Topos-theoretic approach to the geometry of physics

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§1. Basic introduction to the classical category theory

1.1. A definition of a category A category, if we are not digress to *the architecture of mathematics*, has quite a simple and intuitive definition capturing the essence of '*mathematical objects*': objects and structure-preserving mappings between them.

Def 1.1 (Category)**.** *Category $\mathcal{C}$ consists of the following data:*

1. *a class of objects,* $\operatorname{Ob}\mathcal{C}$*,*
2. *for any pair of objects $X, Y : \operatorname{Ob}\mathcal{C}$, a class of morphisms (arrows) between them,* $\operatorname{Hom}(X, Y)$ *(often called hom-set or hom-class); the class of all morphisms is* $\operatorname{Mor}\mathcal{C}$*;*
3. *for any triple of objects $X, Y, Z : \operatorname{Ob}\mathcal{C}$ there is a composition law:* $(-)\circ(-) : \operatorname{Hom}(Y, Z) \times \operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(X, Z)$,

object to the following axioms:

1. *for any object X,* $\operatorname{Hom}(X, X)$ *contains a distinguished element* id_X *(called identity morphism) which are left and right identities to the composition law, in the sense that for any morphism $X \xrightarrow{f} Y \in \operatorname{Hom}(X, Y)$: $1_Y \circ f = f \circ 1_X = f$.*
2. *the composition law is associative, i.e. for any compitable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$ it holds that $(h \circ g) \circ f = h \circ (g \circ f)$.*

Category $\mathcal{C}$ is called *small* if $\operatorname{Mor}\mathcal{C}$ is indeed a set, *locally small* if $\operatorname{Hom}(X, Y)$ is a set for any $X, Y \in \operatorname{Ob}\mathcal{C}$. From now on all categories are assumed to be locally small.

1.2. Simple categorial constructions Before proceeding, we need to set up several common categorial constructions.

Def 1.2 (Opposite category)**.** *Let $\mathcal{C}$ be a category. Then the category $\mathcal{C}^\circ$ (or $\mathcal{C}^{op}$) in which objects are the same as $\mathcal{C}$ but the direction of all arrows is reversed (in other words, $\mathrm{Hom}_{\mathcal{C}^\circ}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(Y,X)$) is called the opposite category of $\mathcal{C}$.*

The *raison d'être* of opposite cateogies is *a posteriori*, though the construction is *a priori tautological* (but still useful, because opposite categories often read as something interesting, as we will see in a minute). It is the framework for *formal (abstract) duality.*

Def 1.3 (Product category)**.** *Given two categories $\mathcal{C}$ and $\mathcal{D}$ one forms a new category $\mathcal{C} \times \mathcal{D}$ given concretely by*

1. $\mathrm{Ob}_{\mathcal{C}\times\mathcal{D}} := \mathrm{Ob}_{\mathcal{C}} \times \mathrm{Ob}_{\mathcal{D}}$,
2. $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((c,d),(c',d')) := \mathrm{Hom}_{\mathcal{C}}(c,c') \times \mathrm{Hom}_{\mathcal{D}}(d,d')$,
3. $(f',g') \circ (f,g) := (f' \circ f, g' \circ g)$.

Def 1.4 (Comma/slice category)**.** *Let $\mathcal{C}$ be a category, $c \in \mathcal{C}$ be a fixed object. Category $c/\mathcal{C}$ is given by (and intuitively means distinguishing an element):*

1. $\mathrm{Ob}\, c/\mathcal{C} := \mathrm{Hom}_{\mathcal{C}}(c, -)$,
2. $\mathrm{Hom}_{c/\mathcal{C}}(c \to x, c \to y)$ *is subset of* $\mathrm{Hom}_{\mathcal{C}}(x,y)$ *that makes the canonical triangle commute.*

1.3. Functors Categories are mathematical objects themselves, so there should be strcuture-preserving map between them. Exactly this is called *a functor.*

Def 1.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of two functions $F_{\mathrm{Ob}} : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ and $F_{\mathrm{Mor}} : \mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{D}$ object to structure-preserving axioms:*

