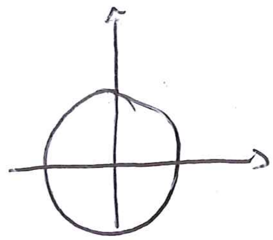
Steklov pb for trig polys ⊥ Muckenhoupt weight ①

(an ode to contractions)

Consider $\mathbb{T} \subseteq \mathbb{C}$



$1, e^{it}, e^{2it}, e^{3it}$

$1, z, z^2, z^3, \dots \quad \perp \quad dm$ (normalized Lebesgue)

$\Big\downarrow$ G.S. $\qquad\qquad \Big\downarrow$

$\varphi_0, \varphi_1, \varphi_2, \dots \quad \perp \quad w\,dm \qquad (w \in L^1)$

where φ_n = orthonormal poly of deg n.

<u>Q:</u> How do $\{\varphi_n(z)\}$ behave? (E.g. ptwise.)

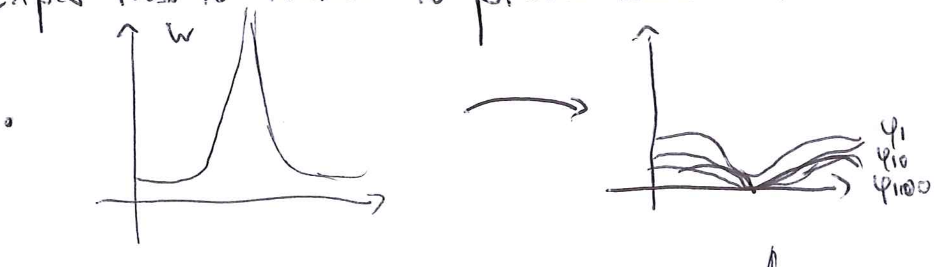
<u>Idea:</u> $\int_{\mathbb{T}} |\varphi_n|^2 w = 1$, so intuitively $|\varphi_n|^2 w \to 1$ or $|\varphi_n| \to \frac{1}{w^{1/2}}$

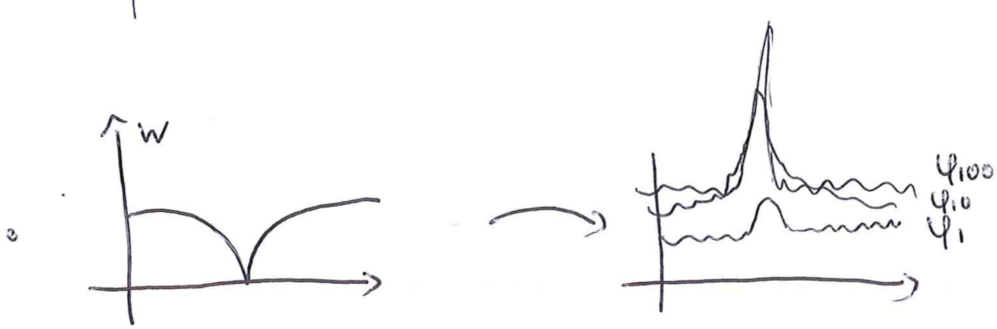
<u>Example:</u> • $\frac{1}{|\varphi_n|^2} \xrightarrow{\text{weak-}*} w \qquad (\to d\sigma)$

• $\log \frac{1}{|\varphi_n|^2} \xrightarrow[\text{and } L^1]{\text{weak}*} \log w$ if $\int \log w > -\infty$.

$\left(\log \frac{1}{|\varphi_n|^2} \to \log \sigma'\right)$

Expect this to translate to ptwise behavior too:





also if σ is Szegő, then $\lim_{n\to\infty} \int |\varphi_n|^2 \, d\sigma_s = 0$

Cor 5.9 in Khrushchev

<u>Steklov's Conjecture:</u> If $w \geq \delta > 0$ then $\{\varphi_n\}$ bded above, i.e. $\sup_n \|\varphi_n\|_{L^\infty} < \infty$.

