Asymptotic Modes with HOM Coupling

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I HOM Coupling in Waveguides

Let us consider the case in which we have a single isolated waveguide that can support a number N_I spatial modes indexed by the value I. We denote by ξ the position along the waveguide and $f_c(\xi)$ the curvature of the waveguide at the point ξ as shown in Fig. 1. By choosing some reference position ξ_0, we define the value $k_I = n_I(\xi_0;\lambda)\frac{2\pi}{\lambda}$ which, for $n_I(\xi;\lambda)$ the effective index of the mode I at the point ξ, represents the wavenumber of the field with wavelength λ at the reference point ξ_0. With this, for some region near the point ξ', we can denote by $\mathbf{r}_\perp$ the coordinates along the plane tangent to the direction of propagation at ξ and expand the displacement field of the waveguide system as

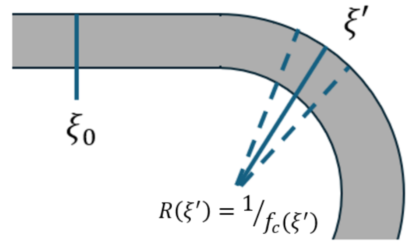

Figure 1: Diagram of a sample waveguide bend.

$$\mathbf{D}(\mathbf{r},t) = \sum_I \int dk_I \sqrt{\frac{\hbar\omega_I(k_I)}{4\pi}} \mathbf{d}_{I,\xi'}(\mathbf{r}_\perp;\lambda) a_{I,\xi'}(k_I,t) e^{i\frac{n_I(\xi';\lambda)}{n_I(\xi_0;\lambda)}k_I\xi}. \tag{1}$$

Here $\mathbf{d}_{I,\xi'}(\mathbf{r}_\perp;\lambda)$ denotes the field distribution along the cross section of the waveguide and the evanescent field extending beyond for a waveguide mode with curvature given by $f_c(\xi')$ and normalization

$$\int d\mathbf{r}_\perp \frac{\mathbf{d}^*_{I,\xi'}(\mathbf{r}_\perp;\lambda)\cdot\mathbf{d}_{I',\xi'}(\mathbf{r}_\perp;\lambda)}{\epsilon_0\varepsilon_1(\mathbf{r}_\perp)} = \delta_{I,I'}, \tag{2}$$

where $\varepsilon_1(\mathbf{r}_\perp)$ is a relative permittivity that we have assumed only varies with $\mathbf{r}_\perp$. The $a_{I,\xi'}(k_I,t)$ is then the corresponding Heisenberg annihilation operator for the mode I with this curvature. Finally, the value $\frac{n_I(\xi';\lambda)}{n_I(\xi_0;\lambda)}k_I$ gives the wavenumber of the field at the expansion point ξ' in terms of the wavenumber at the reference point ξ_0.

We can further decompose the displacement field by introducing a set of disjoint frequency ranges $\{R_I(J)\}_{J\in\mathcal{J}}$ such that each $R_I(J)$ is disjoint with all other $R_I(J')$ for $J\neq J'$, but in which

$$n_I(\xi';\lambda) \cong n_{I,J}(\xi') \equiv n_I(\xi';\lambda_J) \qquad \textit{and} \qquad \mathbf{d}_{I,\xi'}(\mathbf{r}_\perp;\lambda) \cong \mathbf{d}_{I,J,\xi'}(\mathbf{r}_\perp) \equiv \mathbf{d}_{I,\xi'}(\mathbf{r}_\perp;\lambda_J) \tag{3}$$

whenever $\omega_I(\lambda)\in R_I(J)$ with λ_J representing some choice of wavelength with $\omega_{I,J}\equiv\omega_I(\lambda_J)\in R_I(J)$. Of course, the choice of ranges $R_I(J)$ will depend on the expansion point ξ', but if we take the curvature of the to be bounded by some maximum value, with the variation of the bend curvature gradual along the length, then we can make the ranges $R_I(J)$ sufficiently small such that the conditions in equation (3) are satisfied for any expansion point ξ'. With this in mind, we write the displacement field as

