Reinforcement Learning

Introduction

Reinforcement Learning

- Agent interacts and learns from a stochastic environment
- Science of sequential decision making
- Many faces of reinforcement learning
    - Optimal control (Engineering)
    - Dynamic Programming (Operations Research)
    - Reward systems (Neuro-science)
    - Classical/Operant Conditioning (Psychology)

Characteristics of Reinforcement Learning

- No supervisor, only reward *signals*
- Feedback is delayed
- Sequential decisions
- Actions effect observations (non i.i.d.)

Examples

- Automated vehicle control
  - An unmanned helicopter learning to fly and perform stunts
- Game playing
  - Playing backgammon, Atari breakout, Tetris, Tic Tac Toe
- Medical treatment planning
  - Planning a sequence of treatments based on the effect of past treatments
- Chat bots
  - Agent figuring out how to make a conversation

Markov Decision Process

(MDP)

Markov Decision Processes (MDP)

- Sequential decisions in round rounds $t = 1, \ldots, T$
- Important concepts
  - State
  - Action
  - Reward
- Markov property: Future is independent of the past given the current state

Markov Decision Processes (MDP)

- Starts at some initial state s_1
- In every round t, the agent
  - observes the current state s_t,
  - take an action a_t, and then
  - observes a reward signal r_t
  - transitions to the next state s_{t+1}
- Markov Property:
  - $\Pr(s_{t+1} = s' |\text{ history till time t}) = \Pr(s_t = s' | s_t = s, a_t = s) =: P_{s,a}(s')$
  - $E[r_t \,|\text{history till time t}] = E[r_t |\, s_t = s, a_t = a] =: \; R_{s,a}$

Markov Decision Processes (MDP)

- Goal: Maximize some form of cumulative reward
- Total reward in finite time T
  - maximize $\sum_{t=1}^{T} r_t$
- Infinite time average reward
  - maximize $\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_t$
- **Discounted sum of rewards**
  - maximize $r_1 + \gamma\, r_2 + \gamma^2 r_3 + \cdots + \gamma^{i-1} r_i + \cdots$
  - where $\gamma < 1$

Summary

- Markov Decision Process (MDP) is a tuple (S, s_1, A, P, R)
- S is a finite set of states
- A is a finite set of actions
- P is a state transition probability matrix of dimension $S \times A \times S$

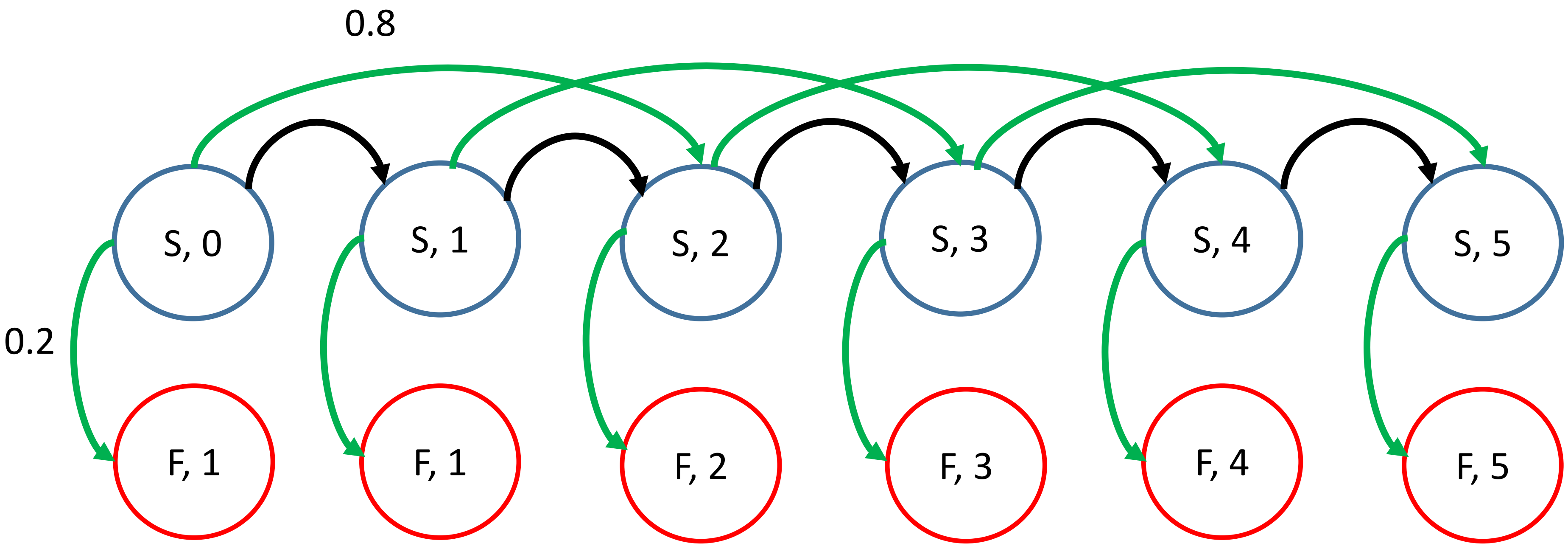
$$P_{s,a}(s') = \Pr(s_{t+1} = s' | \, s_t = s, a_t = a)$$

- R is a reward function

$$R_{s,a} = Ex[r_t | \, s_t = s, a_t = a]$$

- Goal definition, discount factor $\gamma \in [0,1)$

Example



Markov Decision Processes

Finding an optimal policy: Value functions

Overview

- Markov Decision Process is a tuple (S, s_1, A, P, R)
- P is a state transition probability matrix of dimension $S \times A \times S$

$$P_{s,a}(s') = \Pr(s_{t+1} = s' | \, s_t = s, a_t = a)$$

- R is a reward function

$$R_{s,a} = E[R_{t+1} | \, s_t = s, a_t = a]$$

- Goal:

