Construction of a Class-Uniformly Resolvable Design With Block Sizes 2 and 5

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Introduction

A Class-Uniformly Resolvable Design (CURD) is a method of grouping objects into multiple parallel classes of “blocks” so that there are no overlaps between blocks in the same class and each class contains the same number of blocks of each size.

Consider the problem of repeatedly dividing *n* students in a class into groups each week until each pair of students is together in a group exactly once.

We will examine a particularly interesting case of this problem, where we divide a **class of 25 into one group of 5 (a 5-group) and the rest groups of 2 (2-groups).** We can determine through a combinatorial analysis that this is possible to accomplish in exactly **15 weeks**, with **each student being in a group of 5 exactly once**.

This problem corresponds to the CURD with 25 vertices (students), 15 resolution classes (weeks), and block sizes (group sizes) 5 and 2. We say that the λ-value for this CURD is 1 since overall, each pair of students is together in the same block exactly once.

Choosing Groups of 5 using Diagonal Sets

In order to construct a solution for the weekly group divisions, we will first establish a method for choosing which students are in the 5-group and which are in the 2-groups for each week.

To visualize this, assign each student to a number from 1 to 25 and arrange them into a 5x5 table. To obtain disjoint sets of students for the weekly block divisions, we will first strategically construct diagonal sets from the table and remove the overlaps.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Fig. 1.1 Example of “overlapping elements” between two diagonal sets.

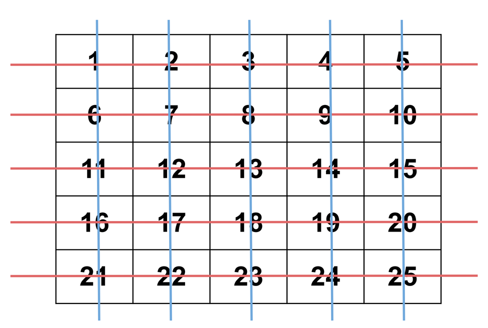


Fig. 1.2 Horizontal (1R 0D) and vertical (0R 1D) diagonal sets.

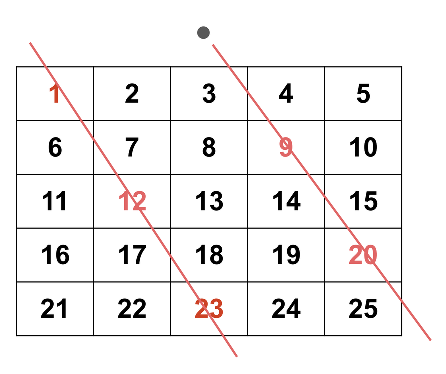


Fig. 1.3 Example of how the 1R 2D diagonal sets are chosen: {1, 12, 23, 9, 20}

We will use this table to extract diagonal sets of varying scale degrees. First, focus on the horizontal and vertical sets (Fig. 1.2).

This set selection ensures that two parallel sets of the same scale degree, eg. two horizontal sets, will be disjoint. But, two sets from different scale degrees will always have exactly one overlapping element (Fig. 1.1).

In each of the weeks below, the elements in red form the 5-group, the elements in blue will be in the 2-groups, and those in purple are the overlaps which cannot be part of 2-groups.

This procedure ensures that no one is in more than one group in any week.

Week 1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	6	11	16	21	2	7	12	17	22	3	8	13	18	23	4	9	14	19	24	5	10	15	20	25

Week 2

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	6	11	16	21	2	7	12	17	22	3	8	13	18	23	4	9	14	19	24	5	10	15	20	25

Week 3

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	6	11	16	21	2	7	12	17	22	3	8	13	18	23	4	9	14	19	24	5	10	15	20	25

Week 4

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	6	11	16	21	2	7	12	17	22	3	8	13	18	23	4	9	14	19	24	5	10	15	20	25

Week 5

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	6	11	16	21	2	7	12	17	22	3	8	13	18	23	4	9	14	19	24	5	10	15	20	25

We will call these first five weeks “Section 1”. Similarly, we can construct the remaining 10 weeks in Section 2 and Section 3 using the same method with different slope degrees to form the diagonal sets (namely 1R 1D, 1R 2D, 1R 3D, 1R 4D).