$$\begin{aligned}\mathbf{D}(\mathbf{r},t) &\cong \sum_{I,J} \int_{R(J)} dk_I \sqrt{\frac{\hbar\omega_{I,J}}{4\pi}} \mathbf{d}_{I,J,\xi'}(\mathbf{r}_\perp) a_{I,\xi'}(k_I,t) e^{i\frac{n_{I,J}(\xi')}{n_{I,J}(\xi_0)}k_I\xi} \\ &= \sum_{I,J} \sqrt{\frac{\hbar\omega_{I,J}}{2}} \mathbf{d}_{I,J,\xi'}(\mathbf{r}_\perp) \psi_{I,J,\xi'}(\xi,t) e^{i\bar{k}_{I,J}\xi},\end{aligned} \tag{4}$$

where $\bar{k}_{I,J} = n_I(\xi_0;\lambda_J)\frac{2\pi}{\lambda_J}$, the operators $\psi_{I,J,\xi'}(\xi,t)$ are defined as

$$\psi_{I,J,\xi'}(\xi,t) = \frac{1}{\sqrt{2\pi}} \int_{R_I(J)} dk_I a_{I,\xi'}(k_I,t) e^{i\left[\frac{n_{I,J}(\xi')}{n_{I,J}(\xi_0)}k_I - \bar{k}_{I,J}\right]\xi} \tag{5}$$

and we have made the approximation $\omega_I(k_I) \cong \omega_{I,J}$ in the square root factor.

Since the mode expansion above, near the expansion point ξ', is equivalent to a field propagating within a waveguide of constant curvature, we immediately find that

$$\frac{\partial}{\partial t} a_{I,\xi'}(k_I,t) = -i\omega_I(k_I) a_{I,\xi'}(k_I,t). \tag{6}$$

Expanding $\omega_I(k_I)$ about the value $\bar{k}_{I,J}$, we write

$$\omega_I(k_I) = \omega_{I,J} + v_{I,J,\xi'} \frac{n_{I,J}(\xi')}{n_{I,J}(\xi_0)} \left(k_I - \bar{k}_{I,J}\right) + \ldots \tag{7}$$

where $v_{I,J,\xi'}$ is the group velocity of the mode I within the range frequency range J at the expansion point ξ'. Neglecting group velocity dispersion within the range $R_I(J)$, we use equations (5) and (7) to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\psi_{I,J,\xi'}(\xi,t) &= -\frac{i}{\sqrt{2\pi}} \int_{R_I(J)} dk_I \omega_I(k_I) a_{I,\xi'}(k_I,t) e^{i\left[\frac{n_{I,J}(\xi')}{n_{I,J}(\xi_0)}k_I - \bar{k}_{I,J}\right]\xi} \\ &= -\frac{i}{\sqrt{2\pi}} \int_{R_I(J)} dk_I \left[\omega_{I,J} + v_{I,J,\xi'}\frac{n_{I,J}(\xi')}{n_{I,J}(\xi_0)}\left(k_I - \bar{k}_{I,J}\right)\right] a_{I,\xi'}(k_I,t) e^{i\left[\frac{n_{I,J}(\xi')}{n_{I,J}(\xi_0)}k_I - \bar{k}_{I,J}\right]\xi} \\ &= -i\tilde{\omega}_{I,J,\xi'}\psi_{I,J,\xi'}(\xi,t) - v_{I,J,\xi'}\frac{\partial}{\partial\xi}\psi_{I,J,\xi'}(\xi,t).\end{aligned} \tag{8}$$

The effective frequency $\tilde{\omega}_{I,J,\xi'}$ at the point ξ' is defined as

$$\tilde{\omega}_{I,J,\xi'} = \omega_{I,J} + v_{I,J,\xi'}\left(1 - \frac{n_{I,J}(\xi')}{n_{I,J}(\xi_0)}\right)\bar{k}_{I,J}, \tag{9}$$

where the second term manifests as a correction to the propagation phase due to our choice of writing $\bar{k}_{I,J}$ in terms of the effective refractive index at the reference point ξ_0 rather than the the expansion point ξ'.

