

# Isogeometric analysis of $C^1/G^1$ dual mortaring and its application for multi-patch Kirchhoff-Love shell

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## 1. Introduction

Isogeometric analysis was introduced by Hughes et. al. [49] in 2005, as a novel discretization technology. Since then, it attracted considerable attentions from the academic world and is enjoying explosive growth. The idea behind isogeometric analysis is to use the same basis functions for the geometric modeling and computational analysis. While the main aim of isogeometric analysis is to eliminate the geometric approximation error, it has been observed that, compared to traditional  $C^0$  finite element, higher regularity Non-uniform Rational B-splines (NURBS) provide higher efficiency per degree of freedom [5, 29, 30]. Meanwhile, high regularity basis functions allow us to solve higher order partial differential equations (PDEs), e.g. the biharmonic equation [66, 56, 53], the Kirchhoff-Love shell problem [58, 57, 59] and the Cahn-Hilliard equation [42, 17, 16].

However, the higher dimensional NURBS basis functions are obtained by a tensor product of one-dimensional NURBS basis functions, which imposes limitations on its feasibility for analysis. Considering a scenario that a refinement is applied to a region of interest, however, for the tensor-product domain, it also introduces control points far from that region, which dramatically increases the problem size.

The adaptive finite element technique try to automatically refine a mesh in an optimal fashion so that a desirable discretization error level is achieved

with the fewest degrees of freedom. Based on the solution from a coarse mesh, a *posteriori* error estimator provides a guidance for deciding where and how to refine a mesh. It can increase the convergence rate, particularly when singularities are present. However, this promising technique can not be applied directly to NURBS mesh, as it does not support local refinement.

Since high smoothness basis function can be used in Isogeometric analysis, the numerical approximation of high order PDEs can be realized in the framework of the standard Galerkin formulation. However, without introducing mesh degenerations, it is impossible to parameterize geometries with sharp corner or kink by high continuity meshes.

## 2. Literature review

To circumvent the shortcomings discussed above, various methods have been proposed. The purpose of this section is to provide an overview of the popular methods that endow B-spline meshes with multi-patch coupling and local refinement abilities.

### 2.1. Local refinable splines

In 1988, Forsey and Bartels [40] introduced the hierarchical B-spline refinement algorithm, which can restrict the influence of refinement to the locality. The algorithm is achieved by a re-representation process that replaces each basis function by an equivalent linear combination of a set of basis functions defined by nested knot vectors. However, due to the lack of a natural control grid, the hierarchical B-spline has not been widely recognized in the CAD society, and a few applications can be found in geometric design. Recently, this technique has been extended to Isogeometric Analysis, by Vuong *et al.* [86]. Owing to the construction strategy, the resulted hierarchical basis function are linearly independent and retain the maximal regularity, which renders the hierarchical B-spline a good candidate for analysis. The numerical tests demonstrate that the use of the hierarchical B-spline lead to a superior performance for problems with corner singularity. A subdivision-based hierarchical B-spline was proposed by Bornemann *et al.* [18], to tackle the intricate algorithms in the software implementation of hierarchical B-splines. The subdivision scheme establishes algebraic relations between the basis functions and their coefficients defined on different refinement level of the mesh and greatly ease the implementation of hierarchical B-splines. Consecutively, the truncated basis for hierarchical splines (THB-spline) was

introduced by Giannelli *et al.* [41]. THB-splines is created by eliminating from the coarse hierarchical basis function the contribution corresponding to the subset of finer basis functions. Besides all the nice properties of hierarchical B-splines, the THB-splines obtain smaller support and form a partition of unity, which lead to sparser matrices and lower condition numbers.

However, all the above hierarchical B-splines are still under the tensor product formulism, which restricts hierarchical B-splines to a global rectangular parametric domain. In order to represent complex topologies, subdivision schemes are widespread in geometry processing and computer graphics. Among the most popular subdivision schemes are the Catmull-Clark [23], Doo-Sabin [31] and Loop's [64] scheme. For Isogeometric Analysis, Wei *et al.* [87] introduced truncated hierarchical Catmull-Clark subdivision (THCCS) that can handle extraordinary nodes involved in complex topologies. THCCS inherits the surface continuity of Catmull-Clark subdivision, namely  $C^1$  continuity at extraordinary points and  $C^2$  continuity elsewhere. Loop subdivision surfaces provides similar regularity properties as THCCS and has been applied to Isogeometric Analysis in [52, 70] to generate triangular meshes. One of the limitations in the implementation of subdivision meshes is that the basis function around the extraordinary point is composed of piecewise polynomial functions with an infinite number of segments, which leads to insufficient integration by Gauss quadrature rule. To deal with this issue, various quadrature rules and adaptive strategies have been examined in [67] for Poisson problem on the disk and in [51] for fourth order PDEs.

In 2003, Sederberg *et al.* [78] introduced T-splines, which allows the existence of T-junctions in the control grid, so that lines of control points need not traverse the entire control grid. Thus, local refinement can be realized by introducing T-junctions around interested region. Since the concept of T-splines is a generalization of NURBS technology, it can be used to merge NURBS surfaces that have different knot-vectors at the intersection. Therefore, the T-splines are also suitable to address trimmed multipatch geometries. Due to the desirable features of T-splines, Bazilevs *et al.* [4] explored this technology in Isogeometric Analysis, and numerical results demonstrated its potential for solving structural and fluid problems. By utilizing the Bézier extraction operator, a finite element data structure for T-splines [77] was developed to ease the incorporation of T-splines into existing finite element codes. However, it has been proven [22] that the original definition of T-splines is not sufficient to ensure the linear independence of the basis functions. To circumvent this issue, analysis suitable T-splines [63]

95 was developed by applying an additional constraint that no two orthogonal  
 96 T-junction extensions are allowed to intersect. Subsequently, the mathemat-  
 97 ical properties of analysis suitable T-splines were studied in [62, 90], and it  
 98 has been successfully applied to the boundary element method [76]. Mean-  
 99 while, an adaptive local h-refinement algorithm with T-splines and a local  
 100 refinement of analysis-suitable T-splines were introduced by Döfel *et al.* [37]  
 101 and Scott *et al.* [75], respectively. However, for both algorithm, the refined  
 102 mesh is not as local as one could hope and this problem might be severe in  
 103 3D.

## 104 2.2. Multi-patch geometrically continuous functions

105 One of the advantages of Isogeometric Analysis is that it provides basis  
 106 functions with high smoothness, *i.e.* for  $p$ -th order splines, they enjoy up  
 107 to  $C^{p-1}$  continuity within a single patch. Thus, it is possible to directly  
 108 discretize differential operators of order higher than 2. However, continuity  
 109 higher than  $C^0$  for multi-patch discretization imposes significant difficulties.  
 110 The conception of geometric continuity is very important in CAD field [72] for  
 111 designing smooth multi-patch domain containing extraordinary vertices [71].  
 112 In the parametric space, the geometric continuity of order  $s$  ( $G^s$  continuity)  
 113 is a weaker continuity constraint as compared to  $C^s$  continuity, while it has  
 114 been proved by Groisser and Peters [43] that  $G^s$  continuity in the parametric  
 115 space is equivalent to  $C^s$  continuity of the basis function after the parametric  
 116 mapping. Thus, the construction of  $C^s$  isogeometric functions over a  $C^0$   
 117 parameterization can be interpreted as geometric continuity  $G^s$  of the graph  
 118 parameterization. Bercovier *et al.* [12] has shown that for multi Bézier  
 119 patches over an unstructured quadrilateral mesh, as long as the order of  
 120 polynomial is high enough, there always exists the minimal determining set  
 121 for a  $C^1$  continuity construction. Moreover, the resulting basis functions do  
 122 not contain subdivisions around extraordinary vertices.

123 The case of  $G^1$  continuous functions on bilinearly parametrized two-patch  
 124 B-spline domains was considered by Kapl *et al.* [56], where the  $C^1$  basis  
 125 functions are constructed and analyzed by numerical tests. It is shown that  
 126 the space dimensionality heavily depends on the parameterization of two  
 127 bilinear patch, and optimal convergence is observed on biharmonic problem.  
 128 However, over-constrained  $C^1$  isogeometric spaces that causes sub-optimal  
 129 convergence is also observed for certain configurations (*e.g.* two-patch non-  
 130 bilinear parameterizations and  $C^{p-1}$  continuity within the patches for  $p$ -th  
 131 order spline space). A theoretical analysis of the causing of  $C^1$  locking is

provided in [25], where the analysis-suitable  $G^1$  geometry parameterization, that allows for optimal approximation of  $C^1$  isogeometric spaces, is identified and testified by numerical examples. The methods in [56] has been extended to bilinearly parameterized multi-patch domains in [53], where the simple explicit formulas for spline coefficients of  $C^1$  basis function is derived and nested  $C^1$  isogeometric spaces are generated. Recently, Kapl *et al.* [55, 54] explored the construction of  $C^2$  isogeometric functions on multi-patch geometries and utilized the  $C^2$  isogeometric spaces for 6-th order PDE.

Although the geometrically continuous functions circumvent the use of subdivisions for domains with extraordinary vertices, the requirement of  $C^0$  parameterization averts local mesh refinement, and lower continuity is required to avoid  $C^1$  locking effect. Thus, its implementation can be complex and it may not be a potential candidate for analysis in more general situations.

### 2.3. Variational approach for domain coupling

Unlike geometric design, where high continuity basis functions along the intersections of neighboring patches are required for the construction of high quality surface; in analysis, these strong point-wise constraints are unnecessarily rigorous, a good approximation of PDEs can be made even if these constraints are applied in the weak sense. Moreover, the non-conforming multi-patch coupling is allowed, which maintains the flexibility for the choice of meshes when multi-patch discretization is needed. Mathematically, the error estimation of the non-conforming finite element approximation is based on Strang's lemma [20, 82], which says that for the non-conforming discretized PDEs, the distance between exact solution to the discrete one is bounded by the sum of the approximation error and the consistency error. The approximation error measures the failure of discretized finite dimensional space to capture the exact solution, while the consistency error measures the inconsistency between the exact equation and the discretized equation. Various methods have been developed to eliminate the consistency error and recover optimal convergence, among them are the mortar method (Lagrange multiplier method), stablized Lagrange multiplier method, the Nitsche's method and the discontinuous Galerkin (dG) method.