Note: See Fig. 1.3 for how the diagonal sets are constructed for more complicated diagonals.

Dividing Students Into Pairs Using Graph Theory

Within the sets that will form the 2-groups for each week, we now need to systematically construct pairs of elements so that each pair is together exactly once.

Continuing to look at the first 5 weeks, look at the elements in the first column of Fig. 1.1.

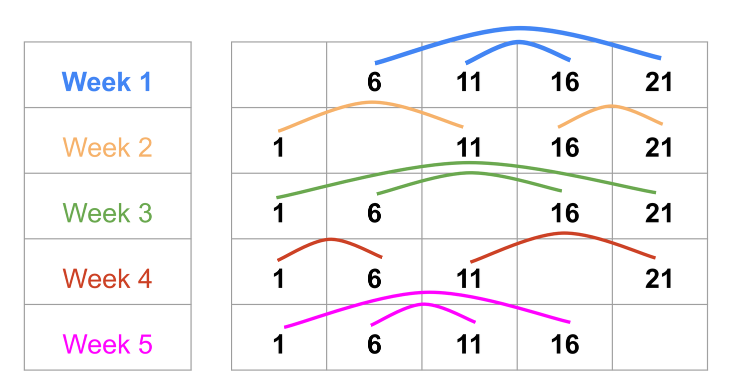
{1} is the overlap with the 5-group in week 1, {6} is the overlap with the 5-group in week 2, …etc.

1	6	11	16	21
1	6	11	16	21
1	6	11	16	21
1	6	11	16	21
1	6	11	16	21

These elements from the set {1, 6, 11, 16, 21} will need to be divided into 2-groups 5 times for each week, with a different element left out each time.

We will represent these pairings using a graph, where the vertices represent students and the edges represent pairings between students. The extra vertex in the center represents the element that is left out of the set that week due to being present in the 5-group.

We use the following pattern to systematically pair the students within this set with no repeated pairings:



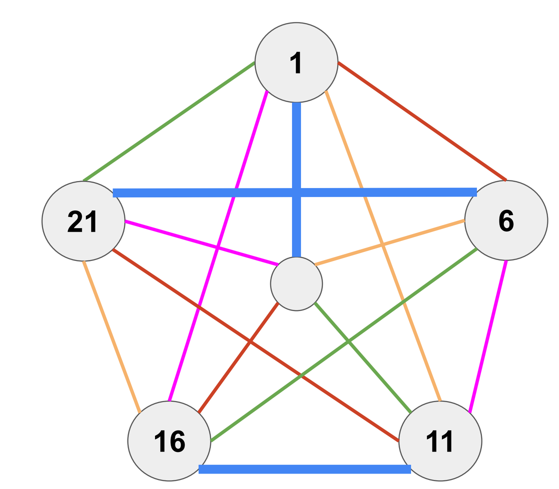
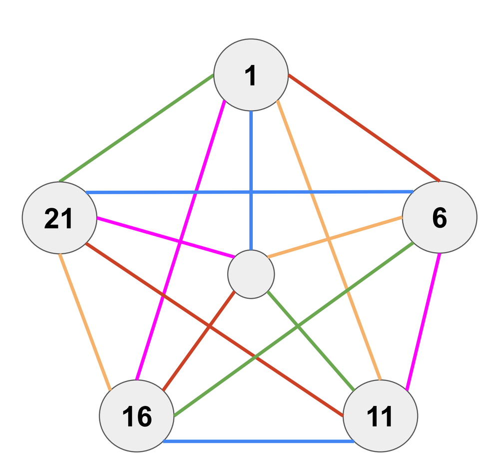


Fig. 2.1 Graph representing the pairings for the set {1, 6, 11, 16, 21}. Week 1 is shown in bold to emphasize the repeated pattern.