Maximize expected discounted reward $\mathrm{E}[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \mid s_1]$

where $r_t = R_{s_t, a_t}$, $\gamma \in [0,1)$ is a discount factor

Policy

- A policy $\pi: S \to A$ is a mapping from state space to action space
- Following a stationary policy π means taking action $a_t = \pi(s_t)$ at all time steps t
- Theorem

  For any discounted MDP, there always exists stationary policy π that is optimal

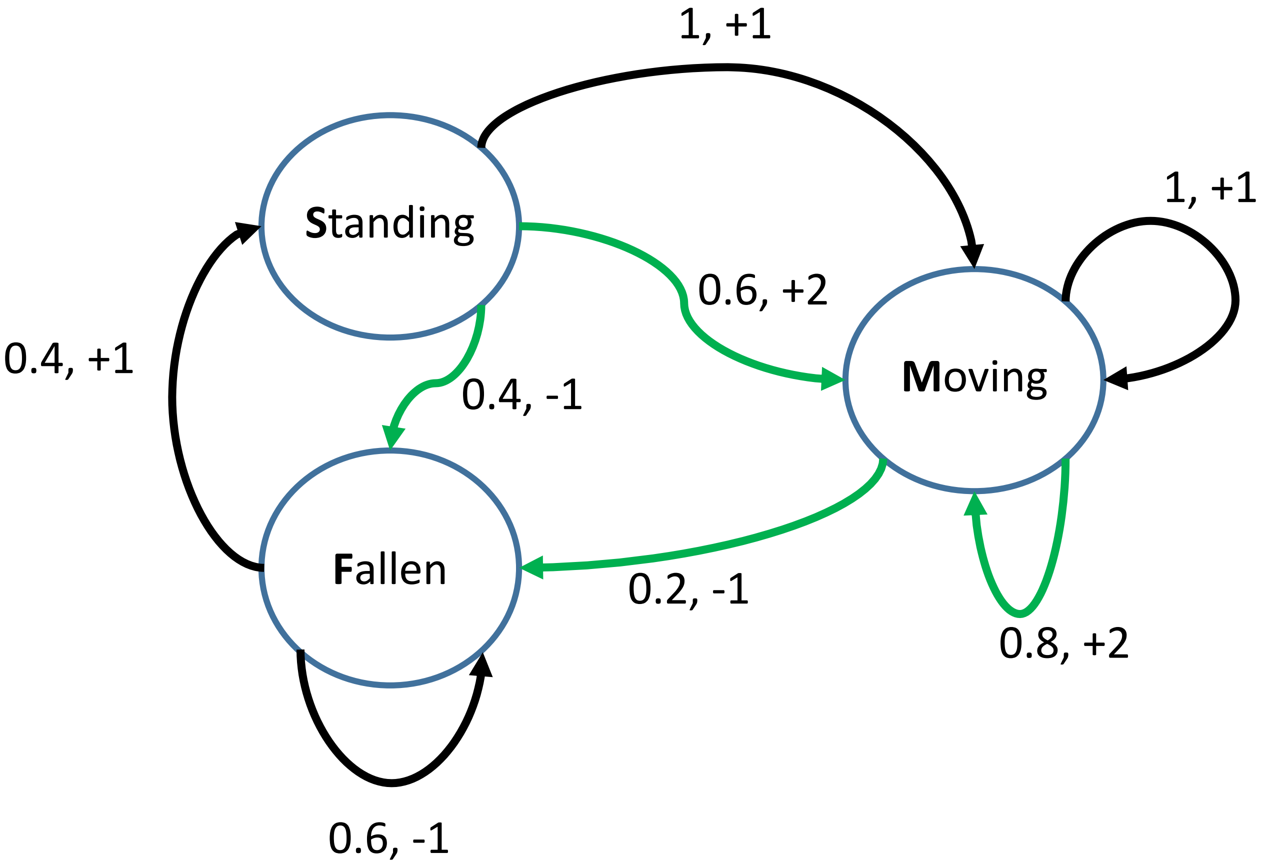
Value function

- Value function $v_\pi(s)$ of a policy π
  - expected reward starting from state s and then following the policy π

$$v_\pi(s) = E[\sum_{t=1}^{\infty} \gamma^{t-1} r_t \mid s_1 = s]$$

where $a_t = \pi(s_t)$, $\mathrm{E}[r_t | s_t, a_t] = R_{s_t,a_t}$, $\Pr(\cdot\, | s_t, a_t) = P_{s_t,a_t}$

Example



Policy: slow action 1 (black) in *Fallen* state, fast action 2 (green) in *Standing* and *Moving* state

Bellman equations

- Value function can be decomposed into immediate reward plus discounted value function of the next state

