Map enumeration and ramified coverings

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Contents

§1. Plancherel measure and Vershik-Kerov-Logan-Shepp curve

1.1. A very first and basic formula from *representation theory of finite groups*,

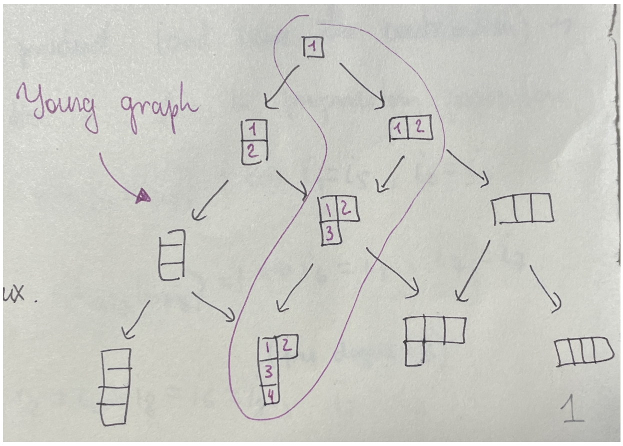
Proposition 1.1. *If G is a finite group, then*

$$\sum_{f \in \hat{G}} (\dim f)^2 = |G|,$$

where $\hat{G}$ is the set of all irreducible representations of G.

induces a natural (probability) measure on $\hat{G}$ by setting $\mu(f) = (\dim f)^2/|G|$. Further we restrict ourselves to the case $G = Sym(\cdot)$ and (very generally) to *the study of the asymptotics of μ as n goes to infinity.* It has its *combinatorial interpretation*:

- irreducible representations of $Sym(n)$ are **Young diagrams** with n cells,
- dimension of a representation f is the number of **standard Young tableaux** with the shape f (in other words, the number of paths in **Young graph** from $\varnothing$ to the vertex f).

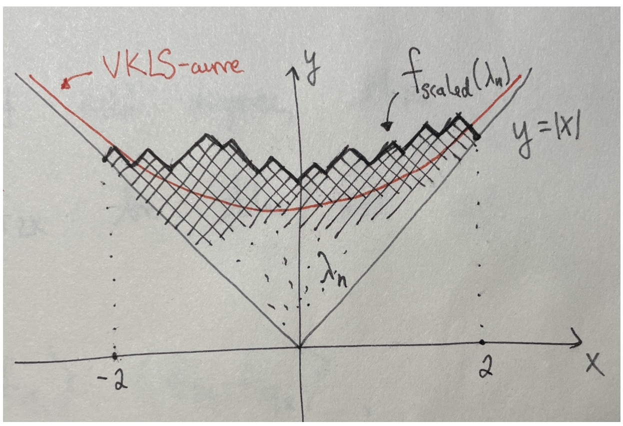


1.2. If we draw an Young diagram (tableau) in the **Russian form** and then scale it properly and consider as a continous piecewise linear function on $[-2, 2]$, it turns out that large random (with respect to Plancherel (probability) measure) Young diagrams tends to be of the form

$$\Omega(x) = \begin{cases} \frac{2}{\pi}(x \arcsin(x/2) + \sqrt{4 - x^2}), & |x| \leq 2 \\ |x|, & |x| \geq 2. \end{cases}$$

This is **Vershik-Kerov-Logan-Shepp curve**, which we shorten to VKLS-curve. Vershik-Kerov and Logan-Shepp independently established the following result

Theorem 1.1. *Under proper technical formalism, we have concentration of (Plancherel) measure theorem near VKLS-curve. In particular, one has that* $\|f_{scaled}(\lambda_n) - \Omega\|_\infty \longrightarrow 0$ *when* $n \to \infty$ *and* λ *is a random (Plancherel) distributed diagram of* n *cells and* f_{scaled} *is the function which describes shape of the diagram under the aforementoined scaling.*



Later, Kerov proved a better result, which gives the next term in asymptotics, roughly speaking

$$f_{scaled}(\lambda_n) = \Omega(x) + \frac{U(x)}{\sqrt{n}} + o(1/n^2),$$

where $U(x)$ is a series, which, unfortunately diverges at the endpoints. Actually, $U(x)$ is a generalised Gaussian random process on the interval $[-2, 2]$.

