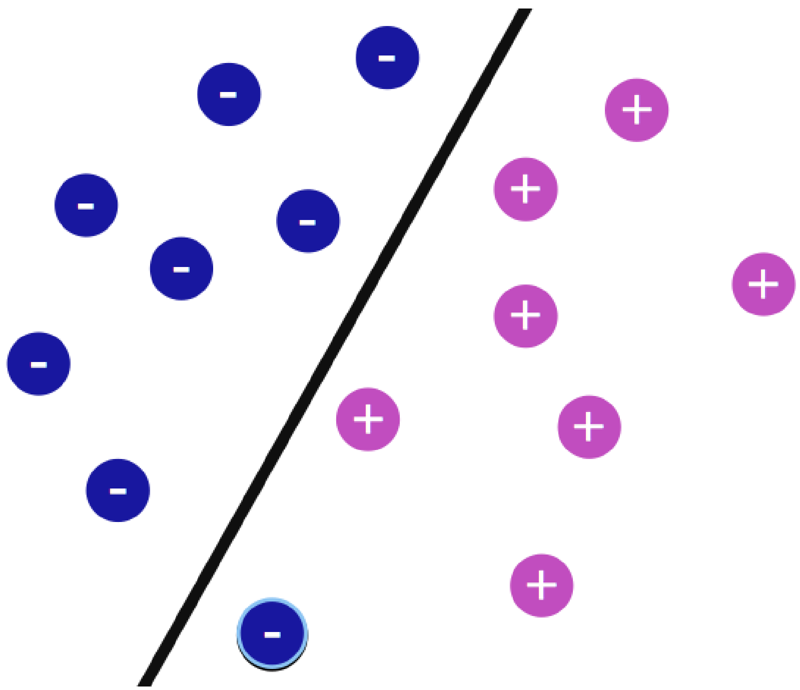
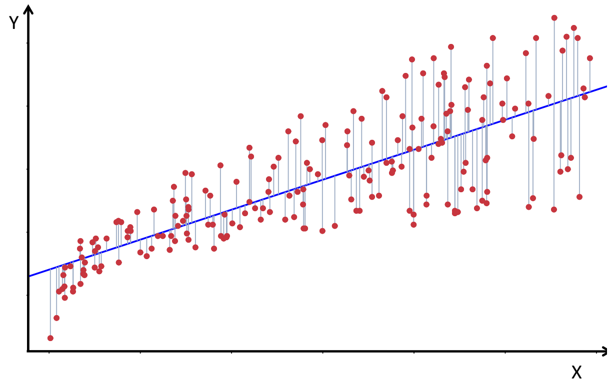
Machine Learning

Linear Models

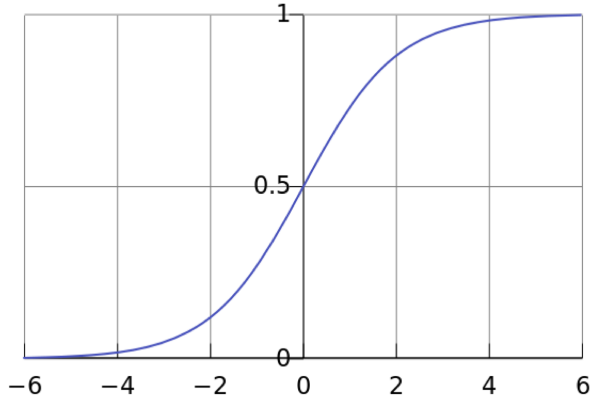


Outline

1. **Linear Regression**



2. **Logistic Regression**



Supervised Learning

Training data: "examples" x with "labels" y.

$$(\vec{x_1}, y_1), \ldots, (\vec{x_n}, y_n) \ / \ \vec{x_i} \in \mathbb{R}^d$$

- **Regression:** y is a real value, $y \in \mathbb{R}$

  $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ $\qquad$ f is called a **regressor**.

- **Classification:** y is discrete. To simplify, $y \in \{-1, +1\}$

  $f : \mathbb{R}^d \longrightarrow \{-1, +1\}$ $\qquad$ f is called a **binary classifier**.

Linear Regression: History

- A very popular technique.
- Rooted in Statistics.
- Method of Least Squares used as early as 1795 by Gauss.
- Re-invented in 1805 by Legendre.
- Frequently applied in **astronomy** to study the large scale of the universe.
- Still a very useful tool today.

