

$$y_2(t) = (4t - 4)e^{-2t} + 4$$

4.1.2 The Direct Method

- Although the substitution method works well for systems of two differential equations, it can become **cumbersome for larger systems**. The following **direct approach** to solving a system of linear differential equations **circumvents this limitation**.

Definition 5. ***A linear system of*** n ***autonomous differential equations is expressed in matrix form*** *as*

$$\frac{dy}{dt} = Ay + b$$

where A is an $n \times n$ matrix of constant coefficients, b is a vector of n constant terms, y is a vector of n variables, and $\frac{dy}{dt}$ is a vector of n derivatives.

Example 6. Write the 2×2 matrix of coefficients and the vector of two constant terms in the case of a system of $n = 2$ linear, autonomous differential equations.

$\Rightarrow$ The matrix of coefficients and the vector of constant terms are respectively

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad b = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$$

4.1.2.1 The Homogeneous Solutions

- The solution to the **complete system of equations** is obtained by adding together the homogeneous solutions and the particular solutions.

⇒ Begin by putting the complete system into its homogeneous form

$$\frac{dy}{dt} = Ay$$

We proceed by "**guessing**" that the homogeneous solutions are of the form

$$y = ke^{rt}$$

where k is an n-dimensional vector of constants and r is a scalar.

⇒ To see if this guess is correct, check that the guessed solution and its first derivative satisfy the differential equation system.

- The derivative of the proposed solution is

$$\frac{dy}{dt} = rke^{rt}$$

- Substitution of these derivatives and the proposed solutions into the original system of equations gives

$$rke^{rt} = Ake^{rt}$$

- Simplifying gives

$$(A - rI)ke^{rt} = 0 \Rightarrow (A - rI)k = 0$$

where I is the identity matrix, and 0 is the zero-vector.

⇒ We wish to find the values of r that solve this equation. These values of r make our guessed solution correct.

⇒ We know that a system of linear homogeneous equations such as equation

$$(A - rI)k = 0$$

has **a nontrivial solution** if and only if **the determinant of** $(A - rI)$ **is identically equal to zero**. Thus, the solution values for r are found by solving

$$|A - rI| = 0$$

which is **a polynomial equation of degree n in the unknown number r**.

⇒ This is known as the **characteristic equation of matrix A** and **its solutions are called the characteristic roots** or the **eigenvalues of the A matrix**.

⇒ A nonzero vector k_1, which is a solution of equation

$$(A - rI)k = 0$$

for a particular eigenvalue, r_1, is called the **eigenvector** of the matrix A corresponding to the eigenvalue r_1.

⇒ In the case $n = 2$, equation $|A - rI| = 0$ becomes

$$|A - rI| = \begin{vmatrix} a_{11} - r & a_{12} \\ a_{21} & a_{22} - r \end{vmatrix} = 0$$

which, after simplifying, gives

$$\Rightarrow (a_{11} - r)(a_{22} - r) - a_{12}a_{21} = 0$$

$$\Rightarrow r^2 - (a_{11} + a_{22})r + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

which is, of course, the **same characteristic equation** obtained using the substitution method.

⇒ Before proceeding, it is useful to pause and make note of the fact that in the case of $n = 2$, the characteristic equation can be written as

$$r^2 - tr(A)r + |A| = 0$$

where $tr(A) = a_{11}+a_{22}$ is **the sum of the diagonal elements** of the coefficient matrix and $|A| = a_{11}a_{22} - a_{12}a_{21}$ is the determinant of the coefficient matrix.

⇒ The solution for the characteristic roots can then be expressed as

$$r_1, r_2 = \frac{tr(A) \pm \sqrt{(tr(A))^2 - 4|A|}}{2}$$

This expression provides a fast way of calculating the characteristic roots directly from the coefficient matrix.

⇒ In general, **there are n equations and n characteristic roots; therefore, there are n solutions to the system of differential equations**. As in the case of a two dimensional system of linear equations, the general solution to the homogeneous form is **a linear combination of n distinct solutions**.

Theorem 10. *If $y^1, y^2, \cdots, y^n$ are linearly independent solutions of the homogeneous system in equation*

$$\frac{dy}{dt} = Ay,$$

then the ***general solution of the system is the linear combination***

$$y(t) = c_1 y^1(t) + c_2 y^2(t) + \cdots + c_n y^n(t)$$

for a unique choice of the constants.

Example 7. *Solve the following 2×2 system of differential equations using the direct method:*

$$\frac{dy}{dt} = \begin{bmatrix} 4 & -1 \\ -4 & 4 \end{bmatrix} y$$

Solutions:

- In this example, $\frac{dy}{dt}$ and y are two-dimensional vectors. The characteristic equation is

$$|A - rI| = \begin{vmatrix} 4-r & -1 \\ -4 & 4-r \end{vmatrix} = 0$$

which becomes

$$r^2 - 8r + 12 = 0$$

$$\iff r^2 - tr(A)r + |A| = r^2 - (4+4)r + (4 \cdot 4 - 4 \cdot 1) = r^2 - 8r + 12 = 0$$

The solutions are $r_1 = 2$ and $r_2 = 6$.