Forward propagating monochromatic fields, for some $\omega \in R_I(J)$, can be found by using the expansion $\psi_{I,J,\xi'}(\xi,t) = \tilde{\psi}_{I,J,\xi'}(\xi;\omega)e^{-i\omega t}$ in equation (8), giving

$$\frac{\partial}{\partial\xi}\tilde{\psi}_{I,J,\xi'}(\xi;\omega) = i\frac{\omega - \tilde{\omega}_{I,J,\xi'}}{v_{I,J,\xi'}}\tilde{\psi}_{I,J,\xi'}(\xi;\omega) = i\beta_{I,J,\xi}(\omega)\tilde{\psi}_{I,J,\xi'}(\xi;\omega), \tag{10}$$

which has a solution of the form

$$\tilde{\psi}_{I,J,\xi'}(\xi;\omega) = \tilde{\psi}_{I,J,\xi'}(\xi';\omega)e^{i\beta_{I,J,\xi'}(\omega)(\xi-\xi')}. \tag{11}$$

To relate the local linear field description developed above to a a global description of the fields along the entire length of the waveguide, we must relate the operators $\psi_{I,J,\xi'}(\xi,t)$ to the operators $\psi_{I,J,\xi''}(\xi,t)$ where ξ' and ξ'' are two choices of expansion points. To do this, we consider the case when $\xi'' = \xi' + \delta\xi$ where $\delta\xi$ is some small displacement. In the case when the bending curvature varies gradually from ξ' to $\xi' + \delta\xi$ such that the set $\{\mathbf{d}_{I,J,\xi'}(\mathbf{r}_\perp)\}_{I,J}$ and $\{\mathbf{d}_{I,J,\xi'+\delta\xi}(\mathbf{r}_\perp)\}_{I,J}$ both provide a complete mode basis for the field propagating with the waveguide, then we can write

$$\mathbf{d}_{I,J,\xi'}(\mathbf{r}_\perp) = \sum_{I'} \alpha^J_{I,I',\xi'}(\delta\xi)\mathbf{d}_{I',J,\xi'+\delta\xi}(\mathbf{r}_\perp), \tag{12}$$

where $\alpha^J_{I,I',\xi'}(\delta\xi)$ is the mode overlap given by

$$\alpha^J_{I,I',\xi'}(\delta\xi) = \int d\mathbf{r}_\perp \frac{\mathbf{d}^*_{I',J,\xi'+\delta\xi}(\mathbf{r}_\perp)\cdot\mathbf{d}_{I,J,\xi'}(\mathbf{r}_\perp)}{\epsilon_0\varepsilon_1(\mathbf{r}_\perp)} \tag{13}$$

Choosing $\delta\xi$ sufficiently small, we can set equal the local displacement field expansion in equation (4) at the expansion points ξ' and $\xi' + \delta\xi$, and use equation (12) to write

$$\begin{aligned}
\mathbf{D}(\mathbf{r},t) &\propto \sum_{I,J} \mathbf{d}_{I,J,\xi'+\delta\xi}(\mathbf{r}_\perp)\psi_{I,J,\xi'+\delta\xi}(\xi'+\delta\xi,t)e^{i\bar{k}_{I,J}(\xi+\delta\xi)} \\
&= \sum_{I,J} \mathbf{d}_{I,J,\xi'}(\mathbf{r}_\perp)\psi_{I,J,\xi'}(\xi'+\delta\xi,t)e^{i\bar{k}_{I,J}(\xi+\delta\xi)} \\
&= \sum_{I,J}\left[\sum_{I'} \alpha^J_{I,I',\xi'}(\delta\xi)\mathbf{d}_{I',J,\xi'+\delta\xi}(\mathbf{r}_\perp)\right]\psi_{I,J,\xi'}(\xi'+\delta\xi,t)e^{i\bar{k}_{I,J}(\xi+\delta\xi)} \\
&= \sum_{I',J} \mathbf{d}_{I',J,\xi'+\delta\xi}(\mathbf{r}_\perp)\left[\sum_{I} \alpha^J_{I,I',\xi'}(\delta\xi)\psi_{I,J,\xi'}(\xi'+\delta\xi,t)e^{i\bar{k}_{I,J}(\xi+\delta\xi)}\right]
\end{aligned} \tag{14}$$

from which it follows that

$$\psi_{I,J,\xi'+\delta\xi}(\xi'+\delta\xi,t) = \sum_{I'} \alpha^J_{I',I,\xi'}(\delta\xi)\psi_{I',J,\xi'}(\xi'+\delta\xi,t)e^{i(\bar{k}_{I',J}-\bar{k}_{I,J})(\xi'+\delta\xi)}. \tag{15}$$