1.3. The question now is, of course, to study the behaviour of Plancherel measure *near the endpoints*. It turns out that,

> **Conjecture (Baik, Deift, Johansson).** Under right scaling, limit distributions of *row lengths* of a random partition and *eigenvalues* of large random Gaussian Hermitian matrices are essentially identical.

This conjecture was proved by A.Yu.Okounkov [1]. Technical explanations:

- By a random Gaussian Hermitian matrix we mean that real and imaginery parts of entries are *independent identically distributed Gaussian* variables with mean 0 and variance 1/2. In other words, let $H = (h_{ij})_{i,j=1}^n$ be a matrix such that $h_{ij} = \bar{h}_{ji}$ and $h_{ij} = x_{ij} + iy_{ij}$ where x_{ij} and y_{ij} are *independent identially distributed* random *Gaussian* variables, normalised to have mean 0 and variance 1/2.
- By the *right scaling* here we mean the following. First of all, we order *eigenvalues* $E_1 \geq E_2 \geq \ldots$ and *rows lengths* $\lambda_1 \geq \lambda_2 \geq \ldots$. Then, we introduce *scaled variables*:

$$x_i = n^{1/3}\left(\frac{\lambda_i}{2n^{1/2}} - 1\right), \quad y_i = n^{2/3}\left(\frac{E_i}{2n^{1/2}} - 1\right)$$

Then, the conjecture can be formalised in

Theorem 1.2. *In $n \longrightarrow \infty$ limit, the joint distribution of $x_1, \ldots, x_k$ is identical to the joint distribution of $y_1, \ldots, y_k$ for all $k \geq 1$.*

In the paper, Okounkov deduces this theorem from more technical statement about *mixed moments*, namely

Theorem 1.3. *If one considers $x_1, x_2, \ldots$ as a random measure on $\mathbf{R}$ with masses 1 at these points, then let $\hat{x}(\xi) = e^{\xi x_1} + e^{\xi x_2} + \ldots$ be its Laplace transform; similarly define $\hat{y}(\xi)$. Then, in the $n \longrightarrow \infty$ limit, one has*

$$\lim_{n\to\infty} \langle \hat{x}(\xi_1) \ldots \hat{x}(\xi_k)\rangle = \lim_{n\to\infty} \langle \hat{y}(\xi_1) \ldots \hat{y}(\xi_k)\rangle$$

for any $k \geq 1$ and $\xi_1, \ldots, x_k > 0$; here $\langle\cdot\rangle$ denotes expected value.

1.4. Vague proof strategy Okounokov's uses analogies between the following three situations:

$$\begin{array}{c}\text{Combinatorial maps}\\ \text{(graphs embeedings)}\end{array} \xrightarrow{\text{glueings}} \begin{array}{c}\text{Topological}\\ \text{surfaces}\end{array} \xleftarrow{evals} \begin{array}{c}\text{Branched}\\ \text{sphere coverings}\end{array}$$

Overall, we evaluate *asymptotics* of some *branched sphere coverings* and some *combinatorial maps* which find their *interpretations* as right- and left-hand parts in the theorem 1.3. Then, we prove that there is *almost* (which means that in the limit the part which is not bijective becomes neglibile) bijection between them, which proved the theorem.

Technically, the core of the proof is the derivation of two **Wick-like formulas** (along with technical machinery, of course) for *coverings* and *combinatorial maps*:

$$\left\langle \prod_{j=1}^{s} \operatorname{tr} H^{k_i} \right\rangle \sim \sum_{S} n^{\chi(S)-s} |Map_S(k_1, \ldots, k_s)|, \tag{1}$$

$$\operatorname{tr} \prod_{j=1}^{s} \tilde{X}_i^{k_i} \sim \sum_{S} n^{(\chi(S)-s)/2} |Cov_S(k_1, \ldots, k_s)|,$$

where S is a *topological surface*, Map denotes the number of certain combinatorial maps and Cov denotes the number of certain coverings. By a fantastic combinatorial argument, Okounkov obtains

$$|Map_g(k_1, \ldots, k_s)| \sim |Cov_g(k_1, \ldots, k_s)|,$$

where g denotes *genus.* All these asymptotic equalities are with respect to $k_i \to \infty$.