$\Rightarrow$ For $r_1 = 2$, we want to compute nontrivial solutions for the eigenvectors

$$[A - rI]k = \begin{bmatrix} 4-2 & -1 \\ -4 & 4-2 \end{bmatrix} k = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

This gives

$$2k_1 - k_2 = 0$$

As we often do with eigenvectors, we set $k_1 = 1$ which gives $k_2 = 2$. Thus the **first set of solutions** is

$$y^1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$$

$\Rightarrow$ For $r_2 = 6$, the eigenvectors are the solution to

$$[A - rI]k = \begin{bmatrix} 4-6 & -1 \\ -4 & 4-6 \end{bmatrix} k = \begin{bmatrix} -2 & -1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

which gives

$$2k_1 + k_2 = 0$$

With $k_1 = 1$, we get $k_2 = -2$. Thus the **second set of solutions** is

$$y^2(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{6t}$$

$\Rightarrow$ Since these two solutions are linearly independent, the general solution is

$$y(t) = C_1 y^1(t) + C_2 y^2(t) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{6t}$$

4.1.2.2 The Particular Solutions

- The steady-state solutions provide the particular solutions we require. Set $\frac{dy}{dt} = 0$ in the complete system of differential equations. This gives

$$\frac{dy}{dt} = A\bar{y} + b = 0$$

for which the solution is

$$\bar{y} = -A^{-1}b$$

provided the inverse matrix A^{-1} exists.

⇒ This matrix exists if and only if A is **nonsingular** (i.e., the determinant must be nonzero).

⇒ We have already assumed this for the case of $n = 2$. We now assume that $|A| \neq 0$ for any n.

Approach Two: The Direct Method

- The coefficient matrix for that system is

$$A = \begin{bmatrix} 1 & -3 \\ 0.25 & 3 \end{bmatrix}$$

The characteristic equation then is

$$|A - rI| = \begin{vmatrix} 1-r & -3 \\ 0.25 & 3-r \end{vmatrix} = 0$$

which becomes

$$r^2 - 4r + 3.75 = 0$$

$$\iff r^2 - tr(A)r + |A| = r^2 - (1+3)r + (1 \cdot 3 + 3 \cdot 0.75) = r^2 - 4r + 3.75 = 0$$

The solutions are $r_1 = 1.5$ and $r_2 = 2.5$.

$\Rightarrow$ For $r_1 = 1.5$, the eigenvector is the solution to

$$[A - rI]k = \begin{vmatrix} 1 - 1.5 & -3 \\ 0.25 & 3 - 1.5 \end{vmatrix} k = \begin{bmatrix} -0.5 & -3 \\ 0.25 & 1.5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

This gives

$$k_1 + 6k_2 = 0$$

With $k_1 = 1$, we get $k_2 = -\frac{1}{6}$. Thus the **first set of solutions** is

$$y^1(t) = \begin{bmatrix} 1 \\ -\frac{1}{6} \end{bmatrix} e^{1.5t}$$

$\Rightarrow$ For $r_2 = 2.5$, the eigenvector is the solution to

$$[A - rI]k = \begin{vmatrix} 1 - 2.5 & -3 \\ 0.25 & 3 - 2.5 \end{vmatrix} k = \begin{bmatrix} -1.5 & -3 \\ 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

which gives

$$k_1 + 2k_2 = 0$$

With $k_1 = 1$, we get $k_2 = -\frac{1}{2}$. Thus the **second set of solutions** is

$$y^2(t) = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} e^{2.5t}$$

$\Rightarrow$ Since these two solutions are linearly independent, the general solution is

$$y(t) = C_1 y^1(t) + C_2 y^2(t) = C_1 \begin{bmatrix} 1 \\ \\ -\frac{1}{6} \end{bmatrix} e^{1.5t} + C_2 \begin{bmatrix} 1 \\ \\ -\frac{1}{2} \end{bmatrix} e^{2.5t} = \begin{bmatrix} C_1 e^{1.5t} + C_2 e^{2.5t} \\ \\ -\frac{C_1}{6} e^{1.5t} - \frac{C_2}{2} e^{2.5t} \end{bmatrix}$$

- Next we identify the particluar solution,

4.2 Stability Analysis

- The **steady-state solutions** to an autonomous system of differential equations are said to **be stable if the system converges to the steady state solutions** and unstable otherwise.

⇒ As in previous sections, we **emphasize the issue of stability here because of its importance in economic applications**.

⇒ We also found that the stability characteristics of differential equations depend on the **signs of the characteristic roots**.

  - **Roots with negative real parts are associated with differential equations that converge to the steady state (stable)**;
  - Roots with positive real parts are associated with differential equations that diverge from the steady state (unstable).

⇒ In this section we will see that **the stability of a system of differential equations also depends on the signs of the roots of the characteristic equation**. Theorem 10 states the conditions for convergence.