Next, we expand the operators above in terms of the monochromatic field solutions and introduce the operator $\bar{\psi}_{I,J}(\xi;\omega)$ defined as

$$\bar{\psi}_{I,J}(\xi;\omega) = \tilde{\psi}_{I,J,\xi}(\xi;\omega) \tag{16}$$

which corresponds to the local field expansion operator evaluated at its expansion point. Inserting this in equation (15) and making use of the monochromatic solution of the local field operators in equation (11), we find

$$\bar{\psi}_{I,J}(\xi'+\delta\xi;\omega) = \sum_{I'} \alpha^J_{I',I,\xi'}(\delta\xi)\bar{\psi}_{I',J}(\xi';\omega)e^{i\beta_{I',J,\xi}(\omega)\delta\xi}e^{i(\bar{k}_{I',J}-\bar{k}_{I,J})(\xi'+\delta\xi)} \tag{17}$$

From this, we solve for the equations of motion of $\bar{\psi}_{I,J}(\xi;\omega)$ as

$$\begin{aligned}\frac{\partial}{\partial\xi}\bar{\psi}_{I,J}(\xi;\omega) &= \lim_{\delta\xi\to 0}\frac{\bar{\psi}_{I,J}(\xi+\delta\xi;\omega)-\bar{\psi}_{I,J}(\xi;\omega)}{\delta\xi} \\ &= -i\beta_{I,J,\xi}(\omega)\bar{\psi}_{I,J}(\xi;\omega) - i\sum_{I'}\gamma^{J}_{I,I'}(\xi)\bar{\psi}_{I',J}(\xi;\omega)e^{-i(\bar{k}_{I,J}-\bar{k}_{I',J})\xi}\end{aligned} \tag{18}$$

where the position dependent coupling factor $\gamma^{J}_{I,I'}(\xi)$ is given by

$$\gamma^{J}_{I,I'}(\xi) = i\left[\frac{\partial}{\partial\delta\xi}\alpha^{J}_{I,I',\xi}(\delta\xi)\right]_{\delta\xi=0} \tag{19}$$

In the form given above, $\gamma^{J}_{I,I'}(\xi)$ would be difficult to solve numerically, needing one to compute the mode distributions at each point ξ and the overlaps of those fields with the field distributions close to the chosen ξ. However, we can note that the position dependence in $\alpha^{J}_{I,I',\xi}(\delta\xi)$ comes only from the variation in the mode profiles due to the change in instantaneous curvature. Indeed, we could instead write

$$\alpha^{J}_{I,I',\xi}(\delta\xi) = \tilde{\alpha}^{J}_{I,I'}(f_c(\xi+\delta\xi)-f_c(\xi);f_c(\xi)) \tag{20}$$

with $\tilde{\alpha}^{J}_{I,I'}$ dependent on the difference in curvature at the points ξ and $\xi+\delta\xi$, and the input curvature, $f_c(\xi)$. Consequentially, we can write

$$\gamma^{J}_{I,I'}(\xi) = i\frac{\partial f_c(\xi)}{\partial\xi}\left[\frac{\partial}{\partial\Delta c}\tilde{\alpha}^{J}_{I,I'}(\Delta c;f_c(\xi))\right]_{\Delta c=0} = i\frac{\partial f_c(\xi)}{\partial\xi}\mu^{J}_{I,I'}(f_c(\xi)), \tag{21}$$

which is formed of a product of the spatial derivative of the waveguide curvature, $\partial_\xi f_c(\xi)$, and what will be henceforth referred to as the 'overlap strength', $\mu^{J}_{I,I'}(f_c)$. The latter of these two quantities quantifies how much the field profile overlaps change by a small perturbation in the waveguide curvature and depends on ξ only through the instantaneous curvature at the point ξ. As a result, for a given waveguide height, width, cladding, etc., the overlap strength can be solve without reference to the shape of the waveguide bends or variation in the waveguide curvature with ξ. In the next section we will discuss how this calculation is performed by *AsymptoticModeSolver* code package. The effect of how rapidly the waveguide bend is implemented on the coupling strength is completely determined by the first factor, $\partial_\xi f_c(\xi)$, and is defined by the given waveguide structure.

