BiLipschitz (in)stability of weighted norm inequalities for singular integral operators

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February 21, 2023

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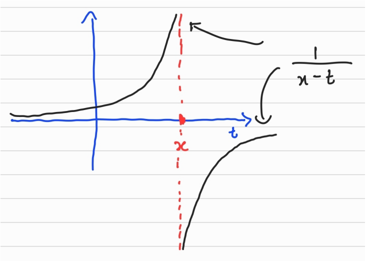

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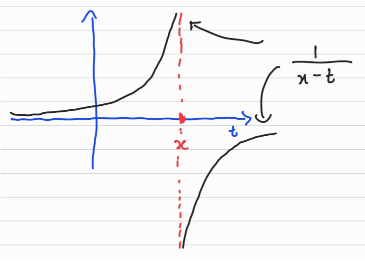

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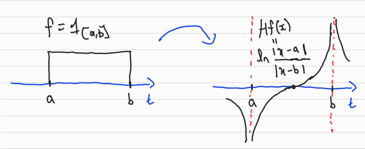

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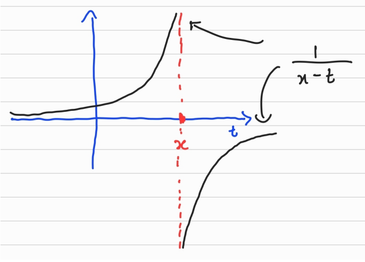

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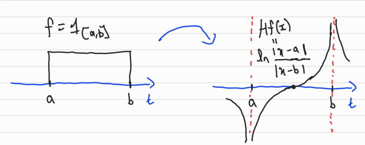

The Hilbert transform is an example of a **Calderón-Zgymund Operator** (CZO).

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- Beurling transform on $\mathbb{C}$, with $K(z,w) = \frac{1}{(z-w)^2}$.

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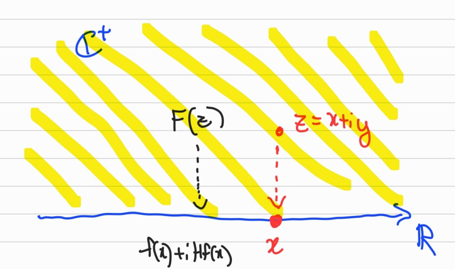

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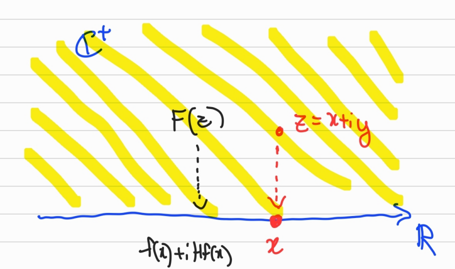

One can check that for $z = x + iy$, as $y \to 0^+$, we get

$$\lim_{y\to 0^+} F(x+iy) = f(x) + iHf(x)\,,$$

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Hilbert transform, Riesz transforms and Beurling transform are all "nice."

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The above inequality holds if and only if $\mu = w(x)\,dx$ is an absolutely continuous measure and has finite A_2 characteristic,

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Calculus Exercise: show that if $w = |x|^\alpha$ on $\mathbb{R}$, then $A_2(w, w^{-1}) < \infty$ if and only if $-1 < \alpha < 1$.

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Assume ω and σ don't have common point masses and a stronger A_2 condition holds where the averages are replaced by Poisson averages. Then

$$\|H_\sigma f(x)\|_{L^2(\omega)} \leqslant C\|f\|_{L^2(\sigma)} \text{ for all } f$$

if and only if

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i.e. morally, to check the norm inequality, it suffices to *test* the norm inequality over indicators of cubes, i.e. check the "testing condition."

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Theorem (Lacey-Sawyer-Uriarte-Tuero, 2012)

$\exists$ measures σ, ω and a biLipchitz map φ such that the norm inequality for H holds w.r.t. (σ, ω), but does not w.r.t. the pushforwards $(\varphi_* \sigma, \varphi_* \omega)$.

