A WILD SURFACE EACH OF WHOSE ARCS IS TAME

By R. H. Bing

1. Introduction. A closed set X in E^3 is *tame* if there is a homeomorphism h of E^3 onto itself such that $h(X)$ is a polyhedron (the sum of a locally finite collection of tetrahedra, triangles, segments, and points). A closed set in E^3 that is not tame is called *wild*. Examples of wild surfaces are found in [1], [9]. Since a tame surface divides space in the same fashion as does a polyhedral surface, it is convenient to have criteria for determining which surfaces are tame and which are wild.

Consider the following question.

Question. Is a 2-sphere in E^3 tame if each arc in it is tame?

In this paper we give a negative answer. Had the answer been in the affirmative, it would have provided us with another characterization of tame 2-spheres. This would have been fortunate since such characterizations are few. We know that a 2-sphere is tame if any of the following conditions are met.

1. It is locally tame [2], [13].
2. Its complement is uniformly locally simply connected [5].
3. It can be homeomorphically approximated from both sides [4].

It is known [6] that each surface (either tame or wild) in E^3 contains many tame arcs. The examples of wild surfaces described in the literature contain wild arcs. These examples are locally tame except at a 0-dimensional set and as pointed out in Theorem 8, any simple closed curve on the 2-sphere that contains this 0-dimensional set is wild.

Suppose S is a 2-sphere in E^3 and J is a simple closed curve in $E^3 - S$ that cannot be shrunk to a point in $E^3 - S$. The following plan was devised for getting a partial affirmative solution to the above question by showing that S contains a wild simple closed curve. Instead of leading to an affirmative solution, the plan backfired and led to a counterexample and a negative answer.

Faulty Plan

1. Consider a decreasing sequence of curvilinear triangulations $T_1, T_2, \cdots$ of S such that the 1-skelton of each triangulation is tame. That there is such a decreasing sequence of triangulations follows from the fact that S contains a sequence of Sierpinski curves $S_1, S_2, \cdots$ such that $S_i \subset S_{i+1}$, each S_i lies on some tame 2-sphere, and each component of $S - S_i$ is of diameter less than $1/i$. See [6].

2. Let D be a 2-simplex of T_1 such that J cannot be shrunk to a point in $E^3 - (S - \text{Int } D)$. Replace D by a disk D' such that Int D' is locally tame

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and $(S - D) + D'$ is a 2-sphere very close to S. That D can be replaced by such a D' very close to D follows from the fact that if f is a continuous non-negative function on S, there is a homeomorphism h of S into E^3 such that for each point s of S, the distance between s and $h(s)$ is no more than $f(s)$ and $h(s)$ is locally tame at $h(s)$ if $f(s) > 0$ [3]. Since D' is locally tame at each of its interior points and Bd D' is tame by Step 1, it follows from [13] that D' is tame.

3. Iterate the procedure started in Step 2 by replacing as many 2-simplexes of T_1 as can be done so that J cannot be shrunk to a point in the complement of the adjusted S. If 2-simplexes D_1, D_2 with a common edge are adjusted, then it follows from [8] that their adjusted sum is tame.

4. Now move to the 2-simplexes of T_2 that have not been adjusted and replace them by tame disks retaining the property that J cannot be shrunk to a point in the complement of the adjusted S.

5. It was hoped to continue this process to get an adjusted 2-sphere S' such that S' is locally tame except at a 0-dimensional set X.

6. The last step was to get a simple closed curve J' in S' such that J' contains X and lies in the sum of X and the various 1-skeletons of the triangulations $T_1, T_2, \cdots$. Then J' would be wild by Theorem 8 and would lie in S as well as in S'. This would have shown that S contains a wild arc.

Just where does the faulty plan break down? It was in Step 5. Not only was it impossible to continue the process but in trying to prove that the process could be carried out, an example was found which showed that it could not even be started as suggested in Step 2.

THEOREM 1. *There is a 2-sphere S in E^3 and a simple closed curve J in $E^3 - S$ such that J cannot be shrunk to a point in $E^3 - S$, but for each disk Y of S, J can be shrunk to a point in $E^3 - Y$.*

Theorem 1 is shown to be true by Example 2 described in §3 of this paper.

An examination of Example 2 also shows that it has the remarkable property that each arc in it is tame. That each arc in 2-sphere S of Example 2 is tame is shown in §5 of this paper.

THEOREM 2. *There is a wild 2-sphere S in E^3 such that each arc in S is tame.*

2. An approximation to the example. Before describing Example 2, we give a simpler example.

EXAMPLE 1. *A 2-sphere S and a simple closed curve J in $E^3 - S$ such that J cannot be shrunk to a point in $E^3 - S$ but for certain large disks Y_i in S, J can be shrunk to a point in $E^3 - Y_i$.*

Suppose S_0 is a tame 2-sphere in E^3 and S_0 has a cellular subdivision such that the disks of the subdivision may be ordered $E_1, E_2, \cdots, E_n$ such that each of $E_1 \cdot E_2, E_2 \cdot E_3, \cdots, E_{n-1} \cdot E_n, E_n \cdot E_1$ is an edge. If S_0 is the boundary of a cube, we could take $n = 6$ and let E_1, E_4 be the top and bottom faces and E_2, E_5 be the front and back faces of the cube.