§2. Random Hermitian matrices and Wick's formula

This section is based on a famous expository paper by A. Zvonkin [4].

2.1. Recall that **standard Gaussian measure on a line**, given by

$$d\mu(x) := \frac{1}{2\pi} e^{-x^2/2} dx$$

has a bunch of remarkable properties; those we are interested in can be expressed as momements, i.e.

$$\langle 1 \rangle = 1, \langle x \rangle = 0, \left\langle x^2 \right\rangle = 1, \left\langle e^{itx} \right\rangle = e^{-t^2/2},$$

where, again, $\langle\cdot\rangle$ denotes **expected value**, defined to be

$$\langle f\rangle_\mu := \int_X f(x)d\mu(x),$$

where $f : X \to \mathbf{R}$. Of course, this measure can be generalised to the *multidimensional* case in a standard fashion.

Definition 2.1. *A measure μ on $\mathbf{R}^n$ is called* **Gaussian** *if its characteristic function is of the form*

$$\int_{R^n} e^{i(t,x)}d\mu(x) = \exp\{i(m,t) - \frac{1}{2}(Ct,t)\},$$

where $(\cdot,\cdot)$ is the usual inner product on $\mathbf{R}^n$, m is the **mean vector** *(i.e. $\langle x_i\rangle = m_i$) and C is called* **covariance matrix** *given by $C_{ij} = \langle(x_i - m_i)(x_j - m_j)\rangle$. We shall restrict ourselves only to case when m is 0. In particular, that means that when C is nondegenerate then the measure has the form*

$$d\mu(x) = \underbrace{(2\pi)^{-n/2}(\det C)^{-1/2}}_{\text{normalisation factor}}\exp\{-1/2(C^{-1}x,x)\}dx \tag{2}$$

2.2. Wick's formula We already know that $\langle x_i\rangle_{Gaussian} = 0$ and $\langle x_i x_j\rangle = C_{ij}$, so it is straightforward to compute $\langle$degree 2 polynomial$\rangle$. One can prove that $\langle$odd degree monomial$\rangle = 0$, so we restrict ourselves only to even degree case. We have the following theorem.

Theorem 2.1. *Let $f_1,\ldots,f_{2k}$ be linear functions. Then,*

$$\langle f_1\ldots f_{2k}\rangle = \sum_{p_1a_1p_2q_2\ldots p_kq_k} \langle f_{p_1}f_{q_1}\rangle\ldots\langle f_{p_k}f_{q_k}\rangle,$$

where the sum is taken over permutation such that $p_1 < \cdots < p_k$ and $p_i < q_i$ for all $i = 1,\ldots,k$. We shall call such pairings (p_i,q_i) **Wick's coupling**.

As expected values of the product of two linear functions usually can be written *explicitly*, this theorem gives us a *combinatorial way* to compute certain integrals.

As a very easy example, we compute $\langle x^4\rangle$. We take $f_i = x$ for $i = 1,\ldots,4$ and then apply Wick's formula:

$$\langle f_1f_2f_3f_4\rangle = \underbrace{\langle f_1f_2\rangle\langle f_3f_4\rangle}_{1234} + \underbrace{\langle f_1f_3\rangle\langle f_2f_4\rangle}_{1324} + \underbrace{\langle f_1f_4\rangle\langle f_2f_3\rangle}_{1423}.$$