Theorem 11. *The steady-state solution of a system of linear, autonomous differential equations is* ***asymptotically stable if and only if the characteristic roots are negative****.*

Proof. Theorem 10 applies regardless of the number of equations in the system; however, we prove it only for the case of two equations. We consider the two possible types of roots that can occur:

Case 1: Real and distinct roots: The solutions to the system of two autonomous, linear differential equations are

$$y_1(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \bar{y}_1$$

$$y_2(t) = \frac{r_1 - a_{11}}{a_{12}} C_1 e^{r_1 t} + \frac{r_2 - a_{11}}{a_{12}} C_2 e^{r_2 t} + \bar{y}_2$$

Under what conditions does $y_1(t)$ converge to $\bar{y}_1$ and $y_2(t)$ converge to $\bar{y}_2$?

⇒ Inspection of the solutions reveals that if r_1 and r_2 are negative, the exponential terms, $e^{r_1 t}$ and $e^{r_2 t}$, tend to zero in the limit as t goes to infinity. Therefore both solutions converge to their steady states as $t \to \infty$ for all values of the constants of integrations.

- **Because $y_i(t)$ converges to $\bar{y}_i (i = 1, 2)$ only as $t \to \infty$**, the path $y_i(t)$ is asymptotic to the value $\bar{y}_i$.
- For this reason the steady state is said to be **asymptotically stable if both roots are negative**.

⇒ If r_1 and r_2 are positive, the exponential terms tend to infinity as $t \to \infty$. Hence both solutions diverge from the steady state, except when the constants of integration are both equal to zero. Therefore the **steady state is unstable when both roots are positive**.

⇒ If one root is negative and the other is positive, the exponential term containing the negative root goes to zero as $t \to \infty$ but the exponential term containing the positive root diverges to infinity as $t \to \infty$.

- Hence **both solutions diverge from the steady state** except when the constant of integration on the divergent exponential term is equal to zero.
- (This special case actually plays an important role in economic applications so we will have much more to say about it.)
- Therefore the steady state is unstable when even one of the roots is positive.

Case 2: Real and equal roots: The solutions to the system of two autonomous, linear differential equations are

$$y_1(t) = C_1 e^{rt} + C_2 t e^{rt} + \bar{y}_1$$

$$y_2(t) = \left[\frac{r - a_{11}}{a_{12}} (C_1 + C_2 t) + \frac{C_2}{a_{12}} \right] e^{rt} + \bar{y}_2$$

The equal roots are either positive or negative.

⇒ If positive, the exponential terms tend to infinity as $t \to \infty$, so both solutions diverge from the steady state.

⇒ If negative, both solutions converge to their steady-state values. The proof of this is identical to the proof of convergence in the case of equal roots for a second-order differential equation in section 3.3.

□

- Theorem 10 says that y_1 and y_2 converge to $\bar{y}_1$ and $\bar{y}_2$, respectively, if the roots are negative, **no matter what values the constants of integration take**.

⇒ Since the constants are determined by initial conditions, we can interpret this result as saying that **no matter what the initial conditions, $y_1(t)$ and $y_2(t)$ will always converge towards the values $\bar{y}_1$ and $\bar{y}_2$ if the roots are negative**.

- In economic models of dynamic optimization, it is common to obtain a system of differential equations in which **one of the characteristic roots is positive and the other is negative**. We examine this important case next.

Theorem 12. *If one of the characteristic roots is positive and the other is negative, the steady state equilibrium is called* ***a saddle-point equilibrium****. It is* ***unstable****. However, $y_1(t)$ and $y_2(t)$ converge to their steady-state solutions if the initial conditions for y_1 and y_2, satisfy the following equation:*

$$y_2 = \frac{r_1 - a_{11}}{a_{12}}(y_1 - \bar{y}_1) + \bar{y}_2$$

where r_1 is the negative root and r_2 is the positive root. The locus of points (y_1, y_2) defined by this equation is known as the ***saddle path****.*

Proof. The characteristic roots are real valued if they are of opposite sign. The solutions are

$$y_1(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \bar{y}_1$$
$$y_2(t) = \frac{r_1 - a_{11}}{a_{12}} C_1 e^{r_1 t} + \frac{r_2 - a_{11}}{a_{12}} C_2 e^{r_2 t} + \bar{y}_2$$

⇒ Without loss of generality, assume that r_1 is the negative root and r_2 is the positive root. Then, $y_1(t)$ **and** $y_2(t)$ **converge to their steady-state solutions if and only if** $C_2 = 0$.

⇒ Solving for C_1 and C_2 in the solutions above gives

$$y_2 = \frac{r_1 - a_{11}}{a_{12}}\left(y_1 - C_2 e^{r_2 t} + \bar{y}_1\right) + \frac{r_2 - a_{11}}{a_{12}} C_2 e^{r_2 t} + \bar{y}_2$$

$$\Rightarrow \frac{r_2 - r_1}{a_{12}} C_2 e^{r_2 t} = y_2 - \bar{y}_2 - (y_1 - \bar{y}_1)\frac{r_1 - a_{11}}{a_{12}}$$

$$\Rightarrow C_2 = \frac{a_{12}(y_2 - \bar{y}_2) - (y_1 - \bar{y}_1)(r_1 - a_{11})}{r_2 - r_1} e^{-r_2 t}$$

where the t arguments for y_1 and y_2 are suppressed to shorten the expressions.