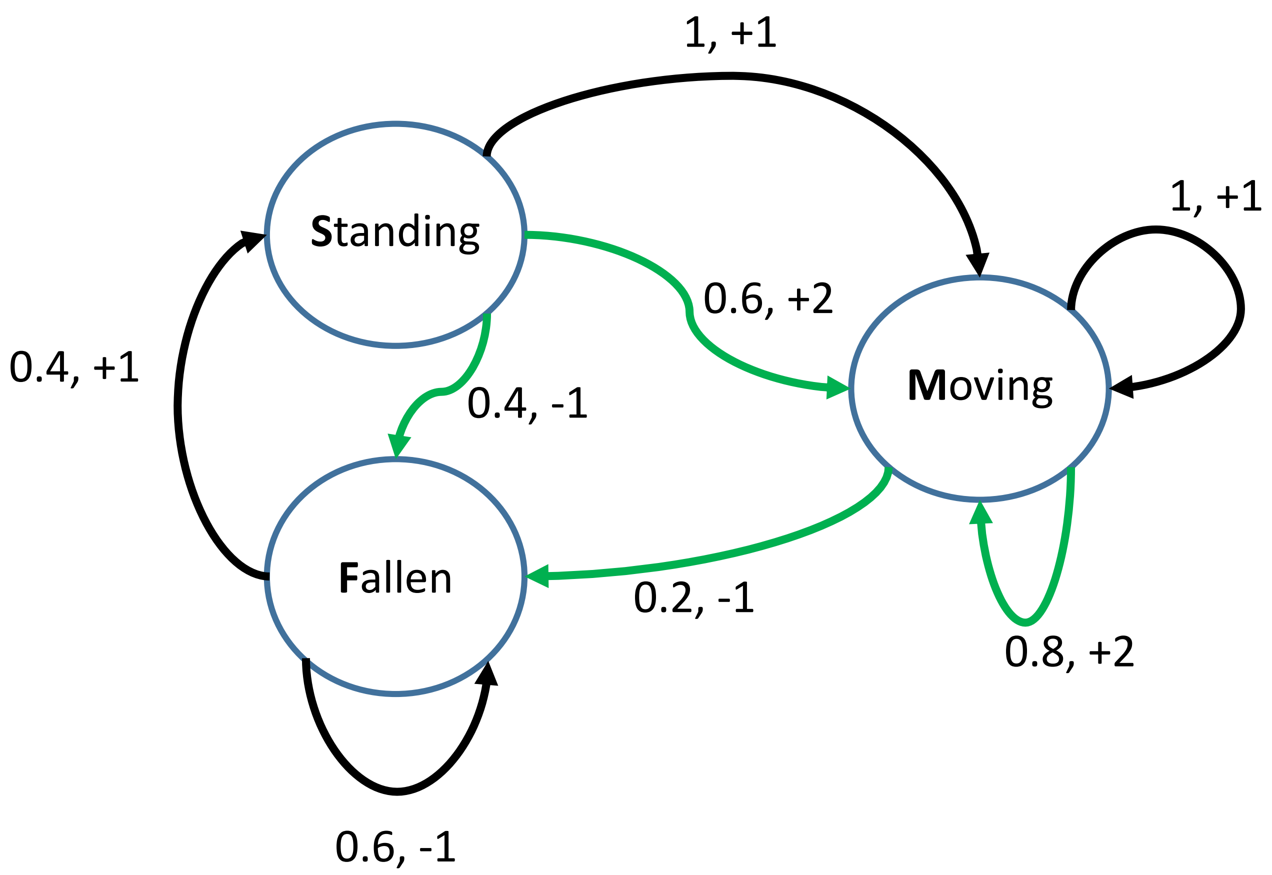
$$v_\pi(s) = \mathrm{R}_{s,\pi(s)} + \gamma \sum_{s\prime} P_{s,\pi(s)}(s') \; v_\pi(s')$$

- Compact matrix notation

$$\boldsymbol{v}_\pi = \boldsymbol{r}_\pi + \gamma P_\pi \boldsymbol{v}_\pi$$

$$\boldsymbol{v}_\pi = (\mathbf{I} - \gamma P_\pi)^{-1} \boldsymbol{r}_\pi$$

Example



Policy: slow action 1 (black) in *Fallen* state, fast action 2 (green) in *Standing* and *Moving* state

Markov Decision Processes

Finding an optimal policy: Iterative methods

Recap

- Value function $v_\pi(s)$ of a policy π

$$v_\pi(s) = E[r_1 + \gamma\, r_2 + \gamma^2\, r_3 + \cdots \,|\, s_1 = s]$$

- Bellman equations

$$\boldsymbol{v}_\pi = \boldsymbol{r}_\pi + \gamma P_\pi \boldsymbol{v}_\pi$$

$$\boldsymbol{v}_\pi = (\mathbf{I} - \gamma P_\pi)^{-1} \boldsymbol{r}_\pi$$

Optimal Policy

- Optimal policy when starting in state s:

$$\underset{\pi}{\operatorname{argmax}} \quad v_\pi(s)$$

Optimal Policy

- Define partial ordering over policies

$$\pi \succcurlyeq \pi' \text{ if } v_\pi(s) \geq v_{\pi'}(s) \text{ for all } s$$

- <u>Theorem</u>
  - There always exists a policy that is better than all other policies

$$\pi \succcurlyeq \pi' \text{ for all } \pi'$$

  Such a policy is called an optimal policy

- All optimal policies achieve the same value function $v_*(s)$ called the optimal value function

Bellman Optimality Equations

- Optimal value functions are recursively related by Bellman optimality equations

$$v_*(s) = \max_{a \in A} \; R_{s,a} + \gamma \sum_{s'} P_{s,a}(s') \; v_*(s')$$

- Matrix notation

$$\boldsymbol{v}_* = \max_{\pi} \; \boldsymbol{r}_\pi + \gamma P_\pi \boldsymbol{v}_*$$

- Optimal policy can be computed by solving Bellman equations

Solving the Bellman optimality equations

- No closed form solution in general
- Iterative solution methods
  - Policy iteration
  - Value Iteration