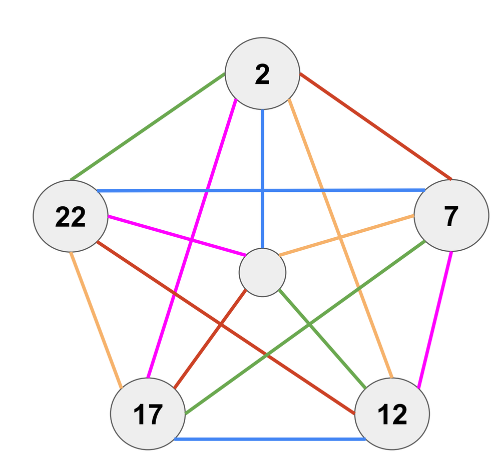
We end up with a complete graph (K_6) in which every pair of vertices is adjacent.

Now, we can apply the same pairing method to each of the other diagonal sets used to get the rest of the 2-groups for these first 5 weeks.

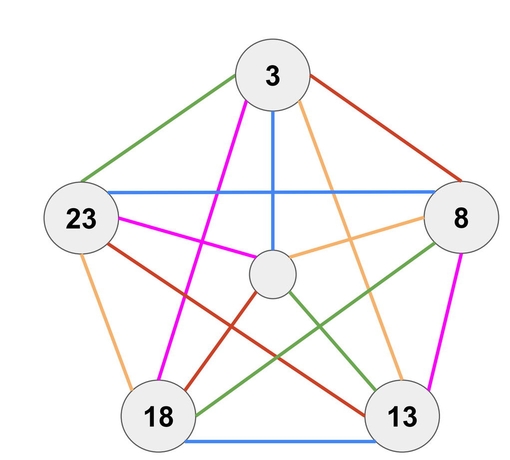
{1, 6, 11, 16, 21}



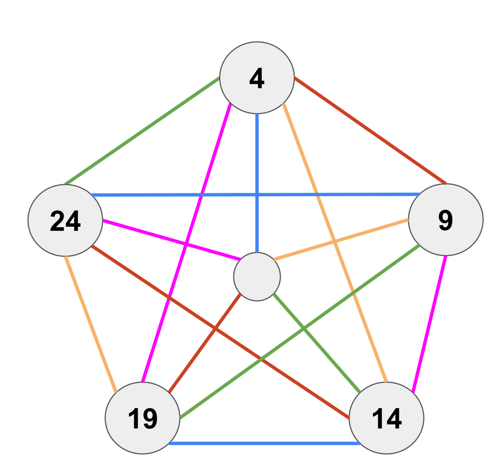
{2, 7, 12, 17, 22}



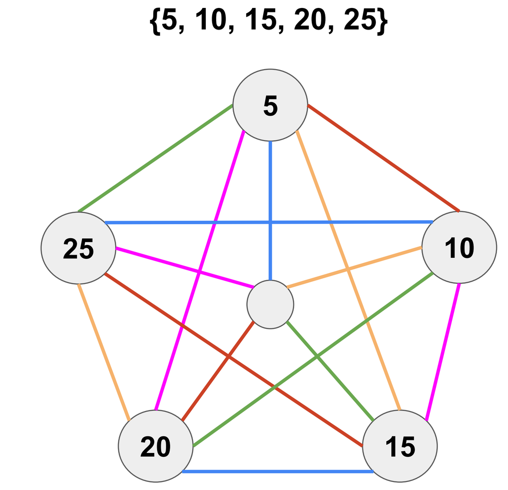
{3, 8, 13, 18, 23}



{4, 9, 14, 19, 24}



{5, 10, 15, 20, 25}



Constructing Week 1 from the Representative Graphs

Looking at the 5 graphs previously shown, each pair of vertices adjacent through a blue edge forms a 2-group in week 1:

6 — 21   11 — 16   7 — 22   12 — 17   8 — 23   13 — 18   9 — 24   14 — 19   10 — 25   15 — 20

Note that each vertex adjacent to the central vertex through a blue edge represents the element from the 5-group {1, 2, 3, 4, 5} which will be left out of the 2-groups in week 1.