To clearly demonstrate these methods, we consider the following Poisson

problem with homogenous Dirichlet boundary conditions

$$\begin{aligned} -\Delta u &= f, & \text{in } \Omega \\ u &= 0, & \text{on } \partial\Omega \end{aligned} \tag{1}$$

165 where  $\Omega$  denote a bounded open domain in  $\mathbb{R}^d$ ,  $d = 2$  or  $3$  being the dimen-  
 166 sion of the problem and its boundary is denoted by  $\partial\Omega$ , in order to simplify  
 167 the presentation we restrict ourselves to the case of two-dimensional com-  
 168 putational domain. The weak form of Equation (1) reads as follow: Find  
 169  $u \in H_0^1(\Omega)$  such that

$$a(u, v) = l(v), \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v d\Omega, \\ l(v) &= \int_{\Omega} f v d\Omega. \end{aligned} \tag{3}$$

170 Using the fact that  $C^0(\Omega) \subset H^1(\Omega)$ , the weak solution can be approximated  
 171 by considering a finite dimensional continuous function space. Now, we as-  
 172 sume that the domain  $\Omega$  is subdivided into  $K$  non-overlapping subdomains  
 173 or patches  $\Omega_k$  for  $1 \leq k \leq K$ , i.e.

$$\bar{\Omega} = \bigcup_{k=1}^K \bar{\Omega}_k \quad \text{and} \quad \Omega_k \cap \Omega_l = \emptyset \quad \forall k \neq l. \tag{4}$$

174 For simplicity, we only consider the case that the intersection of two patches  
 175 is either empty or vertex or the entire edge, which rules out the possibility  
 176 of hanging nodes. We denote the common interface of two neighboring sub-  
 177 domains  $\Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$  so that  $\Gamma_{kl} = \emptyset$  if  $\Omega_k$  is not a neighbor of  $\Omega_l$  and  
 178 define the skeleton  $\mathbf{S} = \bigcup_{k,l \in K, k \neq l} \Gamma_{kl}$  as the union of all interfaces. A rep-  
 179 resentative example of geometry is presented in Figure 1. We can associate  
 180 each subdomain a bijective geometric mapping as

$$\mathbf{F}_k(\xi_k, \eta_k) : \hat{\Omega}_k \mapsto \Omega_k \in \mathbb{R}^d, \tag{5}$$

181 where  $\hat{\Omega}_k$  is the parametric domain of  $k^{th}$  patch associated with coordi-  
 182 nates  $(\xi_k, \eta_k)$ . For the simplicity and without loss of generality, we assume

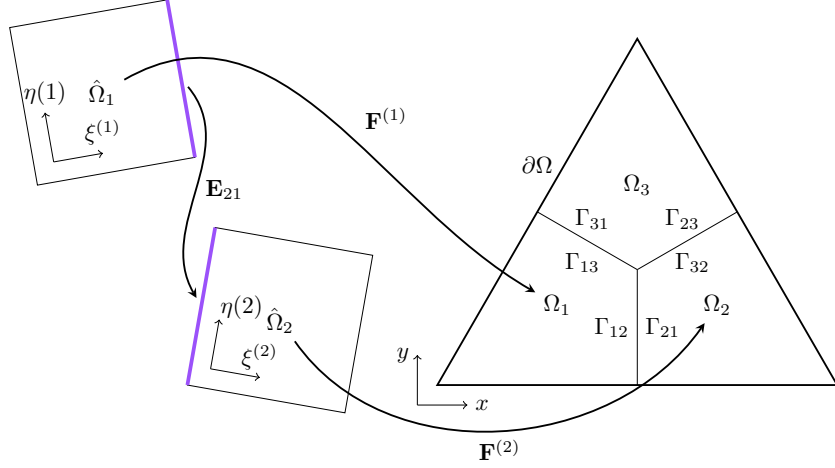


Figure 1: An example of domain decomposition, patches are defined on different parametric domains and are connected via geometric mapping.

183  $\hat{\Omega}_k = [0, 1] \times [0, 1]$  for all patches. Due to the difference in the patch pa-  
 184 rameterizations, a physical point on the interface can be mapped to different  
 185 parametric domains with different coordinates. Owing to non-singular pa-  
 186 rameterization, we can establish a bijective transformation from the shared  
 187 edge of  $\hat{\Omega}_k$  to that of  $\hat{\Omega}_l$  by

$$\mathbf{E}_{kl} = (\mathbf{F}_l)^{-1} \circ \mathbf{F}_k. \quad (6)$$

188

189 For each  $\Omega_k$ , we introduce the function space

$$H_*^1(\Omega_k) := \left\{ u \in H^1(\Omega_k) : u = 0 \quad \text{on } \partial\Omega \cap \partial\Omega_k \right\}, \quad (7)$$

190 now we can define the broken Sobolev space

$$\mathcal{X} := \left\{ u \in L^2(\Omega) : u|_{\Omega_k} \in H_*^1(\Omega_k) \right\}. \quad (8)$$

191 Now, the question is how to approximate the weak solution of Equa-  
 192 tion (2) from a finite dimensional subspace of  $\mathcal{X}$ . Since functions in  $\mathcal{X}$  can be  
 193 discontinuous on the skeleton  $\mathbf{S}$ ,  $a(u, u)$  is no longer coercive (or V-elliptic)  
 194 on  $\mathcal{X}$ . As a result, directly using a finite dimensional subspace of  $\mathcal{X}$  to  
 195 discretize Equation (2) will lead to a non-invertible stiffness matrix. Mod-  
 196 ifications to the weak form is needed, and we will review some of the most  
 197 popular methods in this section.

198 *2.3.1. Lagrange multiplier method*

199 The Lagrange multiplier method (or sometimes called mortar method)  
 200 is a domain decomposition technique that allows the coupling of different  
 201 discretization schemes or of non-matching triangulation along interior inter-  
 202 faces. The inter-element continuity condition is enforced weakly by Lagrange  
 203 multipliers. For the Poisson problem, the  $C^0$  continuity constraint is required  
 204 on the intersections, in other words, the jump on the skeleton

$$[u]_{\Gamma_{kl}} := u_k - u_l = 0, \quad \forall \Gamma_{kl} \in \mathbf{S}, \quad (9)$$

205 where  $u_k = u|_{\Omega_k}$ . In order to apply the constraint to the weak form, we  
 206 introduce the potential energy functional:

$$\Pi(v) := \frac{1}{2}a(v, v) - l(v). \quad (10)$$

207 The Equation (2) is equivalent to the minimization problem:

$$\inf_{v \in H_0^1(\Omega)} \Pi(v). \quad (11)$$

208 Then, given a function space  $\mathcal{M}$  defined on the skeleton, a Lagrange multi-  
 209 plier  $\mu \in \mathcal{M}$  is used to add the constraint (9) to the potential energy func-  
 210 tional (10), and the resulted the potential energy functional for the Lagrange  
 211 multiplier method reads

$$\Pi_{LM}(v, \mu) := \Pi(v) + b(\mu, v), \quad (12)$$

212 where

$$b(\mu, v) = \sum_{\Gamma \in \mathbf{S}} \int_{\Gamma} \mu [u]_{\Gamma} d\Gamma. \quad (13)$$

213 The variational formulation of the Lagrange multiplier problem can be de-  
 214 rived from the saddle point problem of the potential energy functional (12)

$$\inf_{v \in X} \sup_{\mu \in M} \Pi_{LM}(v, \mu), \quad (14)$$

215 as, find  $(u, \lambda) \in \mathcal{X} \times \mathcal{M}$  such that

$$\begin{cases} a(u, v) + b(v, \lambda) = l(v) & \forall v \in \mathcal{X}, \\ b(u, \mu) = 0 & \forall \mu \in \mathcal{M}. \end{cases} \quad (15)$$



216 The solution of the variational formulation is the infimum in  $v$  and the supremum in  $\mu$ , in other words, it is still a minimization problem in terms of the primary variable  $v$  and any function that violate the constraint will be eliminated by the Lagrange multiplier  $\mu$ . This is the reason why it is called the saddle point problem. We also denote that the physical meaning of the Lagrange multiplier  $\mu$  for (12) is the flux of  $v$  over the skeleton. A comprehensive study of the mixed problem (15) can be found in [15].

223 In the discretized problem, for a given discrete space  $\mathcal{X}_h$ , the choice of the discrete Lagrange multiplier space  $\mathcal{M}_h$  plays a fundamental role for the stability of the saddle point problem and the optimality of the discretization scheme. To ensure the optimality, the function space for Lagrange multiplier should be judiciously chosen so that the consistency error should converges at the same rate as that of the approximation error. The feasibility of the discrete space pair  $\mathcal{X}_h \times \mathcal{M}_h$  can be measured by the inf-sup test. The inf-sup condition is also referred to as the Ladyzhenskaya-Babuska-Brezzi condition (or simply LBB). It is a crucial condition to ensure the solvability, stability and optimality of a mixed problem. For the problem (15), the inf-sup condition is [15], for  $v \neq 0$  and  $\mu \neq 0$

$$\inf_{\mu \in \mathcal{M}} \sup_{v \in \mathcal{X}} \frac{|b(v, \mu)|}{\|v\|_{\mathcal{X}} \|\mu\|_{\mathcal{M}}} \geq \beta > 0. \quad (16)$$

234 Since the approximation error of problem (15) is given as

$$\|u - u^h\|_{\mathcal{X}} + \|\lambda - \lambda^h\|_{\mathcal{M}} \leq C \left( \inf_{u^h \in \mathcal{X}^h} \|u - u^h\|_{\mathcal{X}} + \inf_{\lambda^h \in \mathcal{M}^h} \|\lambda - \lambda^h\|_{\mathcal{M}} \right), \quad (17)$$

235 where  $C$  is a constant that depends on variables including  $\beta$  but is independent of the mesh size  $h$ . Hence, in a discretized problem, the inf-sup condition requires the variable  $\beta$  to be a constant that is independent of the mesh size.

239 It is well-known that in order to satisfy the LBB-condition a number of possible natural choices for the approximation space pair  $\mathcal{X}_h \times \mathcal{M}_h$  must be discarded. In particular, the trace space of slave side, specially convenient from the computational point of view, often do not satisfy the LBB-condition and can activate pathologies such as spurious oscillations. To remedy this problem, the most widely used method in the finite element framework is reducing the dimension of Lagrange multiplier space by two (for  $2^{nd}$  order PDEs). Specifically, the degree of Lagrange multiplier basis functions at both

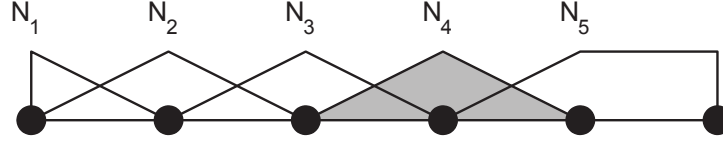


Figure 2: Lagrange multiplier basis functions for the piecewise linear elements, modification on the right end (from Zienkiewicz [91]).