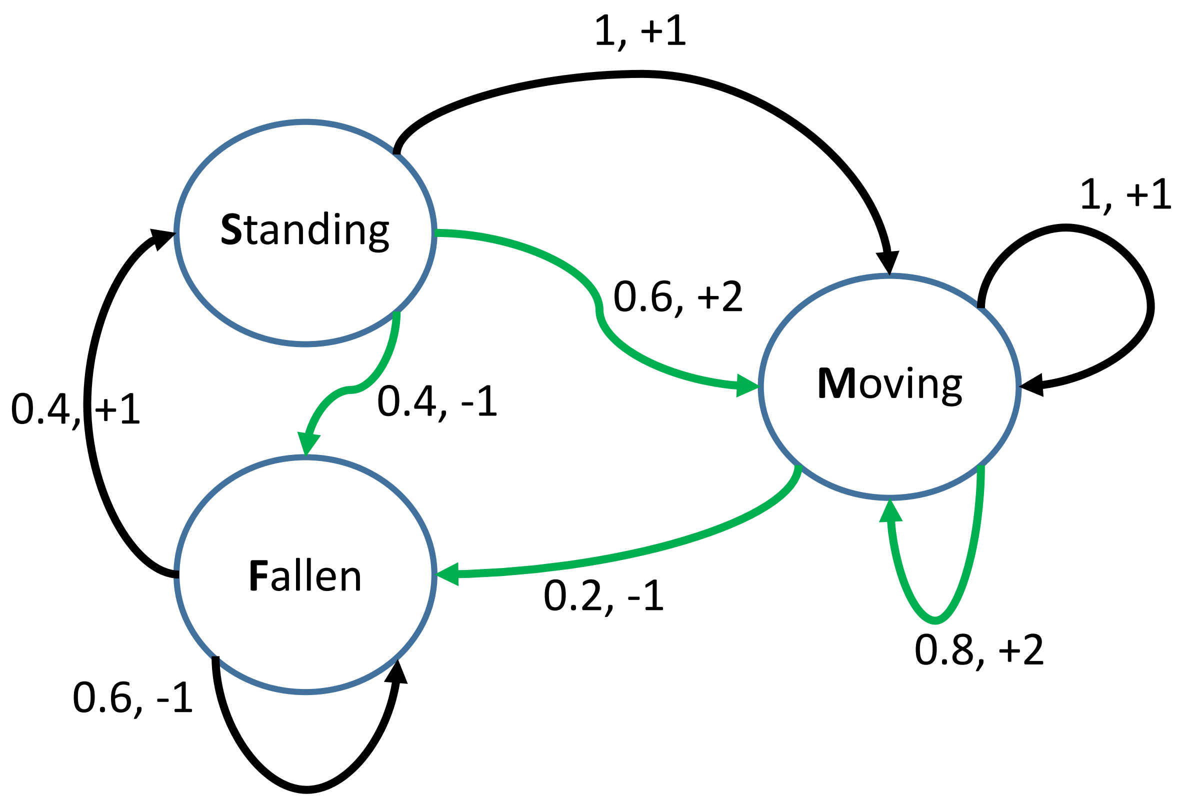
Policy Iteration

- Start with a random policy π

In every iteration,

- Evaluate the policy
  - Compute the value vector for $\boldsymbol{v}_\pi = (\mathrm{I} - \mathrm{P}_\pi)^{-1}\mathbf{r}_\pi$
- Improve the policy
  - New policy: $\pi'(s) = \arg\max_a R_{s,a} + \gamma P_{s,a}\boldsymbol{v}_\pi$
- Stop if no strict improvement ($\boldsymbol{v}_\pi = \boldsymbol{v}_{\pi'}$)

$$v_\pi(s) = \max_a R_{s,a} + \gamma P_{s,a}\boldsymbol{v}_\pi \quad ,\forall s$$



Starting Policy: always slow action

$$r_\pi = \begin{bmatrix} -0.2 \\ 1 \\ 1 \end{bmatrix} \quad P_\pi = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\gamma = 0.1$$

- Iteration 1

$$v_\pi = (I - P_\pi)^{-1} r_\pi = \begin{bmatrix} -0.1655 \\ 1.1111 \\ 1.1111 \end{bmatrix}$$

Improve policy:

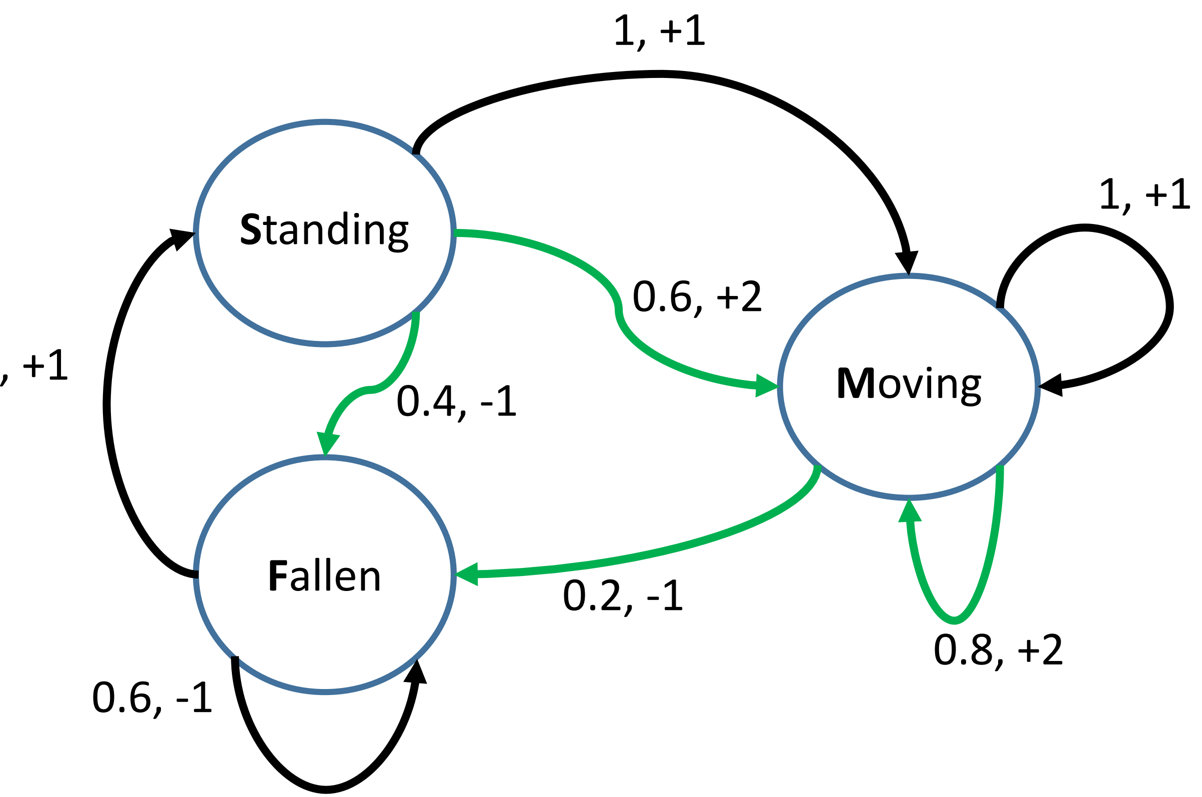
Compute $\arg\max_{a} \; R_{s,a} + \gamma P_{s,a} v_\pi$

