Levi-Civita-Symbol a.k.a. epsilon symbol

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We will proof this very useful identity:

$$\sum_{k=1}^{n} \epsilon_{i_1\, i_2\, \dots\, i_{n-1}\, k} \epsilon_{j_1\, j_2\, \dots\, j_{n-1}\, k} = \begin{vmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \delta_{i_1 j_3} & \cdots & \delta_{i_1 j_{n-1}} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \cdots & \cdots & \delta_{i_2 j_{n-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \delta_{i_{n-1} j_1} & \delta_{i_{n-1} j_2} & \cdots & \cdots & \delta_{i_{n-1} j_{n-1}} \end{vmatrix} \tag{1}$$

We begin by considering the euclidean standard basis B in $\mathbb{R}^3$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2}$$

The metric of B is $g_{ij} = \mathbf{e}_i \mathbf{e}_j = \delta_{ij}$, the Kronecker-Delta. Which is another way of saying that its an orthonormal basis. Let's have a look at the determinants of B.

$$\left| \mathbf{e}_i\ \mathbf{e}_j\ \mathbf{e}_k \right| = \begin{cases} 0 & \text{if} \quad i = j,\ i = k, \text{or} \quad j = k \\ 1 & \text{if} \quad ijk \in \{123, 231, 312\} \\ -1 & \text{if} \quad ijk \in \{132, 213, 321\} \end{cases} \tag{3}$$

Note: Flipping the order of two neighbouring indices (first and last are neighbours) flips the sign of the determinant. These are exactly the properties of the epsilon symbol.

$$\epsilon_{ijk} = \left| \mathbf{e}_i\ \mathbf{e}_j\ \mathbf{e}_k \right| \tag{4}$$

A product of two epsilon symbols becomes just a product of two determinants:

$$\epsilon_{ijk}\epsilon_{lmn} = \left| \mathbf{e}_i\ \mathbf{e}_j\ \mathbf{e}_k \right| \cdot \left| \mathbf{e}_l\ \mathbf{e}_m\ \mathbf{e}_n \right| \tag{5}$$

Note: the determinant of a square matrix is equal to the determinant of its transpose. Further is the determinant of the product of two square matrices equal to the product of the determinants of these matrices: $\det AB = \det A \det B$. Hence:

$$\left| (\mathbf{e}_i\ \mathbf{e}_j\ \mathbf{e}_k)^t (\mathbf{e}_l\ \mathbf{e}_m\ \mathbf{e}_k) \right| = \det \begin{pmatrix} \delta_{il} & \delta_{im} & \delta_{ik} \\ \delta_{jl} & \delta_{jm} & \delta_{jk} \\ \delta_{kl} & \delta_{km} & \delta_{kk} \end{pmatrix} = \det \begin{pmatrix} \delta_{il} & \delta_{im} & 0 \\ \delta_{jl} & \delta_{jm} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Note that for matrix multiplication we have two repeat one index in each matrix to get proper row-column products.

The zeroes in the last row and the last column arise from the fact that we set $i \neq j$, $j \neq k$ and $l \neq k$, $m \neq k$. Otherwise there is not much to calculate because per definition $\epsilon_{ijk} = 0$ and $\epsilon_{lmk} = 0$ if one index is repeated.

So we get:

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{7}$$

A generalization to n indices leads to equation (1).

0.1 Useful example

Its very convenient to write vector equations in tensor notation.

$$
\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= a^i b^i \equiv \sum_{i=1}^{n} a^i b^i \\
\mathbf{a} \times \mathbf{b} &= \epsilon_{ijk}\; a^i b^j \; \mathbf{e}_k \\
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= \epsilon_{klm}\; (\epsilon_{ijk}\; a^i b^j) c^l \; \mathbf{e}_m \\
&= \epsilon_{klm}\; \epsilon_{ijk}\; a^i b^j c^l \; \mathbf{e}_m \\
&= (\delta_{il}\; \delta_{jm} - \delta_{jl}\; \delta_{im}) a^i b^j c^l \; \mathbf{e}_m \\
&= \delta_{il}\; \delta_{jm}\; a^i b^j c^l - \delta_{jl}\; \delta_{im} a^i b^j c^l \; \mathbf{e}_m \\
&= (\delta_{il}\; a^i)\; (\delta_{jm}\; b^j)\; c^l - (\delta_{jl}\; b^j)\; (\delta_{im} a^i)\; c^l \; \mathbf{e}_m \\
&= a^l\; b^m\; c^l - b^l\; a^m\; c^l \; \mathbf{e}_m \\
&= (\mathbf{ac})\mathbf{b} - (\mathbf{bc})\mathbf{a}
\end{aligned} \tag{8}
$$

Note: It does not make any difference for the epsilon symbol if we write indices up or down. We would have to multiply the components by the metric tensor or its inverse respectively, if we decided to do so, which is in both cases the identity matrix.

Happy index gymnastics!