247 ends are reduced by one. This modification has been successfully adopted in  
 248 [14, 13, 7, 10, 8, 65, 60, 9]. An example of the modified Lagrange multiplier  
 249 basis functions are illustrated in Figure 2, where the basis function  $N_5$  is  
 250 constant in the right end.

251 In the context of Isogeometric Analysis, the patch coupling problem has  
 252 been firstly studied by Hesch and Betsch [47], where the coupling of La-  
 253 grangian elements and NURBS elements for 3D nonlinear elastic problem  
 254 is validated. To avoid an over constrained linear system, Hesch and Betsch  
 255 used a linear Lagrange multiplier space for higher order NURBS coupling. In  
 256 [21], the choice of the Lagrange multiplier space has been extensively studied,  
 257 it testifies that for equal order pairing, a local degree reduction at extraor-  
 258 dinary vertices is required, and another possibility is reducing the degree of  
 259 Lagrange multiplier space by two compared to the trace space of slave side.  
 260 These choices of Lagrange multiplier spaces are proven to be inf-sup stable  
 261 by various numerical examples. In addition to the constraint on the inter-  
 262 patch displacement, Bouclier *et al.* [19] considered the constraint on the  
 263 traction and claimed that this strategy enables to present a  $C^1$  behavior. In  
 264 the numerical test, smoother displacement fields and smoother stress fields  
 265 are observed.

266 Another drawback of the implementation of mortar methods is that most  
 267 of them introduce Lagrange multipliers as additional variables to enforce  
 268 interface constraints weakly, increasing the problem size. Moreover, different  
 269 physical fields are involved in the weak form, deteriorating the conditioning  
 270 of the global matrix if no appropriate pre-conditioner is applied (detailed  
 271 discussion about preconditioning for saddle point problem can be found in  
 272 [39, 11, 83]).

273 *2.3.2. Dual mortar method*

274 To circumvent the increase of problem size, we considering the minimiza-  
 275 tion problem

$$\inf_{v \in \mathcal{K}} \Pi(v), \quad (18)$$

276 where the function space  $\mathcal{K} = \{v \in \mathcal{X} : b(v, \lambda) = 0, \forall \lambda \in \mathcal{M}\}$ . The mini-  
 277 mization problem (18) is indeed equivalent to the saddle point problem (15),  
 278 the proof can be found in [15]. Note that, since  $K \subset X$ , the introduce of  
 279 Lagrange multiplier indeedly reduces the problem size of (18). Meanwhile,  
 280 the symmetric positive definite structure of the resulting stiffness matrix is  
 281 preserved. But the construction of the function space  $K$  is not a trivial task.

282 To reduce the cost of constructing the function space  $\mathcal{K}$ , we use the dual  
 283 basis functions of the trace space of the slave side as the discrete Lagrange  
 284 multiplier space. For a given basis function  $N_i$ , the dual basis function  $\hat{N}_j$  is  
 285 defined to satisfy

$$\int_{\Gamma} N_i \hat{N}_j d\Gamma = \delta_{ij} \int_{\Gamma} N_i d\Gamma, \quad (19)$$

286 where  $\delta_{ij}$  is a Kronecker delta function. Of special interest, are biorthogonal  
 287 basis functions with compact support, especially

$$\text{supp } \hat{N}_i = \text{supp } N_i. \quad (20)$$

288 Due to the biorthogonality, the discrete bilinear form  $b(v, \mu)$  forms a diagonal  
 289 matrix on the slave side, and forms a sparse matrix on the master side.  
 290 The function space  $\mathcal{K}$  can be formulated without additional efforts and all  
 291 the slave degree of freedom are eliminated in the resulting linear system.  
 292 Moreover, owing to the local support property the resulting stiffness matrix  
 293 is a symmetric positive definite sparse matrix. Thus, the dual basis functions  
 294 are very attractive in the perspective of computational efficiency.

295 Figure 3 shows an example of dual basis functions corresponding to the  
 296 basis functions in Figure 2. Again, order reduction is made at the right end.  
 297 The dual mortar method was first introduced in [88] for first order finite  
 298 element. This method has been extended to higher order degree elements in  
 299 [61], to three-dimensional problem in [89] and to contact problem [48, 73].

300 In isogeometric analysis framework, a master-slave type mortar method  
 301 has been suggested by Dornisch *et al.* [34], where the weakly applied con-  
 302 straint is represented as a master-slave relation and the the slave interface  
 303 degrees of freedom (DOF) can be condensed out of the global linear system.

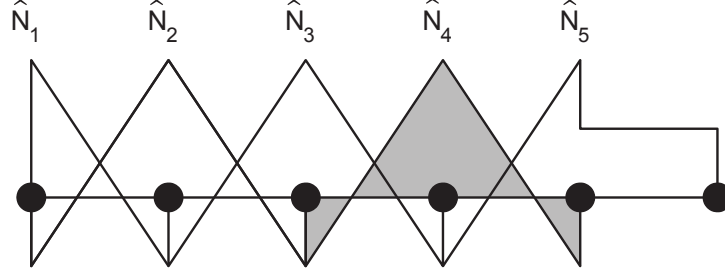


Figure 3: Dual Lagrange multiplier basis functions for the piecewise linear elements, modification on the right end (from Zienkiewicz [91]).

304 Recently, Dornisch *et al.* extended this research to multiple patch coupling in  
 305 [36, 33], where different types of dual basis functions are applied as the basis  
 306 of Lagrange multipliers. The numerical results demonstrate that the approx-  
 307 imate dual basis functions yield accurate result and generate sparse global  
 308 matrix due to the local support. The concept of dual mortar methods is also  
 309 utilized in [79] for contact problem in Isogeometric analysis framework. Coox  
 310 *et al.* [27] proposed an interesting approach to establish the master-slave mor-  
 311 tar method and implemented this approach in [26] to form boundary element  
 312 analysis on complex manifold. In this approach, the master-slave relation are  
 313 formed by knot insertion algorithm and pseudo-inverse.

### 314 2.3.3. Perturbed Lagrangian method

315 Applying constraint by Lagrange multiplier leads to a saddle point prob-  
 316 lem, of which the discrete Lagrange multiplier basis functions cannot be  
 317 chosen independently of that of primal variable and special treatment is  
 318 required on the cross point to ensure the solvability and optimality of the  
 319 discretized system. The stiffness matrix for the discrete problem arising from  
 320 the Lagrangian multiplier method always contains both positive and negative  
 321 eigenvalues, for which iterative methods are known to be less efficient than for  
 322 symmetric positive definite systems. To ensure the invertibility of the stiff-  
 323 ness matrix, a quadratic penalty term is added to the energy functional (12),  
 324 as

$$\Pi_{PLM}(v, \mu) := \Pi_{LM}(v, \mu) - \frac{1}{2\epsilon} \sum_{\Gamma \in \mathbf{S}} \int_{\Gamma} \mu^2 d\Gamma, \quad (21)$$

325 where the penalty term is scaled by a parameter  $\epsilon$ . The resulted func-  
 326 tional (21) is referred to as perturbed Lagrangian and the last term is often

327 called stablization term. The resulted variational formulation is stated as

$$\begin{cases} a(u, v) + b(v, \lambda) = l(v) & \forall v \in \mathcal{X}, \\ b(u, \mu) - \frac{1}{2\epsilon} \sum_{\Gamma \in \mathbf{S}} \int_{\Gamma} \mu \lambda d\Gamma = 0 & \forall \mu \in \mathcal{M}. \end{cases} \quad (22)$$

328 As  $\epsilon \rightarrow \infty$ , the solution obtained from (22) will converge to the solution ob-  
 329 tained by the classical Lagrange multiplier method. For  $0 < \epsilon < \infty$ , any solu-  
 330 tion that inconsistent with the constraint will not be fully prohibited, but will  
 331 be penalized by the stability term. And the rank of discrete stiffness matrix is  
 332 preserved no matter whether the discrete space pair  $\mathcal{X}_h \times \mathcal{M}_h$  fulfills the inf-  
 333 sup condition or not. However, for a moderate  $\epsilon$ , the perturbed Lagrangian  
 334 method is inconsistent with the classical Lagrange multiplier method, and  
 335 the increase of  $\epsilon$  will deteriorate the conditioning of stiffness matrix.

336 The perturbed Lagrangian method has been utilized in [80] for contact  
 337 problem and [35, 1] for domain decomposition problem in isogeometric anal-  
 338 ysis framework.

#### 339 2.3.4. Stablized Lagrange multiplier method

340 To fully circumvent the inf-sup condition for imposing Dirichlet boundary  
 341 by Lagrange multiplier, Barbosa et. al. [3] added a new penalty like term  
 342 to the energy functional (12) to enhance the stability. Unlike perturbed  
 343 Lagrangian method where the penalty term is inconsistent with the original  
 344 problem, the new term proposed by Barbosa maintaining the consistency.  
 345 The energy functional of stablized Lagrange multiplier method is given as

$$\Pi_{SLM}(v, \mu) := \Pi_{LM}(v, \mu) - \sum_{\Gamma \in \mathbf{S}} \frac{h}{2\gamma} \int_{\Gamma} (\mu + \left\{ \frac{\partial v}{\partial n} \right\})^2 d\Gamma, \quad (23)$$

346 where  $n$  is the normal vector of the interface,  $h$  is the mesh size on the  
 347 intersection,  $\gamma$  is a user defined constant, the average operator

$$\{u\}_{\Gamma_{kl}} := \frac{1}{2}u_k + \frac{1}{2}u_l. \quad (24)$$

348 Since the physical meaning of the Lagrange multiplier is the flux on the  
 349 intersection, the stablization term in (23) is consistent with the original  
 350 problem. The resulted variational formulation is stated as

$$\begin{cases} a(u, v) + b(v, \lambda) - \frac{h}{\gamma} \sum_{\Gamma \in \mathbf{S}} \int_{\Gamma} \frac{\partial v}{\partial n} (\lambda + \left\{ \frac{\partial u}{\partial n} \right\}) d\Gamma = l(v) & \forall v \in \mathcal{X}, \\ b(u, \mu) - \frac{h}{\gamma} \sum_{\Gamma \in \mathbf{S}} \int_{\Gamma} \mu (\lambda + \left\{ \frac{\partial u}{\partial n} \right\}) d\Gamma = 0 & \forall \mu \in \mathcal{M}. \end{cases} \quad (25)$$

351 The stabilization parameter  $\gamma$  needs to be carefully chosen. If  $\gamma$  is too large,  
 352 the method degrades to a penalty-type method, with sub-optimal accuracy  
 353 in the asymptotic limit. If  $\gamma$  is too small, the method becomes unstable.  
 354 Recall the trace inequality

$$\|h^{\frac{1}{2}} \frac{\partial u}{\partial n}\|_{\partial\Omega_k}^2 \leq C \|\nabla u\|_{\Omega_k}^2. \quad (26)$$

355 It has been shown [50] that the mixed formulation (25) fulfills the inf-sup  
 356 condition if  $\gamma > 2C$ . The constant  $C$  can be approximated by discretize  
 357 the norms in the inequality (26) and solve the resulting discrete eigenvalue  
 358 problem.