Carl Friedrich Gauss

Linear Regression

Given: Training data: $(\vec{x_1}, y_1), \ldots, (\vec{x_n}, y_n)$ / $\vec{x_i} \in \mathbb{R}^d$ and $y_i \in \mathbb{R}$

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example $\vec{x_1} \rightarrow$	x_{11}	x_{12}	$\ldots$	x_{1d}	$y_1 \leftarrow$ label
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
example $\vec{x_i} \rightarrow$	x_{i1}	x_{i2}	$\ldots$	x_{id}	$y_i \leftarrow$ label
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Task: Learn a regression function:

$$f : \mathbb{R}^d \longrightarrow \mathbb{R}$$
$$f(\vec{x}) = y$$

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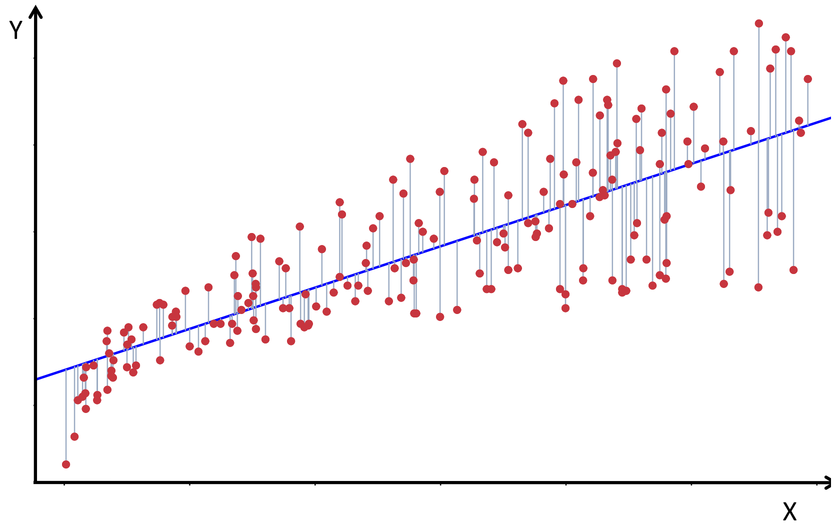
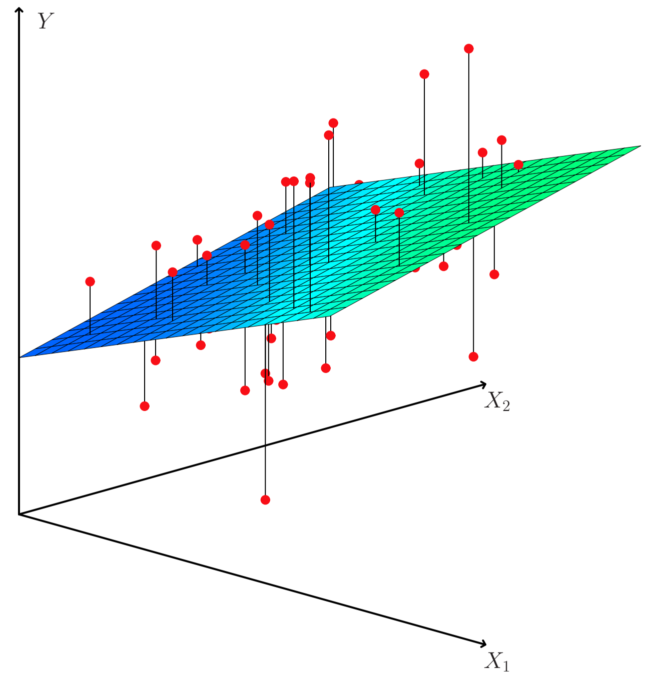
Notations: $\vec{x_i}$ or x_i is the ith example. x_j is the jth feature. x_{ij} is the jth feature for the ith example.

Linear Regression





$d = 1$, line in $\mathbb{R}^2$

$d = 2$, hyperplane is $\mathbb{R}^3$

Credit: Introduction to Statistical Learning.

Linear Regression

Linear Regression Model:

$$f(\vec{x}_i) = \beta_0 + \sum_{j=1}^{d} \beta_j x_{ij} \quad \text{with} \quad \beta_j \in \mathbb{R}, \quad j \in \{1, \ldots, d\}$$

x_j 's are the features.

β 's are called parameters or coefficients or weights.

