

1 Intro

1.1 Isometries of sesquilinear metrics

Consider a vector space V with a basis $B = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_k\}$ and its corresponding metric $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$. After a change of the Basis B to a Basis B' this becomes

$$g'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j = \overline{A^k_i} \mathbf{e}_k \cdot \mathbf{e}_l A^l_j = \overline{A^l_j} g_{kl} A^k_i \quad (1)$$

Note: In case of the field of scalars of B being real or complex, multiplication is commutative. In case of quaternions, $F = \mathbb{H}$, this is not the case.

If $F = \mathbb{R}$ the complex conjugate is a real number, so we get a bilinear metric.

Let's assume that we are looking for transformations A that keep certain properties of V invariant under transformations of its basis. For example if we were to look for transformations that preserves volume we would demand the determinant of the basis matrix being unchanged, that is $\epsilon_{ijk} \mathbf{e}'_i \mathbf{e}'_j \mathbf{e}'_k = \epsilon_{ijk} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k$.

If we are interested in transformations that keeps the metric unchanged, in other words transformations that keeps inner products unchanged, we get

$$g'_{ij} = g_{ij} \quad g_{ij} = \overline{A^l_j} g_{kl} A^k_i \quad (2)$$

Multiplying both sides with the inverse metric tensor g^{jl} gives us

$$\delta_i^j = \overline{A^k_i} A^j_k \quad (3)$$

In matrix notation this is just $A^\dagger A = \mathbb{1}$. Where A^\dagger is the complex conjugate transpose of A .

1.2 Quaternions

To get a grasp of what this means we consider a 2-dimensional complex vector space. Let $a, b, c, d \in \mathbb{C}$ and separate their real and complex part e.g. $a = a_r + ia_i$. Where $a_r, a_i \in \mathbb{R}$. This is the most general transformation:

$$A = \begin{bmatrix} a_r + ia_i & b_r + ib_i \\ c_r + ic_i & d_r + id_i \end{bmatrix} \quad (4)$$

The condition $A^\dagger A = \mathbb{1}$ leads to $A^\dagger = A^{-1}$, so

$$\begin{bmatrix} a_r - ia_i & c_r - ic_i \\ b_r - ib_i & d_r - id_i \end{bmatrix} = \begin{bmatrix} d_r + id_i & -b_r - ib_i \\ -c_r - ic_i & a_r + ia_i \end{bmatrix} \quad (5)$$

Comparing the matrix entries one by one, we get the most general form of a matrix satisfying the above equation. We will however call U the set of matrices A satisfying this condition (U stands for unitary).

$$A = \left\{ A \mid \begin{bmatrix} a_r - ia_i & -b_r - ib_i \\ b_r - ib_i & a_r + ia_i \end{bmatrix} \right\} \quad (6)$$

We can write this as

$$A = a_r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_i \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} + b_r \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + b_i \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \quad (7)$$

$$\sigma_0 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_1 \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 \equiv \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \quad \sigma_3 \equiv \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \quad (8)$$

With the matrices defined this way an element of \mathbf{U} has the form $A = a^k \sigma_k$. The set of matrices satisfying $A^\dagger A = \mathbb{1}$ build a subspace of the 4-dimensional space of general complex 2×2 -matrices. In fact they also are a subgroup. To show this their products have to meet certain conditions that define a group:

$$\text{Closure: } A * B \in \mathbf{U} \quad (9a)$$

$$\text{Identity: } A * \mathbb{1} = \mathbb{1} * A = A \quad (9b)$$

$$\text{Inverse: } A * B = B * A = \mathbb{1} \quad (9c)$$

$$\text{Associativity: } A * (B * C) = (A * B) * C \quad (9d)$$

To show that we first build the products of the base matrices:

$$\begin{aligned} \sigma_0 \cdot \sigma_i &= \sigma_i \cdot \sigma_0 = \sigma_i \\ \sigma_1 \cdot \sigma_2 &= \sigma_3, \quad \sigma_2 \cdot \sigma_3 = \sigma_1, \quad \sigma_3 \cdot \sigma_1 = \sigma_2 \\ \sigma_1 \cdot \sigma_3 &= -\sigma_2, \quad \sigma_2 \cdot \sigma_1 = -\sigma_3, \quad \sigma_3 \cdot \sigma_2 = -\sigma_1 \\ \sigma_1 \cdot \sigma_1 &= \sigma_2 \cdot \sigma_2 = \sigma_3 \cdot \sigma_3 = -\sigma_0 \end{aligned} \quad (10)$$

With this results we can now multiply two arbitrary elements of \mathbf{U} .

$$\begin{aligned} A * B &= (a^0 \sigma_0 + a^1 \sigma_1 + a^2 \sigma_2 + a^3 \sigma_3) * (b^0 \sigma_0 + b^1 \sigma_1 + b^2 \sigma_2 + b^3 \sigma_3) \\ &= (a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3) \sigma_0 \\ &\quad + (a^0 b^1 + a^1 b^0 + a^2 b^3 - a^3 b^2) \sigma_1 \\ &\quad + (a^0 b^2 - a^1 b^3 + a^2 b^0 + a^3 b^1) \sigma_2 \\ &\quad + (a^0 b^3 + a^1 b^2 - a^2 b^1 + a^3 b^0) \sigma_3 \end{aligned} \quad (11)$$