⇒ Setting $C_2 = 0$ and simplifying implies that

$$a_{12}(y_2 - \bar{y}_2) - (y_1 - \bar{y}_1)(r_1 - a_{11}) = 0$$

$$\Rightarrow y_2 = \frac{r_1 - a_{11}}{a_{12}}(y_1 - \bar{y}_1) + \bar{y}_2$$

⇒ Theorem 11 tells us that **if even one of the characteristic roots is positive, the solutions will not converge to the steady state from arbitrarily chosen initial conditions**; on the other hand, **if the initial conditions for y_1 and y_2 happen to satisfy the equation given in theorem 11, the solutions will converge**.

⇒ This is called **a saddle-point equilibrium** and **can only occur for the case of real and distinct roots**. It plays quite an important role in economic dynamics so we will have more to say about it throughout this topic.

□

4.3 Linear Phase Diagrams

- The phase diagram proved to be a useful tool for conducting a qualitative analysis of a single nonlinear differential equation in section 3.2. It will prove to be equally valuable in the analysis of a system of two differential equations.

⇒ We explain the construction of a phase diagram, beginning with a system of two linear differential equations.

⇒ The method, and the interesting variety of trajectory systems that can arise, carry over to the analysis of systems of two nonlinear differential equations.

4.3.1 Phase Diagram for Both Roots Negative (Stable Node)

- Solve the following differential equation system, and draw its phase diagram:

$$\frac{dy_1}{dt} = -2y_1 + 2$$

$$\frac{dy_2}{dt} = -3y_2 + 6$$

Solutions: Since these differential equations are actually **independent of one another** and so can be solved separately as single equations. The solutions are thus

$$y_1(t) = C_1 e^{-2t} + 1$$

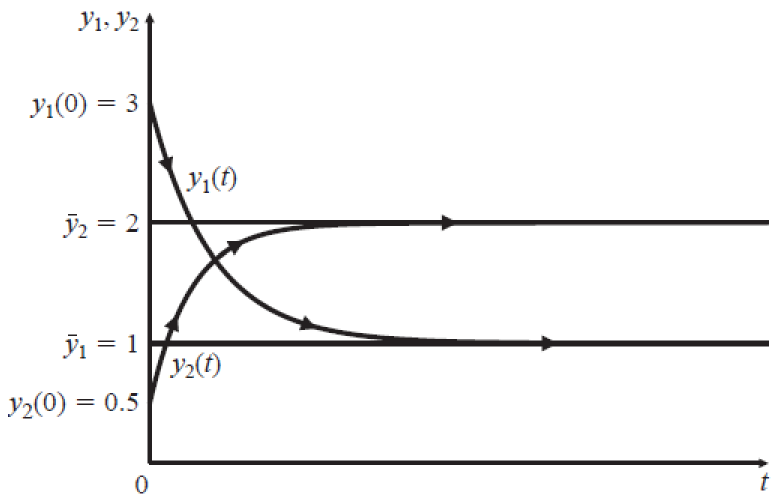
$$y_2(t) = C_2 e^{-3t} + 2$$

where C_1 and C_2 are arbitrary constants of integration.

⇒ In this system, it is clear that $y_1(t)$ converges to its steady-state solution $\bar{y}_1 = 1$ and $y_2(t)$ converges to its steady-state solution $\bar{y}_2 = 2$ because the exponential terms go to zero as $t \to \infty$.

⇒ **The steady-state point (1, 2) is therefore a stable equilibrium**.

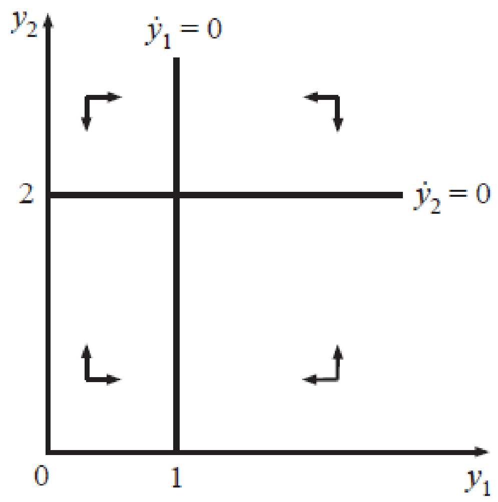
⇒ Suppose that we choose initial values of $y_1^0 = 3$ and $y_2^0 = 1/2$. Figure 7 shows the trajectories for $y_1(t)$ and $y_2(t)$ that emanate from these initial conditions.



The $y_1(t)$ trajectory falls (since it starts above its steady-state solution) as it converges and the $y_2(t)$ trajectory rises (since it starts below its steady-state solution) as it converges.

