How to represent a function in a quantum computer

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Question: Given a function $f : [0,1] \to [-1,1]$, does there exist a sequence of coefficients $\{\psi_k\}_{k=0}^{\infty}$ for which

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Making sense of of the matrix product

Consider the truncated product

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Mutliplying two such matrices together yields another such matrix. Indeed

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So $U_d(x, \{\psi_k\}_{k\geqslant 0}) = \begin{pmatrix} poly(x) & poly(x)\sqrt{1-x^2} \\ poly(x)\sqrt{1-x^2} & poly(x) \end{pmatrix}$.

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Theorem (A.-Lin-Mnatsakanyan-Thiele-Wang, 2024)

If $f : [0,1] \to [-1,1]$ satisfies $\int_0^1 \log(1 - f(x)^2) \frac{dx}{\sqrt{1-x^2}} > -\infty$, then there exists a unique coefficient sequence $\{\psi_k\}_{k \geqslant 0}$ such that the imaginary parts of the upper left entries of $U_d(x, \{\psi_k\}_{k \geqslant 0})$ converge to f in $L^2\left(\frac{dx}{\sqrt{1-x^2}}\right)$, and we have the nonlinear Plancherel identity

$$\sum_k \log(1 + \tan^2 \psi_{|k|}) = -\frac{2}{\pi} \int_0^1 \log\left(1 - f(x)^2\right) \frac{dx}{\sqrt{1-x^2}} .$$

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The key idea here is to use nonlinear Fourier analysis!

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The NLFT of a sequence $(F_k)_k$ is the matrix function

$$\prod_{j=-\infty}^{\infty} \frac{1}{\sqrt{1+|F_j|^2}} \begin{pmatrix} 1 & F_j z^j \\ -\overline{F_j} z^{-j} & 1 \end{pmatrix}$$

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Recall $SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \ : \ |\alpha|^2 + |\beta|^2 = 1 \right\}$.

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$$\begin{pmatrix} a(z) & b(z) \\ -\overline{b(z)} & \overline{a(z)} \end{pmatrix} := \prod_{j=-\infty}^{\infty} \frac{1}{\sqrt{1+|F_j|^2}} \begin{pmatrix} 1 & F_j z^j \\ -\overline{F_j} z^{-j} & 1 \end{pmatrix}$$

where the matrix product should be read left to right as j increases. NLFT is well-defined for compactly supported $\{F_k\}$, since only finitely many matrices above are not the identity.

Recall $SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \ : \ |\alpha|^2 + |\beta|^2 = 1 \right\}$.

Thus the NLFT of F is an $SU(2)$-valued function, and so the determinant condition $|a|^2 + |b|^2 = 1$ then holds for $z \in \mathbb{T}$.

The $SU(2)$-nonlinear Fourier transform (NLFT)

Let z denote an element of the unit circle $\mathbb{T} \subset \mathbb{C}$:

linear Fourier transform	:	$(F_k)_k$	$\mapsto$	Laurent series $\sum_k F_k z^k$
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We abbreviate this $SU(2)$ matrix as the pair (a, b), and call (a, b) the NLFT of $(F_k)_k$.

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$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} = \left(\prod_j (1 + |F_j|^2)^{-\frac{1}{2}} \right) \prod_j \begin{pmatrix} 1 & F_j z^j \\ -\overline{F_j} z^{-j} & 1 \end{pmatrix}$$

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$n = 1$:

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$n = 0:$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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This expansion also shows that $\int_{\mathbb{T}} a = \prod_j \frac{1}{\sqrt{1+|F_j|^2}}$

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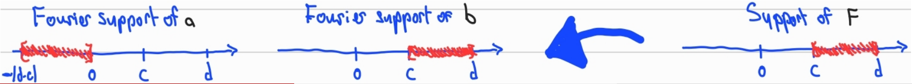
$b = \quad \sum_j F_j z^j +$ higher order terms

This expansion also shows that $\int_{\mathbb{T}} a = \prod_j \frac{1}{\sqrt{1+|F_j|^2}}$, and the support of F dictates the frequency supports of a, b.



The L^2 theory for the NLFT

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A contour integral of $2\log|a(z)|$ yields

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where $\{z_k\}$ are the zeros of the analytic extension of $\overline{a}$ in $\mathbb{D}$. This is known as nonlinear Plancherel.

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$$(a, b) := (a_-, b_-)(a_+, b_+)$$

where $(a_-, b_-)\in\mathbf{H}_{<0}$ is the NLFT of $F_{<0}$, and $(a_+, b_+)\in\mathbf{H}_{\geqslant 0}$ is the NLFT of $F_{\geqslant 0}$.

From QSP to the NLFT

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Recall the truncated QSP

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$$e^{ik\theta Z}e^{i\psi_k X}e^{-ik\theta Z} = \frac{1}{\sqrt{1+|F_k|^2}}\begin{pmatrix} 1 & F_k z^k \\ -\overline{F_k}z^{-k} & 1 \end{pmatrix} \text{ where } z = e^{2i\theta}.$$

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- So (A, B) exists and is always unique! One can show then that (a_+, b_+) exists and is always unique too!

Thank you for Listening!