Bakery model. A baker decided to bake q goods every. Goal is to explore how adding the option to sell leftoover goods at the end of a day will increase the output of the baker. Suppose the per unit production cost is a constant c.

Easiest model: Price p is fixed. Assume that the number of goods demanded at price p on a given day is a random variable D. This has a distribution function $\rho(x)$. For example, we could have

$$\rho(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{100-x}{2\sigma}\right)^2}$$

This says that the average number of customers is 100, and the standard deviation is σ. We can be more creative with $\rho(x)$ but for now let's keep it as above.

First we write down the profit : On a given day, D goods are demanded, q are produced. The number sold will be the minimum of D and q. If the price is p and the cost c, profit π is given by

$$\pi = \min(D, q)p - qc.$$

Now D is a random variable, so we need to compute the expected value of π :

$$\begin{aligned} E[\pi] &= E[\min(D, q)p - qc] \\ &= E[\min(D, q)]\, p - qc \\ &= \begin{pmatrix} E[D : D < q] \Pr\{D \geq q\} \\ +E[q : D \geq q] \Pr\{D \geq q\} \end{pmatrix} p - qc \end{aligned}$$

Which is

$$= \begin{pmatrix} \int_0^q x\rho(x)dx \\ q\int_q^\infty \rho(x)dx \end{pmatrix} p - qc$$

Now observe that when $q = 0$ the expected profit is zero. Also observe that when q becomes large $\int_q^\infty \rho(x)dx \to 0$ and so the expected profit becomes negative. Somewhere in the middle is where profit is maximized. In order to find the q we take the derivative

$$\begin{aligned} \frac{d}{dq}E[\pi] &= \frac{d}{dq}\left(\begin{pmatrix} \int_0^q x\rho(x)dx \\ q\int_q^\infty \rho(x)dx \end{pmatrix} p - qc\right) \\ &= \left(q\rho(q) - q\rho(q) + \int_q^\infty \rho(x)dx\right) p - c \end{aligned}$$

where we have used the Fundamental Theorem of calculus to differentiate the integral.

Setting this to zero, we get

$$p\int_q^\infty \rho(x)dx - c = 0$$

or

$$\int_q^\infty \rho(x)dxc = \frac{c}{p}$$

This means the following: To maximize profit, the bakery chooses q so that the probability of the demand exceeding q is equal to the ratio of the cost to the price. For example, if the cost per unit is \$1 and the price charged is \$3, then quantity will be chosen so that the probability that all good sold will be $\frac{1}{3}$: More often than not the goods will be overproduced.

Next, we modify this to the case where we can sell all of the quantity at a positive price. Note that if the positive price is larger than cost, the bakery will create infinitely many goods. So we need to assume it's smaller than costs. Suppose this default price is β. Then

$$\begin{aligned} \pi &= Dp + (q-D)\beta - cq \text{ for } D \leq q \\ \pi &= Dq - cq : \text{ for } q \leq D \end{aligned}$$

So now we compute

$$\begin{aligned} E[\pi] &= \int_0^q (px + (q-x)\beta)\rho(x)dx \\ &\quad + pq\int_q^\infty \rho(x)dx - cq \end{aligned}$$

Take ther derivate as before.

$$\begin{aligned} \frac{d}{dq}E[\pi] &= (pq + (q-q)\beta)\rho(q) \\ &\quad + \int_0^q \beta\rho(x)dx \\ &\quad + p\int_q^\infty \rho(x)dx - pq\rho(q) \\ &\quad - c \end{aligned}$$

Simplifying

$$\frac{d}{dq}E[\pi] = \beta\int_0^q \rho(x)dx + +p\int_q^\infty \rho(x)dx - c$$

Letting χ

$$\chi = \int_q^\infty \rho(x) = \text{ probability that all goods are sold}$$

we get

$$\frac{d}{dq}E[\pi] = \beta(1-\chi) + p\chi - c = 0$$

Solving for χ

$$\chi = \frac{c-\beta}{p-\beta}$$

This isn't that complicated. If the price is \$3 and the and cost is \$1, a "dump price" of \$0.50 means that

$$\chi = \frac{0.5}{2.5} = \frac{1}{5}$$

So the quantity would be increased from the quantity such that $\frac{1}{3}$ or the time the goods will be sold to the higher quantity such that $\frac{1}{5}$ of the time the goods with be sold.

We may be interested in the marginal effect of β

$$\frac{d\chi}{d\beta} = \frac{d}{d\beta}\frac{c-\beta}{p-\beta} = -\frac{p-c}{(p-\beta)^2}$$

As $\beta \to c$ then it makes sense to produce infinitely many goods. What we haven't explored yet is a more natural situation when the the capacity to buy the excess is bounded. For a capacity C, the profit becomes

$$\begin{aligned}
\pi &= Dp + \max\left((q-D), C\right)\beta - cq \text{ for } D \leq q \\
\pi &= Dq - cq : \text{ for } q \leq D
\end{aligned}$$

$$\begin{aligned}
E[\pi] &= \int_0^q \left(px + \max\left((q-x), C\right)\beta\right)\rho(x)dx \\
&\quad + pq\int_q^\infty \rho(x)dx - cq
\end{aligned}$$

and take derivative

$$\begin{aligned}
\frac{d}{dq}E[\pi] &= \left(pq + (q-q)\beta\right)\rho(q) \\
&\quad + \int_0^C \beta\rho(x)dx \\
&\quad + p\int_q^\infty \rho(x)dx - pq\rho(q) \\
&\quad - c
\end{aligned}$$

Setting this to zero

$$\frac{d}{dq}E[\pi] = \beta\int_0^C \rho(x)dx + p\int_q^\infty \rho(x)dx = c$$

But now we have

$$\begin{aligned}
\int_0^C \rho(x)dx &= \int_0^q \rho(x)dx - \int_C^q \rho(x)dx \\
&= 1 - \chi - \int_C^q \rho(x)dx
\end{aligned}$$

and we get, assuming that $q > C$

$$\left(1 - \chi - \int_C^q \rho(x)dx\right)\beta + p\chi = c$$

Solving for χ

$$\begin{aligned}
\chi &= \frac{c + \left(\int_C^q \rho(x)dx - 1\right)\beta}{p - \beta} \\
&= \frac{c - \beta}{p - \beta} + \frac{\beta \int_C^q \rho(x)dx}{p - \beta}
\end{aligned}$$

Analyzing this requires more attention to the actually distribution function $\rho(x)$ for D but for C small and q large that expression $\int_C^q \rho(x)dx$ approaches 1.

Remember that $\chi(q)$ represents that probability that all q produced goods will be sold.