But, of course, $\langle f_i f_j \rangle = \langle x^2 \rangle = 1$, so one compute the integral

$$\int_{\mathbf{R}} x^4 e^{-x^2/2} dx = 3\sqrt{2\pi}.$$

Generalising the argument one obtains that

$$\int_{\mathbf{R}} x^{2k} e^{-x^2/2} dx = (2k-1)!!\sqrt{2\pi}.$$

2.3. As we started our story about *Hermitian* matrices, we go on. As we already mentioned any Hermitian matrix is described by N^2 real numbers, namely *real diagonal entries* (N numbers), and *complex subdiagonal entries* ($2(N^2-N)$ numbers). We actually have that $H_N \cong \mathbf{R}^{N^2}$, where H_N is the space of Hermitian $(N \times N)$-matrices. Thus, one can introduce *Lesbegue measure.* Then one defines **Gaussian measure** by (2) when dx is that Lesbegue measure. We need matrix C which is induced from the quadratic form

$$\operatorname{tr}(H^2) = \sum_{i=1}^{N} x_{ii}^2 + 2\sum_{i<j}(x_{ij}^2 + y_{ij}^2),$$

where x_{ii}, x_{ij} and y_{ij} are diagonal entries, real and imaginery parts of subdiagonal entries. Then, C is the inverse of the diagonal matrix which contains N diagonal entries of 1 and the rest is 2. One can easy compute the determinant then, and hence, we obtain the expression for our Gaussian measure.

Definition 2.2. *The Gaussian measure on the space of Hermitian matrices is given by*

$$d\mu(H) = (2\pi)^{-N^2/2} 2^{N(N-1)/2} \exp\{-1/2 \operatorname{tr} H^2\} dH.$$

Thus, from now we can do Wick-like stuff on the space of Hermitian matrices. First of all, we note that elements of Hermitian matrices can be considered itself as *linear functions* of coordinates x_{ii}, x_{ij}, y_{ij}. So, as we pointed out before, for the Wick's formula we have only to compute expected values of product of two linear functions. The remarkable thing here is the following obvious proposition.

Proposition 2.1. *One has* $\langle h_{ij} h_{ji} \rangle = 1$ *and* $\langle h_{ij} h_{kl} \rangle = 0$ *when* $(i,j) \neq (k,l)$.

2.4. As in the rudimentary example of $\langle x^2 k \rangle$ we compute $\langle \operatorname{tr} H^{2k} \rangle$. Note that

$$\operatorname{tr} H^{2k} = \sum_{i_1,\dots,i_{2k}=1}^{N} h_{i_1 i_2} h_{i_2 i_3} \dots h_{i_{2k-1} i_{2k}} h_{i_{2k} i_1}.$$

Then, $\left\langle \operatorname{tr} H^{2k} \right\rangle$ is the sum over Wick's couplings of such summands. This is the place, when a very interesting combinatorial construction arises. If we consider an arbitrary Wick's coupling

$$\left\langle h_{p_{1i}p_{1j}} h_{p_{2i}q_{2j}} \right\rangle \ldots \left\langle h_{p_{ki}p_{kj}} h_{p_{ki}q_{kj}} \right\rangle$$

which vanished in all cases but all the factors are nonvanishing. That leads to *chains of equalities*, which, as we shall we later, are actually *glueings of polygons*, i.e. *one-face combinatorial maps.* To make the idea clearer, let us consider an example when $k = 4$.

Take, for instance, a term when $k = 4$

$$h_{i_1 i_2} h_{i_2 i_3} h_{i_3 i_4} h_{i_4 i_5} h_{i_5 i_6} h_{i_6 i_7} h_{i_7 i_8} h_{i_8 i_1}$$

and Wick-couple it,

$$\langle h_{i_1 i_2} h_{i_4 i_5} \rangle \langle h_{i_2 i_3} h_{i_5 i_6} \rangle \langle h_{i_3 i_4} h_{i_8 i_1} \rangle \langle h_{i_6 i_7} h_{i_7 i_8} \rangle$$

and deduce that this term is non-zero if and only if,

$$\begin{array}{lcl} \langle h_{i_1 i_2} h_{i_4 i_5} \rangle = 1 & \Longleftrightarrow & i_1 = i_5 \text{ and } i_2 = i_4 \\ \langle h_{i_2 i_3} h_{i_5 i_6} \rangle = 1 & \Longleftrightarrow & i_2 = i_6 \text{ and } i_3 = i_5 \\ \langle h_{i_3 i_4} h_{i_8 i_1} \rangle = 1 & \Longleftrightarrow & i_3 = i_1 \text{ and } i_4 = i_8 \\ \langle h_{i_6 i_7} h_{i_7 i_8} \rangle = 1 & \Longleftrightarrow & i_6 = i_8 \text{ and } i_7 = i_7 \end{array}$$