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Estimation with Least squares:

Use least square loss: $\ell oss(y_i, f(\vec{x_i})) = (y_i - f(\vec{x_i}))^2$

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Estimation with Least squares:

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We want to minimize the loss over all examples, that is minimize the *risk or cost function* R:

$$R = \frac{1}{2n} \sum_{i=1}^{n} (y_i - f(\vec{x_i}))^2$$

Linear Regression

A simple case with one feature ($d = 1$):

$$f(x) = \beta_0 + \beta_1 x$$

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A simple case with one feature ($d = 1$):

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Find β_0 and β_1 that minimize:

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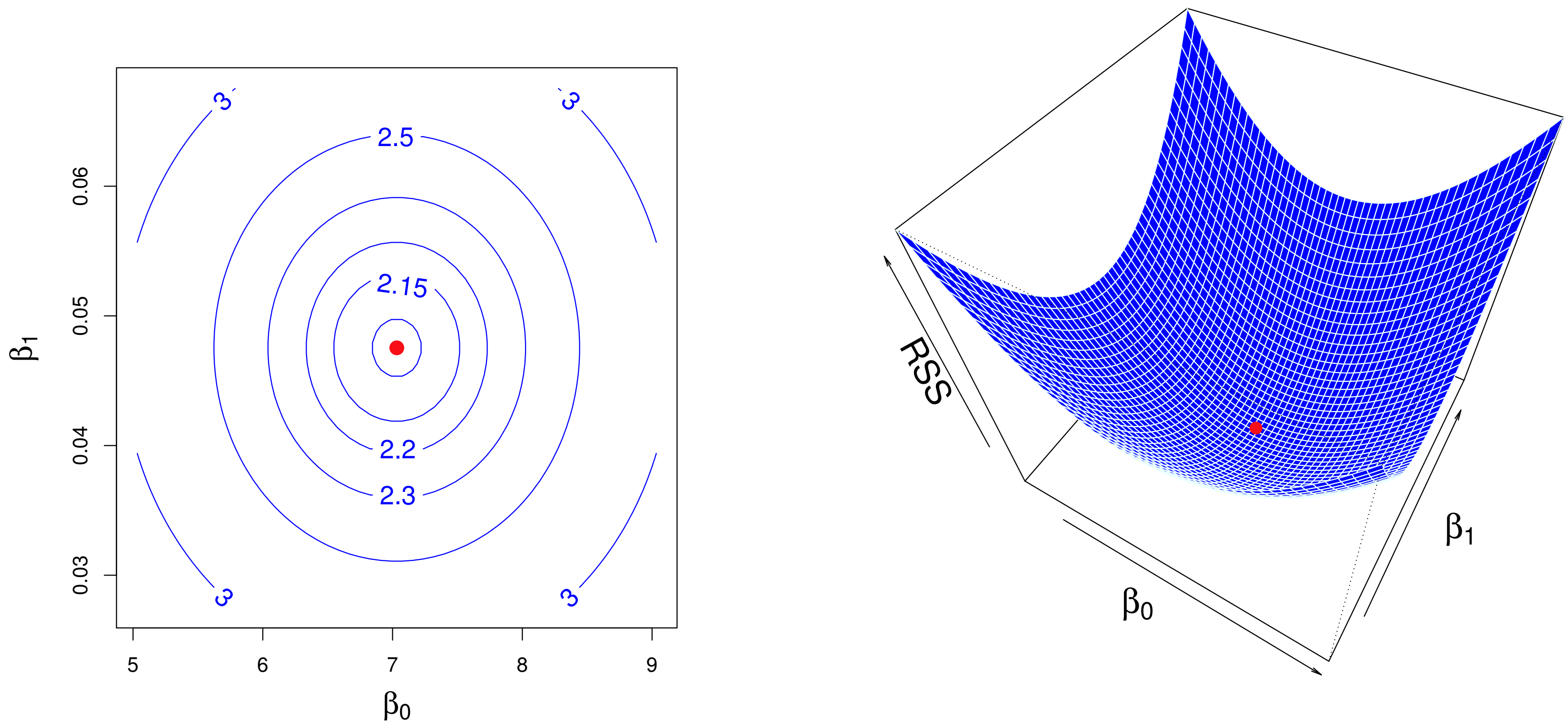

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Linear Regression

Find β_0 and β_1 so that:

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$$\frac{\partial R}{\partial \beta_0} = 2 \times \frac{1}{2n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i) \times \frac{\partial}{\partial \beta_0}(y_i - \beta_0 - \beta_1 x_i)$$

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$$\beta_0 = \frac{1}{n}\sum_{i=1}^{n} y_i - \beta_1 \frac{1}{n}\sum_{i=1}^{n} x_i$$

Linear Regression

$$\frac{\partial R}{\partial \beta_1} = 2 \times \frac{1}{2n} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) \times \frac{\partial}{\partial \beta_1} (y_i - \beta_0 - \beta_1 x_i)$$

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Plugging β_0 in β_1:

$$\beta_1 = \frac{\sum_{i=1}^{n} y_i x_i - \frac{1}{n} \sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2 - \frac{1}{n} \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i}$$