From the center of each face E_i erect a solid feeler with a solid loop on the far end of it. The feeler and loop are three-dimensional as shown in Figure 1

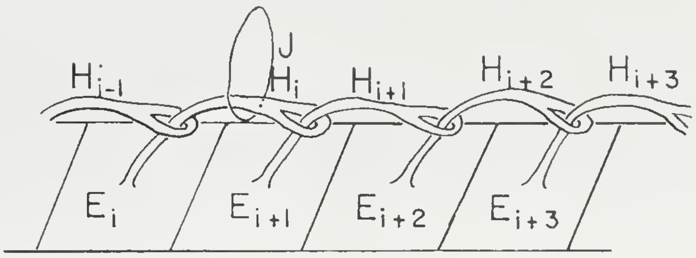

Figure 1

and resemble a bent eye bolt. We shall denote the "eye bolt" from E_i as H_i. Although it is topologically a solid torus, it more nearly resembles a solid torus on a curved stem. The loop in H_1 circles around the stem of H_2, the loop of H_2 circles around the stem of H_3, $\cdots$, and the loop of H_n circles around the stem of H_1 as shown in Figure 1.

Suppose J is a simple closed curve that circles around one of the solid feelers as shown in Figure 1. By using Theorem 11 it can be shown that J cannot be shrunk to a point in the complement of $S_0 + \sum H_i$. To show this, let C^i be a cube which contains the loop of H_i such that Bd $C^i \cdot H_i = B_3^i$ is a disk which is a cross section of the stem of H_i, Bd $C^i \cdot H_{i+1} = B_2^i + B_2^i$ is the sum of two disks which are cross sections of the stem of H_{i+1}, and C_i intersects $S_0 + \sum H_i$ only in a torus of H_i and a cylinder of H_{i+1}. The C^i's do not intersect each other. Then C^i, B_1^i, B_2^i, B_3^i, $C^i \cdot H_i$, $C^i \cdot H_{i+1}$ correspond to the C, B_1, B_2, B_3, T, C_1 of Theorem 11 as shown in Figure 7. If S_0' denotes the sum of S_0 and the component of $E^3 - S_0$ missing the H_i's, $S_0' + \sum H_i + \sum C^i$ is a cube with handles such that J circles one of these handles as does each B_1^i and each B_2^i. We use Theorem 11 to replace C^1 by $C^1 \cdot (H_1 + H_2)$ in $S_0 + \sum H_i + \sum C^i$, then replace C^2 by $C^2 \cdot (H_2 + H_3)$, $\cdots$, and finally C^n by $C_n \cdot (H_n + H_1)$. At each stage we find that J cannot be shrunk to a point in the complement of what is left. Neither can any B_j^i $(j = 1, 2)$ be shrunk to a point in the sum of B_j^i and the complement of what is left so we can continue to apply Theorem 11 at the next step. After the n-th step we are left only with $S_0' + H_i$ and J cannot be shrunk to a point in its complement.

If H_{i+1} is deleted from $S_0 + \sum H_i$, the simple closed curve J shown in Figure 1 can be shrunk to a point over the right end of H_i.

If instead of deleting H_{i+1}, we omitted H_{i+2}, it is a bit trickier to shrink J to a point without hitting S_0 or any remaining H. However, this may be accomplished as follows. Let J bound a disk which extends over the right end of H_i and which intersects the stem of H_{i+1} in two disks, one above the loop of H_i and one below. The two disks are adjusted so that they reach over the right end of H_{i+1}. The adjusted lower disk goes through the hole in H_i.

Continuing in this fashion it may be shown that if H_j is any eye bolt whatever, J may be shrunk to a point in

$$E^3 - (S_0 + \sum_{k \neq j} H_k).$$

Although $\sum (E_i + H_i)$ is not a 2-sphere, it is an approximation to the 2-sphere of Example 1. Of course a 2-sphere does not contain loops as shown in Figure 1 but Alexander's horned sphere does contain loops of a certain sort. If each $E_i + H_i$ is replaced by a wild disk D_i as shown in Figure 2, there would result

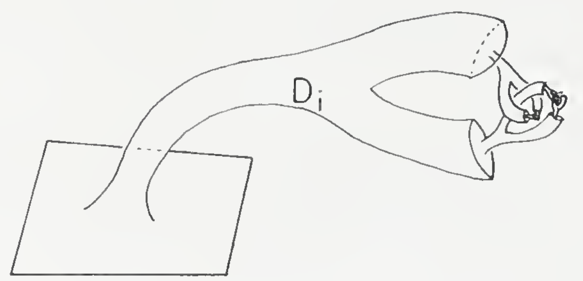

Figure 2

a 2-sphere $S = \sum D_i$ such that J cannot be shrunk to a point in $E^3 - S$. However if $Y_j = \sum_{i \neq j} D_i$, J can be shrunk to a point in $E^3 - Y_j$.