which leads to the following promised **chains of equalities**

$$\begin{array}{l} i_1 = i_5 = i_7 = i_1, \\ i_2 = i_4 = i_8 = i_2, \\ i_7 = i_7. \end{array}$$

from what we deduce that number of *free variables* is three. For *computer-science oriented minds*, it is **2-SAT problem**, and the graphs shall work with are very similar in nature with those which arise in 2-SAT.

§3. Map enumeration

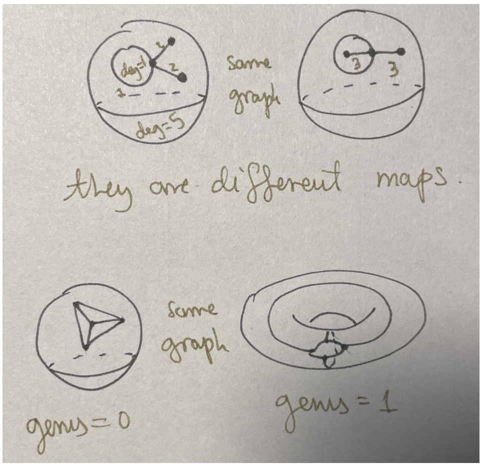
3.1. Before we proceed to the interesting part of the story, we recall (fix) basic definitions:

- Graphs are *allowed* to have loops and multiple edges. We consider only compact oriented 2-dimensional surfaces without boundary (spheres with g handles, where g is the *genus*).
- **Combinatorial map** is a graph Γ embeeded into a surface S such that:

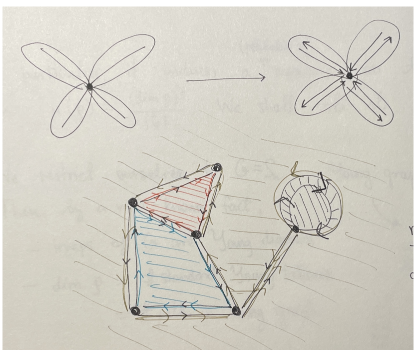
1. the edges do not intersect,
2. $S \setminus \Gamma$ is a *disjoint union of open disks (faces).* In particular, that implies that Γ is *connected.*

- Let **degree of a vertex** v be the number of edges incident to v. The same definition is for **degree of a face** (but note that *edges can be counted twice* unlike in the vertex degree). Also, we have **Euler's formula**: $vertices - edges + faces = 2 - 2g$.

Here are some examples:



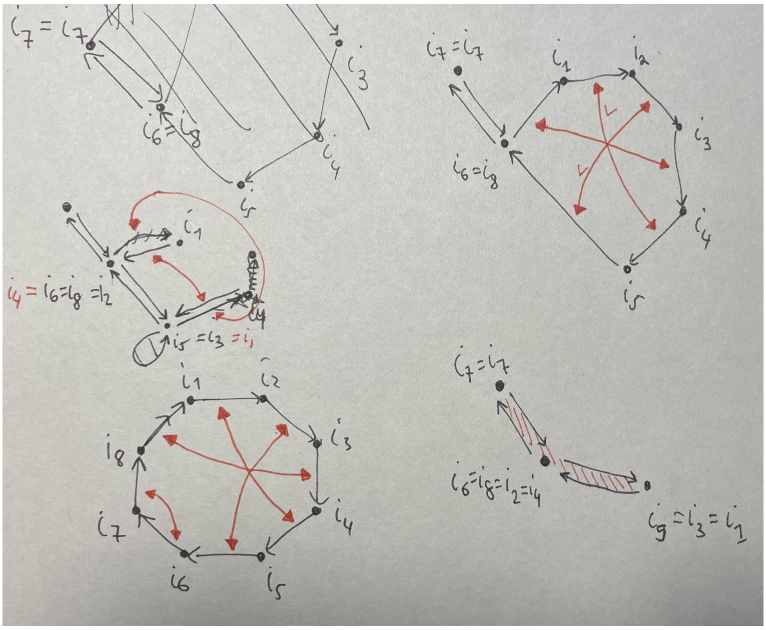
The next step is to **combinatorialise** this description (this is rather *topological* one in flavour). To do so, we *treat all edges as 2-cycles*, and this, in particular, is a nice *formalism* for faces — they are just *cycles* in this interpretation. In particular, this works due to a result of Edmons, which says that *any cyclic order* on edges here induces a *correct* picture. On the other (or maybe the same) hand, such graphs can be generally obtained by glueings.