1. *let X an object of $\mathcal{C}$, 1_X be its identity morphism; then, $F_{\mathrm{Mor}}(1_X) = 1_{F_{\mathrm{Ob}}(X)} \in \mathrm{Mor}\,\mathcal{D}$ (or, abusing notation, $F(1_X) = 1_{F(X)}$);*
2. *if $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $F_{\mathrm{Mor}}(g \circ f) = F_{\mathrm{Mor}}(g) \circ F_{\mathrm{Mor}}(f)$.*

Here are some basic examples of functors:

fund-l. group	$\mathrm{Top}_* \xrightarrow{\pi_1(-,-)} \mathrm{Grp}$	fund-l. groupoid	$\mathrm{Top}_* \longrightarrow \mathrm{Top} \xrightarrow{\Pi_1(-)} \mathrm{Grp}$
Hodge theory	$\mathrm{CKahler} \xrightarrow{H^n(-,\mathbf{C})} \mathrm{PureHS}_n^\circ$	Mixed Hodge theory	$\mathrm{AlgVar}_{\mathbf{C}} \xrightarrow{H^n(-,\mathbf{C})} \mathrm{MixedHS}_n^\circ$
Continuity	$\mathrm{Top}^\circ \xrightarrow{\mathrm{Ouv}(-)} \mathrm{Poset}$	Zariski spectra	$\mathrm{CRing}^\circ \xrightarrow{\mathrm{Spec}(-)} \mathrm{Top}$

A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* if all its hom-functions $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ are *surjective*, *faithful* if *injective* and *fully faithful* if *bijective* (then $\mathcal{C}$ is a *full* subcategory).

1.4. Function algebras duality, Gel'fand-Kolmogorov Let X and Y be two compact Hausdorff topological spaces (i.e. assumed to be *nice enough*).

> It is *natural* to expect that if $C(X, \mathbf{R})$ and $C(Y, \mathbf{R})$ are *isomoprhic*, then X and Y are *homeomorphic.*

And this indeed holds; this statement is (usually) called *Gel'fand-Kolmogorov theorem.*

> *L'importance philosophique:* compact Hausdorff topological spaces is *fully* described by its *algebra of continous functions.*

One should notice additionally that any morphism (continous map) $X \xrightarrow{f} Y$ induces a homomorphism of algebras $C(Y) \xrightarrow{(-)\circ f} C(X)$ given by *precomposition.* Quite the same proof of the Gelfand-Kolmogorov theorem yields

$$\mathrm{Hom}_{\mathrm{TopHC}}(X, Y) = C(X, Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{CAlg}_{\mathbf{R}}}(Y, X),$$

stated categorially renders as *a fully faithful functor*

$$\mathrm{TopHC} \xhookrightarrow{C(-,\mathbf{R})} \mathrm{CAlg}_{\mathbf{R}}^{\circ}.$$

Very similar functor works for smooth manifolds (commonly called *Milnor's exercise*):

$$\mathrm{SmthMfd} \xhookrightarrow{C^\infty(-,\mathbf{R})} \mathrm{CAlg}^\circ_{\mathbf{R}}.$$

However, there are some examples where these notions play in a more involved way. First of these, a generalisation of *Gel'fand-Kolmogorov* is called *Gel'fand duality*:

$$\mathrm{TopHC} \xrightarrow{\simeq} \mathrm{C}^*\mathrm{Alg}^\circ_{\mathrm{comm}}$$

This duality gives us many analogies which usually cause good flashbacks:

geometry	algebra	geometry	algebra
cpt. Haus. topspace	comm. unital. C^*-alg.	loc. cpt. Haus. topspace	comm. C^*-alg.
point	maximal ideal	Alexandrov cptn.	unitalisation
homeo.	automorphism	Stone-Cech cptn.	multiplier alg.
disjoint union $\sqcup$	product $\times$	$\bullet$	$\bullet$
product $\times$	complete t.p. $\hat{\otimes}$	$\bullet$	$\bullet$

The general notion for that is *Isbell duality*, which is a basic construction of so-called *enriched category theory.*

1.5. Study case: affinoid spaces (local pieces of rigid analytic spaces)

(Reference: P. Achinger, Introduction to Non-Archimedean Geometry)

Sometimes people even *define* the category of spaces as the opposite category of algebras. Here we model *analytic space* over a complete *non-archimedian* field $(K, \|\cdot\|)$.

