Isometry Groups

Michael Gehricke

May 8, 2019

1

1.1

Consider an inner product (.,.) of two elements of a vector space V with Basis B = $\mathbf{e_i}$.

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \tag{1}$$

Linearly combining the $\mathbf{e}_i$ leads to a new Basis B'. Note: Over same indices upper and lower has to be summed (Einstein sum convention).

$$\mathbf{e'}_i = A^k{}_i \mathbf{e}_k \tag{2}$$

Substituting (2) into (1) leads to

$$g'_{ij} = A^k{}_i \mathbf{e}_k \cdot A^l{}_j \mathbf{e}_l = A^k{}_i \cdot A^l{}_j \cdot g_{kl} \tag{3}$$

Now suppose we want the transformation not to effect the metric g_{ij}

$$g_{ij} = g'_{ij} \quad \implies A^k{}_l \cdot A^l{}_m = \delta^k_m \quad \text{, with} \quad \delta^i_j \equiv \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

In matrix notation this looks like $A^t A = I$. The transpose of matrix A times itself is the identity matrix. This result was obtained by choosing a bilinear metric. If the field of scalars are not the reals but complex or quaternion there is another possibility to set a metric for V: the sesquilinear metric. This then leads to the condition $A^\dagger A = 1$. That is the conjugate transpose of A times A is the identity matrix.

1.2 The Unitary Group U(2,$\mathbb{C}$)

For example let's take a general complex 2×2-matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_r + ia_i & b_r + ib_i \\ c_r + ic_i & d_r + id_i \end{bmatrix} \tag{5}$$

Note: the last notation is separating the real and the imaginary part of a, b, c, d.

Applying the metric preserving condition $A^\dagger = A^{-1}$ to these matrices we obtain

$$\begin{bmatrix} a_r - ia_i & c_r - ic_i \\ b_r - ib_i & d_r - id_i \end{bmatrix} = \begin{bmatrix} d_r + id_i & -b_r - ib_i \\ -c_r - ic_i & a_r + ia_i \end{bmatrix} \tag{6}$$

This leads to the general form of a matrix meeting this condition

$$A \in U(2, \mathbb{C}), \quad \mathrm{A} = \begin{bmatrix} a_r - ia_i & -b_r - ib_i \\ b_r - ib_i & a_r + ia_i \end{bmatrix} \tag{7}$$

Renaming $a_r \to a_0$, $b_r \to a_1$, $a_i \to a_2$, $b_i \to a_3$ and setting $a_i = 0$ for all $i \neq j$ and $a_i = a_j$ for $i = j$ with $i, j \in \{0, 1, 2, 3\}$, we can get a coordinate form of the above A.

$$A(a) = a_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_1 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + a_2 \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} + a_3 \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \tag{8}$$

Setting the symbols **e** (identity), **i**, **j** and **k** for our basis vectors

$$\mathbf{e_0} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{e_1} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{e_2} \equiv \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad \mathbf{e_3} \equiv \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \tag{9}$$

we can do pairwise products of this four matrices by following matrix multiplication rules and find

1. $\mathbf{e_1} \times \mathbf{e_2} = \mathbf{e_3}$, $\mathbf{e_2} \times \mathbf{e_3} = \mathbf{e_1}$, $\mathbf{e_3} \times \mathbf{e_1} = \mathbf{e_2}$
2. $\mathbf{e_2} \times \mathbf{e_1} = -\mathbf{e_3}$, $\mathbf{e_3} \times \mathbf{e_2} = -\mathbf{e_1}$, $\mathbf{e_1} \times \mathbf{e_3} = -\mathbf{e_2}$
3. ${\mathbf{e_1}}^2 = {\mathbf{e_2}}^2 = {\mathbf{e_3}}^2 = -\mathbf{e_0}$

As an overview here is the resulting multiplication table:

$$\begin{array}{cccc} \mathbf{e_0} & \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ \mathbf{e_1} & -\mathbf{e_0} & \mathbf{e_3} & -\mathbf{e_2} \\ \mathbf{e_2} & -\mathbf{e_3} & -\mathbf{e_0} & \mathbf{e_1} \\ \mathbf{e_3} & \mathbf{e_2} & -\mathbf{e_1} & -\mathbf{e_0} \end{array} \tag{10}$$

A product of to element $\mathbf{a} = a^i \mathbf{e}_i$ and $\mathbf{b} = b^j \mathbf{e}_j$ satisfies the following rule

$$\begin{aligned} \mathbf{a} &= a^i \mathbf{e_i} \cdot a^j \mathbf{e_j} \\ &= (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) \mathbf{e_0} \\ &+ (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2) \mathbf{e_1} \\ &+ (a_0 b_2 - a_1 b_3 + a_2 b_0 + a_3 b_1) \mathbf{e_2} \\ &+ (a_0 b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0) \mathbf{e_3} \end{aligned}$$

Maxwell's equations:

$$B' = -\nabla \times E, \tag{12a}$$
$$E' = \nabla \times B - 4\pi j, \tag{12b}$$

$$\mathbf{A}^{\mathbf{ij}} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 7 & 8 & 9 \end{bmatrix} \tag{13}$$

$$\begin{matrix} a & b \\ c & d \end{matrix}$$