- Figure 8 provides a useful way to **"see" the solutions** for $y_1(t)$ and $y_2(t)$. However, it is **not always possible** to construct a diagram like figure 8 with y_1 and y_2 plotted as explicit functions of t **because explicit solutions cannot always be obtained (i.e., for nonlinear differential equations)**.

⇒ A phase diagram circumvents this problem and, at the same time, provides a different way to "see" the solution. **A phase diagram for a system of two differential equations** is drawn with y_2 on the vertical axis and y_1 on the horizontal axis. The y_1, y_2 plane is referred to as the **phase plane**.



⇒ We construct the phase diagram for this example in two steps:

$$y_1(0) = C_1 + 1 = 3 \Rightarrow C_1 = 2 \Rightarrow y_1(t) = 2e^{-2t} + 1$$

$$y_2(0) = C_2 + 2 = \frac{1}{2} \Rightarrow C_2 = -\frac{3}{2} \Rightarrow y_2(t) = -\frac{3}{2}e^{-3t} + 2$$

⇒ **Step 1: Determine the motion of y_1 in the phase plane**.

– Begin by graphing the locus of points for which $\frac{dy_1}{dt} = 0$. To find these points, set $\frac{dy_1}{dt} = 0$. This gives

$$\frac{dy_1}{dt} = -2y_1 + 2 = 0 \Rightarrow y_1 = 1$$

In figure 8, a vertical line is drawn at $y_1 = 1$ to show this locus of points.

- This line is called the **y_1 isocline**; **it divides the phase plane into two regions or isosectors**.
- In the region to the right of the isocline ($y_1 > 1$), the differential equation for y_1 shows that **$\frac{dy_1}{dt}$ is negative**.
- In the region to the left of the isocline ($y_1 < 1$), it shows that **$\frac{dy_1}{dt}$ is positive.**

⇒ We have established the motion of y_1 in the two regions separated by the y_1 isocline: y_1 **is decreasing to the right** (because $\frac{dy_1}{dt} < 0$) of the isocline and **increasing to the left** (because $\frac{dy_1}{dt} > 0$) of the isocline. To indicate this motion in the diagram, we **draw horizontal arrows** pointing in the appropriate directions.

⇒ **Step 2**: **Determine the motion of y_2 in the phase plane**.

- Begin by graphing the y_2 isocline. Setting $\frac{dy_2}{dt} = 0$ gives

$$\frac{dy_2}{dt} = -3y_2 + 6 = 0 \Rightarrow y_2 = 2$$

which is a **horizontal line** at $y_2 = 2$ in figure 8.

- **The y_2 isocline too divides the plane into two regions**. Inspection of the differential equation for y_2 shows that $\frac{dy_2}{dt}$ is negative above the isocline (for $y_2 > 2$) and positive below the isocline (for $y_2 < 2$).
- To indicate the motion of y_2 in the diagram, we **draw vertical arrows** pointing in the appropriate directions.

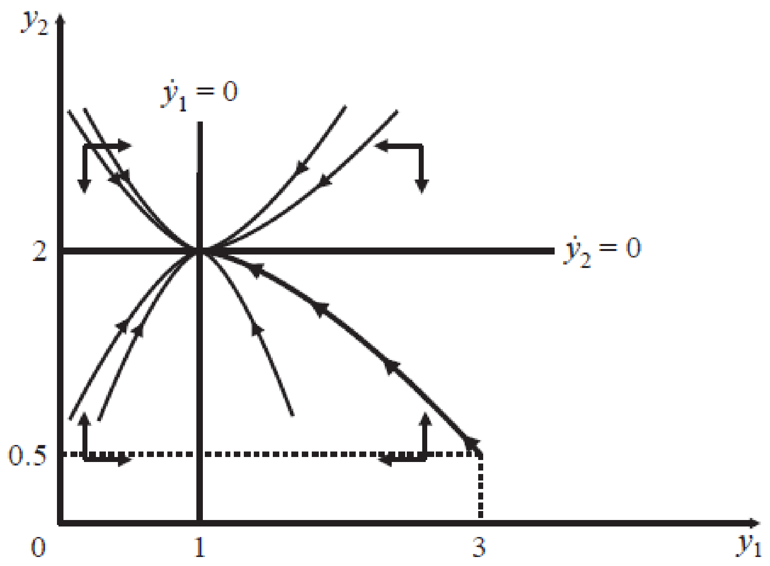
⇒ The **two isoclines intersect** where both $\frac{dy_1}{dt}$ and $\frac{dy_2}{dt}$ equal zero. By definition, this is the steady-state point. In figure 8, this occurs at the point (1, 2).

⇒ **The arrows of motion give a rough picture of what the trajectories** in the phase plane **look like and whether they move towards the steady state**.

- For example, to the southeast of the steady-state solution, the arrows of motion indicate that **trajectories in this sector of the phase plane move in a northwesterly direction**, meaning that $y_1(t)$ decreases and $y_2(t)$ increases over time.
- **Trajectories in the southwestern sector** of the phase plane **move in a northeasterly direction**.
- **Trajectories in the northwest move southeast and trajectories in the northeast move southwest**.