Tate algebra $K\langle T_1, \ldots, T_n\rangle$ is defined as a subring of the ring of formal power series $K[\![T_1, \ldots, T_n]\!]$ such that all its elements $\sum_J a_J T^J$ are strictly convergent in the sense that for any $\varepsilon > 0$ there are only finitely many J such that $|a_J| > \varepsilon$.

We should think of Tate algebra as *the ring of functions on an unit polydisque.*

Tate algebra is Banach when endowned with *the Gauss norm*: $\|\sum_J a_J T_J\| := \max_J |a_J|$.

A Banach algebra X is an *affinoid algebra* if it is $K\langle T_1, \ldots, T_n\rangle/I$ for some n and (closed) ideal I (any ideal of Tate algebra is closed, see Proposition 3.4.5).

The category of affinoid spaces is the opposite category of *affinoid algebras and bounded homomorphisms between them.*

To define *rigid-analytic spaces* we proceed classically (with the notorious technical adjustments due to the topological defects that appear in the non-archmedian setting):

$$\underset{\text{Tate algebras}}{K\langle X_1, \ldots, X_n\rangle} \longrightarrow \underset{\text{affinoid spaces}}{\{K\langle X_1, \ldots, X_n\rangle/I\}^\circ} \xrightarrow{G\text{-gluing}} \underset{\text{rigid-analytic spaces}}{X = (X, \mathcal{O}_X)}$$

1.6. Natural transformations We have already defined categories and morphisms between them — that is, functors. Expectedly, small categories form a category, called Cat.

A quite natural question arise: *what is a morphism between functors?*

Def 1.6 (Natural transformation). *Let $\mathcal{C}, \mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ are functors between them.* Natural transformation between F and G *is a collection of arrows $\alpha_X : F(X) \to G(X)$ between corresponding images of functors such that for any arrow $f : X \to Y \in \operatorname{Mor}\mathcal{C}$ the following diagram is commutative (in $\mathcal{D}$):*

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

In this case, one writes $\alpha : F \implies G$ (F and G are functors). Graphically,

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$$

All functors from $\mathcal{C}$ to $\mathcal{D}$ with natural transformations form *functor category* $[\mathcal{C}, \mathcal{D}]$.

1.7. Universal properties and constructions Many objects can be defined by their *determining (universal) property.* We provide several important examples.
Product. This construction patches together several objects $(X_i)_{i\in I}$ into a large object $\prod X_i$. Of course, we should have projections onto each factor: $\prod X_i \to X_i$.
Coproduct. Formal dual of *product.* Gives an analogy of disjoint union.

$$\begin{tikzcd} X_i & \\ \prod X_i \arrow[u, "\pi_i"] & \mathcal{U} \arrow[ul, "f_i"'] \arrow[l, dashed, "\exists!"'] \end{tikzcd} \qquad \begin{tikzcd} X_i \arrow[d, "\iota_i"'] \arrow[dr, "f_i"] & \\ \coprod X_i \arrow[r, dashed, "\exists!"] & \mathcal{U} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{U} \arrow[dl, "f"'] \arrow[d, dashed, "\exists!"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y \end{tikzcd}$$

Fiber product (pullback). Describes '*base change*', the most general object which makes a commutative square. Also, can be thought as a 'product respecting the base'.
Fiber coproduct (pushout). Formal dual of *pullback.* (eg: $*$ with amalgamation).
Limits, colimits. The universal cone; the formal dual is *colimit.*
Kan extensions... Generalises all these.
There are (at least) two good books which shows these concepts in **Set** and **Top**:

Author	Name
Lawvere, W, Rosebrugh, R.	Sets for Mathematics
Bryson, T, Terilla, J, Bradley, T.-D.	Topology: A Categorical Approach

1.8. Internalisation and internal language (logic) of categories Consider the classical definition of the group, namely

> a group is *a set* G endowned with three functions (*neutral element*) $e : 1 \to G$, (*group operation*) $(-)\cdot(-) : G\times G \to G$ and (*inverse*) $(-)^{-1} : G \to G$ satisfying well-known axioms of group.