To construct week 2 from the graphs, we can look at the orange edges, and so on.

Assembling the Blocks Each Week

We have now systematically assembled all of the 5-groups and 2-groups for each week! We have explicitly demonstrated how to do this for the first 5 weeks, and the rest of the weeks are assembled using the same methods.

The full solution set is shown in the table below, where each row corresponds to a week.

Group sizes:	5					2		2		2		2		2		2		2		2		2		2	
	1	2	3	4	5	6	21	11	16	7	22	12	17	8	23	13	18	9	24	14	19	10	25	15	20
	6	7	8	9	10	1	11	16	21	2	12	17	22	3	13	18	23	4	14	19	24	5	15	20	25
Section 1 (1R 0D, 0R 1D)	11	12	13	14	15	1	21	6	16	2	22	7	17	3	23	8	18	4	24	9	19	5	25	10	20
	16	17	18	19	20	1	6	11	21	2	7	12	22	3	8	13	23	4	9	14	24	5	10	15	25
	21	22	23	24	25	1	16	6	11	2	17	7	12	3	18	8	13	4	19	9	14	5	20	10	15
	1	7	13	19	25	12	20	23	9	2	24	10	16	3	17	14	6	4	15	21	18	5	8	11	22
	2	8	14	20	21	1	9	12	23	13	16	24	10	3	25	6	17	4	18	15	7	5	11	22	19
Section 2 (1R 1D, 1R 2D)	3	9	15	16	22	1	20	12	9	2	10	13	24	14	17	25	6	4	21	7	18	5	19	11	8
	4	10	11	17	23	1	12	23	20	2	13	24	16	3	6	14	25	15	8	21	7	5	22	8	19
	5	6	12	18	24	1	23	9	20	2	16	13	10	3	14	25	17	4	7	15	21	11	19	22	8
	1	17	8	24	15	22	10	18	14	19	6	2	23	3	20	11	7	4	12	25	16	5	9	21	13
	2	18	9	25	11	1	18	14	10	23	6	19	15	3	24	20	7	4	16	12	8	5	13	21	17
Section 3 (1R 3D, 1R 4D)	3	19	10	21	12	1	10	22	14	2	6	23	15	24	7	20	11	4	25	16	8	5	17	13	9
	4	20	6	22	13	1	22	18	10	2	15	23	19	3	7	24	11	25	8	16	12	5	21	17	9
	5	16	7	23	14	1	14	22	18	2	19	15	6	3	11	24	20	4	8	25	12	21	9	17	13

Each week, we have *C(5, 2) = 10* new pairs of students made in the group of 5, and *10* extra new pairs for each of the 2-groups. So in total, we have *20*15 = 300* pairings. For each pair to be together once in the class of 25, we need *C(25, 2) = 300* pairings.

Thus, we have obtained all possible pairings through this method if there are no repeats.

We know that within the 5-groups, there are no repeated pairs since each 5-group contains the elements of a different diagonal set from the 5x5 table. Within the 2-groups, there are no repeated pairs as shown previously through the representative graphs.

Lastly, there is no intersection between pairs made in the 5-groups and pairs made in the 2-groups since these students come from different diagonal sets from the 5x5 table, which cannot have two overlapping elements.

Thus, each pair of students is together exactly once.

Generalization to $n = p^2$ where p is an odd prime

The same method can be used to solve the problem of dividing a class of p^2 students into one group of *p* and the rest groups of 2. This works when *p* is prime because the diagonal sets from a p x p table are disjoint.

This grouping corresponds to the CURD with p^2 vertices, $\frac{p(p-1)}{2}$ resolution classes, 1 block of size p, the rest blocks of size 2, and λ-value of 1.

References

(1) P. Danziger, B. Stevens, *Class-Uniformly Resolvable Designs*, 2000.

(2) P. Danziger, B. Stevens, *Class-Uniformly Resolvable Group Divisible Structures I: Resolvable Group Divisible Designs*, 2004.