The 2-sphere S of Example 1 is $\sum D_i$. Although it has the property that J can be shrunk to a point in the complement of the sum of any proper subcollections of the D_i's, it is not a good enough example to establish Theorem 1. However, it points the way.

Although J can be shrunk to a point in each $E^3 - Y_i$, we could get a disk Y by deleting a small tame subset of some D_i and we could not shrink J to a point in $E^3 - Y$. Although we could make the mesh of the subdivision of S_0 very small, and the corresponding D_i's also small, this would not help since one is permitted to choose the Y last and our example cannot be made to have the property claimed by Theorem 1. However, the example does provide a clue as to how to proceed. It is an approximation to a better example. The better example is obtained by choosing different D_i's.

3. The example. In this section we describe a 2-sphere with the following properties.

Example 2. *A 2-sphere S and a simple closed curve J in $E^3 - S$ such that J cannot be shrunk to a point in $E^3 - S$ but for each disk Y in S, J can be shrunk to a point in $E^3 - Y$.*

This 2-sphere is a modification of the one given in Example 1. As before we let E_1 , E_2 , $\cdots$, E_n be a cellular decomposition of a tame 2-sphere S_0 such that adjacent E_i's in the sequence have an edge in common (the first and the last also share a common edge). Before adding to the "eye bolt" H_i as in

Example 1, we thicken E_i first so that the thickened E_i plus the eye bolt H_i is topologically a solid torus T_i. We call T_i a cube-with-eye-bolt.

The 2-sphere S of Example 2 lies in $\sum T_i = W_1$ and intersects $\sum \text{Bd } T_i$ only in $\sum \text{Bd } E_i$. It follows from Theorem 11 that the simple closed curve J of Example 1 cannot be shrunk to a point in $E^3 - W_1$.

A slice is removed from the loop of T_i to form a topological cube K_i. Then Bd E_i separates Bd K_i into two disks one of which has a "hook" in it. The interior of this hooked disk is pushed slightly into Int T_i so as to form a disk D_i as shown in Figure 3. The disk D_i lies except for its boundary Bd D_i =

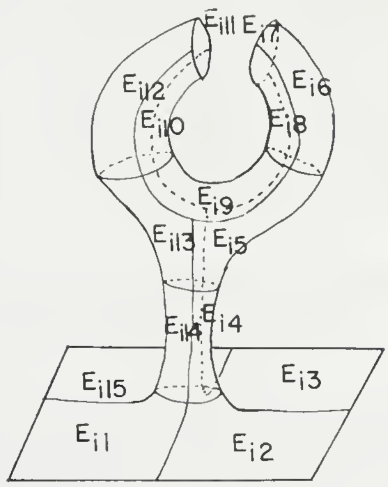

Figure 3

Bd E_i in Int T_i. The 2-sphere $S_1 = \sum D_i$ is an approximation to the 2-sphere S of our example.

Each D_i is given a cellular subdivision into small disks $E_{i1}, E_{i2}, \cdots, E_{i15}$ as shown in Figure 3 such that adjacent D_{ij}'s in the sequence share an edge. It is not essential that there be exactly 15 elements in the subdivision and the scheme suggested is only for convenience.

Each E_{ij} is thickened and an eye bolt added to it to form a cube-with-eye-bolt T_{ij}. The loop in the eye bolt T_{ij} goes around the stem of the eye bolt H_{ij+1} where $j + 1$ is computed mod 15. Also, the stems of T_{i7} and T_{i11} intertwine as shown in Figure 4 so that a simple closed curve linking T_i cannot be shrunk to a point in the complement of $\sum_{j=1}^{15} T_{ij}$. It follows from Theorem 9 followed by Theorem 11 that no such shrinking is possible. The eye bolts run so close to D_i that $\sum_{j=1}^{15} T_{ij} \subset T_i$.

Just as a disk D_i is placed in each T_i, a disk D_{ij} is placed in each T_{ij}. Let $S_2 = \sum D_{ij}$ and $W_2 = \sum T_{ij}$.

Perhaps we have gone far enough for the reader to anticipate the description of S. It is to be the limit of a sequence of 2 spheres $S_0, S_1, S_2, S_3 \cdots$ of which we have already described S_0, S_1, S_2. Also, it is to be the intersection of a

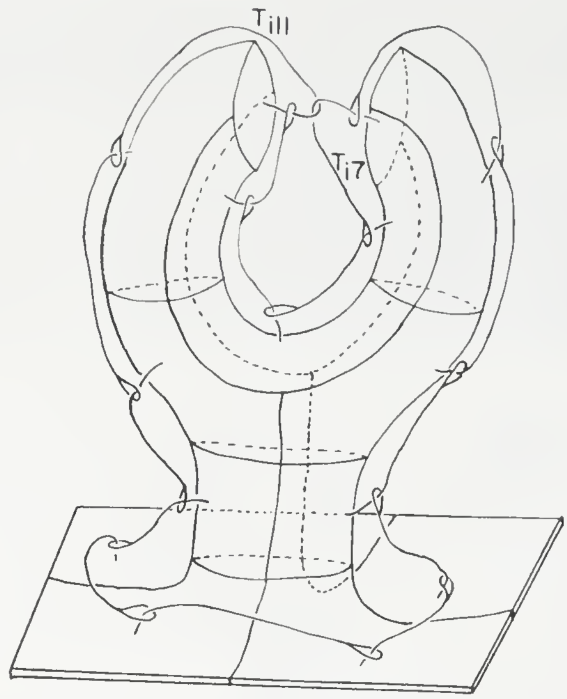

Figure 4

decreasing sequence of 3 manifolds with boundaries W_1, W_2, W_3, $\cdots$ of which we have already described W_1, W_2.