⇒ Overall, the arrows of motion tell us that **no matter what sector of the phase plane we start in, trajectories always move towards the steady state**. The phase diagram then **provides a good indication that the steady state is globally stable**.

- To get a precise picture of the trajectories requires that we actually plot them using the solutions to the two differential equations. We do this in figure 9.



A trajectory in the y_1, y_2 phase plane is the path followed by the pair y_1, y_2.

⇒ For example, suppose we start at the point (3, 0.5), the same initial point that was used in figure 7. The trajectory emanating from this initial point travels northwest over time.

⇒ Compare this trajectory of (y_1, y_2) to the trajectories of y_1 and y_2 plotted against t in figure 7. Although time is not explicitly shown in the phase diagram, it is definitely **implicit in the trajectories**.

- For example, at $t = 0$, the trajectory is at the point $(3, 1/2)$.
- After some time has passed, the pair $y_1(t)$ and $y_2(t)$ have moved along the trajectory to the northwest. This means $y_1(t)$ has decreased while $y_2(t)$ has increased, as shown explicitly in figure 7.
- After more time has passed, the pair is further along the trajectory which means $y_1(t)$ has decreased further while $y_2(t)$ has increased further, as shown in figure 7.
- As $t \to \infty$, the pair $y_1(t)$ and $y_2(t)$ converges to the point (1, 2) in figure 9 just as $y_1(t)$ and $y_2(t)$ converge to $\bar{y}_1$ and $\bar{y}_2$ in figure 7.

⇒ An important characteristic of a dynamic system in which **both roots are negative** is that **no matter what the initial values of y_1 and y_2, their paths converge to the steady state**.

⇒ This is demonstrated in figure 9 where a number of representative trajectories are drawn. All trajectories in the phase plane **converge asymptotically to the steady state**. This kind of equilibrium is called **a stable node**.

4.3.2 Phase Diagram for Both Roots Positive (Unstable Node)

- Solve and graph the phase diagram for the differential equation system

$$\frac{dy_1}{dt} = 2y_1 - 2$$

$$\frac{dy_2}{dt} = 3y_2 - 6$$

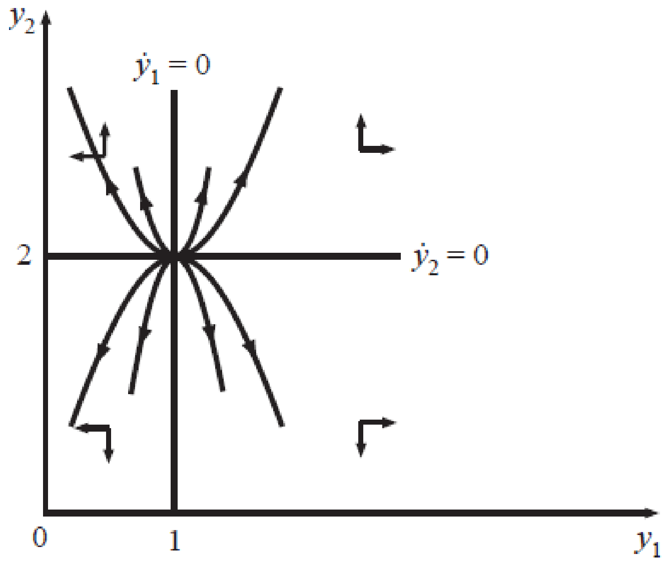
Solutions: The only differences between these equations and those in the previous section are the **signs of the coefficients and terms**. The solutions are

$$y_1(t) = C_1 e^{2t} + 1$$

$$y_2(t) = C_2 e^{3t} + 2$$

⇒ The phase diagram is constructed in the same way as for the system in the previous example. It has **the same isoclines as that system but the motion is exactly opposite**.

⇒ Thus, the trajectories in the phase diagram, drawn in figure 10 **have exactly the same shape but go in the opposite direction** to the trajectories for the previous example.



⇒ As a result, we see clearly that **the system diverges from the steady state from all points in the phase plane except the steady state itself**.

⇒ The steady state in this case is called **an unstable node**.

• In the case of real and equal roots, the dynamic system is called an **improper node**. We do not show a phase diagram here for this case; it is sufficient for our purposes to note that, **if the repeated root is negative, all trajectories converge to the steady state and if the repeated root is positive, all trajectories diverge from the steady state**.