359 It has been demonstrated that there is a close connection with the sta-  
 360 bilized Lagrange multiplier method and Nitsche's method in the context of  
 361 setting the Dirichlet boundary conditions [81] and in the context of domain  
 362 decomposition [46, 45, 50]. Tur et. al. [85] utilized this method to solve  
 363 both small and large deformation contact problems and obtained optimal  
 364 convergence rate for linear elements. To our knowledge, this method has not  
 365 been applied in the isogeometric analysis framework yet.

### 366 2.3.5. Discontinuous Galerkin method

367 Discontinuous Galerkin method (or Nitsche's method) was introduced in  
 368 1971 [69] for handling Dirichlet boundary conditions in the weak sense. Dis-  
 369 continuous Galerkin method resembles a mesh-dependent penalty method.  
 370 Unlike the standard penalty method, which is not consistent unless the  
 371 penalty coefficient goes to infinity, discontinuous Galerkin method is consis-  
 372 tent with the original problem. Moreover, no additional unknown (Lagrange  
 373 multiplier) is needed and no discrete inf-sup condition must be fulfilled, con-  
 374 trarily to mixed methods. Meanwhile, additional term are added into the  
 375 weak form to ensure the ellipticity of the problem.

376 To develop the weak form of discontinuous Galerkin method for homoge-  
 377 neous Poisson problem, we start by multiplying (1) by a test function  $v \in X$   
 378 and integrating by parts, we obtain

$$a(u, v) - \sum_{\Gamma \in \mathbf{S}} \int_{\Gamma} \left\{ \frac{\partial u}{\partial n} \right\} [v] d\Gamma = l(v). \quad (27)$$

379 However, if we consider the right-hand side as a bilinear form, it is not  
 380 coercive. In other words, this problem is not well-posed, since coercive implies

381 the uniqueness of solution. Meanwhile, this bilinear form is not symmetric.  
 382 To recover the symmetry and coercivity of the bilinear form, additional terms  
 383 are needed. To maintain the consistency, the added terms must vanish for  
 384 the true solution. This lead to the following weak form: find  $u \in X$  such  
 385 that

$$\begin{aligned}
 a(u, v) - \sum_{\Gamma \in \mathbf{S}} \int_{\Gamma} \left\{ \frac{\partial u}{\partial n} \right\} [v] d\Gamma - \epsilon \sum_{\Gamma \in \mathbf{S}} \int_{\Gamma} \left\{ \frac{\partial v}{\partial n} \right\} [u] d\Gamma + \\
 \sum_{\Gamma \in \mathbf{S}} \frac{\gamma}{h} \int_{\Gamma} [u] [v] d\Gamma = l(v) \quad \forall v \in \mathcal{X}.
 \end{aligned} \tag{28}$$

386 Since  $[u] = 0$  on the intersections, the above formulation is consistent with (27).  
 387 Furthermore, and as already stated in [74] the parameter  $\epsilon$  can be set to some  
 388 particular values, namely:

- 389 • For  $\epsilon = +1$ , the resulting method is called the symmetric interior  
 390 penalty Galerkin (SIPG) method. The stiffness matrix of SIPG is sym-  
 391 metric.
- 392 • If  $\epsilon = 0$ , we obtain the incomplete interior penalty Galerkin (IIPG)  
 393 method. It involves only a few terms and is of easiest implementation.
- 394 • If  $\epsilon = -1$ , the resulting method is called the nonsymmetric interior  
 395 penalty Galerkin (NIPG) method. It admits one unique solution and  
 396 converges optimally irrespectively of the value of  $\gamma > 0$ .

397 For  $\epsilon = 0$  and  $\epsilon = +1$ , the bilinear form is coercive if  $\gamma > C$  and  $\gamma > 2C$ ,  
 398 respectively [74]. Similar to the stablized Lagrange multiplier method, the  
 399 discontinuous Galerkin method also requires to solve an eigenvalue problem  
 400 to determine the value of  $\gamma$ .

401 Discontinuous Galerkin method has been widely studied in various as-  
 402 pects, including imposing boundary condition [45], domain decomposition  
 403 [6] and contact problem [24]. In the field of Isogeometric analysis, Discon-  
 404 tinuous Galerkin method has been utilized to imposing Dirichlet boundary  
 405 condition for trimmed spline meshes [38]. The first article discussing discon-  
 406 tinuous Galerkin method based domain decomposition strategy was written  
 407 by Apostolatos *et al.* [2]. Nguyen *et al.* extended it to three-dimensional  
 408 problems in [68]. Guo *et al.* [44] proposed a Nitsche's method for cou-  
 409 pling Kirchhoff-Love NURBS shell patches. Since the governing equation for

Table 1: Property comparison of Lagrange multiplier, dual mortar, perturbed Lagrange multiplier, stablized Lagrange multiplier and discontinuous Galerkin methods.

Methods	well-defined	inf-sup	symmetry	positive definite	size
<b>Lagrange multiplier</b>	depends	depends	yes	no	enlarged
<b>Dual mortar</b>	yes	depends	yes	yes	reduced
<b>Perturbed Lagrange multiplier</b>	yes	depends	yes	no	enlarged
<b>Stablized Lagrange multiplier</b>	yes	yes	yes	no	enlarged
<b>Discontinuous Galerkin</b>	yes	yes	depends	yes	same

410 Kirchhoff-Love shell is 4-th order PDE,  $C^1$  continuity constraint is imposed  
411 weakly in the method.

412 Although discontinuous Galerkin method does not introduce additional  
413 DOF and does not need the judicious choice of multiplier function space,  
414 the value of the constants in the stabilizing term need to be determined.  
415 Normally, they are determined by solving an eigenvalue problem on the domain  
416 of the combination of all intersections, which leads to extra computational  
417 cost. Meanwhile, the additional stabilizing terms reduce the sparsity of the  
418 global linear system. For higher order PDEs, discontinuous Galerkin method  
419 becomes more complex as higher order derivatives exist in the tractions.

420 A comparison of the variational coupling methods discussed above is  
421 shown in Table. 1.

### 422 3. Research Objectives

423 My dissertation research focuses on the construction of NURBS basis  
424 functions among multi-patches that are analysis-suitable for 4<sup>th</sup> order PDEs.  
425 The coupling constraints are applied weakly by using the dual mortar method.  
426 The dual basis functions are constructed based on the Bézier projection  
427 technology proposed in [84].

### 428 4. Preliminaries

429 This section provides the formulation of univariate basis functions, its  
430 extension to higher dimensional space, and representations of geometries in  
431 the context of Isogeometric Analysis. For a detailed explanation we refer to.



432 *4.1. Univariate B-spline basis functions*

433 A univariate B-spline is peicewise polynomial curve represented as a linear  
 434 combination of B-spline basis functions. Basis functions of  $p^{th}$  order B-spline  
 435 with  $n$  degrees of freedom can be defined by a non-decreasing set of real  
 436 numbers

$$\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}, \quad (29)$$

which is called knot vector. B-splines that are interpolatory at the ends can be achieved by requiring the multiplicity of  $p + 1$  for the first and the last knot. Associated B-spline basis functions are defined using the Cox-de Boor recursion formula:

$$N_{i,0}(\xi) = \begin{cases} 1 & \xi_i \leq \xi \leq \xi_{i+1} \\ 0 & otherwise \end{cases} \quad (30)$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \quad (31)$$

437 *4.2. Univariate NURBS basis functions*

438 The univariate Non-Uniform Rational B-spline (NURBS) can describe  
 439 objects that cannot be represented by polynomial basis, such as circular arcs.  
 440 NURBS are built from B-splines by dividing each B-spline basis functions by  
 441 a weight function

$$W(\xi) = \sum_{j=1}^n w_j N_{j,p} \quad (32)$$

442 and multiplying each B-spline basis functions by the associated weight coefficient for the partition of unity. Thus, the NURBS basis functions are defined  
 443 as:  
 444

$$R_{i,p}(\xi) = \frac{w_i N_{i,p}}{W(\xi)} \quad (33)$$

445 *4.3. Multivariate basis functions*

446 For higher dimensional spaces, the B-spline and NURBS basis functions  
 447 can be formed by the Kronecker product of vectors of univariate basis functions.  
 448 For a two-dimensional parametric space, given polynomial orders of  
 449  $p_\xi, p_\eta$  and degrees of freedom  $n_\xi, n_\eta$  in  $\xi, \eta$  direction, the bivariate B-spline  
 450 basis functions are defined as:

$$N_{a,\mathbf{p}}(\xi, \eta) = N_{i,p_\xi}(\xi) N_{j,p_\eta}(\eta), \quad (34)$$

451 where the index  $a$  is defined by the map

$$a = n_\eta i + j. \quad (35)$$

452 The bivariate NURBS basis functions are defined as

$$R_{a,\mathbf{p}}(\xi, \eta) = \frac{w_a N_{a,\mathbf{p}}}{\sum_{i=1}^n w_i N_{i,\mathbf{p}}}, \quad (36)$$

453 where  $n = n_\xi \times n_\eta$ . With some abuse of notation, we will drop the dependency  
 454 on the polynomial order and use  $N_i$  to denote both NURBS basis functions  
 455 and B-spline basis functions in the rest of the paper.