To get S_{n+1} from S_n, each of the special disks D in S_n is subdivided into 15 small special subdisks, each of the special subdisks is thickened into a topological cube and a set topologically equivalent to an eye bolt is added as suggested by Figure 4. The cube-with-eye-bolts T shown in Figure 4 are solid even though the figure shows the eye bolts as linear and the thickened subdisks as thin. The 3 manifold with boundary W_{n+1} is the sum of the T's at the $(n + 1)$-st stage and lies in W_n. A slice is cut in the loop of each T at the $(n + 1)$-st stage and a disk D like that shown in Figure 3 is put in each sliced T. The sum of these D's is the 2-sphere S_{n+1}.

The limit of the S_n's is a 2-sphere because each boundary of a special disk D at the n-th stages lies in each of the following S's (and hence in the limit) and if two D's at the n-th stage fail to intersect each other, their replacement at following stages lies in the W's at the n-th stages containing the D's so the limits of the replacements do not intersect each other.

J cannot be shrunk to a point in $E^3 - S$.

Let J be the simple closed curve shown in Figure 1. If J could be shrunk to a point in $E^3 - S$, it could be shrunk to a point in some $E^3 - W_n$. We show that this cannot occur.

Let W_i' be the set obtained from W_i by filling each hole in each eyebolt in W_i. Let W_i'' be the set obtained by removing slices from the eyebolts of W_i

and then adding back to each of these eye bolts the intertwining stems of the 7-th and 11-th eye bolts at the next stage.

Since W_1' is a hollow ball with handles and J circles one of these handles exactly once, J cannot be shrunk to a point in $E^3 - W_1'$. It follows from Theorem 11 that J cannot be shrunk to a point in $E^3 - W_1$. It follows from Theorem 9 that J cannot be shrunk to a point in $E^3 - W_1''$. Since W_2' is like W_1'' (after a slight isotopy shrinking W_1'') with more handles added, J cannot be shrunk to a point in $E^3 - W'$. It follows from Theorem 9 that J cannot be shrunk to a point in $E^3 = W_2$.

Suppose it has been shown that J cannot be shrunk to a point in $E^3 - W_i$. It follows from Theorem 9 that J cannot be shrunk to a point in $E^3 - W_i''$, by a slight isotopy of E^3 with the adding of handles that J cannot be shrunk to a point in $E^3 - W_{i+1}'$, and from Theorem 11 that J cannot be shrunk to a point in $E^3 - W_{i+1}$.

It is shown in Theorem 6 that for each disk Y in S. J can be shrunk to a point in $E^3 - Y$.

4. Cubes about subsets of the 2-sphere S of Example 2. We introduce the notion of associated annuli and associated cubes to prove some additional things about the 2-sphere S of Example 2. In this section we use D's, E's, T's and S's with the same meaning that they had in the preceding section.

Associated annulus. Recall that S_n is the sum of a collection of special disks D_i and that each D_i lies except for its boundary on the interior of a cube-with-eye-bolt T_i. Each T_i intersects the sum of the other T_i's in an annulus ring on Bd T_i. This annulus is called the annulus associated with T_i. The D's are defined at each succeeding stage so that each S_{n+k} intersects the annulus in exactly Bd D_i. In fact, the regularity of the annuli A_i associated with T_i is maintained so that A_i may be regarded as the Cartesian product $S^1 \times [0, 1]$ with Bd D_i corresponding to $S^1 \times [1/2]$ and if A' is an annulus at a later stage intersecting A_i, then there is an $\epsilon > 0$ and a subset X of S^1 such that $A' \cdot A_i$ corresponds to $X \times [1/2 - \epsilon, 1/2 + \epsilon]$.

Associated cube. Suppose C_i is a tame cube in the cube-with-eye-bolt T_i such that Bd $C_i \cdot$Bd T_i is the annulus associated with T_i. This cube C_i is called a cube associated with T_i. If $T_{i1}, T_{i2}, \cdots, T_{i15}$ are the cubes-with-eye-bolts in T_i, there is no cube associated with T_i that contains $\sum_{j=1}^{15} T_{ij}$ since one of the meridianal simple closed curves linking the eye of T_i can be shrunk to a point in the complement of any cube associated with T_i but not in the complement of $\sum_{j=1}^{15} T_{ij}$. However, we have the following result.