With the equations of motion of the operators $\bar{\psi}_{I,J}(\xi;\omega)$ and the overlap strength in hand, one can solve for the evolution numerically and write the displacement field at all points within the structure as

$$\mathbf{D}(\mathbf{r},t) = \sum_{I,J}\int_{R_I(J)} d\omega\,\mathbf{d}_{I,J,\xi}(\mathbf{r}_\perp)\bar{\psi}_{I,J}(\xi;\omega)e^{-i\omega t}e^{i\bar{k}_{I,J}\xi} \tag{22}$$

Importantly, this no longer depends on the expansion point ξ' and thus gives a global description of the fields within the waveguide system.

II Some Notes on Computing the Overlap Strength Numerically

This *AsymptoticModeSolver* package includes, in addition to the asymptotic mode solver portion, methods and sample scripts to solve for mode properties, such as the mode overlap strength, through the use of the open source mode package *femwell*. In particular, mode profiles for a variety of waveguide and system parameters are generated through *femwell* and saved to data set of mode properties. These data sets can then be passed to a fitting method, producing fitted function parameters usable to interpolate quantities in the asymptotic mode solver. Some of these values are directly available from *femwell* such as the effective refractive index and radiative loss parameters, but other quantities like the overlap strength require additional methods provided in this package. In this section, we will outline some of the details when computing the overlap strength numerically, and provide a brief discussion of the data set fitting.

To begin, recall equation (21) which defines the overlap strength in terms of a directional derivative of the mode overlap function $\tilde{\alpha}^{J}_{I,I'}(\Delta c; f_c(\xi))$. Through the use of a mode solver such as *femwell*, one can compute the field distributions corresponding to the input and output curvature values for any choice of I and I', generating 2D mesh of points of the overlap function. To extract the directional derivative of the function to obtain the overlap strength, we look to fit a sufficiently populated mesh of overlap points, with which an analytic derivative of the approximate overlap fit can be computed. However, there are a few important details one need to concern themselves with when performing the overlap fit.

First, we can note that the derivation presented in Section I assumed a fixed value of the parameters of the waveguide (waveguide cross-sectional shape, width, height, core and cladding material, etc.), but we could just as easily introduce a set of parameters $\vec{\zeta}$ describing all these contributions, and write the resulting mode distributions as $\mathbf{d}_{I,J,\xi}(\mathbf{r}_\perp; \vec{\zeta})$. Indeed, this will be important to include if one would like to interpolate the values of the overlap strength due to perturbations in the waveguide properties. Furthermore, as long as we are not within the parameter regimes in which modes of interest can be lost due to small modifications of the waveguide parameters or curvature, we should expect that we can construct such a set of modes $\{\mathbf{d}_{I,J,\xi}(\mathbf{r}_\perp; \vec{\zeta})\}_{I,J,\xi}$ which varies gradually with ξ (more specifically with the curvature at ξ) as well as each of the components of $\vec{\zeta}$. Consequentially, we would expect that, for such a choice of the field distributions, that the overlap function $\tilde{\alpha}^{J}_{I,I'}(\Delta c; f_c(\xi), \vec{\zeta})$ would vary gradually in each of its arguments. However, when solving for each mode within the mesh of points individually, one instead receives such a field mode distribution up to some constant phase over the plane given by the $\mathbf{r}_\perp$ argument. For simplicity, we can focus on a single resonance J and drop this subscript to write the field mode obtained from the mode solver as

$$\bar{\mathbf{d}}^{(0)}_{I,c}(\mathbf{r}_\perp; \vec{\zeta}) = \mathbf{d}_{I,c}(\mathbf{r}_\perp; \vec{\zeta}) e^{i\theta_I(c;\vec{\zeta})} \tag{23}$$

where we have also replace the argument ξ with c, denoting the curvature of the bend, to be more explicit about how the field distribution varies with the given parameters. Here $\theta_I(c; \vec{\zeta})$ is an arbitrary phase which may not vary smoothly with in the argument c or $\vec{\zeta}$. Using $\bar{\mathbf{d}}^{(0)}_{I,c}(\mathbf{r}_\perp; \vec{\zeta})$ to define the mode overlap would result in

