1 Intro

$$\partial_i(A^j\mathbf{e}_j) = \partial_i A^j \mathbf{e}_j + A^j \partial_i \mathbf{e}_j \tag{1}$$

$$\partial_i \mathbf{e}_j \equiv \Gamma^k{}_{ij}\mathbf{e}_k \quad \text{Definition} \tag{2}$$

$$\nabla_i(A^j) = \partial_i A^j + A^m \Gamma^j{}_{im} \tag{3}$$

$$\nabla_i A^j = \partial_i A^j + \Gamma^j{}_{im} A^m \tag{4a}$$
$$\nabla_i B_j = \partial_i B_j - \Gamma^m{}_{ij} B_m \tag{4b}$$

$$\begin{aligned}
\nabla_i(A^j B_k) &= \nabla_i A^j B_k + A^j \nabla_i B_k && (5a)\\
&= \partial_i(A^j B_k) + \Gamma^j{}_{im} A^m B_k - \Gamma^m{}_{ik} A^j B_m && (5b)\\
&= \partial_i A^j B_k + A^j \partial_i B_k + \Gamma^j{}_{im} A^m B_k - \Gamma^m{}_{ik} A^j B_m && (5c)\\
& && (5d)
\end{aligned}$$

Replacing A^j by ∇_j leads to

$$\begin{aligned}
\nabla_i(\nabla_j B_k) &= \partial_i \nabla_j B_k + \nabla_j \partial_i B_k + \Gamma^j{}_{im} \nabla_m B_k - \Gamma^m{}_{ik} \nabla_j B_m && (6a)\\
&= \partial_i(\partial_j B_k - \Gamma^m{}_{jk} B_m) + \nabla_j \partial_i B_k + \Gamma^j{}_{im} \nabla_m B_k - \Gamma^m{}_{ik} \nabla_j B_m && (6b)\\
& && (6c)
\end{aligned}$$