Note that we do not really need G to be *a set*, but an element of a category $\mathcal{C}$ subject to the existence of *a terminal object* and *finite products.* Now we give the full defintion.

> Let $\mathcal{C}$ be a category with finite products. *An (internal) group* G in $\mathcal{C}$ is an object of $\mathcal{C}$ equipped with $e : pt \to G$, $(-)\cdot(-) : G \times G \to G$ and $(-)^{-1} : G \to G$ such that unitality, associativity and inverses hold.

What one finds as a reader?

$\mathcal{C}$	group object	$\mathcal{C}$	group object
Set	(vanilla) group	Cat	2-group
Top	topological group	SmthMfd	Lie group
AlgVar_k	algebraic k-group	Sch_k	group k-schemes
Grp, Ab	abelian group	$[\mathcal{C}, \mathcal{D}]$	group functor

In more general terms there is the following slogan:

> the more an (ambient) category additional features has, the richer its internal logic (language) becomes.

Moreover, it turns out that *the happy medium* are presented by (elementary) *topoi* (or, $(\infty, 1)$-topoi, if one wishes): its internal language allows to do all constructive mathematics (i.e. no exluded middle or AC involved). This adds a new layer to our understanding:

$$\underset{\text{internal logic}}{\text{syntax}} \longrightarrow \underset{\text{category of context}}{\text{semantics}} \longrightarrow \underset{\text{concrete manf-s}}{\text{model}}$$

Probably the best book to start is HoTT:

> Voevodsky, V, Awodey, S, Coquand, T, et al. *Homotopy Type Theory: Univalent Foundations of Mathematics*

Also good things to read are:

Author	Name
Mac Lane, S, Moerdijk, I.	Sheaves in Geometry and Logic
Johnstone, P.	Sketches of an Elephant (B2,D)

§2. Smooth sets and (Grothendieck) topoi

2.1. Probes In physics we study the world by *interacting* with (or *discovering*) it. This can be folded in the following paradigm:

> You work at a particle accelerator. You want to understand some particle. All you can do are throw other particles at it and see what happens. If you understand how your mystery particle responds to all possible test particles at all possible test energies, then you know everything there is to know about your mystery particle. — Ravi Vakil

More concrete version of the above is

> To study a space we need to *probe* it, i.e. to consider the collection of mappings $\mathrm{Plots}(-, X)$ from *probes* (say, $\mathbf{R}^n$) to our (would-be) space X. Now, *knowing everything about plots, we know everything about X.*

Note that we have some (toy) analogues of *branes* (from string theory) by now: p-branes are interpreted as a set $\mathrm{Plots}(\mathbf{R}^p, X)$. Of course, if we could recover a whole space when given plots being just random data, our approach would definitely fail. This is why we need some *consistency conditions*: functoriality and gluing (being a *sheaf*).

This is the motivation for the definition of topos as an element of the sequence:

$$\text{Euclidean spaces} \longrightarrow \text{Metric spaces} \longrightarrow \text{Topological spaces} \longrightarrow \\ \longrightarrow \textbf{Grothendieck sites} \longrightarrow (\infty, 1)\text{-sites}$$

In this sense, to work with (generalised) spaces we need only good (open) covers. This motivates the definition of *Grothendieck topology*:

Let $\mathcal{C}$ be a category, $\mathcal{T}$ be a family of morphisms $\{\phi_i : U_i \to U\}$, s.t.

1. (Trivial cover is a cover) If $\phi : U' \to U$ is an isomorphism, then $\{\phi\} \in \mathcal{T}$.
2. (Open cover of an open cover is an open cover) If $\{U_i \to U\} \in \mathcal{T}$ and $\{V_{ij} \to U_i\} \in \mathcal{T}$, then $\{V_{ij} \to U\} \in \mathcal{T}$ as well.
3. (Base change saves open covers) If $\{U_i \to U_i\} \in \mathcal{T}$ and $V \to U$ is a morphism, then $V \times_U U_i$ exist and $\{V \times_U U_i \to U\} \in \mathcal{T}$.

