Probability Theory

Set Theory

Set: A set is defined as a collection of objects.

D = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}

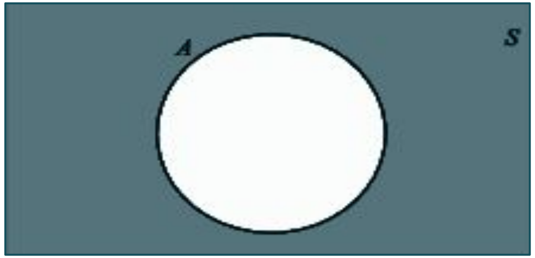
Element: **Each individual object is called an element of that set.**

Tuesday is an element of the set D. Also, written as Tuesday $\in$ D

Null/Void Set: A set which has no elements. Denoted by ϕ.

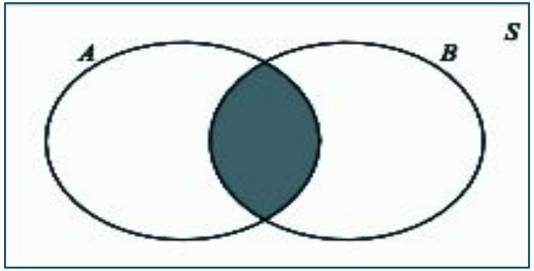
Universal Set: Set containing all elements and of which all other sets are subsets. Denoted by U.

Venn Diagrams



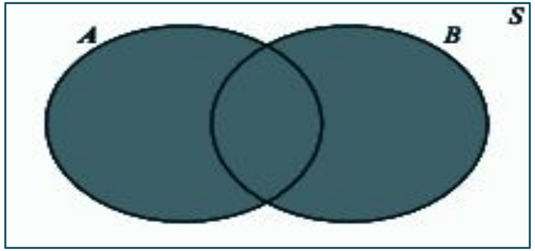
Complement of A (denoted by A' or A^C)

Everything that is not in A



Intersection of A and B (denoted by $A \cap B$)

Everything in A <u>and</u> B



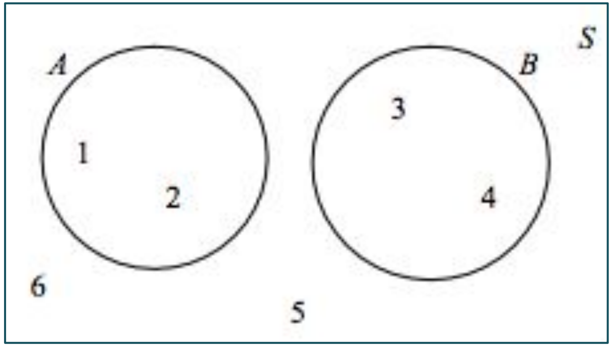
Union of A and B (denoted by $A \cup B$)

Everything in A <u>or</u> B <u>or</u> both

Mutually Exclusive Sets

If no element is common between two sets, we say that they are mutually exclusive.

If $A \cap B = \phi$, then A and B are mutually exclusive.



Counting

Permutation is the different arrangements of a given number of elements taken one by one, or some, or all at a time. **Order matters.**
For example, if we have two elements A and B, then there are two possible arrangements, AB and BA.

$$ {}_nP_r = \frac{n!}{(n-r)!} \qquad C(n,r) = \frac{n!}{r!(n-r)!} $$

Combination is the different selections of a given number of elements taken one by one, or some, or all at a time. **Order doesn't matter.**
For example, if we have two elements A and B, then there is only one way select two items, we select both of them.

Basic Probability

An **experiment** can be infinitely repeated and has a well-defined set of possible outcomes, called the **sample space**. **Event** is a subset of the sample space.

<u>Experiment</u>: Rolling a die => <u>Sample Space</u>, S = {1, 2, 3, 4, 5, 6}

<u>Event</u>: Getting an odd number, A = {1, 3, 5}

Probability is a numerical way of describing how likely (or not) an event is to happen. If each of the elements in the sample space are equally likely, then we can define the probability of event A as **P(A) = n(A) / n(S)**

Probability Axiom #1

The relative frequency of event that is certain to occur must be 1.

If S contains all possible outcomes, the probability of S must be 1,

i.e, $P(S) = 1$

Example : The probability of getting any one of the numbers 1 to 6 on a dice is certain! So this result is saying that we give certain events a probability of 1.

Probability Axioms #2

The relative frequency of occurrence of any event must not be negative, that is, probabilities can never be negative.

So the probability of an impossible event is 0.

Example : The probability of getting number 0 on a dice is impossible! So this result is saying that we give certain events a probability of 0.

So rules 1 and 2 together are telling us that probabilities lie between 0 (impossible) and 1 (certain).

Probability Axiom #3

If two events cannot occur simultaneously, because they are mutually exclusive, the probability of an event defined by their union is equal to the sum of the probabilities of the two events. This property is known as additivity.

$$P(A \cup B) = P(A) + P(B) \quad \text{if } A \cap B = \phi$$

Addition Rule

Axiom #3 is often known as the special addition rule. For a more general case, where two sets are not necessarily mutually exclusive, the rule can be extended as follows:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Axiom 3 is a special case of this rule when A and B are mutually exclusive.

Conditional Probability

A conditional probability is the probability of an event, given some other event has already occurred. It is denoted by P(A|B) inferred as probability of A given B.

$$P(A \mid B) = P(A \cap B) / P(B)$$

Multiplication Rule

The probability that Events A and B both occur is equal to the probability that Event B occurs times the probability that Event A occurs, given that B has occurred.

$$P(A \cap B) = P(B) * P(A \mid B)$$

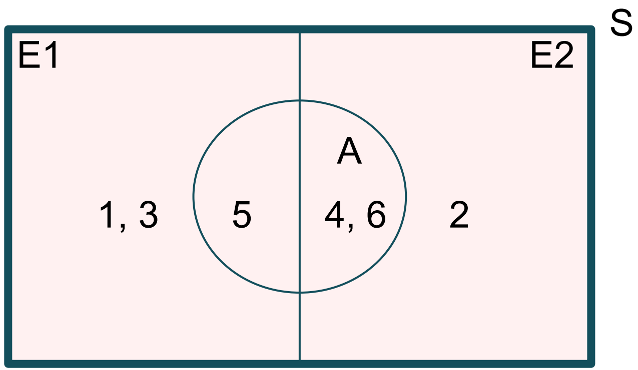
If A and B are independent,

$$P(A \cap B) = P(B) * P(A)$$

Bayes' Theorem

Let E1, E2,..., En be a partition of S and define event A as the subset of S.

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum_{j=1}^{n} P(E_j)P(A \mid E_j)}, \qquad i = 1, 2, 3, \dots, n$$



Bayes' formula allows us to "turnaround" conditional probabilities, i.e, calculate P(Ei | A) given information about P(A | Ei) only.

P(Ej) -> **Prior Probabilities**

P(Ei | A) -> **Posterior Probability**