$$\bar{\alpha}^{(0)}_{I,I'}(\Delta c; c, \vec{\zeta}) = \tilde{\alpha}_{I,I'}(\Delta c; c, \vec{\zeta}) e^{i(\theta_I(c,\vec{\zeta}) - \theta_{I'}(c+\Delta c,\vec{\zeta}))} \tag{24}$$

due to the arbitrary nature of $\theta_I(c, \vec{\zeta})$, each of the points on the $\bar{\alpha}^{(0)}_{I,I'}(\Delta c; c, \vec{\zeta})$ mesh could have phases which vary significantly between neighboring points regardless of the resolution of the mesh. As a result, $\bar{\mathbf{d}}^{(0)}_{I,c}(\mathbf{r}_\perp; \vec{\zeta})$ and the related mode overlap function $\bar{\alpha}^{(0)}_{I,I'}(\Delta c; c, \vec{\zeta})$ would not be a good choice of mode basis to expand the fields in order to extract the overlap strength and propagate the field numerically in the way presented previously.

Note that in the discussion above, the particular choice of $\mathbf{d}_{I,c}(\mathbf{r}_\perp; \vec{\zeta})$ which would lead to a smoothly varying overlap function $\tilde{\alpha}_{I,I'}(\Delta c; c, \vec{\zeta})$ is arbitrary. Indeed, there is an infinite number of related mode distributions $\mathbf{d}_{I,c}(\mathbf{r}_\perp; \vec{\zeta}) e^{i\beta_I(c,\vec{\zeta})}$ which would also lead to a smoothly varying overlap function suitable for fitting, provided the function $\beta_I(c; \vec{\zeta})$ itself varies smoothly with the curvature c and $\vec{\zeta}$. As such, we need only be able to solve for one such choice of mode distribution, given knowledge only of the modes produced from the mode solver, $\bar{\mathbf{d}}^{(0)}_{I,c}(\mathbf{r}_\perp;\vec{\zeta})$. Luckily, this can be done quite easily with the given setup by defining the angle

$$\phi^{(0)}_I(c;\vec{\zeta}) = phase\{\bar{\alpha}^{(0)}_{I,I}(-c;c,\vec{\zeta})\} = \varphi_I(c;\vec{\zeta}) + \left[\theta_I(c;\vec{\zeta}) - \theta_I(0;\vec{\zeta})\right] \tag{25}$$

where $\varphi_I(c;\vec{\zeta})$ corresponds to the phase of $\tilde{\alpha}_{I,I}(c;0,\vec{\zeta})$, which we have assumed is smooth in c and $\vec{\zeta}$. Recalling the definition of $\tilde{\alpha}_{I,I'}(\Delta c;c,\vec{\zeta})$, $\phi^{(0)}_I(c;\vec{\zeta})$ would correspond to the phase of the overlap of an unbent (straight waveguide) mode with that of a mode with a bend curvature of c. From this, we construct a new mode basis as

$$\bar{\mathbf{d}}^{(1)}_{I,c}(\mathbf{r}_\perp;\vec{\zeta}) = \bar{\mathbf{d}}^{(0)}_{I,c}(\mathbf{r}_\perp;\vec{\zeta})e^{-i\phi^{(0)}_I(c;\vec{\zeta})} \tag{26}$$

with a corresponding mode overlap function

$$\begin{aligned}\bar{\alpha}^{(1)}_{I,I'}(\Delta c;c,\vec{\zeta}) &= \bar{\alpha}^{(0)}_{I,I'}(\Delta c;c,\vec{\zeta})e^{-i(\phi^{(0)}_I(c;\vec{\zeta})-\phi^{(0)}_{I'}(c+\Delta c;\vec{\zeta}))}\\ &= \tilde{\alpha}_{I,I'}(\Delta c;c,\vec{\zeta})e^{i(\theta_I(c;\vec{\zeta})-\theta_{I'}(c+\Delta c;\vec{\zeta}))}e^{-i(\phi^{(0)}_I(c;\vec{\zeta})-\phi^{(0)}_{I'}(c+\Delta c;\vec{\zeta}))}\\ &= \tilde{\alpha}_{I,I'}(\Delta c;c,\vec{\zeta})e^{-i(\varphi_I(c;\vec{\zeta})-\varphi_{I'}(c+\Delta c;\vec{\zeta}))}e^{i(\theta_I(0;\vec{\zeta})-\theta_{I'}(0;\vec{\zeta}))}.\end{aligned} \tag{27}$$