## 456 5. Weak- $C^1$ coupling for two-patch planar domains

457 To ground our approach in a practical example, we consider a biharmonic  
 458 problem on a two-patch planar domain, as demonstrated in Figure. 4. The  
 459 domain  $\Omega$  is decomposed to the slave subdomain  $\Omega_s$  (with finer mesh on the  
 460 interface) and the master subdomain  $\Omega_m$  (with coarser mesh on the interface).

In order of focusing on the coupling algorithm itself, we assume the bound-  
 aries that neighboring to the common intersection to be homogeneous Neu-  
 mann boundaries (north and south of  $\Omega_s$  and east and west of  $\Omega_m$ ) and the  
 rest to be homogeneous Dirichlet boundaries (west of  $\Omega_s$  and south of  $\Omega_m$ ),  
 denoted by  $\Gamma_N$  and  $\Gamma_D$  respectively. Then, the strong form of the two-patch  
 biharmonic boundary value problem writes:

$$\begin{aligned} \Delta^2 u &= f, & \text{in } \Omega, \\ u &= \frac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } \Gamma_D, \\ \Delta u &= \frac{\partial \Delta u}{\partial \mathbf{n}} = 0, & \text{on } \Gamma_N. \end{aligned} \quad (37)$$

### 461 5.1. Continuity constraints

The weak solution of the biharmonic problem (37) is in the space  $H^2(\Omega)$ .  
 Due to the inclusion  $C^1(\Omega) \subset H^2(\Omega)$ , we can use  $C^1$ -continuous functions  
 to approximate the solution. For the two multi-patch domain, constraints

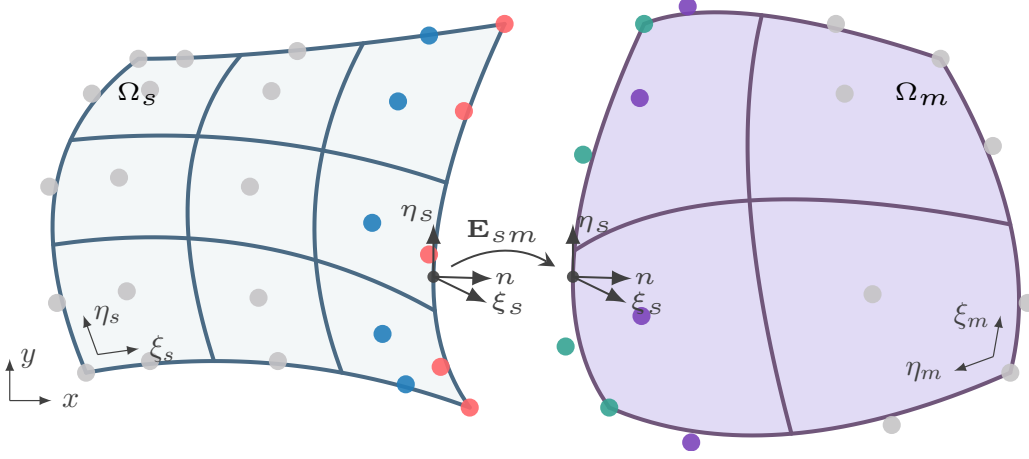


Figure 4: A two-patch planar domain constituted by  $\Omega_m$  and  $\Omega_s$ .

should be added to compromise the discontinuity along the intersection. In general, the following two constraints are requested for  $u$  to be  $C^1$ -continuous

$$[u]_{\Gamma_{sm}} = 0, \quad (38a)$$

$$\left[\frac{\partial u}{\partial \mathbf{n}}\right]_{\Gamma_{sm}} = 0, \quad \text{with } \mathbf{n} = \mathbf{n}_s = -\mathbf{n}_m \quad (38b)$$

where  $\mathbf{n}_k$  is the outward normal direction of  $\partial\Omega_k$ .

Whereas the constraint (38a) can easily fit into the framework of dual mortar method, the constraint (39) can not be directly imposed. First of all, the existence of dual basis functions of  $\frac{\partial N_i}{\partial \mathbf{n}}|_{\Gamma_{sm}}$  is doubtful. Even if they exist, as they are biorthogonal to the normal derivative of NURBS, their formulation must depend on the parameterization of  $\Gamma_{sm}$ , which violates the virtue of simplicity of dual basis functions. Hence, we need the following result, of which the derivatives are defined in the parametric domain and the dual basis functions can be formulated in an elegant manner.

**Lemma 1.** *Given two differentiable bijective geometric mappings  $\mathbf{F}_s: \hat{\Omega}_s \rightarrow \Omega_s$  and  $\mathbf{F}_m: \hat{\Omega}_m \rightarrow \Omega_m$ , a  $C^0$ -continuous function  $u$  is  $C^1$ -continuous in the physical domain if and only if*

$$\left[\frac{\partial u}{\partial \xi_s}\right]_{\Gamma_{sm}} = 0 \text{ and } \left[\frac{\partial u}{\partial \eta_s}\right]_{\Gamma_{sm}} = 0. \quad (39)$$

*Proof.* It suffices to consider two neighboring patches as shown in Figure. 4.  $u$  is  $C^0$ -continuous function implies  $[u]_{\Gamma_{sm}} = 0$ . For the  $C^1$ -continuity of  $u$ ,

we have the following relation

$$\begin{cases} \frac{\partial u_s}{\partial x} = \frac{\partial u_m}{\partial x} \\ \frac{\partial u_s}{\partial y} = \frac{\partial u_m}{\partial y} \end{cases} \xleftrightarrow{[\frac{\partial u}{\partial \eta_s}]_{\Gamma_{sm}}=0} \begin{cases} \frac{\partial u_s}{\partial \xi_s} \frac{\partial \xi_s}{\partial x} = \frac{\partial u_m}{\partial \xi_s} \frac{\partial \xi_s}{\partial x} \\ \frac{\partial u_s}{\partial \xi_s} \frac{\partial \xi_s}{\partial y} = \frac{\partial u_m}{\partial \xi_s} \frac{\partial \xi_s}{\partial y} \end{cases} \quad \text{on } \Gamma_{sm} \quad (40)$$

474 Since the geometric mapping  $\mathbf{F}_s$  is bijective, there exist an inverse mapping  
 475  $\mathbf{F}_s^{-1}$  and  $\det(\mathbf{F}_s^{-1}) \neq 0$ . Thus,  $[\frac{\partial u}{\partial \xi_s}]_{\Gamma_{sm}} = 0$ . This concludes the proof.  $\square$

The derivatives of  $u_m$  w.r.t.  $\xi_s$  and  $\eta_s$  can be obtained following the chain rule, as

$$\begin{bmatrix} \frac{\partial u_m}{\partial \xi_s} \\ \frac{\partial u_m}{\partial \eta_s} \end{bmatrix} = J(\mathbf{E}_{sm})^T \cdot \begin{bmatrix} \frac{\partial u_m}{\partial \xi_m} \\ \frac{\partial u_m}{\partial \eta_m} \end{bmatrix}, \quad (41)$$

476 where  $J(\cdot)$  is the Jacobian of the mapping in the argument. The Jacobian of  
 477 the composition mapping  $\mathbf{E}_{sm}$  can be written as

$$J(\mathbf{E}_{sm}) = J((\mathbf{F}_m)^{-1} \circ \mathbf{F}_s) = J((\mathbf{F}_m)^{-1}) \cdot J(\mathbf{F}_s) = J(\mathbf{F}_m)^{-1} \cdot J(\mathbf{F}_s). \quad (42)$$

## 478 5.2. Lagrange multiplier formulation and dual mortar formulation

We introduce two Lagrange multiplier spaces:  $M_0$  is devoted to the  $C^0$  constraint (38a) and  $M_1$  is devoted to the  $C^1$  constraint (39). The Lagrange multiplier formulation of the weak problem of (37) reads: find  $u \in X_b$ ,  $\lambda_0 \in M_0$  and  $\lambda_1 \in M_1$  such that:

$$\begin{cases} a_b(u, v) + b_0(\lambda_0, v) + b_1(\lambda_1, v) = l(v), & \forall v \in X_b; \\ b_0(\mu_0, u) = 0, & \forall \mu_0 \in M_0; \\ b_1(\mu_1, u) = 0, & \forall \mu_1 \in M_1; \end{cases} \quad (43)$$

with

$$a_b(u, v) = \int_{\Omega} \Delta u \Delta v d\Omega, \quad (44)$$

$$b_0(\mu, u) = \int_{\Gamma_{sm}} \mu [u]_{\Gamma} d\Gamma, \quad (45)$$

$$b_1(\mu, u) = \int_{\Gamma_{sm}} \mu \left[ \frac{u}{\xi_s} \right]_{\Gamma} d\Gamma. \quad (46)$$

479 The broken Sobolev space for the biharmonic problem is given as

$$\mathcal{X}_b := \{u \in L^2(\Omega) : u|_{\Omega_k} \in H_*^2(\Omega_k)\}, \quad (47)$$

480 with

$$H_*^2(\Omega_k) := \left\{ u \in H^2(\Omega_k) : u = 0 \text{ and } \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Gamma_D \cap \partial\Omega_k \right\}. \quad (48)$$

481 By moving the constraints from the problem statement to the definition of the  
 482 trial and test function spaces, we obtain the following variational problem:  
 483 find  $u \in \mathcal{K}_b$ , such that

$$a_b(u, v) = l(v), \quad \forall v \in \mathcal{K}_b, \quad (49)$$

484 where

$$\mathcal{K}_b := \{ u \in \mathcal{X}_b : b_0(u, \mu_0) = 0 \text{ and } b_1(u, \mu_1) = 0 \quad \forall (\mu_0, \mu_1) \in \mathcal{M}_0 \times \mathcal{M}_1 \}. \quad (50)$$

485 On one hand, the absence of the Lagrange multipliers  $\lambda_0$  and  $\lambda_1$  reduces the  
 486 size of the discretized problem and recovers the symmetric positive definite  
 487 structure of the stiffness matrix. As a result, efficient iterative solvers can be  
 488 applied for the problem. On the other hand, in the standard formalism,  $\mathcal{M}_0$   
 489 and  $\mathcal{M}_1$  are discretized by the trace of one side of the intersection so that  
 490 the construction of  $\mathcal{K}_b^h$  requires a factorization of a global constraint matrix,  
 491 which is not a trivial task.

492 However, in the following section, we will show that, with the help of dual  
 493 basis functions, the discretized function space  $\mathcal{K}_b^h$  can be formulated in an  
 494 elegant manner.