4.3.3 Phase Diagram for Roots of Opposite Sign (Saddle Point)

- Draw the phase diagram for

$$\frac{dy_1}{dt} = y_2 - 2$$

$$\frac{dy_2}{dt} = \frac{1}{4}y_1 - \frac{1}{2}$$

Solutions: The characteristic equation is

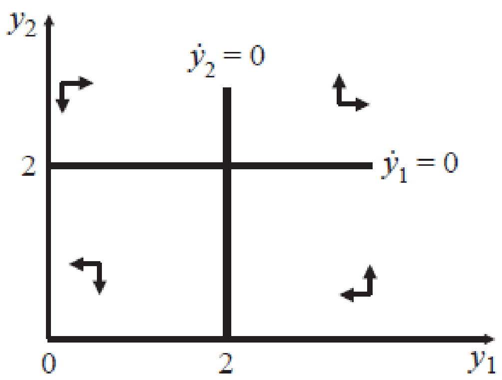
$$|A - rI| = \begin{vmatrix} 0 - r & 1 \\ \frac{1}{4} & 0 - r \end{vmatrix} = 0$$

$$\iff r^2 - \frac{1}{4} = 0$$

for which the solutions are

$$\Rightarrow r_1 = -\frac{1}{2}, \quad r_2 = \frac{1}{2}$$

Since the roots are of opposite sign, the steady-state solution is a saddle-point equilibrium. Next we construct the phase diagram:



$\Rightarrow$ **Step 1: Determine the motion of y_1.**

– Begin by graphing the **y_1 isocline**: setting $\frac{dy_1}{dt} = 0$ to find the isocline gives the horizontal line $y_2 = 2$.

– Next, we note that $\frac{dy_1}{dt} < 0$ below this isocline (when $y_2 < 2$) and $\frac{dy_1}{dt} > 0$ above the isocline (when $y_2 > 2$).

– The appropriate **horizontal arrows of motion** are shown in figure 11.

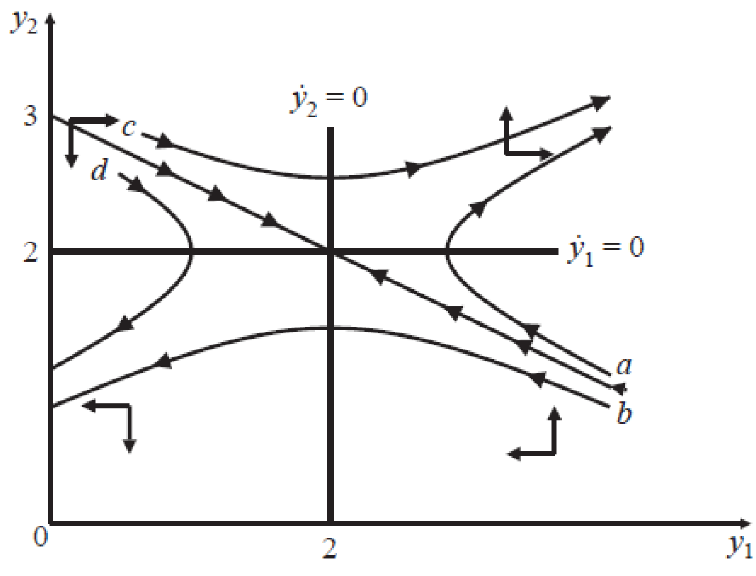
⇒ **Step 2: Determine the motion of y_2.**

– Begin by graphing the **y_2 isocline**: setting $\frac{dy_2}{dt} = 0$ to find the isocline gives the vertical line $y_1 = 2$.

– Next, we note that to the right of this line ($y_1 > 2$), $\frac{dy_2}{dt} > 0$ and to the left of it ($y_1 < 2$), $\frac{dy_2}{dt} < 0$.

– The appropriate **vertical arrows of motion** are shown in figure 11.

- The arrows of motion in figure 11 indicate that **trajectories in the southwest and northeast sectors of the phase plane definitely move away from the steady state**. But the arrows of motion in the **northwest and southeast sectors show that trajectories move toward the steady state**. What do the trajectories actually look like? Figure 12 shows some representative trajectories.



⇒ Consider an arbitrary starting point in figure 12 such as point a. At this point, the arrows of motion indicate that y_1 is decreasing and y_2 is increasing. The motion is northwesterly, therefore.

⇒ Follow this trajectory along its path. As it gets close to the y_1 isocline where $\frac{dy_1}{dt} = 0$, the motion of y_1 slows down but y_2 continues to increase. As a result the trajectory bends upward.

⇒ As it crosses the y_1 isocline, y_1 is stationary for an instant even though y_2 keeps increasing. As a result the **trajectory must be vertical at the crossing**.

⇒ From there it proceeds into a new isosector in which both y_1 and y_2 are increasing. Thus the **trajectory bends back and goes in a northeasterly direction**. It stays in that isosector, **traveling ever farther and farther away from the steady state**.

- Consider a trajectory starting at point b.

⇒ Like the trajectory that started at a, the motion is northwesterly; however, this time the trajectory gets close to the y_2 isocline where $\frac{dy_2}{dt} = 0$, so the motion of y_2 slows down while y_1 continues to decrease. As a result the trajectory bends to the left.

⇒ As it crosses the y_2 isocline, it is **horizontal** because y_2 is stationary at that point, even though y_1 keeps decreasing.

⇒ In the new isosector, the trajectory turns southwesterly and continues in that direction, traveling away from the steady state.

- Trajectories starting from points c and d are also shown. These have the mirror-image properties of the trajectories starting from points a and b.