THEOREM 3. *Suppose $T_{i1}, T_{i2}, \cdots, T_{i15}$ are the cubes-with-eye-bolts in T_i and C_{i1} is a cube associated with T_{i1}. Then there is a cube C_i associated with T_i that contains $C_{i1} + \sum_{j=2}^{15} T_{ij}$.*

Proof. Let X_{ij} $(j > 1)$ be the solid cylinder in the eye of T_{ij}. Then $X_{ij} + T_{ij}$ is a tame 3-cell in T_i, X_{ij} misses T_{ik} $(k \neq j, j + 1)$ and $X_{ij} \cdot T_{ij+1}$ is a solid cylinder whose boundary consists of two disks on Bd X_{ij} and an annulus on Bd T_{ij+1}. With no loss of generality we suppose that C_{i1} and each T_{ij}, X_{ij} are polyhedral and Bd $X_{i15} \cdot$ Bd C_{i1} is the sum of a collection of mutually exclusive simple closed curves.

In a fashion reminiscent of the manner used in §2 to show that J can be shrunk to a point if we omitted H_{i+2} instead of H_{i+1}, we describe a cube in T_i containing $C_{i1} + T_{i15}$. However, we obtain this cube by pushing things over the far end of C_{i1} rather than over the far end of H_{i+1} as was done in §2.

Let J be a component of Bd $X_{i15} \cdot$ Bd C_{i1} that is an inside simple closed curve on Bd C_{i1} in that it bounds a disk D_J on Bd C_{i1} such that Int D_J does not intersect Bd X_{i15} + Bd D_{i1}. Let E_J be the disk in Bd X_{i15} which is bounded by J and misses T_{i15} $\cdot$ Let X'_{i15} be the 3-cell whose boundary is (Bd $X_{i15} - E_J) + D_J$. Suppose X''_{i15} is obtained from X'_{i15} by shoving it slightly to one side of Bd C_{i1} near D_J so that Bd $X''_{i15} \cdot$ Bd C_{i1} has fewer components than does Bd $X_{i15} \cdot$ Bd C_{i1}.

An iteration of the procedure described in the preceding paragraph yields a polyhedral 3-cell Y_{i15} in Int T_i such that

$$Y_{i15} \cdot T_{i15} = X_{i15} \cdot T_{i15} \subset \text{Bd } Y_{i15} \,.$$

$$Y_{i15} \cdot C_{i1} = 0, \quad \text{and}$$

$$Y_{i15} \cdot \sum_{j=2}^{14} (T_{ij} + X_{ij}) = 0.$$

Next we replace X_{i14} by a polyhedral 3-cell Y_{i14} which misses $C_{i1} + T_{i15} + Y_{i15} + \sum_{j=2}^{13} (T_{ij} + X_{ij})$. Only two steps are needed in making this replacement since Bd $X_{i14} \cdot$(Bd $T_{i15} + Y_{i15} + C_{i1}$) is the sum of only two simple closed curves on Bd T_{i15}.

We continue to replace the X_{ij}'s with Y_{ij}'s. Now $C_{i1} + \sum_{j=2}^{15} (T_{ij} + Y_{ij})$ is a polyhedral cube in T_i that contains $C_{i1} + \sum_{j=2}^{15} T_{ij}$. It contains a middle part of the annulus associated with T_i and it is a simple matter to expand $C_{i1} + \sum_{j=2}^{15} (T_{ij} + Y_{ij})$ until it contains all of this associated annulus.

THEOREM 4. *Suppose T is a cube-with-eye-bolt at some stage in the description of S and T' is a cube-with-eye-bolt at a later stage which lies in T. For each cube C' associated with T' there is a cube associated with T such that C contains C' and each cube-with-eye-bolt other than T' at the stage of T' which lies in T.*

Proof. We suppose that $T = T_1, T_2, \cdots, T_n = T'$ is a sequence such that the cube-with-eye-bolt T_i at one stage contains the cube-with-eye-bolt T_{i+1} at the next stage. For the stage containing T_i, let Z_i denote the sum of the T's at this stage other than T_i.

Theorem 3 shows that there is a cube C_{n-1} associated with T_{n-1} that contains $C' + Z_n \cdot T_{n-1}$. Another application of Theorem 3 shows that there is a cube C_{n-2} associated with T_{n-2} which contains $C_{n-1} + Z_{n-1} \cdot T_{n-2}$. We note that C_{n-2} also contains $C' + Z_n \cdot T_{n-2}$.

A succession of $n - 1$ applications of Theorem 3 shows that there is a cube $C_1 = C$ associated with $T_1 = H$ which contains $C' + Z_n \cdot T_1$.

The following theorem is an immediate consequence of Theorem 4.

THEOREM 5. *For each cube-with-eye-bolt H at some stage and each closed proper subset Y of $H \cdot S$ there is a tame cube C in H which contains Y.*

THEOREM 6. *For each closed proper subset Y of S of Example 2, the simple closed curve J shown in Figure 1 can be shrunk to a point in $E^3 - Y$.*

Proof. There is an integer n and a cube-with-eye-bolt H_n at the n-th stage that misses Y. Let H_1 be the cube-with-eye-bolt at the first stage containing H_n. It follows from Theorem 5 that there is a cube C_1 associated with H_1 that contains every cube-with-eye-bolt at the n-th stage in H_1 other than H_n.