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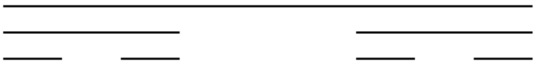

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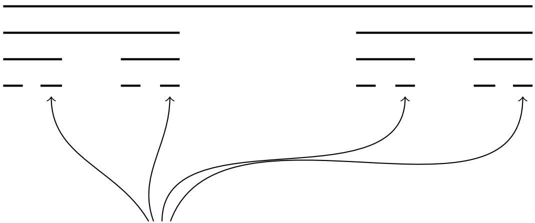

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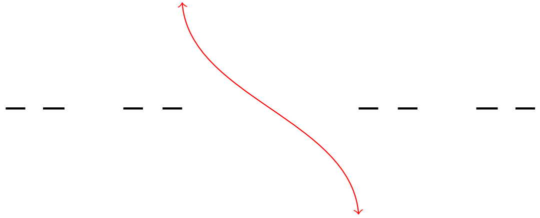

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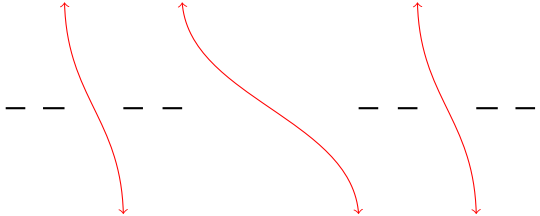

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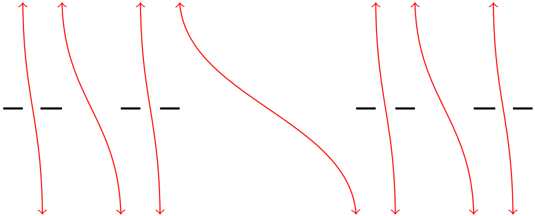

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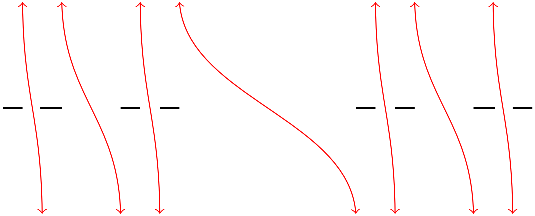

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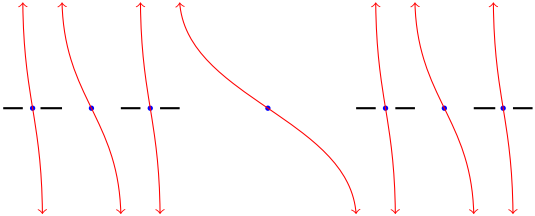

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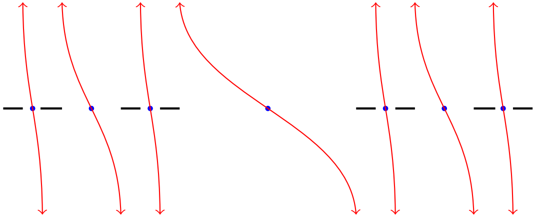

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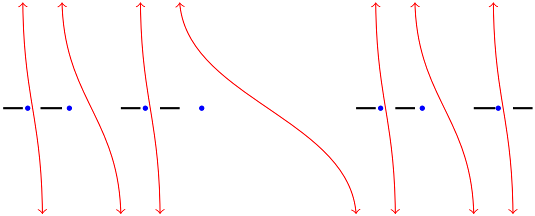

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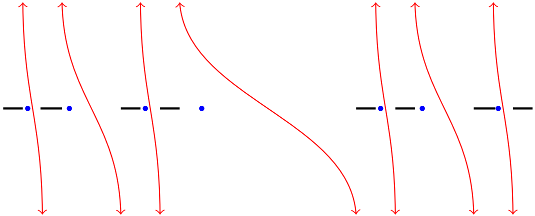

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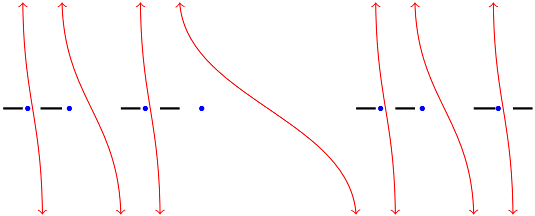

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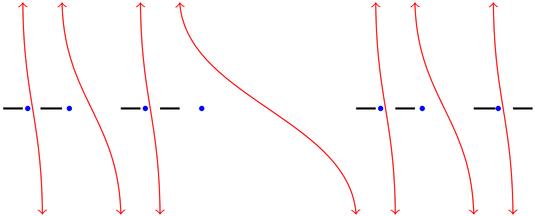

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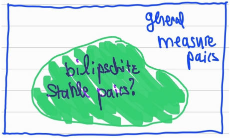

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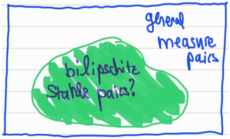

