Levi-Civita-Symbol a.k.a. epsilon symbol

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Flipping the order of two indices in a permutation flips its sign. Doing it twice is equal to no flip. In a 3-number sequence a double flip is equal to a rotatation of the sequence, no matter wether rotated left or right. So if we have a sequence 1234, rotating the last three digits keeps the sign. We can do that two times, the third rotation gives us the original sequence. In order to get all odd permutations we flip the second and the third digit to get 1324 and rotate the last three digits to get the two remaining sequences. Then all rotations of 4 digits form the complete set of sequences.

Permutations of ϵ_{ijkl} that are not zero.

$$\begin{array}{cccccc} 1 & 1 & 1 & -1 & -1 & -1 \\ 1234 & 1342 & 1423 & 1324 & 1243 & 1432 \\ 2341 & 2134 & 2314 & 2413 & 2431 & 2143 \\ 3412 & 3421 & 3142 & 3241 & 3124 & 3214 \\ 4123 & 4213 & 4231 & 4132 & 4312 & 4321 \end{array} \tag{1}$$

$\boldsymbol{\sigma}$-matrices:

$$\boldsymbol{\sigma}_1 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{\sigma}_2 \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \boldsymbol{\sigma}_3 \equiv \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \quad \boldsymbol{\sigma}_4 \equiv \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \tag{2}$$

Here is the multiplication table for the $\boldsymbol{\sigma}$-matrices:

$$\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j = \begin{pmatrix} \boldsymbol{\sigma}_1 & \boldsymbol{\sigma}_2 & \boldsymbol{\sigma}_3 & \boldsymbol{\sigma}_4 \\ \boldsymbol{\sigma}_2 & -\boldsymbol{\sigma}_1 & \boldsymbol{\sigma}_4 & -\boldsymbol{\sigma}_3 \\ \boldsymbol{\sigma}_3 & -\boldsymbol{\sigma}_4 & -\boldsymbol{\sigma}_1 & \boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_4 & \boldsymbol{\sigma}_3 & -\boldsymbol{\sigma}_2 & -\boldsymbol{\sigma}_1 \end{pmatrix} \tag{3}$$

$$\begin{aligned} \Sigma_{ij} &= \epsilon_{ijkl}\ \boldsymbol{\sigma}_k\boldsymbol{\sigma}_l \\ &= \begin{pmatrix} 0 & \boldsymbol{\sigma}_3\,\boldsymbol{\sigma}_4 - \boldsymbol{\sigma}_4\,\boldsymbol{\sigma}_3 & \boldsymbol{\sigma}_4\,\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_4 & \boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_3 - \boldsymbol{\sigma}_3\,\boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_3\,\boldsymbol{\sigma}_4 - \boldsymbol{\sigma}_4\,\boldsymbol{\sigma}_3 & 0 & \boldsymbol{\sigma}_4\,\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_1\,\boldsymbol{\sigma}_4 & -(\boldsymbol{\sigma}_1\,\boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_3\,\boldsymbol{\sigma}_1) \\ \boldsymbol{\sigma}_4\,\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_4 & -(\boldsymbol{\sigma}_4\,\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_1\,\boldsymbol{\sigma}_4) & 0 & \boldsymbol{\sigma}_1\,\boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_3 - \boldsymbol{\sigma}_3\,\boldsymbol{\sigma}_2 & \boldsymbol{\sigma}_1\,\boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_3\,\boldsymbol{\sigma}_1 & -(\boldsymbol{\sigma}_1\,\boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_2\,\boldsymbol{\sigma}_1) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2\boldsymbol{\sigma}_2 & 2\boldsymbol{\sigma}_3 & 2\boldsymbol{\sigma}_4 \\ 2\boldsymbol{\sigma}_2 & 0 & 2\boldsymbol{\sigma}_4 & -2\boldsymbol{\sigma}_3 \\ 2\boldsymbol{\sigma}_3 & -2\boldsymbol{\sigma}_4 & 0 & 2\boldsymbol{\sigma}_2 \\ 2\boldsymbol{\sigma}_4 & 2\boldsymbol{\sigma}_3 & -2\boldsymbol{\sigma}_2 & 0 \end{pmatrix} \\ &= \frac{1}{2}\boldsymbol{\sigma}_k\boldsymbol{\sigma}_l - \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \boldsymbol{\sigma}_1 \end{aligned} \tag{4}$$

We obtain an interesting result:

$$\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j = 2\ \epsilon_{ijkl}\ \boldsymbol{\sigma}_k \boldsymbol{\sigma}_l + \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \boldsymbol{\sigma}_1 \tag{5}$$

For the actual product of two arbitrary elements of $U(2)$ we get

$$a^i \boldsymbol{\sigma}_i b^j \boldsymbol{\sigma}_j = a^i a^j \left[2\ \epsilon_{ijkl}\ \boldsymbol{\sigma}_k \boldsymbol{\sigma}_l + \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \boldsymbol{\sigma}_1 \right] \tag{6}$$