### 495 5.3. Discretization

In order to approximate the solution of the variational problem, we use the NURBS basis functions  $N_i^{(s)}$   $i \in I_s$  and  $N_j^{(m)}$   $j \in I_m$  to discretize coupled patches  $\Omega_s$  and  $\Omega_m$ , respectively. An appropriate offset has been made so that there is no overlapping between index sets  $I_s$  and  $I_m$  (given  $n_s$  basis functions in  $\Omega_s$ , we can assume the starting index in the index set  $I_m$  is  $n_s + 1$ ). The discretized geometrical mappings are represented by

$$\mathbf{F}_s = \sum_{i \in I_s} \mathbf{P}_i^s N_i^s, \quad (51)$$

$$\mathbf{F}_m = \sum_{i \in I_m} \mathbf{P}_i^m N_i^m, \quad (52)$$

where the control points  $\mathbf{P}_i^s, \mathbf{P}_i^m \in \mathbb{R}^2$ . The same basis functions are also used to discretize the test function  $u$  in broken Sobolev space  $\mathcal{X}_b$ , as

$$u^h = \sum_{i \in I_s + I_m} U_i N_i, \quad (53)$$

with

$$N_i = \begin{cases} N_i^s, & i \in I_s; \\ N_i^m, & i \in I_m. \end{cases} \quad (54)$$

As compared to the standard formalism that utilizes the trace of the slave patch on the intersection as the discretization of Lagrange multipliers, in this research, we construct Lagrange multipliers by using Bézier dual basis. We first classify NURBS basis functions into five different kinds, as shown in Figure. 4, namely:

1. The basis functions  $N_i^{(s)}$  such that  $\text{supp}(N_i^{(s)}) \cap \Gamma_{sm} = \emptyset$  and  $\text{supp}(\frac{\partial N_i^{(s)}}{\partial \xi_s}) \cap \Gamma_{sm} \neq \emptyset$ , whose indices are in the index set  $I_i$ . (denoted by blue dots)
2. The basis functions  $N_i^{(s)}$  such that  $\text{supp}(N_i^{(s)}) \cap \Gamma_{sm} \neq \emptyset$ , whose indices are in the index set  $I_{ii}$ . (denoted by red dots)
3. The basis functions  $N_i^{(m)}$  such that  $\text{supp}(N_i^{(m)}) \cap \Gamma_{sm} = \emptyset$  and  $\text{supp}(\frac{\partial N_i^{(m)}}{\partial \xi_s}) \cap \Gamma_{sm} \neq \emptyset$ , whose indices are in the index set  $I_{iii}$ . (denoted by green dots)
4. The basis functions  $N_i^{(m)}$  such that  $\text{supp}(N_i^{(m)}) \cap \Gamma_{sm} \neq \emptyset$ , whose indices are in the index set  $I_{iv}$ . (denoted by purple dots)
5. All basis functions neither the supports of themselves nor the supports of their first order derivatives in  $\xi_s$  direction intersect with  $\Gamma_{sm}$ , whose indices are in the index set  $I_v$ . (denoted by grey dots)

For the basis functions of the second kind, their restrictions on the intersection  $\Gamma_{sm}$  are one dimensional NURBS basis functions  $N_{i,p_\eta}^{(s)}$   $i \in \{1, 2, \dots, n_\eta^s\}$  that are used to discretize the tensor-product domain  $\Omega_s$  in  $\eta$  direction. In order to obtain an Identity matrix on the slave side of the discretized bilinear form  $b_0$ , the associated dual basis functions of  $N_{i,p_\eta}^{(s)}$  are used to discretize the Lagrange multiplier space  $\mathcal{M}_0$ , as

$$\lambda_0^h = \sum_{i=1}^{n_\eta^s} \Lambda_i^0 \hat{N}_{i,p_\eta}^{(s)}. \quad (55)$$

520 For the basis functions of the first kind, the restrictions of their first order  
 521 derivatives on the intersection  $\Gamma_{sm}$  can be written as  $cN_{i,p_\eta}^{(s)}$   $i \in \{1, 2, \dots, n_\eta^s\}$ ,  
 522 with  $c = N_{n_\xi^s-1,p_\xi}^{(s)'}(1)$ . Hence, the Lagrange multiplier space  $\mathcal{M}_1$  can be  
 523 discretized by

$$\lambda_1^h = \sum_{i=1}^{n_\eta^s} \Lambda_i^1 \tilde{N}_i, \quad \text{with } \tilde{N}_i = \frac{1}{c} N_{i,p_\eta}^{(s)}. \quad (56)$$

524 We denote the basis functions in  $\mathcal{M}_0^h$  and  $\mathcal{M}_1^h$  as the dual basis functions of  
 525 the second and the first kinds of NURBS basis functions, respectively.

526 By substituting these NURBS approximations into the Lagrange multi-  
 527 plier formulation (43), we obtain the following linear system:

$$\begin{bmatrix} \mathbf{K} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \Lambda_0 \\ \Lambda_1 \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix}, \quad (57)$$

528 where  $\mathbf{K}$  and  $\mathbf{F}$  are the stiffness matrix and the load vector for the uncoupled  
 529 problem, respectively.  $\mathbf{B}$  is the constraint matrix discretized from the bilinear  
 530 forms  $b_0$  and  $b_1$ . In order to construct the finite element space  $\mathcal{K}_b^h$ , we need to  
 531 solve the constraint matrix  $\mathbf{B}$ 's null space  $\mathbf{C} = \ker(\mathbf{B})$ . Since the structure of  
 532 the constraint matrix  $\mathbf{B}$  depends on the index sets  $I_s$  and  $I_m$  and the ordering  
 533 of Lagrange multiplier basis functions, without adding any constraints on the  
 534 indices, we introduce two permutation matrices  $\mathbf{P}_c$  and  $\mathbf{P}_r$  (this step is not  
 535 necessary from the implementation point of view, but is very convenient  
 536 for the demonstration, especially for multi-patch problem). We define the  
 537 column permutation matrix  $\mathbf{P}_c$  as

$$\begin{bmatrix} \mathbf{I}_i \\ \mathbf{I}_{ii} \\ \mathbf{I}_{iii} \\ \mathbf{I}_{iv} \\ \mathbf{I}_v \end{bmatrix} = \mathbf{P}_c \begin{bmatrix} \mathbf{I}_s \\ \mathbf{I}_m \end{bmatrix}, \quad (58)$$

538 where  $\mathbf{I}_i$  is the vector form of the index set  $I_i$ . For an appropriate row  
 539 permutation matrix  $\mathbf{P}_r$ , the modified constraint matrix can be written in a  
 540 partitioned form as

$$\mathbf{B}_p = \mathbf{P}_r \mathbf{B} \mathbf{P}_c^T = \begin{bmatrix} \mathbf{B}_1^1 & \mathbf{B}_1^2 & \mathbf{B}_1^3 & \mathbf{B}_1^4 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2^2 & \mathbf{B}_2^3 & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (59)$$

541 where  $\mathbf{B}_i^j$  corresponds to the contribution of the inner product of the basis  
 542 functions of the  $j^{th}$  kind with the dual basis functions of the  $i^{th}$  kind.  $\mathbf{P}_r$   
 543 can be defined as a row permutation matrix such that the resulted block  
 544 sub-matrices  $\mathbf{B}_1^1$  and  $\mathbf{B}_2^2$  are both identity matrices. Under a rank-preserving  
 545 transformation  $\mathbf{T}$ , we can eliminate the block sub-matrix  $\mathbf{B}_1^2$ , as

$$\mathbf{TB}_p = \begin{bmatrix} \mathbf{I} & \begin{bmatrix} \mathbf{B}_1^3 - \mathbf{B}_1^2\mathbf{B}_2^3 & \mathbf{B}_1^4 & \mathbf{0} \\ \mathbf{B}_2^3 & \mathbf{0} & \mathbf{0} \end{bmatrix} \end{bmatrix}. \quad (60)$$

546 We may now take

$$\mathbf{C}_p := \ker(\mathbf{B}_p) = \begin{bmatrix} \mathbf{B}_1^2\mathbf{B}_2^3 - \mathbf{B}_1^3 & -\mathbf{B}_1^4 & \mathbf{0} \\ -\mathbf{B}_2^3 & \mathbf{0} & \mathbf{0} \\ \hline & \mathbf{I} & \end{bmatrix}. \quad (61)$$

547 Hence, the null space of  $\mathbf{B}$  can be taken as

$$\mathbf{C} = \mathbf{P}_c^T \mathbf{C}_p. \quad (62)$$

548 Now we can discretize functions in  $\mathcal{K}_b^h$  as

$$u^h = \mathbf{N}^T \mathbf{U}, \quad \text{with } \mathbf{U} = \mathbf{C} \tilde{\mathbf{U}}, \quad (63)$$

549 where  $\mathbf{N}$  is the vector form of basis functions of  $\mathcal{X}_b^h$ , and  $\tilde{\mathbf{U}}$  is the control  
 550 point vector. By substituting the above discretization in the weak form (50),  
 551 we obtain the following linear system to be solved:

$$\mathbf{C}^T \mathbf{K} \mathbf{C} \tilde{\mathbf{U}} = \mathbf{C}^T \mathbf{F}. \quad (64)$$

552 Because of the biorthogonality of the NURBS basis functions and their dual  
 553 basis functions, the constrained solution space  $\mathcal{K}_b^h$  can be constructed very  
 554 efficiently, leading to a sparse, symmetric positive definite formulation.