<u>A:</u> False; can make w oscillate a ton to make φ_n blow-up

Maybe asking for too much regularity.

↳ Is there $p>2$ s.t. $\sup_n \|\varphi_n\|_{L^p(w)} < \infty$?

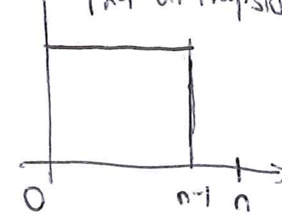
<u>Rk:</u> If $\int \log w > -\infty$ (zeros not too deep), then $\varphi_n(z) \sim \Phi_n(z)$ ptwise, where Φ_n = monic orthogonal poly of deg n (makes things easier).

<u>Prop:</u> Spose $w, w^{-1} \in L^\infty$. Then $\exists\, p>2$ s.t. $\sup_n \|\Phi_n\|_{L^p(dm)} < \infty$.

<u>Pf:</u> By rescaling, assume wlog $\varepsilon \le w \le 1$. Have

$$\begin{cases} ① \ \Phi_n = z^n + P_{n-1}\Phi_n & \text{(monic poly of deg } n) \\ ② \ P_{n-1}\Phi_n w = 0 & (\perp_w;\ \text{by G.S. have } \Phi_n \perp_w 1, z, z^2, \dots, z^{n-1}) \end{cases}$$

P_{n-1} on freq. side



where P_{n-1} = projection onto $\mathrm{Span}\{1, z, z^2, \dots, z^{n-1}\}$

$$① - ② \Rightarrow \Phi_n = z^n + P_{n-1}(1-w)\Phi_n$$

$$\Leftrightarrow \Big(I - \underbrace{P_{n-1}(1-w)}_{A_n}\Big)\Phi_n = z^n.$$

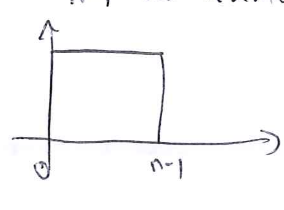
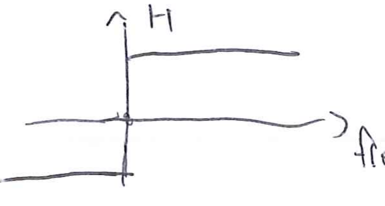
<u>Idea:</u> if we can show $\|A_n\|_{L^p \to L^p} < 1-\delta$ uniformly in n, then by geometric sum formula

$$\Phi_n = \sum_{k=0}^{\infty} (A_n)^k z^n$$

so $\| \ \|_{L^p} \le \sum_{k=0}^{\infty} (1-\delta)^k < \infty$.

So just need uniform L^p-estimate on A_n for some $p>2$

$A_n = P_{n-1}\underbrace{(1-w)}_{0 \le \cdot \le 1-\varepsilon}$, and P_{n-1} is a linear combo of Hilbert transforms and I.

Can get $\|P_{n-1}\|_{p\to p} \lesssim \|H\|_{p\to p}$.

Since $\|H\|_{2\to 2} = 1$, can choose p near 2 s.t. ~~Hol~~ $\|P_{n-1}\| \lesssim \|H\|_{p\to p} \le 1+\varepsilon$,

so $\|A_n\|_{p\to p} \le 1-\varepsilon^2 < 1$. $\square$

~~Also~~ Next? Need bddedness of Hilbert transform here.

Thm (Coiffman-Rochberg-Weiss): If $b \in BMO$, T a SIO, then

$\|[b,T]\|_{p\to p} \lesssim \|b\|_{BMO}$.

Thm (Denisov-Rush, '16): If $w, w^{-1} \in BMO$ then

$\exists p > 2$ s.t. $\{\Phi_n\}$ bded in $L^p(dx)$.