We can write the coefficients a^k as elements of $\mathbb{R} \times \mathbb{R}^3$, so dividing it into a real part and a real 3-vector part $A \in \mathbb{R} \times \mathbb{R}^3$, $A := (a, \mathbf{a})$ we can write the product of A and B like this:

$$A * B = (a, \mathbf{a}) * (b, \mathbf{b}) = (ab - \mathbf{a}\mathbf{b}, a\mathbf{b} + b\mathbf{a} + \mathbf{a} \times \mathbf{b}) \quad (12)$$

The scalar parts basis is simple the identity matrix σ_0 , whereas the vector parts basis is $\{\sigma_1, \sigma_2, \sigma_3\}$.

Obviuosly $A * B$ is an element of \mathbf{U} . The identity element is $U_{id} = (1, 0)$. To find an inverse element we need the vector part to be zero and can then divide through the scalar part to make it 1.

$$ab - \mathbf{a}\mathbf{b} = 1 \quad \text{and} \quad (13a)$$

$$a\mathbf{b} + b\mathbf{a} + \mathbf{a} \times \mathbf{b} = 0 \quad (13b)$$

This can only be the case if $\mathbf{b} = \lambda \mathbf{a}$ which implies $b = -\lambda a$, with $\lambda \in \mathbb{R}$. This leads to the final form of the inverse element

$$B = A^{-1} = \frac{1}{aa + \mathbf{a}\mathbf{a}}(a, -\mathbf{a}) \quad (14)$$

Final point on our list is to check for associativity. This is a rather lengthy but straightforward calculation which shows that the product defined in equation (12) is in fact also associative. Conclusion: \mathbf{U} forms a subgroup of the general linear group $\mathbf{GL}(2, \mathbb{C})$.

With this calculation rules for the coefficients of the σ -matrices in our toolbox we can nearly forget about the basis they are referring to. But we should not! Members of the group are complex 2×2 -matrices that have the property $A^\dagger A = \mathbb{1}$. Their coefficients q still refer to the basis of σ -matrices.

$$A = (q, \mathbf{q}) = q\sigma_0 + \mathbf{q}\vec{\sigma} \quad \text{with } \vec{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\} \quad (15)$$

We should keep that in mind!

1.3 Properties of Quaternions

$$\begin{pmatrix} a \\ \mathbf{a} \end{pmatrix} * \begin{pmatrix} b \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} ab - \mathbf{a}\mathbf{b} \\ a\mathbf{b} + b\mathbf{a} + \mathbf{a} \times \mathbf{b} \end{pmatrix} \quad \begin{matrix} a, b \in \mathbb{R} \\ \text{and } \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \end{matrix} \quad (16)$$

This equation shows that the quaternion product in general is not commutative because the cross product term in the vector part will flip sign.

$$\Rightarrow \begin{pmatrix} a \\ \mathbf{a} \end{pmatrix} * \begin{pmatrix} a \\ -\mathbf{a} \end{pmatrix} = \begin{pmatrix} aa + \mathbf{a}\mathbf{a} \\ a\mathbf{a} - a\mathbf{a} - \mathbf{a} \times \mathbf{a} \end{pmatrix} = \begin{pmatrix} aa + \mathbf{a}\mathbf{a} \\ \mathbf{0} \end{pmatrix} \quad (17)$$

This is a projection onto the positive real numbers. So it's a good choice for a norm. But propose we scale some q by a factor s , this norm would become $s^2 q$. Applying a square root will normalize this.

$$\|a\| = \sqrt{a\bar{a}} = \sqrt{\begin{pmatrix} a \\ \mathbf{a} \end{pmatrix} * \begin{pmatrix} a \\ -\mathbf{a} \end{pmatrix}} = \sqrt{aa + \mathbf{a}\mathbf{a}} \quad (18)$$

$$\begin{aligned} \left[\begin{pmatrix} a \\ \mathbf{a} \end{pmatrix} * \begin{pmatrix} b \\ \mathbf{b} \end{pmatrix} \right] * \begin{pmatrix} c \\ \mathbf{c} \end{pmatrix} &= \begin{pmatrix} ab - \mathbf{a}\mathbf{b} \\ a\mathbf{b} + b\mathbf{a} + \mathbf{a} \times \mathbf{b} \end{pmatrix} * \begin{pmatrix} c \\ \mathbf{c} \end{pmatrix} \\ &= \begin{pmatrix} abc - a\mathbf{b}\mathbf{c} - b\mathbf{a}\mathbf{c} - c\mathbf{a}\mathbf{b} - (\mathbf{a} \times \mathbf{b})\mathbf{c} \\ (ab - \mathbf{a}\mathbf{b})\mathbf{c} + (ac + \mathbf{a}\mathbf{c})\mathbf{b} + (bc - \mathbf{b}\mathbf{c})\mathbf{a} + a(\mathbf{b} \times \mathbf{c}) - b(\mathbf{c} \times \mathbf{a}) + c(\mathbf{a} \times \mathbf{b}) \end{pmatrix} \end{aligned} \quad (19)$$