It follows from the methods used in the proof of Theorem 3 that there is a tame hollow ball W (Cartesian product of a 2-sphere and an arc) in $E^3 - J$ that contains C_1 plus the cubes-with-eye-bolts at the first stage other than H_1. Since each component of $E^3 - W$ is simple connected, J can be shrunk to a point in $E^3 - W$ and hence in $E^3 - Y$.

5. Each simple closed curve in S is tame. Harrold, Griffith, and Posey have shown [10] that a simple closed curve J in E^3 is tame if it has the following two properties at each point p.

Property P. For each $\epsilon > 0$ there is a 2-sphere K of diameter less than ϵ such that p lies in the bounded component of $E^3 - K$ and $J \cdot K$ is a set of two points.

Property Q. An open subset of J containing p lies on a disk in E^3.

Originally, the conditions were imposed that the 2-sphere K mentioned in the definition of Property P was locally polyhedral mod the two points of $J \cdot K$ and the disk mentioned in the definition of Property Q were locally polyhedral mod the intersection of J and the disk. However, as pointed out by Harrold in [11], it follows as a consequence of [3] that these extra conditions are not necessary. Harrold [12] has called a simple closed curve locally peripherally unknotted if it has Property P and locally unknotted if it has Property Q.

Each simple closed curve J in S has Property Q at each point since the required disks can be found on S. Hence, we only need to show that Property P is satisfied at each point p of J.

THEOREM 7. *For each point p of each simple closed curve J' in 2-sphere S of Example 2, J' has Property* P *at p.*

Proof. Let D be an $\epsilon/2$ disk in S whose interior contains p and whose boundary meets J' in exactly two points. We use Theorem 3 or 4 to adjust D to get the required 2-sphere K.

First we consider the case where D lies in the cube-with-eye-bolt H_i, Bd D lies on Bd H_i, diameter $H_i < \epsilon$, and J' misses H_{i1}. In this case Theorem 3 gives the result immediately. We let C_{i1} be any cube whatsoever associated with H_{i1} and find from Theorem 3 that there is a cube C_i associated with H_i that contains $C_{i1} + \sum_{j=2}^{15} H_{ij}$. Then $J' \cdot D \sum_{j=2}^{15} H_{ij} \subset C_i$ and $J' \cdot \mathrm{Bd}\, C_i$ is exactly two points. In this case we let $K = \mathrm{Bd}\, C_i$.

If we omit the assumption that J' misses H_{i1}, it is not much more difficult to show that there is a required 2-sphere K since we only use Theorem 4 instead of Theorem 3. As J' does not fill the part of S in H_i and since the H's at later stages become very small, there is an H' at a later stage which lies in H_i but which misses J'. Theorem 4 shows that there is a cube C_i associated with H_i which contains all of the cubes-with-eye-bolts at this later stage which lie in H_i and which intersect J'. Again we let $K = \mathrm{Bd}\, C_i$.

We now consider the general case where D is not necessarily the intersection of S and a small H. Consider the eubes-with-eye-bolts at the various stages that intersect S only in Int D. Let $H_1, H_2, \cdots$ be an infinite collection of these which cover Int D and are such that each H_i is of diameter less than $\epsilon/4$, no one of the H's is a subset of any other H, and the collection of H's is locally finite at each point of Int D. It follows from Theorem 4 that there is a cube C_i associated with H_i that covers all of J' in H_i. Then Bd $D + \sum C_i$ is a 3-cell C of diameter less than ϵ which contains $D \cdot J'$. Furthermore, Bd C intersects J' only in the end points of $D \cdot J'$.

THEOREM 8. *If a wild 2-sphere is locally tame except at each point of a set X, each simple closed curve containing X in the 2-sphere is wild.*

Proof. Let D_1, D_2, be the two disks on the 2-sphere bounded by the simple closed curve. Each is locally tame except possibly on its boundary.

If Bd D_1 = Bd D_2 were tame, it would follow from [13] that each D_i is tame. In turn this implies [13] that the 2-sphere $D_1 + D_2$ is tame.

6. Another example. Suppose Example 2 is modified so that T_{i+15} loops around the stem of $T_{i+1,1}$ instead of around the stem of T_{i+1} where it is understood if T_n is the last T, $T_{n+1} = T_1$. If this change is made at each stage, there results a wild 2-sphere S such that for each disk D in S and each simple closed curve J in $E^3 - S$, J can be shrunk to a point $E^3 - D$.

Question. It is not evident that the arcs in the two sphere of the above example are tame. Is a 2-sphere S in E^3 tame if it has the following two properties?

1) Each arc in S is tame.

2) For each disk D in S and each simple closed curve J in $E^3 - S$, J can be shrunk to a point in $E^3 - D$.

In fact, I know of no wild 2-sphere in E^3 such that each of its arcs is tame and each of its complementary domains is simply connected.

7. Reducing sets without simplifying their complements. In this section we show that certain reductions may be made in a set without trivializing any of the elements of the fundamental group of its complement.