A pair $(\mathcal{C}, \mathcal{T})$ is *a site*.

Note that $(\mathrm{Ouv}(X), \{\text{open covers of } X\})$ recovers the classical topological space. The definition of *a sheaf* can be generalised: $\mathcal{F}$ is a sheaf when for any open cover $\{U_i \to U\}$:

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j) \quad \text{is exact.}$$

So, our consistency conditions are:

1. functoriality: $\mathrm{Plots}(-, \mathrm{X}) : \mathrm{CartSp}^\circ \to \mathrm{Set}$,
2. gluing: let $\{U_i\}$ be an open cover, $p_1 : U_i \to X$ and $p_2 : U_j \to X$ are plots; then they agree on $U_i \cap U_j$.

We can regard *generalised (smooth) spaces* X as *sheaves over Cartesian spaces*, more presicely, *the sheaf of its probes* so in order to introduce the *topos* we need to define morphisms as natural transformations of underlying presheaves. Finally, our *(Grothendieck) topos* is

$$\boxed{\mathrm{SmthSet} := \mathrm{Sh}\big(\mathrm{CartSp}, \mathrm{Set}\big)}$$

which will be our "context" for the further study of (bosonic) classical field theory. Finally, in the best French tradition we call presheaves *pre-smooth sets*:

$$\mathrm{PreSmthSet} := \mathrm{PSh}(\mathrm{CartSp}, \mathrm{Set}) \xhookrightarrow{\text{gluing cdn.}} \mathrm{Sh}(\mathrm{CartSp}, \mathrm{Set}).$$

A posteriori, due to *Yoneda embedding*, we know that

$$\mathrm{SmthMfd} \xhookrightarrow{\text{Plots:=}C^\infty} \mathrm{SmthSet}.$$

2.2. Fields as smooth sets In the context of classical physics, a field assigns some physical quantity (say, scalar: temperature, vector: velocity) to points of some geometric space:

$$\underset{\text{field}}{\Phi} : \underset{\text{space}}{X} \xrightarrow{\textit{smooth}} \underset{\text{coefficient space}}{K}$$

This leads to the usual definition, namely: Fields $:= \Gamma^{\infty}(X, K)$. This works in our topos. Now, we need to introduce action functional, off-shell and on-shell fields (configurations):

$$S : \mathrm{Fields}(X, K) \xrightarrow{\textit{smooth}} \mathbf{R} \qquad \text{(action functional)},$$

such that on its critical locus $\delta S = 0$ we have our *on-shell* configurations. It can be made rigorous in this topos by making a proper *pullback*.

Author	Name	Description
Blohmann, C.	Lagrangian Field Theory	general (diffg) account
Giotopoulos, G.	Field Theory via Higher Geometry I: Smooth Sets of Fields	the perspective of smooth sets

§3. Synthetic (differential) geometry

3.1. An example of reasoning with infinitesimals What follows shows a rudimentary example *why it is natural to work with infinitely small quantities:* 1D heat equation.

Take a rod under reasonable assumptions (homogenity, *infinitesimal* cross section S) to *model* $[0,1]$ (1D object) on it. Introduce:

$$u(x,t) := \text{the temperature at the point } x \text{ at the time } t.$$

Consider *infinitesimally thin* piece $(x, x+dx)$ so that its temperature is *constant* and equals $u(x,-)$. The quantitiy of heat is

$$Q(x,dx;t) = cmu(x,t) = c(\rho V)u(x,t) = c\rho(Sdx)u(x,t).$$

After some small time dt:

$$dQ(x,dx;t,dt) := Q(x,dx;t+dt) - Q(x,dx;t) = c\rho S[u(x,t+dt) - u(x,t)].$$

For some physical reasons (so-called *Fourier's law of thermal conduction*):

$$\partial Q/\partial t = -\kappa \cdot \partial u/\partial x.$$

Thus, in our situation it reads as

$$dQ(x, dx; t, dt)/dt = -\kappa \cdot \left[\partial u/\partial x\big|_x - \partial u/\partial x\big|_{x+dx} \right]$$