3.2. Given an $2k$-gon with orientation, we can pair its edges and glue. Then, we obtain so-called **one-face maps**.

Such **glueings** decrease the number of vertices by identifying the vertices. If now we put our indexes $i_1, \ldots, i_{2k}$ into correspondence with vertices of our $2k$-gon, *glueings* will give us chains of equalities, we obtained earlier.

It is clear that any Wick's coupling gives a glueing, and each glueing gives us a Wick's coupling. Thus, they are in a bijection. The number of vertices in such a graph is given by $k+1-2g$, where g is a genus (this is from *Euler characteristic* formula $V - E + F = 2 - 2g$, where F, of course, equals 1) and note that the number of vertices in such a graph is the number of *free variables* in the system of chains of equalities.



This contributes N^V to the sum (in the Wick's formula). By far we have derived the following

Theorem 3.1. *In above notations,*

$$\langle \operatorname{tr} H^{2k} \rangle = \sum_{g=0}^{k/2} N^{k+1-2g} |Map_g(k)|,$$

where $Map_g(k)$ is **the set of one-face maps of genus g with k edges.** *Note that the formula can be rewritten as the* **ordinary generating function of genus** *for the numbers $|Map_g(k)|$, where the formal parameter is $1/N^2$.*

$$\langle \operatorname{tr} H^{2k} \rangle = N^{k+1} \sum_{g=0}^{k/2} |Map_g(k)| (1/N^2)^g.$$

Also, it has the generalised form of (1) which holds for several faces.

§4. Details on the proof strategy

4.1. Disclaimer: throughout the following points, we have a number of *simple exceptional cases* to be handled separetely.

4.2. Understanding the asymptotics of Map First of all, we restrict ourselves to only connected surfaces, hence a genus g, so we can consider Map_g instead of Map_S, where S is (not neccessarily connected) surface. Then Okounkov introduces a function $map_g(\xi_1, \dots, \xi_s)$, s.t.

$$\frac{1}{2^{|k|}} |Map_g(k_1, \dots, k_s)| \overset{k\to\infty}{\sim} map_g(\xi_1, \dots, \xi_s) t^{3g-3+3s/2}, \qquad \xi_i = k_i/t.$$

and obtains, therefore,

$$\frac{1}{2^{|k|} n^{|k|/2}} \left\langle \prod_{j=1}^{s} \operatorname{tr} H^{k_i} \right\rangle \longrightarrow \sum_S map_S(\xi_1, \dots, \xi_s), \qquad \xi_i \sim k_i n^{3/2};$$

one extends *map* to surfaces multiplicatively. Now, the key point is that we want $map_g(\xi_1, \dots, \xi_s)$ to be nice, in the sense we can work with it analytically reasonably well. We can do that via **Laplace transform** (again) and **correlation functions**.

4.3. Understanding map By analytic (probabilistic) considerations, we have

$$\sum_S map_S(\xi_1, \dots, \xi_s) = \sum_{S \subset \{1,\dots,n\}} H(\xi_i)_{i \in S} H(\xi_i)_{i \notin S},$$

where

$$H(\xi_1, \dots, \xi_\ell) = \sum_{\alpha \in Partitions(\{1,\dots,\ell\})} R(\xi_\alpha),$$