⇒ These four arbitrarily chosen trajectories **verify that most trajectories end up diverging from a steady state which is a saddle-point equilibrium**.

⇒ These trajectories also **demonstrate the very important property** that **trajectories must obey the arrows of motion and must be horizontal when they cross the y_2 isocline and vertical when they cross the y_1 isocline**.

⇒ **Since the steady state is a saddle-point equilibrium, we know that some trajectories do converge to the steady state**, provided they start from initial conditions satisfying theorem 11. By theorem 11, the saddle path is given by

$$\frac{dy_1}{dt} = y_2 - 2 \Rightarrow a_{11} = 0, a_{12} = 1, \bar{y}_2 = 2$$

$$\frac{dy_2}{dt} = \frac{1}{4}y_1 - \frac{1}{2} \Rightarrow \bar{y}_1 = 2$$

$$\Rightarrow y_2 = \frac{r_1 - a_{11}}{a_{12}}(y_1 - \bar{y}_1) + \bar{y}_2 = \frac{-1/2 - 0}{1}(y_1 - 2) + 2 = -\frac{1}{2}y_1 + 3$$

- This equation is graphed in figure 12 as well; it is a straight line with intercept 3 and slope $-1/2$.
- If the pair of initial conditions for y_1 and y_2 lie anywhere on this line, the pair $y_1(t)$ and $y_2(t)$ converge to the steady state.
- Since trajectories beginning anywhere along this line do converge, we can think of the line itself as a trajectory, but one with the special property that **it is the only path that converges to the saddle-point equilibrium**. This is why it is called the **saddle path**.

⇒ Let's formally work out the solutions for the saddle path given the system of differential equations using substitution method:

$$\frac{dy_2}{dt} = \frac{1}{4}y_1 - \frac{1}{2}, \quad \frac{dy_1}{dt} = y_2 - 2$$

- Step 1: Homogeneous solution

$$\Rightarrow \frac{d^2y_1}{dt^2} = \frac{dy_2}{dt} = \frac{1}{4}y_1 \Rightarrow r^2 - \frac{1}{4} = 0 \Rightarrow r_1 = -\frac{1}{2}, r_2 = \frac{1}{2}$$

$$y_1(t) = C_1e^{-\frac{1}{2}t} + C_2e^{\frac{1}{2}t} \Rightarrow \frac{dy_1}{dt} = y_2(t) = -\frac{1}{2}C_1e^{-\frac{1}{2}t} + \frac{1}{2}C_2e^{\frac{1}{2}t}$$

- Step 2: Particular solution

$$\Rightarrow \bar{y}_1 = 2, \quad \bar{y}_2 = 2$$

– Step 3: General solution

$$y_1(t) = C_1 e^{-\frac{1}{2}t} + C_2 e^{\frac{1}{2}t} + 2$$

$$y_2(t) = -\frac{1}{2} C_1 e^{-\frac{1}{2}t} + \frac{1}{2} C_2 e^{\frac{1}{2}t} + 2$$

– Step 4: Setting $C_2 = 0$ to obtain the saddle path.

$$y_2(t) = -\frac{1}{2}\left(y_1(t) - C_2 e^{\frac{1}{2}t} - 2\right) + \frac{1}{2} C_2 e^{\frac{1}{2}t} + 2 = -\frac{1}{2} y_1(t) + 3$$

4.4 Determining Stability from the Coefficient Matrix

- Calculating the values of the characteristic roots allows us to determine if a steady state is stable or unstable; however, it would be helpful if there were a **faster and more direct way of determining stability**. It turns out there is: **stability can be determined directly from the coefficients of the differential equations**.

Theorem 13. *Let $|A| = a_{11}a_{22} - a_{12}a_{21}$ be the determinant of the coefficient matrix A in a system of two linear differential equations, and assume that $|A| \neq 0$. Let $tr(A) = a_{11} + a_{22}$ be the trace of A.* ***The stability properties of the steady-state equilibrium of the system are determined*** *as follows:*

(i) If $|A| < 0$, ***the characteristic roots are real and of opposite sign****. In this case the steady state is a* ***saddle-point equilibrium****.*

(ii) If $|A| > 0$, ***the characteristic roots are of equal sign if real-valued****.*

(a) If $tr(A) < 0$, the real parts of ***both roots are negative****, giving an* ***asymptotically stable steady state****.*

(b) If $tr(A) > 0$, the real parts of ***both roots are positive****, giving an* ***unstable*** *steady state.*

Proof. The proof uses the following two properties of characteristic roots (eigenvalues): the sum of the two roots of matrix A is equal to the trace of A,

$$r_1 + r_2 = tr(A)$$

and the product of the two roots of A is equal to the determinant A,

$$r_1 r_2 = |A|$$

(i) Since $|A| = r_1 r_2$, then if $|A| < 0$, r_1 and r_2 must be of opposite sign.

(ii) If $|A| > 0$, then r_1 and r_2 must be either both negative or both positive, if real valued. Since $tr(A) = r_1 + r_2$, they are both negative if $tr(A) < 0$ and both positive if $tr(A) > 0$.

□