This now exhibits a smoothly varying phase in both c and Δc due to $\varphi_I(c;\vec{\zeta})$ being smooth, as well as the function $\tilde{\alpha}_{I,I'}(\Delta c;c,\vec{\zeta})$. However, the overlap functions $\bar{\alpha}^{(1)}_{I,I'}(\Delta c;c,\vec{\zeta})$ and the corresponding overlap strengths still may not be smooth in the components of $\vec{\zeta}$

Before continuing, it is important to note that adding a relative phase to the mode distributions, as done above, can change the magnitude or phase of the overlap strength. Indeed, both $\bar{\alpha}^{(1)}_{I,I'}(\Delta c;c,\vec{\zeta})$ and $\tilde{\alpha}_{I,I'}(\Delta c;c,\vec{\zeta})$ are slowly varying in phase with $|\bar{\alpha}^{(1)}_{I,I'}(\Delta c;c,\vec{\zeta})| = |\tilde{\alpha}_{I,I'}(\Delta c;c,\vec{\zeta})|$, but

$$\begin{aligned}\mu^{(1)}_{I,I'}(c,\vec{\zeta}) &= \left[\frac{\partial}{\partial\Delta c}\bar{\alpha}^{(1)}_{I,I'}(\Delta c;c,\vec{\zeta})\right]_{\Delta c=0}\\ &= \begin{cases}\mu_{I,I'}(c,\vec{\zeta}) + i\frac{\partial}{\partial c}\varphi_{I'}(c,\vec{\zeta}) & \textit{for } I = I'\\ \mu_{I,I'}(c,\vec{\zeta})e^{-i(\varphi_I(c,\vec{\zeta})-\varphi_{I'}(c,\vec{\zeta}))}e^{i(\theta_I(0,\vec{\zeta})-\theta_{I'}(0,\vec{\zeta}))} & \textit{for } I \neq I'\end{cases}\end{aligned} \tag{28}$$

The effect of this on the evolution of the corresponding field operators can be understood as an extra propagation phase factor taking into account the misalignment of the mode distributions at neighboring positions. Indeed, if one were to replace $\mathbf{d}_{I,\xi}(\mathbf{r}_\perp;\vec{\zeta})$ with $\bar{\mathbf{d}}^{(1)}_{I,\xi}(\mathbf{r}_\perp;\vec{\zeta})$ in the mode expansion in equation (22), then the additional phase accumulated on the operator $\bar{\psi}^{(1)}_I(\xi;\omega)$ due to the modification of the overlap strength would exactly cancel the position dependent phase introduced on the distribution $\bar{\mathbf{d}}^{(1)}_{I,\xi}(\mathbf{r}_\perp;\vec{\zeta})$.

In the case when we consider a set of two modes ($I = 1,2$, corresponding to the TE00 and TE01 modes, for example), we can further define the angle

$$\begin{aligned}\eta(\vec{\zeta}) &= phase\{\mu^{(1)}_{1,2}(0,\vec{\zeta})\}\\ &= phase\{\mu_{1,2}(0,\vec{\zeta})\} - (\varphi_1(0,\vec{\zeta}) - \varphi_2(0,\vec{\zeta})) + (\theta_1(0,\vec{\zeta}) - \theta_2(0,\vec{\zeta}))\\ &= \lambda_{1,2}(\vec{\zeta}) - (\varphi_1(0,\vec{\zeta}) - \varphi_2(0,\vec{\zeta})) + (\theta_1(0,\vec{\zeta}) - \theta_2(0,\vec{\zeta}))\end{aligned} \tag{29}$$

where $\lambda_{1,2}(\vec{\zeta})$ is the phase of $\mu_{1,2}(0,\vec{\zeta})$ which, from our assumptions, would vary smoothly with $\vec{\zeta}$. Then, we construct a new mode set as