State *Standing*,

Slow Action: $= 1.11$

Fast Action: $0.8 +$

$$0.1 \begin{bmatrix} 0.4 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} -0.1655 \\ 1.1111 \\ 1.1111 \end{bmatrix} = 0.86$$



Starting Policy: always slow action

$$r_\pi = \begin{bmatrix} -0.2 \\ 1 \\ 1 \end{bmatrix} \quad P_\pi = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- Iteration 1

$$v_\pi = (I - P_\pi)^{-1} r_\pi = \begin{bmatrix} -0.1655 \\ 1.1111 \\ 1.1111 \end{bmatrix}$$

Improve policy:

Compute $\arg\max_a \; R_{s,a} + \gamma P_{s,a} v_\pi$

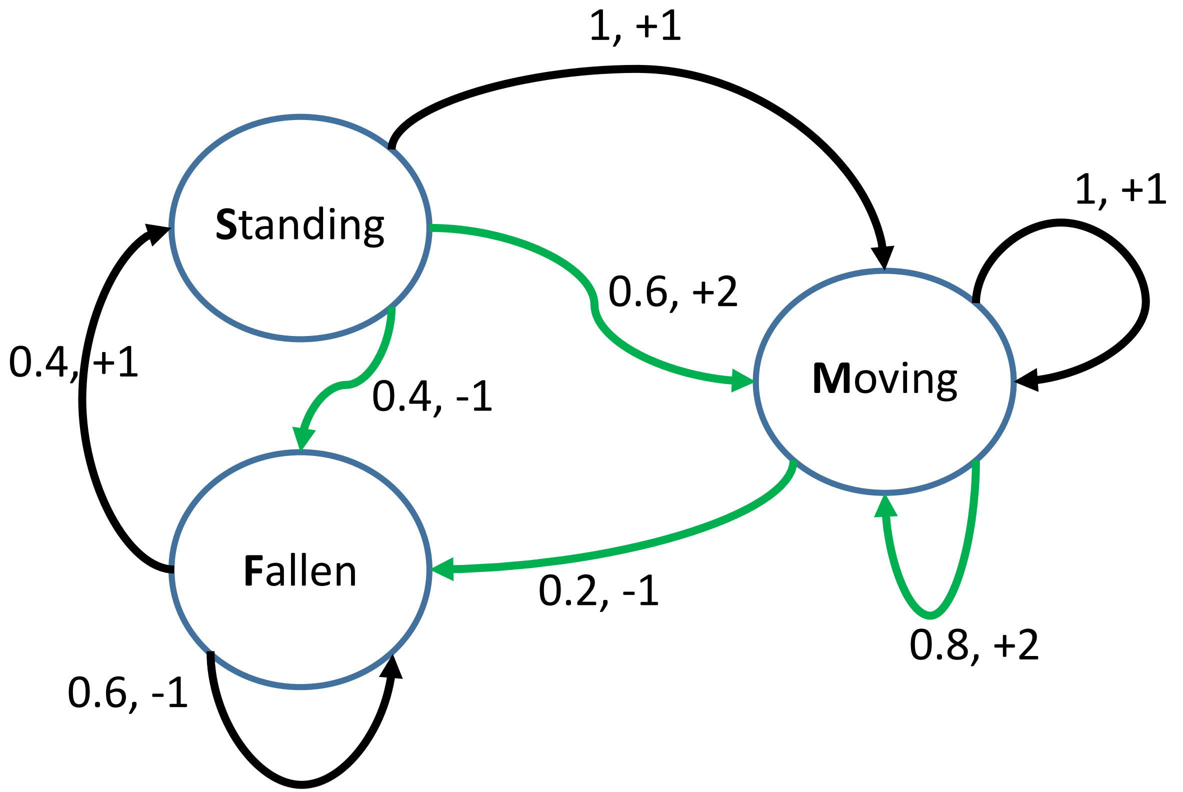
State *Standing*, slow action

State *Moving*

Slow Action: $= 1.1111$

Fast Action: $1.4 \; +$

$$0.1 \begin{bmatrix} 0.2 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} -0.1655 \\ 1.1111 \\ 1.1111 \end{bmatrix} \simeq 1.48$$



Starting Policy: always slow action

$$r_\pi = \begin{bmatrix} -0.2 \\ 1 \\ 1 \end{bmatrix} \quad P_\pi = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- Iteration 1