THEOREM 9. *Suppose C is a cylinder as shown in Figure 5 with bases B_1, B_2; T_1, T_2 are two linked topological tori imbedded in C as shown such that $T_i \cdot$ Bd*

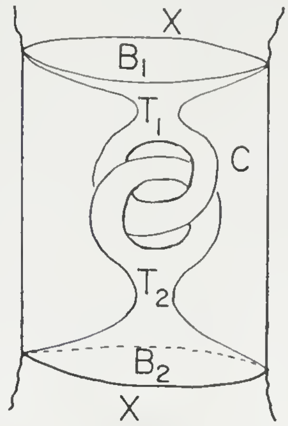

Figure 5

$C = B_i$; and X is a closed subset of E^3 such that $X \cdot C = B_1 + B_2$. If J is a simple closed curve in $E^3 - (X + C)$ that cannot be shrunk to a point in $E^3 - (X + C)$, it cannot be shrunk to a point in $E^3 - (X + T_1 + T_2)$.

Proof. Assume J can be shrunk to a point in $E^3 - (X + T_1 + T_2)$. Then there is a map f of a disk D into $E^3 - (X + T_1 + T_2)$ that takes Bd D homeomorphically onto J.

If f^{-1} (Bd C) $= 0$, $f(D)$ misses C and J can be shrunk to a point in $E^3 - (X + C)$. This is contrary to hypothesis.

We suppose that f^{-1} (Bd C) is the sum of a finite number of simple closed curves. Suppose furthermore that n is the least integer for which there is a map f of D into $E^3 - (X + T_1 + T_2)$ that takes Bd D homeomorphically onto J and such that f^{-1} (Bd C) is the sum of n simple closed curves. We finish the proof of Theorem 9 by showing that we can reduce n if it is not already 0.

Let J_1 be a simple closed curve in f^{-1} (Bd C) such that J_1 bounds a disk D_1 in D whose interior misses f^{-1} (Bd C). If we show that $f(J_1)$ can be shrunk to a point on the lateral boundary of C (that is, the map f of J_1 can be extended to map D_1 into the lateral boundary of C), then we can adjust f on a disk slightly larger than D_1 to reduce the number of components of f^{-1} (Bd C). We finish the proof of Theorem 9 by showing that $f(J_1)$ can be shrunk to a point on the lateral boundary of C. Either $f(\text{Int } D_1)$ belongs to Int C or Ext C and we treat each case separately.

Case 1. $f(\text{Int } D_1) \subset \text{Int } C$. Now $M_1 = C - (T_1 + T_2)$ is a 3-manifold with boundary. The fundamental group of M_1 is equivalent to the fundamental group of the complement of the finite graph G shown in Figure 6. This group has five generators a, b, c, d, e where a generator is represented by a path that starts at the eye (starting point), goes under G in the direction indicated by the arrow, and returns to the eye. The only relations are given at the crossing points and the branch points and these (reading from top to bottom of Figure 6)

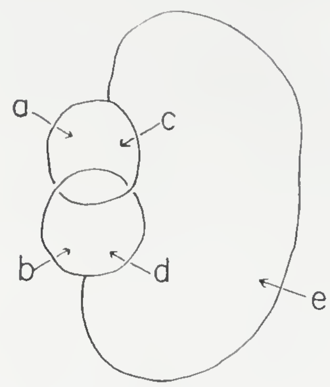

Figure 6

are $a = ec$, $cb = dc$, $ab = bc$, and $b = ed$. The four relations tend to eliminate the generators c, d, e and show that the fundamental group of M_1 is a free group on two generators a and b. Solving for e in terms of a and b we find that $e = ab^{-1}\, a^{-1}b$ which is nontrivial in the free group. Any loop that circles Bd C an integral number of times different from zero cannot be shrunk to a point in M_1. Any loop on Bd M_1 that can be shrunk to a point in M_1 can be shrunk to a point on Bd M_1.

Case 2. $f(\text{Int } D_1) \subset \text{Ext } C$. Let M_2 be the 3-manifold with boundary $E^3 - (X + \text{Int } C)$. If there is a singular closed curve on Bd M_2 that can be shrunk to a point in M_2 but not in Bd M_2, it follows from [14] that there is a simple closed curve with the same property. Such a simple closed curve could only be one which circles the lateral boundary of C once. The existence of such a simple closed curve would show that any loop on Bd M_2 could be shrunk to a point in M_2.

If any loop on Bd M_2 can be shrunk to a point in M_2, we arrive at the contradiction that f can be adjusted to shrink J to a point in $E^3 - (X + C)$. The first approximation to the adjusted f is unchanged on the component E of $D - f^{-1}(C)$ containing Bd D. This first approximation of f takes each component of $D - E$ into M_2. This first approximation to f takes certain points of D into Bd C but a pushing out gives an adjusted f which shrinks J to a point in $E^3 - (X + C)$.

Blankinship and Fox showed [7] that the exterior of a solid Alexander horned sphere [1] is not simple connected by computing its fundamental group. However, it is not necessary to compute this fundamental group to show the complement is not simply connected. We offer an alternate proof.