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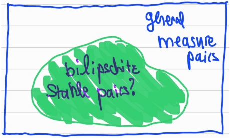

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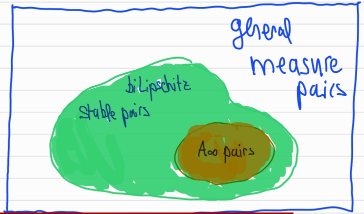

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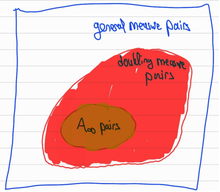

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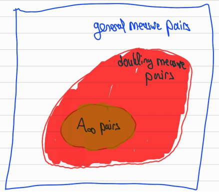

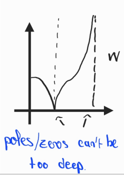

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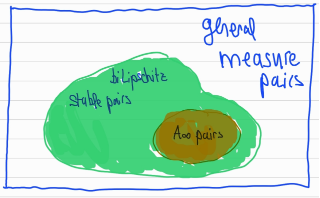

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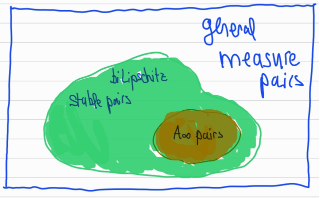

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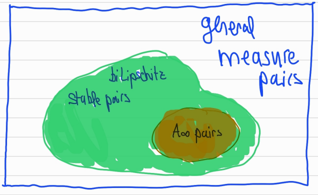

Theorem (A.-Luna-Garcia-Sawyer-Uriarte-Tuero, 2022)

*Let R_j be a Riesz transform. For every $\epsilon > 0$, there exists an angle $\theta \in (0, \epsilon)$ and doubling measures σ, ω such that the norm inequality for R_j holds w.r.t. (σ, ω), but does not w.r.t. the rotated measures $(\sigma_\theta, \omega_\theta)$.**

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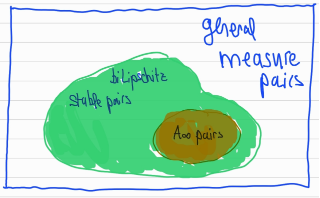

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Let R_j be a Riesz transform. For every $\epsilon > 0$, there exists an angle $\theta \in (0, \epsilon)$ and doubling measures σ, ω such that the norm inequality for R_j holds w.r.t. (σ, ω), but does not w.r.t. the rotated measures $(\sigma_\theta, \omega_\theta)$. *

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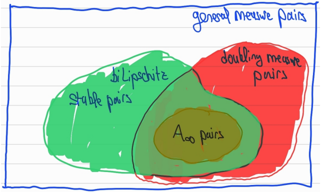

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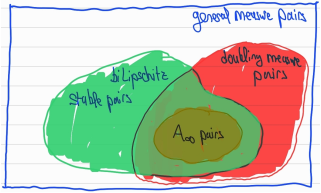

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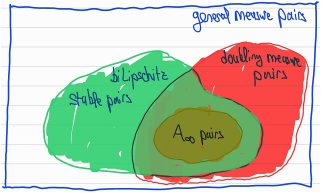

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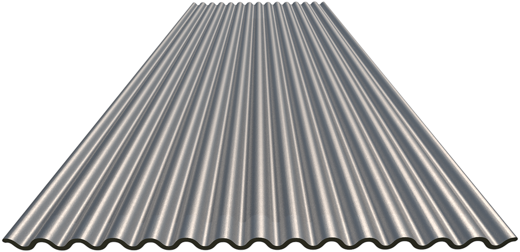

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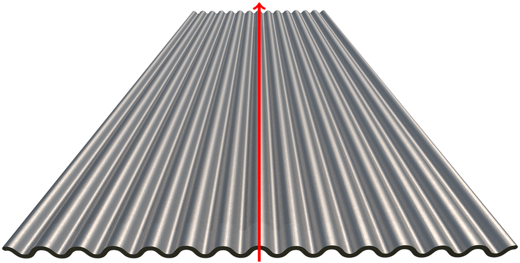

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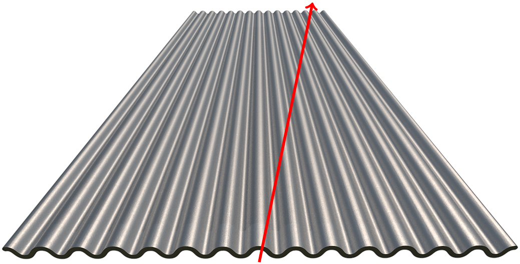

Start with R_2: it points vertically along the level curves of the sheet metal. Rotate R_2 slightly: it sees all the ripples all of a sudden.