Now, equating both expressions:

$$c\rho S[u(x, t + dt) - u(x, t)] = \kappa \cdot \left[\partial u/\partial x\big|_x - \partial u/\partial x\big|_{x+dx} \right] dt$$

Diving by $dxdt$ we obtain

$$\frac{u(x, t + dt) - u(x, t)}{dt} = \frac{\kappa}{c\rho S} \cdot \frac{\partial u/\partial x\big|_{x+dx} - \partial u/\partial x\big|_x}{dx}$$

And by using *the meaning of derivative* (i.e. df/dx when df and dx are treated *infinitesimally*),

$$\boxed{\partial u/\partial t = \text{const} \cdot \partial^2 u/\partial x^2.}$$

We worked with $dx, dt, \ldots$ as with *absolutely legal* variables. It would be great to develop a *rigorous theory* for which such reasonings is just a *syntactic sugar*.

3.2. Formalisations of infinitesimality To formalise the previous arguments, we could apply first-year analysis, and consider *vanishing sequences/functions* instead of *infinitely small numbers*. Similar arguments works in the generic case of curves, tangent vectors, etc.

"model" of a geometry	first-order infsml.	formal (jet-like) infsml.
smooth functions (curves)	derivative (tangent vectors)	Taylor series (jets)
algebras of functions	square-zero extensions	adic completion
Lie geometry	Lie algebra	formal group(?)

As a *Taylor series* contains all the information about derivatives of all orders, *jets* do the same but in the setting of smooth manifolds (and in a coordinate-free way): for *a surjective submersion* $E \xrightarrow{p} X$ we define $J^k P \to X$ to be the bundle which fibers over x are germs of p considered equal when they coincide at first k derivatives. Then *the jet bundle* $J^\infty E$ is the projective limit over them, i.e.

$$J^\infty E \to X := \varprojlim J^n E \quad \text{("intersection" of the finite-order bundles)}$$

Lie algebras are commonly known to be tangent spaces at the unit element of the corresponding Lie group. Moreover, there is a correspondence between them due to the *third Lie theorem* (i.e., the canonical functor LieGrp $\to$ LieAlg is *essentially surjective*).

The second row shows what happens in the algebraic realm under *the function-algebra duality* discussed earlier. We shall exploit similar construction to introduce *infinitesimals* directly (and whence define a topos).

Let R be a (commutative) ring (to be thought as *algebras of functions*). A *square-zero extension* is an extension $p : R' \twoheadrightarrow R$ such that $(\ker p)^2 = 0$. A particular case is given by *the dual numbers* $p : R[\varepsilon]/(\varepsilon^2) \twoheadrightarrow R$. Jets are given by $\widehat{R} = J^\infty R := \varprojlim_n R[\varepsilon]/(\varepsilon)^n$.

> This naturally summons a discussion of *deformation theory.*

Given a morphism of spaces (e.g., schemes) $f : X \to Y$ and *infinitesimal thickenings* of these spaces $\widetilde{X}, \widetilde{Y}$ we are interested in classifying possible maps $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ to make the square commute. Such maps are called *infinitesimal deformations.*

> The obstructions to the existence of such *deformations* is measured by the (hyper)cohomologies (Ext) of spaces with respect to the *cotangent complex*.

The exact construction of cotangent complex is usually given either by *derived* or *higher* category theory. One major application is *p-adic geometry* (e.g., arithmetic (Galois) deformations). Among physical "applications", *Kontsevich-Deligne theorem* (string theory is a deformation of point-particle theory) and *(formal) deformation quantization* (adjoining $\hbar$, '*the Plank constant*') seems to be the most celebrated.