Just to show that the quaternion product is associative:

$$\begin{aligned} \begin{pmatrix} a \\ \mathbf{a} \end{pmatrix} * \left[\begin{pmatrix} b \\ \mathbf{b} \end{pmatrix} * \begin{pmatrix} c \\ \mathbf{c} \end{pmatrix} \right] &= \begin{pmatrix} a \\ \mathbf{a} \end{pmatrix} * \begin{pmatrix} bc - \mathbf{b}\mathbf{c} \\ b\mathbf{c} + c\mathbf{b} + \mathbf{b} \times \mathbf{c} \end{pmatrix} \\ &= \begin{pmatrix} abc - a\mathbf{b}\mathbf{c} - b\mathbf{a}\mathbf{c} - c\mathbf{a}\mathbf{b} - \mathbf{a}(\mathbf{b} \times \mathbf{c}) \\ (ab - \mathbf{a}\mathbf{b})\mathbf{c} + (ac + \mathbf{a}\mathbf{c})\mathbf{b} + (bc - \mathbf{b}\mathbf{c})\mathbf{a} + a(\mathbf{b} \times \mathbf{c}) - b(\mathbf{c} \times \mathbf{a}) + c(\mathbf{a} \times \mathbf{b}) \end{pmatrix} \end{aligned} \quad (20)$$

Now consider the third factor being the conjugate of the first:

$$\begin{aligned} \left[\begin{pmatrix} a \\ \mathbf{a} \end{pmatrix} * \begin{pmatrix} b \\ \mathbf{b} \end{pmatrix} \right] * \begin{pmatrix} a \\ -\mathbf{a} \end{pmatrix} &= \begin{pmatrix} ab - \mathbf{a}\mathbf{b} \\ a\mathbf{b} + b\mathbf{a} + \mathbf{a} \times \mathbf{b} \end{pmatrix} * \begin{pmatrix} a \\ -\mathbf{a} \end{pmatrix} \\ &= \begin{pmatrix} aba + a\mathbf{b}\mathbf{a} + b\mathbf{a}\mathbf{a} - a\mathbf{a}\mathbf{b} + (\mathbf{a} \times \mathbf{b})\mathbf{a} \\ (-ab + \mathbf{a}\mathbf{b})\mathbf{a} + (aa - \mathbf{a}\mathbf{a})\mathbf{b} + (ba + \mathbf{b}\mathbf{a})\mathbf{a} - a(\mathbf{b} \times \mathbf{a}) + b(\mathbf{a} \times \mathbf{a}) + a(\mathbf{a} \times \mathbf{b}) \end{pmatrix} \\ &= \begin{pmatrix} a^2b + b\mathbf{a}\mathbf{a} + (\mathbf{a} \times \mathbf{b})\mathbf{a} \\ 2(\mathbf{a}\mathbf{b})\mathbf{a} + (a^2 - \mathbf{a}\mathbf{a})\mathbf{b} \end{pmatrix} \\ &= \begin{pmatrix} (a^2 + \mathbf{a}\mathbf{a})b \\ 2(\mathbf{a}\mathbf{b})\mathbf{a} + (a^2 - \mathbf{a}\mathbf{a})\mathbf{b} + 2a(\mathbf{a} \times \mathbf{b}) \end{pmatrix} \end{aligned} \quad (21)$$

$$\begin{aligned} R: \mathbb{H} &\mapsto \mathbb{H} \\ b &\mapsto a(b) = ab\bar{a} \\ a(b) &= \begin{pmatrix} (a^2 + \mathbf{a}\mathbf{a})b \\ 2(\mathbf{a}\mathbf{b})\mathbf{a} + (a^2 - \mathbf{a}\mathbf{a})\mathbf{b} + 2a(\mathbf{a} \times \mathbf{b}) \end{pmatrix} \end{aligned} \quad (22)$$

Setting $b = (0, \mathbf{b})$ gives:

$$a(b) = \begin{pmatrix} 0 \\ 2(\mathbf{a}\mathbf{b})\mathbf{a} + (a^2 - \mathbf{a}\mathbf{a})\mathbf{b} + 2a(\mathbf{a} \times \mathbf{b}) \end{pmatrix} \quad (23)$$

So an element of the vector subspace is mapped to a vector of the same subspace:

$$\begin{aligned} M: \mathbb{R}^3 &\mapsto \mathbb{R}^3 \\ \mathbf{M}_a(\mathbf{b}) &= 2(\mathbf{a}\mathbf{b})\mathbf{a} + (a^2 - \mathbf{a}\mathbf{a})\mathbf{b} + 2a(\mathbf{a} \times \mathbf{b}) \end{aligned} \quad (24)$$