## 555 6. Weak- $C^1$ coupling for multi-patch planar domains

556 To generate complex geometries, we need to decompose domain to mul-  
 557 tiple patches. Unfortunately, we cannot apply directly the results of two-  
 558 patch coupling to this more general mortar situation. The main issue is  
 559 the so-called cross points. For a multi-patch decomposition, at least three



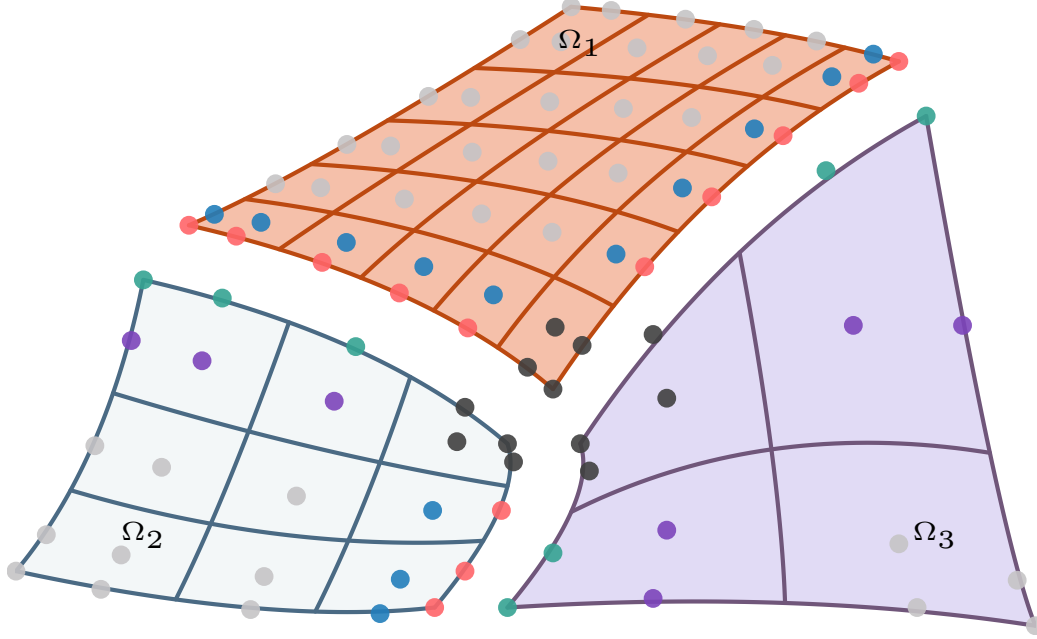


Figure 5: A three-patch planar domain constituted by  $\Omega_1$ ,  $\Omega_2$  and  $\Omega_3$ .

subdomains meet at an interior crosspoint and several interfaces can share this cross point as a common endpoint (Figure. 5). If we use the discretized Lagrange multipliers proposed in the previous section, owing to the presence of cross points, some of the control points will serve as both slave points (indices in  $I_1$  and  $I_2$ ) and master points (indices in  $I_3$  and  $I_4$ ). Hence, there is no permutation matrices such that the constraint matrix  $\mathbf{B}$  can be modified to the form as equation (60), of which the null space can be found in a trivial way. Even more, although the constraint matrices defined on each interfaces are full row rank, the assembled constraint matrix  $\mathbf{B}$  may not be full row rank in most cases, which renders the linear system (57) to be rank deficient. As a result, either modifications to the Lagrange multipliers or to the method itself is required so that the proposed method can be generalized to a setting where a domain can be decomposed to an arbitrary number of patches. Before we start this section, we would like to introduce the sixth kind of NURBS basis function that associated with the cross point  $v$ , as

6. The basis function  $N_i$  such that  $\text{supp}(N_i) \cap v \neq 0$ , or  $\text{supp}(\frac{\partial N_i}{\partial \xi}) \cap v \neq 0$ , or  $\text{supp}(\frac{\partial N_i}{\partial \eta}) \cap v \neq 0$  or  $\text{supp}(\frac{\partial^2 N_i}{\partial \xi \partial \eta}) \cap v \neq 0$ , whose indices are in the index set  $I_{vi}$ . (denoted by black dots in Figure. 5)

578 The rest five kinds of NURBS basis function remain the same except their  
 579 intersection with the sixth kind are excluded, that is

$$I_k = I_k - I_{vi} \bigcap I_k, \quad k \in \{i, ii, \dots, v\}. \quad (65)$$

580 The basis functions in  $\mathcal{M}_0^h$  and  $\mathcal{M}_1^h$  can be classified as the dual basis func-  
 581 tions of the NURBS basis function of the 1<sup>st</sup>, 2<sup>nd</sup> and 6<sup>th</sup> kind, respectively.  
 582 The domains on the two sides of each interface can still be considered as slave  
 583 and master based on the same rule as for the two-patch coupling case.

#### 584 6.1. Cross point modification

585 Since using the discretization of Lagrange multipliers proposed for two-  
 586 patch coupling case directly results in over-constrained constraint matrix  $\mathbf{B}$   
 587 for the control points round the crosspoints, we can remedy this issue by re-  
 588 ducing the dimension of the Lagrange multiplier spaces. Roughly speaking,  
 589 we have to remove the two degrees of freedom of the Lagrange multiplier  
 590 spaces associated with each cross point so that the resulted Lagrange multi-  
 591 plier spaces on each interface should have the same dimension as  $\mathcal{X}_b^h|_{\Omega_s} \bigcap H_0^2(\Gamma_{sm})$ .

592 This can be achieved by two ways: we can either coarse the mesh for the  
 593 Lagrange multiplier in the neighborhood of the cross point, or reduce their  
 594 polynomial degree in the neighborhood of the cross point. An example for  
 595 quadratic 1-D B-spline basis functions is shown in Figure. 6. The coarsed  
 596 basis functions are achieved by replacing the first three and last three basis  
 597 functions by their summations, while we can construct degree reduced basis  
 598 functions by reduce the polynomial degree of the first and last two elements  
 599 by one while retaining the inter-element continuity.

600 Although the degree reduction is not a trivial task for dual basis, the  
 601 summation trick can be applied to coarse dual basis functions. After the  
 602 coarsen procedure, the dual basis functions associated with the basis func-  
 603 tions of 6<sup>th</sup> kind are all eliminated so that no inter-dependencies will happen  
 604 in the neighborhood of the cross point. We can define a column permutation  
 605 matrix

$$\begin{bmatrix} \mathbf{I}_i \\ \mathbf{I}_{ii} \\ \mathbf{I}_{iii} \\ \mathbf{I}_{iv} \\ \mathbf{I}_{vi} \\ \mathbf{I}_v \end{bmatrix} = \tilde{\mathbf{P}}_c \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix}. \quad (66)$$

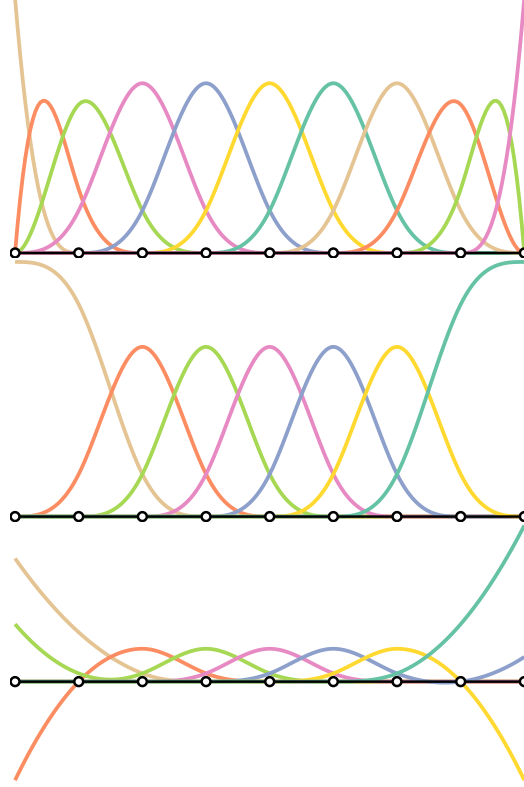


Figure 6: Quadratic basis functions and their cross point modifications. Top: original quadratic basis functions. Middle: coarsened basis functions. Bottom: degree reduced basis functions

606 With a suitable row permutation matrix  $\tilde{\mathbf{P}}_r$ , the constraint matrix  $\mathbf{B}$  can be  
 607 modified as

$$\tilde{\mathbf{B}}_p := \tilde{\mathbf{P}}_r \mathbf{B} \tilde{\mathbf{P}}_c^T = \begin{bmatrix} \mathbf{B}_1^1 & \mathbf{B}_1^2 & \mathbf{B}_1^3 & \mathbf{B}_1^4 & \mathbf{B}_1^6 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2^2 & \mathbf{B}_2^3 & \mathbf{0} & \mathbf{B}_2^6 & \mathbf{0} \end{bmatrix}, \quad (67)$$

608 with  $\mathbf{B}_1^1$  and  $\mathbf{B}_2^2$  be identity matrices. Hence, its null space can be found as

$$\tilde{\mathbf{C}}_p := \ker(\tilde{\mathbf{B}}_p) = \left[ \begin{array}{ccccc} \mathbf{B}_1^2 \mathbf{B}_2^3 - \mathbf{B}_1^3 & -\mathbf{B}_1^4 & -\mathbf{B}_1^6 & \mathbf{0} \\ -\mathbf{B}_2^3 & \mathbf{0} & -\mathbf{B}_2^6 & \mathbf{0} \\ \hline & \mathbf{I} & & \end{array} \right]. \quad (68)$$

609 Although the boundary modification by coarsening eliminates the inter-dependency  
 610 in the neighborhood of cross points, numerical tests demonstrate sub-optimality.

611 *6.2. Explicitly solve the null space*

612 Instead of modifying the Lagrange multipliers, we can solve the null space  
 613 of the over-constrained constraint matrix  $\mathbf{B}$  directly. Several matrix factoriza-  
 614 tion methods can be used to solve the null space, including LU, QR, SVD. For  
 615 example, a rank-revealing QR factorization over a rank-deficiency constraint  
 616 matrix  $\mathbf{B}$  yields

$$\mathbf{B}\mathbf{P} = \mathbf{Q} \begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_2 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (69)$$

617 where  $\mathbf{P}$  is a permutation matrix,  $\mathbf{Q}$  is an unitary matrix,  $\mathbf{R}_1$  is a upper  
 618 triangular matrix and  $\mathbf{R}_2$  is a rectangular matrix. The null space can be  
 619 taken as

$$\ker(\mathbf{B}) = \mathbf{P} \begin{bmatrix} -\mathbf{R}_1^{-1}\mathbf{R}_2 \\ \mathbf{I} \end{bmatrix}. \quad (70)$$

620 However, this requires a factorization of the entire constraint matrix  $\mathbf{B}$  and  
 621 we fail to utilize the advantage of the Bézier dual basis. Even more, the  
 622 sparsity of the constrained stiffness matrix might be impacted as the inverse  
 623 of  $\mathbf{R}_1$  is a dense matrix. This type of global factorization has been utilized  
 624 for patch coupling problem in [27, 28, 32].