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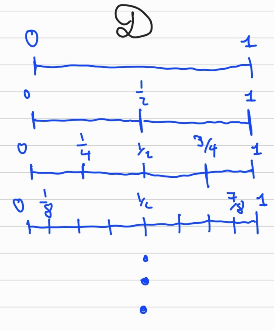

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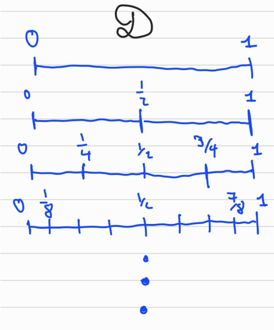

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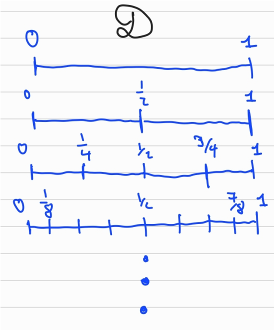

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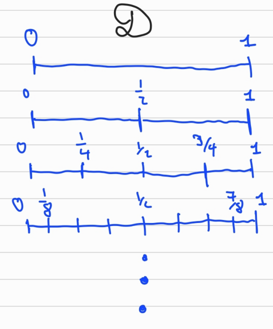

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(Note: virtually only way I know to get doubling weight counter-examples is via this method.)

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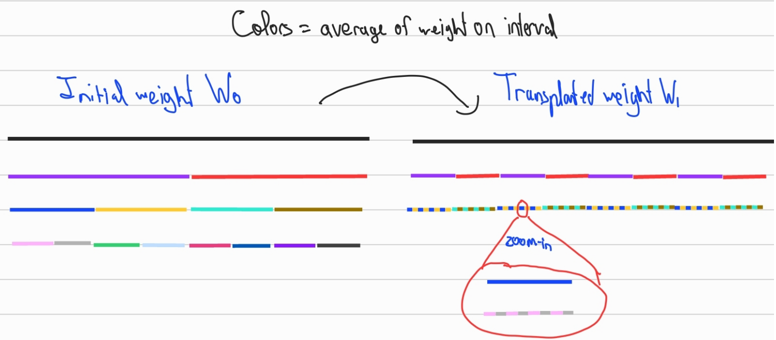

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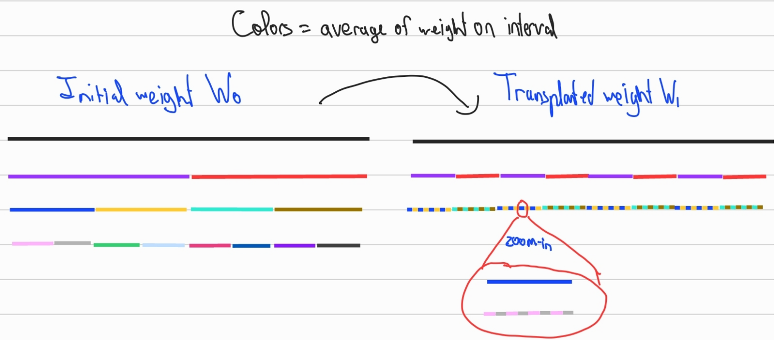

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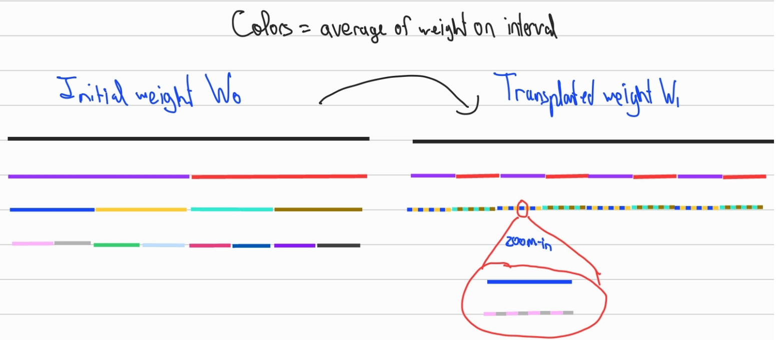