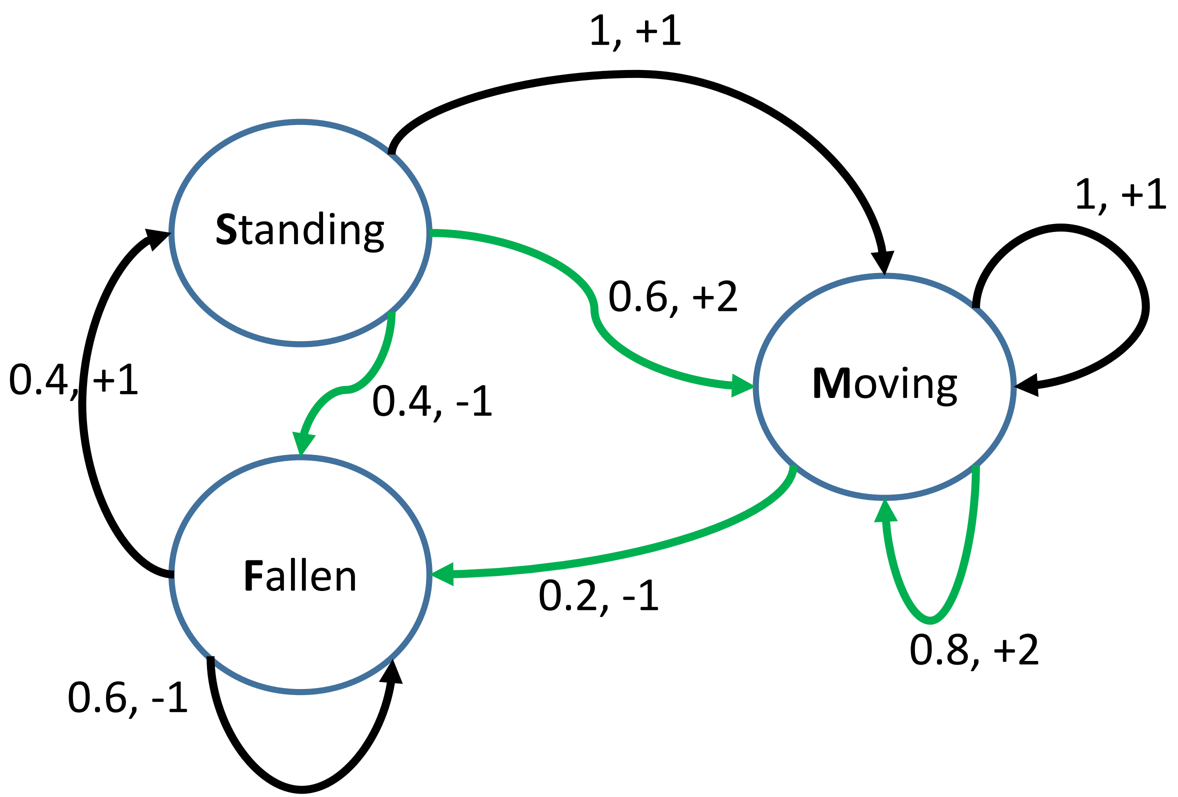
$$v_\pi = (I - P_\pi)^{-1} r_\pi = \begin{bmatrix} -0.1655 \\ 1.1111 \\ 1.1111 \end{bmatrix}$$

Improve policy:

Compute $\arg\max_{a} \; R_{s,a} + \gamma P_{s,a} v_\pi$

State *Standing*, SLOW action

State *Moving*, FAST action



New Policy: fast action in moving state, slow elsewhere

$$r_\pi = \begin{bmatrix} -0.2 \\ 1 \\ 1.4 \end{bmatrix} \quad P_\pi = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0 & 0 & 1 \\ 0.2 & 0 & 0.8 \end{bmatrix}$$

- Iteration 2

$$v_\pi = (I - P_\pi)^{-1} r_\pi = \begin{bmatrix} -0.1638 \\ 1.1518 \\ 1.5182 \end{bmatrix}$$

Improve policy:

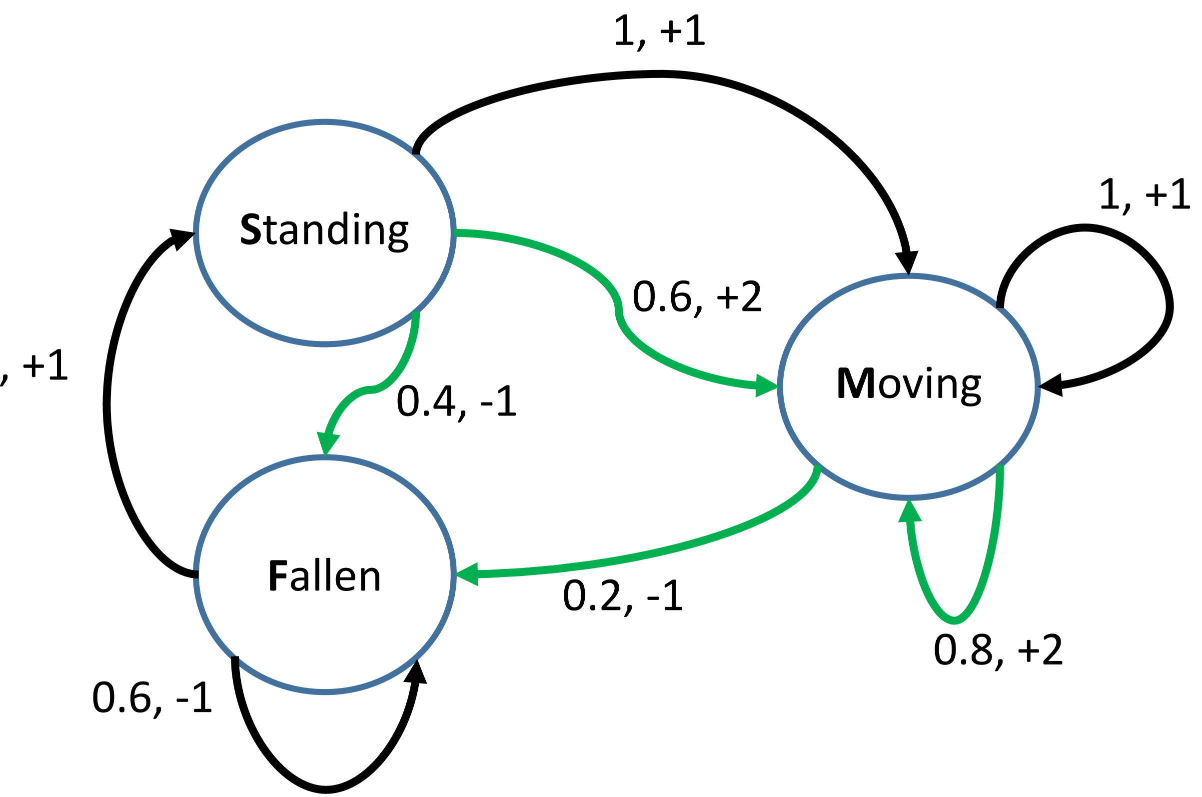
Compute $\arg\max_{a} \; R_{s,a} + \gamma P_{s,a} v_\pi$