Linear Regression

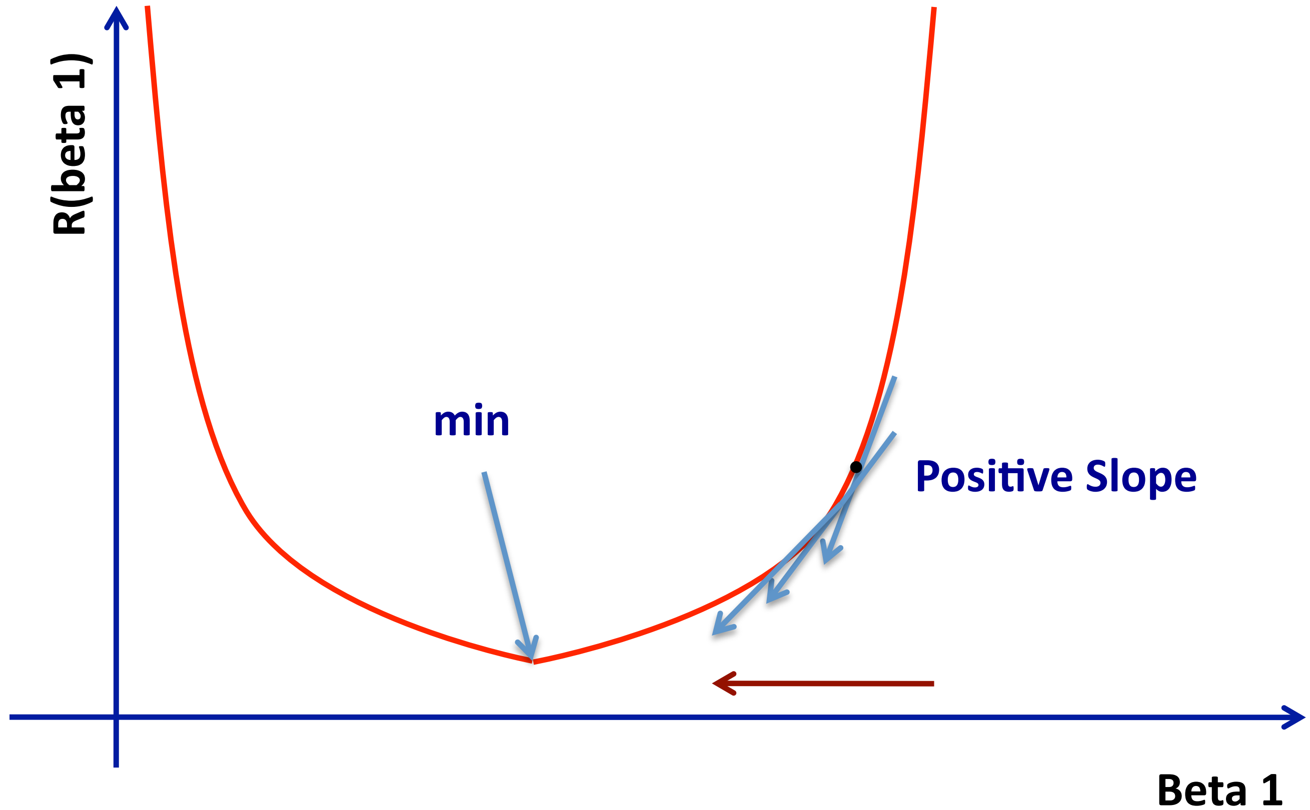
With more than one feature:

$$f(x) = \beta_0 + \sum_{j=1}^{d} \beta_j x_j$$

Find the β_j that minimize:

$$R = \frac{1}{2n} \sum_{i=1}^{n} (y_i - \beta_0 - \sum_{j=1}^{d} \beta_j x_{ij}))^2$$

Gradient descent



Gradient descent

Gradient Descent is an optimization method.

Repeat until convergence:

Update **simultaneously** all β_j for ($j = 0$ and $j = 1$)

$$\beta_0 := \beta_0 - \alpha \frac{\partial}{\partial \beta_0} R(\beta_0, \beta_1)$$

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α is a learning rate.

Gradient descent

In the linear case:

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Let's generalize it!

Gradient descent

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$$\beta_0 := \beta_0 - \alpha\frac{1}{n}\sum_{i=1}^{n}(\beta_0 + \beta_1 x_i - y_i)$$

$$\beta_1 := \beta_1 - \alpha\frac{1}{n}\sum_{i=1}^{n}(\beta_0 + \beta_1 x_i - y_i)(x_i)$$

Practical considerations

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3. **R should decrease** after each iteration.

4. **Declare convergence** if it start decreasing by less ϵ

Credit

- The elements of statistical learning. Data mining, inference, and prediction. 10th Edition 2009. T. Hastie, R. Tibshirani, J. Friedman.
- Machine Learning 1997. Tom Mitchell.