Why do this? The weight W_0 on the left can be written as

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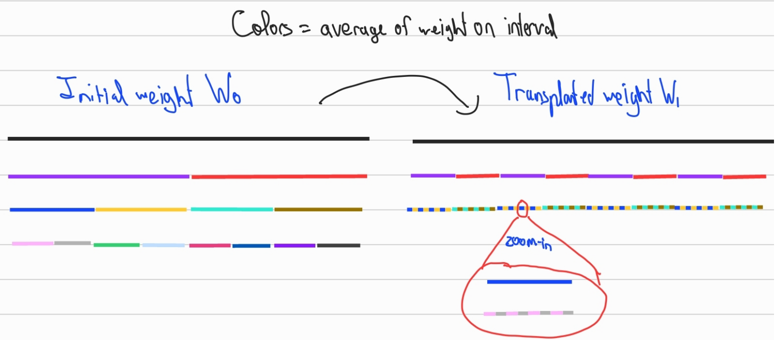

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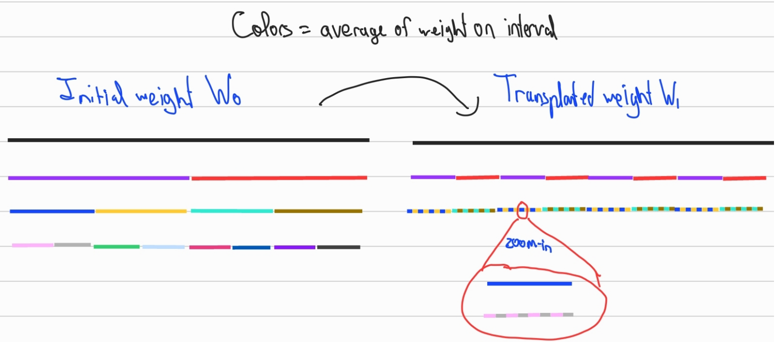

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By re-arranging the weight W_0 into W_1, we preserve the basic data of W_0 (has same averages on cubes), but shift all the Haar wavelets to frequencies of drastically different scales, ensuring they are well-localized and don't "interfere" with each other.

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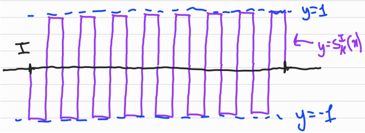
By relocating Haar data and increasing its frequency, to get $W_1 = E_{[0,1]} W_1 + \sum_{I \in \mathcal{S}} c_I s^I_{k_I}$, where s^I_k is an alternating function of frequency 2^k on I, and $\mathcal{S}$ is a set of intervals at sparse scales.



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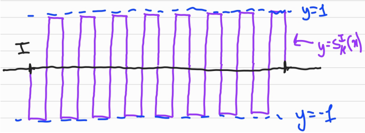

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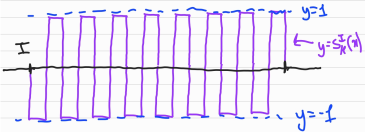

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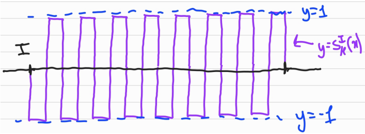

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- Since the real part is going to 0, then $(Hs_k^I)^2 \to \mathbf{1}_I$.

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$$s_k^Q(x) \equiv \left(s_k^{Q_1} \otimes \mathbf{1}_{Q_2}\right)(x) = s_k^{Q_1}(x_1)\mathbf{1}_{Q_2}(x_2)\,, \qquad \text{where cube } Q \equiv Q_1 \times Q_2\,.$$

Proposition (A.-Luna-Garcia-Sawyer-Uriarte-Tuero, 2022)

1. $R_1\left(s_k^{Q_1} \otimes \mathbf{1}_{Q_2}\right) \to \left(Hs_k^{Q_1}\right) \otimes \mathbf{1}_{Q_2}$ *weakly.*
2. $R_2\left(s_k^{Q_1} \otimes \mathbf{1}_{Q_2}\right) \to 0$ *in* $L^p(\mathbb{R}^2)$.

Tensoring weights, and the needed reductions

Now tensor $1D$ weights with $\mathbf{1}_{[0,1]}(x_2)$ to get "sheet metal" weights σ, ω.

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By the first property, we can apply the same techniques as for the Hilbert transform and get norm for R_1 to blow-up.

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By the first property, we can apply the same techniques as for the Hilbert transform and get norm for R_1 to blow-up.

If we rotate our measures 90 degrees, our oscillatory functions are killed off by R_2, and so the norm of R_2 is under control!

Thank you for Listening!