(for example,

$$H(\xi_1, \xi_2, \xi_3) = R(\xi_1, \xi_2, \xi_3) + R(\xi_1 + \xi_2, \xi_3) + R(\xi_1 + \xi_3, \xi_2) + \\ + R(\xi_2 + \xi_3, \xi_1) + R(\xi_1 + \xi_2 + \xi_3))$$

and

$$R(\xi_1, \dots, \xi_\ell) = \int_{\mathbf{R}^\ell} e^{(\xi, y)} \rho_{det}(y_1, \dots, y_\ell) dy_1 \dots dy_\ell.$$

is the *Laplace transform*, where $\rho_{det}(y_1, \dots, y_\ell)$ is given by *determinantal form* $\det(K_{Airy}(y_i, y_j))_{i,j=1}^{\ell}$, and the **Airy kernel** is given by

$$K_{Airy}(x, y) = \frac{Ai(2x)Ai'(2y) - Ai'(2x)Ai(2y)}{x - y}.$$

From the celebrated **Tracy-Widom article** [3] (in which we, in a somewhat similar fashion study the **analogies** between **sine kernel** and this **Airy kernel**, it follows that

$$n^{-2s/3}\rho\left(1+\frac{y_1}{n^{2/3}},\ldots,1+\frac{y_s}{n^{2/3}}\right)\xrightarrow{n\to\infty}\rho_{det}(y_1,\ldots,y_s),$$

where ρ denotes the ordinary correlation function of *scaled eigenvalues* $E_i/2n^{1/2}$. All these (especially correlation functions), of course, tighly connected to mixed moments of $\hat{y}$.

4.4. Counting maps Further, Okounkov uses nice combinatorial argument, which, first, shares some similarities with that *Tracy-Widom article* (so it is highly suggested to read that article beforehand) and, second, works as **level set decomposition**, which, in analysis represent formulas like

$$\int_E f(x)d\mu(x)=\int_0^\infty \mu\{x\in E:|f(x)|>t\}dt,$$

or

$$\int_E f(x)d\mu(x)=\int\limits_{0\le|f(x)|<1} fd\mu+\int\limits_{1\le|f(x)|<2} fd\mu+\int\limits_{2\le|f(x)|<4} fd\mu+\ldots$$

which are induced by maps

$$\begin{array}{lcl} E & \xrightarrow{\text{level set decomposition}} & \bigcup_t\left\{x\in E:|f(x)|>t\right\} \\ E & \xrightarrow{\text{dyadic decomposition}} & \bigcup_k\left\{x\in E:2^k\le|f(x)|<2^{k+1}\right\} \end{array}$$

and here we use *level set decomposition* more combinatorial in flavour, in particular,

$$\Phi: Map_g(k_1,\ldots,k_s)\longrightarrow\left\{(\Gamma,\ell)\right\},$$

where Γ is a **ribbon graph** of genus g and s marked vertices (very imprecise here!) and ℓ is a **graph metric** (it is better to think about **lengths of ribbons** in my opinion) on the boundary $\partial\Gamma$ (which is a usual graph). Then, Φ, first of all, *collapses* all the **univalent vertices** in the following way:

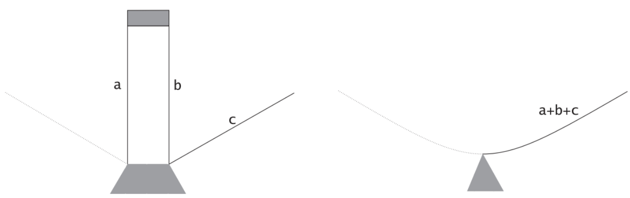


Figure 8: Collapsing of univalent vertices

Then he handle somehow, first, the structure of the graph and, second, desired asymptotics. Note that, we actually do not need such strong and explicit statements he makes. He does it because of the **connection with the moduli spaces**.

4.5. Then, using the machinery of so-called **Jucys-Murphy elements** [2], he develops the theory of **JM coverings**, which definition are more or less clear from his *JM-approach to representation theory of symmetric groups*. In particular, he derives some equation in $S(n)$ which reflects on *the structure of ramifications* on the sphere. Then, it turns out that we can transfer (unfortunately, not faithfully) this special structure on the previous construction and vice versa. This makes those '*almost bijection*' we mentioned in our vague description.

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