This *deformation quantisation* actually adds a a row to our table of analogies:

"model" of a geometry	first-order infsml.	formal (jet-like) infsml.
symplectic geometry	Poisson manifolds	(formal) deformation quantisation

Generally, good *general* references for *deformation theory* are:

Author	Name
Hartshorne, R.	Deformation theory
Sernesi, E.	Deformation of algebraic schemes
The Stacks Project	Part 6: Deformation theory, Chapters 90-91
Schlessinger, M.	Functors of Artin Rings
Mazur, B.	Deforming Galois representations

The classical articles on *the deformation quantisation* include

Author	Name
Kontsevich, M, Soibelman, Y.	Deformations of algebras over operads and Deligne's conjecture
Kontsevich, M.	Deformation quantization of Poisson manifolds

3.3. Synthetic differential geometry We finished the discussion of *deformation theory* at this point and proceed to developing the *synthetic* language for differential geometry. As we already pointed out, *Milnor's exercise* gives us a *fully faithful functor*

$$\mathrm{SmthMfd} \xhookrightarrow{C^\infty(-,\mathbf{R})} \mathrm{CAlg}_{\mathbf{R}}^{\circ}.$$

We apply the idea which is very close-knit to that of *deformation theory*:

> Define the category of infinitesimally thickened Cartesian spaces as a full subcategory given by the function algebra duality:
>
> $$\begin{array}{ccc} \mathrm{ThCartSp} & \xhookrightarrow{C^\infty(-,\mathbf{R})} & \mathrm{CAlg}_{\mathbf{R}}^{\circ} \\ \mathbf{R}^k \times \mathbf{D}_r^m & \mapsto & C^\infty(\mathbf{R}^k, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{R}[\varepsilon_1, \ldots, \varepsilon_m]/(\varepsilon^{r+1}). \end{array}$$
>
> It has Grothendieck (pre-)topology given by disjoining infinitesimal haloe from the classical part. Thus, one obtains a topos
>
> $$\boxed{\mathrm{FormalSmthSet} := \mathrm{Sh}\big(\mathrm{ThCartSp}, \mathrm{Set}\big)} \qquad \text{(Cahiers topos)}$$

Meanwhile SmthSet provided us with *differential topology*, FormalSmthSet *formally* encodes information about *infsml. neighbourhoods* thus allows to do *differential geometry*.

We expect something like

$$\begin{aligned} \text{SmthMfd} \hookrightarrow \text{HilbertMfd} \hookrightarrow \text{BanachMfd} \hookrightarrow \text{FréchetMfd} \hookrightarrow \\ \hookrightarrow \text{SmthSet} \hookrightarrow \textbf{FormalSmthSet} \end{aligned}$$

Now, for example a tangent bundle $TX \to X$ now becomes canonical $[\mathbf{D}^1_1, X] \to [pt, X]$. It should sound plausible that *classical Lagrangian field theory* takes place in our topos FormalSmthSet via variational calculus on jet bundles.

For some physical reasons, fermionic wave functions have vanishing square and they anticommute, i.e. $\psi_1\psi_2 = -\psi_2\psi_1$. It is natural to introduce the algebra of smooth functions on 'fermionic' $\mathbf{R}^{0|q}$ to be the Grassmann algebra $\Lambda(\mathbf{R}^q)^*$ in order to implement this. Then, we add $\mathbf{Z}/2$-grading, obtaining

$$C^\infty(\mathbf{R}^{n|q}) := C^\infty(\mathbf{R}^n) \otimes_{\mathbf{R}} \Lambda(\mathbf{R}^q)^*, \quad \text{SuperCartSp} \xhookrightarrow{C^\infty(-)} \text{sCAlg}_{\mathbf{R}}{}^\circ.$$

Now we effortlessly define a topos of *super smooth sets*. Clearly, we can obtain *synthetic geometry*, what gives us:

$$\text{SmthSet} \hookrightarrow \text{FormalSmthSet} \hookrightarrow \textbf{SuperFormalSmthSet},$$

more or less the place where we want to do (differential geometry of) physics.

§4. Some references

We did not discuss higher structures in physics, but very general idea is:

> *gauges* are homotopies, and *higher category theory* is (more or less) the study of *homotopies*.

Crucial references are

Author	Name
Schreiber, U.	Differential cohomology in a cohesive ∞-topos.
Schreiber, U, Sati, H.	Mathematical Foundations of Quantum Field and Pertubative String Theory

Good books on category theory:

Author	Name
Riehl, E.	Survey on categorial concepts
Riehl, E.	Category theory in context
Mac Lane, S.	Categories for the working mathematician
Borceux, F.	Handbook of Categorial Algebra I-III
Richter, B.	From categories to homotopy theory

References on ∞-category theory:

Author	Name
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