State *Standing*,

Slow Action: $= 1.1518$

Fast Action: $0.8 +$

$$0.1 \begin{bmatrix} 0.4 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} -0.1638 \\ 1.1518 \\ 1.5182 \end{bmatrix} \simeq 0.88$$



New Policy: fast action in moving state, slow elsewhere

$$r_\pi = \begin{bmatrix} -0.2 \\ 1 \\ 1.4 \end{bmatrix} \quad P_\pi = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0 & 0 & 1 \\ 0.2 & 0 & 0.8 \end{bmatrix}$$

- Iteration 2 Improve policy:

$$v_\pi = (I - P_\pi)^{-1} r_\pi = \begin{bmatrix} -0.1638 \\ 1.1518 \\ 1.5182 \end{bmatrix}$$

Compute $\arg\max_a \; R_{s,a} + \gamma P_{s,a} v_\pi$

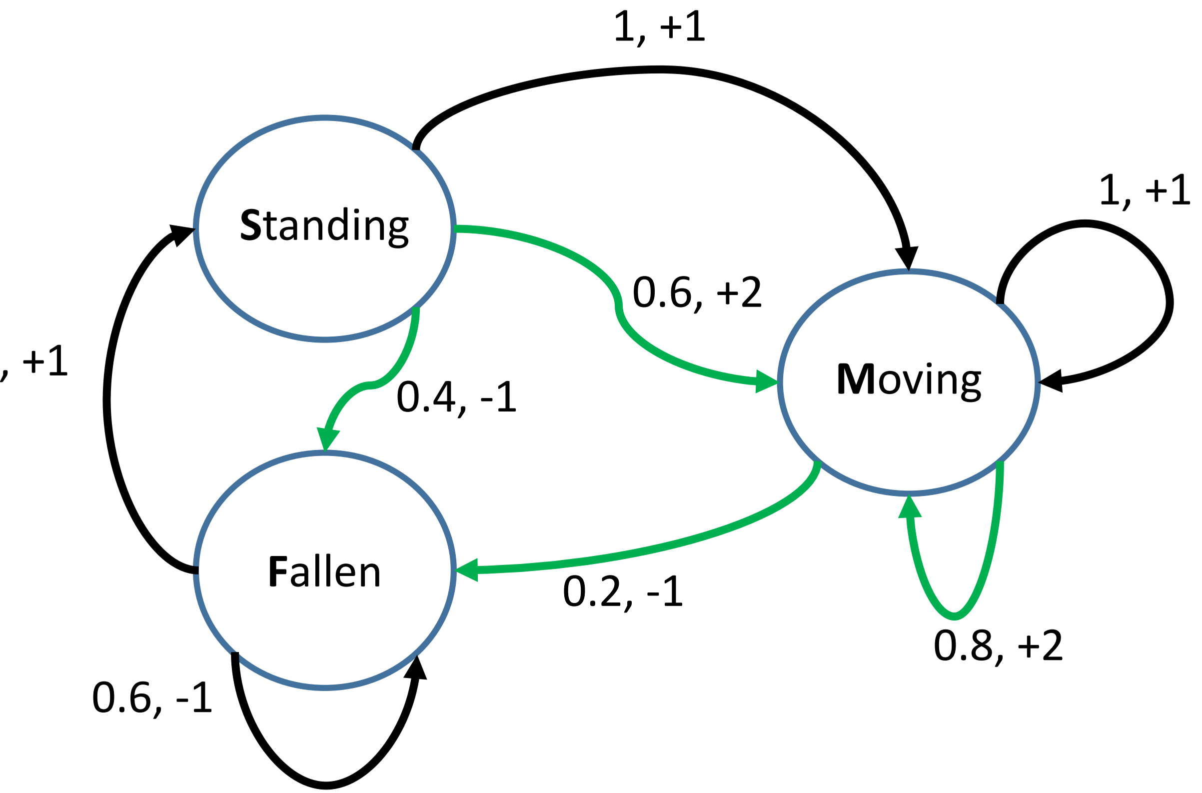
State *Standing*: SLOW action

State *Moving*,

Fast Action: $=1.5182$

Slow Action: $1 +$

$$0.1\,[0 \quad 0 \quad 1] \begin{bmatrix} -0.1638 \\ 1.1518 \\ 1.5182 \end{bmatrix} \simeq 1.1518$$