THEOREM 10. *The complement of a solid Alexander horned sphere is not simply connected.*

Proof. The solid Alexander horned sphere may be described as the intersection of a decreasing sequence of 3-manifolds with boundaries $H_1, H_2, \cdots$. H_1 is a torus. H_2 is obtained by cutting a slice in H_1 and replacing the slice with two solid tori as shown in Figure 5. H_3 is obtained by cutting slices in each of these two solid tori and replacing each slice with two solid tori as indicated in Figure 5. The process is continued. If J is a simple closed curve in $E^3 - H_1$ that cannot be shrunk to a point in $E^3 - H_1$, it follows from repeated use of Theorem 9 that J cannot be shrunk to a point in any $E^3 - H_i$. Hence it cannot be shrunk to a point in the complement of the solid Alexander horned sphere.

THEOREM 11. *Suppose C, B_1, B_2, B_3, C_1, T are as shown in Figure 7 where C is a topological cube; B_1, B_2, B_3 are three mutually exclusive disks on* Bd C;

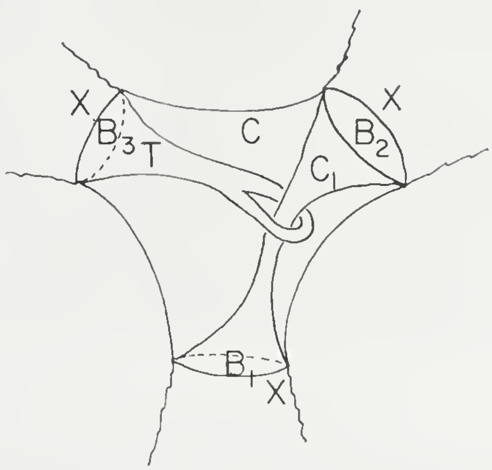

Figure 7

C_1 is a cube which lies except for disks B_1, B_2 on Int C; *and T is a solid torus which intersects* Bd C *in B_3 and circles C_1 as shown. Suppose X is a closed set which intersects C only in $B_1 + B_2 + B_3$ and is such that for $i = 1, 2$,* Bd B_i *cannot be shrunk to a point in $(E^3 - (X + C))$ +* Bd B_i. *Then if J is a simple closed curve in $E^3 - (X + C)$ that cannot be shrunk to a point in $E^3 - (X + C)$, then J cannot be shrunk to a point in $E^3 - (X + C_1 + T)$.*

Proof. If Bd B_3 can be shrunk to a point in $(E^3 - (X + C))$ + Bd B_3, J cannot even be shrunk to a point in $E^3 - (X + C_1)$ so we consider only the case in which Bd B_3 cannot be shrunk to a point in $(E^3 - (X + C))$ + Bd B_3. To see that this is true, consider a thin cylinder C' obtained by slicing off C slightly to one side of B_3. Then Bd C' intersects Bd C − Int B_3 in an annulus A. If J can be shrunk to a point in $E^3 - (X + C_1)$, it can be shrunk to a point in $E^3 - (X +$ Int C' + Int $A)$. However, if Bd B_3 can be shrunk to a point in $(E^3 - (X + C))$ + Bd B_3, each loop in Int A can be shrunk to a point in $(E^3 - (X + C))$ + Int A so there is no need to intersect Int A in shrinking J

to a point in $E^3 - (X + \text{Int } C' + \text{Int } A)$. If Int A is avoided, J is shrunk in $E^3 - (X + C)$.

Let $M_1 = C - (T + C_1)$. It is a 3-manifold with boundary. We compute its fundamental group is a fashion similar to that used in computing the fundamental group of M_1 of Theorem 9 and find that it is a free group on two generators.

Each simple closed curve in $A = \text{Bd } C - (\text{Int } B_1 + \text{Int } B_2 + \text{Int } B_3)$ either bounds a disk in A or is homotopic in A to either Bd B_1, Bd B_2, or Bd B_3. Since each Bd B_i corresponds to a nontrivial element of the fundamental group of $M_1 + \sum \text{Bd } B_i$, it follows from a result of Papakyriakopoulos [13] that each loop on Bd M_1 that can be shrunk to a point in M_1 can be shrunk to a point in Bd M_1.

Let $M_2 = E^3 - (X + \text{Int } C)$. Using Papakyriakopoulos' result again, the fact that each simple closed curve in A either bounds a disk in A or is homotopic in A to some Bd B_i ($i = 1, 2, 3$), and the conditions that no Bd B_i ($i = 1, 2, 3$) can be shrunk to a point in $M_2 + \text{Bd } B_i$, it follows that a loop on Bd M_2 can be shrunk to a point on Bd M_2 if it can be shrunk to a point in M_2.

The proof is completed in a manner analogous to the way in which the proof of Theorem 9 was finished. We suppose that there is a map f of a disk D into $E^3 - (X + C_1 + T)$ that maps Bd D homeomorphically onto J. We consider the number of components of f^{-1} (Bd C) and show that f can be adjusted to reduce this number to zero. But this adjusted f shows that J could be shrunk to a point in $E^3 - (X + C)$.

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