Math 218: Combinatorics

HOMEWORK 6 : DUE SEPTEMBER 24

1. Recall that in the Binomial Theorem, the coefficient of each term represents the number of ways to rearrange the variables in each term. For example, the binomial coefficient of $a^2b = aab$ would be $\binom{3}{2}$, corresponding to the 3 ways to arrange the two as among three terms (or equivalen tly, the 3 ways to arrange one b among three terms).

   In the Multinomial Theorem, we generalize from working with just two summands a and b to any number of them, $x_1, x_2, \ldots, x_m$. Accordingly, when we expand out the left hand side of the equation, the coefficient of each term will be the number of ways of rearranging the variables in the term where any number of the variables can repeat. For example, the coefficient for the term $x_1^2 x_2^1 x_3^0$ would be $\binom{3}{2,1,0} = \frac{3!}{2!1!0!} = 3$.

   More generally, for any arbitrary $k_1, k_2, \ldots, k_m$ such that $k_1 + k_2 + \cdots + k_m = n$, we can see that the coefficient for $x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$ will be the number of ways to arrange the terms $x_1, x_2, \ldots, x_m$ such that each may correspondingly repeat $k_1, k_2, \ldots, k_m$ times. Using this fact and the distributive property of multiplication, we see that the expansion of the left hand side of the equation will be the sum of the terms $x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$, each with the coefficient $\binom{n}{k_1, k_2, \ldots, k_m}$. $\square$

2. We know by Example 4.1.1 in Morris that since $|A| = 10$, there are $2^{10} = 1024$ distinct subsets of size 10 of A.

   Let s be the sum of any size-10 subset of A. We know that

   $$\begin{aligned} 1 + 2 + \cdots + 10 \leq s &\leq 91 + 92 + \cdots + 100 \\ 55 \leq s &\leq 955, \end{aligned}$$

   and so there are 901 possibilities for the value of s. Consider the objects to be size-10 subsets of A and categories to be their sum. Since $1024 > 901$, by the Pigeonhole Principle, there must be at least 2 size-10 subsets of A that have the same sum. $\square$