New Policy: fast action in moving state, slow elsewhere

$$r_\pi = \begin{bmatrix} -0.2 \\ 1 \\ 1.4 \end{bmatrix} \quad P_\pi = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0 & 0 & 1 \\ 0.2 & 0 & 0.8 \end{bmatrix}$$

- Iteration 2 Improve policy:

$$v_\pi = (I - P_\pi)^{-1} r_\pi = \begin{bmatrix} -0.1638 \\ 1.1518 \\ 1.5182 \end{bmatrix}$$

Compute $\arg\max_{a} \; R_{s,a} + \gamma P_{s,a} v_\pi$

State *Standing*: SLOW action

State *Moving*, FAST action

New policy is the same as the old policy

STOP!

Value Iteration method

- Finding optimal value function
  - No explicit policy
- In every iteration k, improve the value vector

$$v^{(k+1)}(s) = \max_a R_{s,a} + \gamma\, P_{s,a}\, \boldsymbol{v}^{(k)}$$

- Converges to $\boldsymbol{v}_*$

$$v^{(k)} \rightarrow v_*$$

- Optimal policy given by $\max_a R_{s,a} + \gamma\, P_{s,a}\, \boldsymbol{v}_*$

Reinforcement Learning

Algorithms

Model free methods

- Reinforcement learning $\equiv$ MDP with unknown transition model and/or reward distribution
- Model is unknown but agent observes samples
- Learn while optimizing the policy

Formulation

- Starts at some initial state s_1

In every round t, the agent

- observes the current state s_t,
- take an action a_t, and then
- observes a reward signal r_t, *and* next state s_{t+1}

$$E[r_t | s_t = s, a_t = a] = R_{s,a}$$

$$\Pr(s_{t+1} = s' | s_t = s, a_t = a) = P_{s,a}(s')$$

$\{R_{s,a}, P_{s,a}\}$ are unknown

Goal

- Find the optimal policy:

  Policy that maximizes expected sum of discounted reward

$\{R_{s,a}, P_{s,a}\}$ are unknown

Q-learning

- Uses "Q-values" instead of value function
- $Q(s, a)$: the value of taking action a in state s
- Formally

$$Q(s,a) = R_{s,a} + \gamma\, E_{s'}[\max_{\mathrm{a}'} \mathrm{Q}(\mathrm{s}', \mathrm{a}')]$$

Immediate expected reward plus the best utility from the next state onwards.

- From Bellman optimality equations, an optimal policy π satisfies

$$Q(s,\pi(s)) = R_{s,\pi(s)} + \gamma\, E_{s'}\big[Q\big(s',\pi(s')\big)\big] = v_*(\mathrm{s})$$

Q-learning

- Proceeds in discrete rounds $t = 1,2, \ldots.$

In every round *t*,

- Choose action greedily using “estimated” Q-values

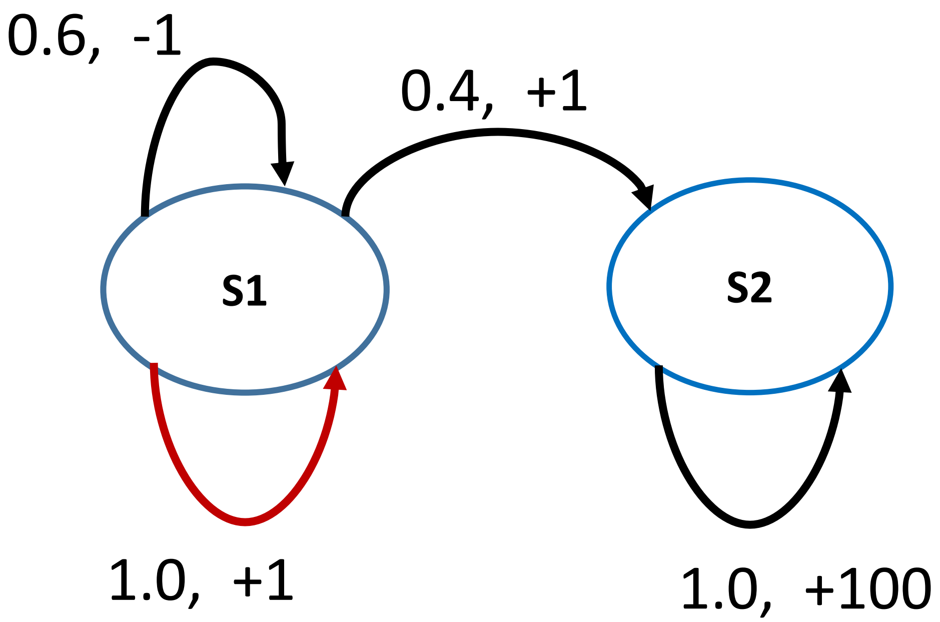
$$a_t = \operatorname*{argmax}_a \hat{Q}(s_t, a)$$

- Take action a_t observe reward r_t, next state s_{t+1}
- Update Q-values for s_t, a_t

$$\hat{Q}(s_t, a_t) = r_t + \gamma \max_a \hat{Q}(s_{t+1}, a)$$

(Compare to $\mathrm{Q(s, a)} = R_{s,a} + \gamma\, E_{s'}[\max_{\mathrm{a'}} \mathrm{Q(s', a')}]$)

The need for Exploration



Epsilon Greedy exploration

- With probability $1 - \epsilon$, use greedy action

$$a_t = \operatorname*{argmax}_a \hat{Q}(s_t, a)$$

- With probability ϵ, play random action