$$\bar{\mathbf{d}}^{(2)}_{I,\xi}(\mathbf{r}_\perp;\vec{\zeta}) = \begin{cases}\bar{\mathbf{d}}^{(1)}_{I,\xi}(\mathbf{r}_\perp;\vec{\zeta}) & \textit{for } I = 1\\ \bar{\mathbf{d}}^{(1)}_{I,\xi}(\mathbf{r}_\perp;\vec{\zeta})e^{i\eta(\vec{\zeta})} & \textit{for } I = 2\end{cases} \tag{30}$$

with a corresponding overlap strength given by

$$\mu^{(2)}_{I,I'}(c,\vec{\zeta}) = \begin{cases} \mu_{I,I'}(c,\vec{\zeta}) + i\frac{\partial}{\partial c}\varphi_{I'}(c,\vec{\zeta}) & \textit{for } I = I' \\ \mu_{I,I'}(c,\vec{\zeta})e^{-i(\varphi_I(c,\vec{\zeta})-\varphi_{I'}(c,\vec{\zeta}))}e^{i(\varphi_I(0,\vec{\zeta})-\varphi_{I'}(0,\vec{\zeta}))}e^{-i\lambda_{I,I'}(\vec{\zeta})} & \textit{for } I \neq I' \end{cases} \tag{31}$$

which varies smoothly with ξ and the waveguide parameters $\vec{\zeta}$. As such, it is the mode distributions $\bar{\mathbf{d}}^{(2)}_{I,\xi}(\mathbf{r}_\perp;\vec{\zeta})$ and the overlap strengths $\mu^{(2)}_{I,I'}(c,\vec{\zeta})$ that we look to compute. Indeed, by following the procedure above, this can be done only with knowledge of $\bar{\mathbf{d}}^{(0)}_{I,\xi}(\mathbf{r}_\perp;\vec{\zeta})$ and $\mu^{(0)}_{I,I'}(c,\vec{\zeta})$ resulting from the mode fitter and does not rely on knowing the underlying smooth $\mathbf{d}_{I,\xi}(\mathbf{r}_\perp;\vec{\zeta})$ and $\mu_{I,I'}(c,\vec{\zeta})$

With $\bar{\alpha}^{(2)}_{I,I'}(\Delta c;c,\vec{\zeta})$ in hand, we now turn to fitting the overlap function to extract the overlap strength. In practice, this can be done quite well with a 2D polynomial over the input curvature, c, and the difference in curvature, Δc, written as

$$h_{I,I'}(\Delta c,c;\vec{\zeta}) = \sum_{p,q} b^{I,I'}_{p,q}(\vec{\zeta})\,(\Delta c)^p\,c^q \tag{32}$$

However, we can note that from the normalization of the mode distributions, we are guaranteed to have that $\bar{\alpha}^{(1)}_{I,I'}(0;c,\vec{\zeta}) = \delta_{I,I'}$. Consequentially, it follows that

$$b^{I,I'}_{0,0}(\vec{\zeta}) = \delta_{I,I'} \qquad \textit{and} \qquad b^{I,I'}_{0,q>0}(\vec{\zeta}) = 0 \tag{33}$$

Furthermore, from equation (13), it follows that

$$\tilde{\alpha}_{I,I'}(\Delta c;c,\vec{\zeta}) = \left[\tilde{\alpha}_{I',I}(-\Delta c;c,\vec{\zeta})\right]^*. \tag{34}$$

Hence, for $I = I'$ we conclude that

$$b^{I,I}_{p,q}(\vec{\zeta}) = \begin{cases} b^{I,I*}_{p,q}(\vec{\zeta}) & \textit{when } p \textit{ is even} \\ -b^{I,I*}_{p,q}(\vec{\zeta}) & \textit{when } p \textit{ is odd} \end{cases} \tag{35}$$

The functions *ComputeOverlapStrength_selfCoupling()* and *ComputeOverlapStrength_crossCoupling()* perform a least-squares fit of the provided overlap meshes for the cases when $I = I'$ and $I \neq I'$ respectively using the conditions on the coefficients $b^{I,I'}_{p,q}(\vec{\zeta})$ outlined above. Optionally, one can provide a maximum polynomial order such that the sum $p+q$ is bounded by the given value.