625 Instead of solving the null space directly, we will localize the constraint  
 626 to each cross point and solve the null space of a localized linear system. For  
 627 the constraint matrix  $\mathbf{B}$  constructed by the Lagrange multipliers without  
 628 modification, we assume there exist a row permutation matrix  $\hat{\mathbf{P}}_r$  such that

$$\hat{\mathbf{B}}_p := \hat{\mathbf{P}}_r \mathbf{B} \tilde{\mathbf{P}}_c^T = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_6^1 & \mathbf{B}_6^2 & \mathbf{B}_6^3 & \mathbf{B}_6^4 & \mathbf{B}_6^6 & \mathbf{0} \\ \mathbf{B}_1^1 & \mathbf{B}_1^2 & \mathbf{B}_1^3 & \mathbf{B}_1^4 & \mathbf{B}_1^6 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2^2 & \mathbf{B}_2^3 & \mathbf{0} & \mathbf{B}_2^6 & \mathbf{0} \end{bmatrix}, \quad (71)$$

629 with  $\mathbf{B}_1^1$  and  $\mathbf{B}_2^2$  be identity matrices. The null space of  $\mathbf{B}_2$  can be found as

$$\mathbf{C}_2 := \ker(\mathbf{B}_2) = \begin{bmatrix} \mathbf{B}_1^2\mathbf{B}_2^3 - \mathbf{B}_1^3 & -\mathbf{B}_1^4 & -\mathbf{B}_1^6 & \mathbf{0} \\ -\mathbf{B}_2^3 & \mathbf{0} & -\mathbf{B}_2^6 & \mathbf{0} \\ \mathbf{I} \end{bmatrix}. \quad (72)$$

Due to the inclusion  $\hat{\mathbf{C}}_p := \ker(\hat{\mathbf{B}}_p) \subset \ker(\mathbf{B}_2)$ , we can find  $\hat{\mathbf{C}}_p$  by solving  $\ker(\mathbf{B}_1\mathbf{C}_2)$ . Since dual basis functions have compact support,  $\mathbf{B}_1$  and  $\mathbf{C}_2$  are all sparse matrices. We can split the columns of  $\mathbf{C}_2$  into two matrices, as

$$\begin{aligned} \mathbf{C}_2^1 &:= \{v \in \mathbf{C}_2 : \mathbf{B}_1 v \neq 0\}, \\ \mathbf{C}_2^2 &:= \{v \in \mathbf{C}_2 : \mathbf{B}_1 v = 0\}. \end{aligned} \quad (73)$$

Such a split can be defined a priori, based on the discretization of each patch.  
 Note that  $\mathbf{C}_2^2 \subset \hat{\mathbf{C}}_p$ ,  $\hat{\mathbf{C}}_p$  can be written as

$$\hat{\mathbf{C}}_p = [\mathbf{C}_2^1 \ker(\bar{\mathbf{B}}_p) \quad \mathbf{C}_2^2], \text{ with } \bar{\mathbf{B}}_p = \mathbf{B}_1 \mathbf{C}_2^1. \quad (74)$$

Compared with the constraint matrix  $\mathbf{B}$  whose size increases as we refine the mesh, the row size of  $\bar{\mathbf{B}}_p$  is fixed and its column size is fixed after certain refinement. A comparison of the size of matrix  $\mathbf{B}$  and matrix  $\bar{\mathbf{B}}_p$  as a function of the degrees of freedom is given in Figure. 7 for the three-patch coupling in Figure. 5 with  $2^{nd}$  order B-spline basis functions. As can be seen, the size of  $\mathbf{B}$  grows rapidly as the mesh being refined. The computational cost of directly solving its kernel will be very expensive. However, owing to the compact support of dual basis function, we can transfer a global, size-varying problem (factorization on  $\mathbf{B}$ ) to a local, size-fixed problem (factorization on  $\bar{\mathbf{B}}_p$ ), and the problem size is very small ( $12 \times 24$  for this case).

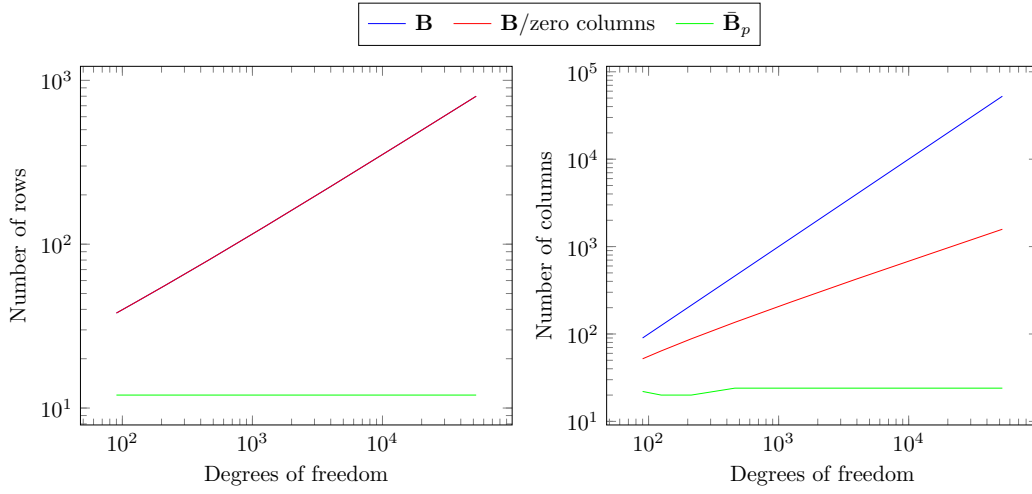


Figure 7: The change of matrix size of  $\mathbf{B}$ ,  $\mathbf{B}$  with zero columns being removed, and  $\bar{\mathbf{B}}_p$ , as a function of the degrees of freedom. The constraint matrix  $\mathbf{B}$  is formulated for the three-patch coupling in Figure. 5 with  $2^{nd}$  order B-spline basis functions.

A graphical comparison of the sparsity patterns among standard Lagrange multiplier with global factorization, Bézier dual basis with global factorization and Bézier dual basis with local factorization is given in Figure. 8. As expected, the standard method yields the lowest sparsity, while Bézier dual basis with local factorization yields the highest sparsity (40% sparser). Meanwhile, the Bézier dual basis does not significantly improve the sparsity if a

648 global factorization is applied to construct  $\mathcal{K}_b^h$ . Moreover, a global factoriza-  
 649 tion yeilds entries with very small absolute values ( $\leq 10^{-14}$ ), especially for  
 650 the constraint matrix formed by Bézier dual basis, while all entries are away  
 651 from zero for the local factorization.

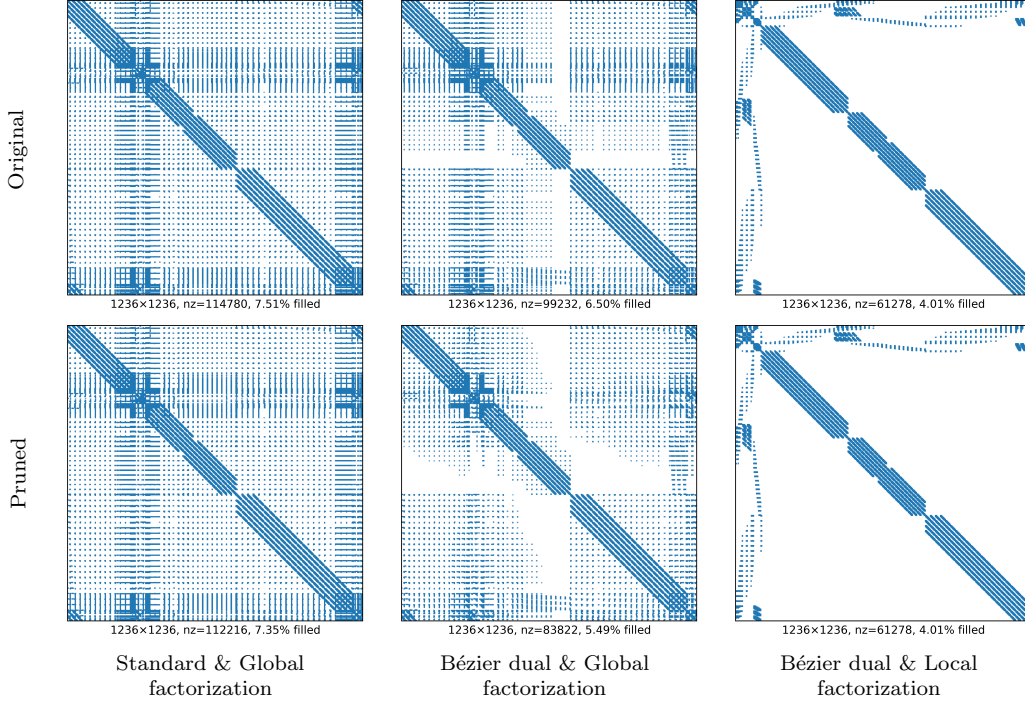


Figure 8: Sparsity patterns of constrained stiffness matrices. Left: standard Lagrange multipliers with global factorization. Middle: Bézier dual basis with global factorization. Right: Bézier dual basis with localized factorization. Top: original matrix. Bottom: small absolute values ( $\leq 10^{-14}$ ) be pruned. All stiffness matrices are formulated for the three-patch coupling in Figure. 5 with  $3^{\text{rd}}$  order B-spline basis functions after 4 refinements. The number of non-zero entries is given by nz.

## 652 7. Schedule

653 Although there are various aspects in weak  $C^1/G^1$  coupling deserve a  
 654 thorough study, our main focus in this stage is to extend our finding in an ab-  
 655 stract problem (biharmonic problem) to practical problems (e.g. Kirchhoff-  
 656 Love shell). Compared to the planar biharmonic problem, Kirchhoff-Love  
 657 shell is a more challenging problem, as the computational domain is in  $\mathbb{R}^3$

658 and the constraint is not isotropically applied in each direction. Hence, a  
659 more generalized constraint is needed to compromise geometries with kinks.  
660 And validations are needed for two-patch and multi-patch Kirchhoff-Love  
661 shells. Meanwhile, although we have implemented two algorithms to solve  
662 multi-patch biharmonic problems, the boundary modification method does  
663 not delivers ideal results while an additional factorization is needed for ex-  
664 plicitly solving the null space. We will still make efforts in finding a feasi-  
665 ble boundary modification for multi-patch coupling. A detailed time line is  
666 shown in Table. 2.

Table 2: A schedule of tasks and stages of my research towards the final dissertation.





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