"Easy Case Pf:" Assume $1 \le w$, $\|w\|_{BMO} < \varepsilon$.

$$\begin{cases} ① \ \Phi_n = z^n + P_{n-1}\Phi_n \\ ② \ P_{n-1} w \Phi_n = 0 \end{cases} \Rightarrow \begin{cases} ① \ \Phi_n = z^n + w^{-1} w P_{n-1} \Phi_n \\ ② \ w^{-1} P_{n-1} w \Phi_n = 0 \end{cases}$$

erase and modify

$$① - ② \Rightarrow \Phi_n = z^n + \underbrace{w^{-1}}_{\le 1} \underbrace{[w, P_{n-1}]}_{\lesssim \varepsilon} \Phi_n$$

For ε sufficiently small, can invert like before. $\square$

Other weights? Natural step is Muckenhoupt weights for Hilbert transform bddedness.

Def: For $1<p<\infty$, define A_p to be the set of all weights w satisfying

$$[w]_{A_p} := \sup_I \langle w \rangle_I \langle w^{-\frac{p'}{p}} \rangle^{\frac{p}{p'}} < \infty.$$

Facts about A_p weights: • $[w]_{A_p} \ge 1$ (Hölder) with equality iff $w = cst$;

• the closer $[w]_{A_p}$ is to 1, the more "flat" w is.

• A_p is increasing, i.e. $A_{p_0} = \bigcup_{p<p_0} A_p$, or $p_0 < p_1 \Rightarrow A_{p_0} \subset A_{p_1}$

and $[w]_{A_{p_1}} \le [w]_{A_{p_0}}$ Hölder

- If $w \in A_p$, then $\log w \in BMO$ and $[w]_{A_p} = 1+\tau \Rightarrow \|\log w\|_{BMO} \lesssim \tau^{1/2}$,
- If T a CZO, then $T: L^p(w) \to L^p(w)$ bdd when $w \in A_p$, (and $\|T\|_{L^p(w)\to L^p(w)} \lesssim [w]_{A_p}^{\max(1, \frac{1}{p-1})}$)
- (Openness) if $w \in A_p$, then $\exists\, \varepsilon > 0$ s.t. $w \in A_{p-\varepsilon}$.

__Examples:__ $|\theta|^\alpha \in A_p$ if $-1 < \alpha < p-1$

For us, $p=2$ is interesting.

__Rk:__ ~~$A: L^p(w)$~~
- $\|A\|_{L^p(w) \to L^p(w)} = \|w^{1/p} A w^{-1/p}\|_{L^p(dm)}$
- If A is self-adjt, then $\|w^{-1/p'} A w^{1/p'}\|_{L^p(dm)} \overset{\text{duality}}{=} \|w^{1/p'} A w^{-1/p'}\|_{L^{p'}(dm)}$

$$= \|A\|_{L^{p'}(w)}.$$

__Prop__ (A.-Aptekarev-Denisov, '19): If $[w]_{A_2} = 1+\tau$, $\tau \le 1$, then $\exists\, p(\tau)$ s.t. $\|\Phi_n\|_{L^{p(\tau)}(w)} < \infty$, with $p(\tau) \nearrow +\infty$ as $\tau \to 0$.

(The flatter the weight, the better the regularity of $\{\Phi_n\}$)

__Pf:__ Since $\|\Phi_n\|_{L^p(w)} = \|\Phi_n w^{1/p}\|_{L^p(dm)}$, want to bd $\Phi_n w^{1/p}$ in $L^p(dm)$.

$$\begin{cases} \Phi_n = z^n + P_{n-1}\Phi_n \\ P_{n-1}\Phi_n w = 0 \end{cases} \Rightarrow \begin{cases} (w^{1/p}\Phi_n) = w^{1/p} z^n + w^{1/p} P_{n-1} w^{-1/p} (w^{1/p}\Phi_n) \\ w^{-1/p'} P_{n-1} w^{1/p'} (w^{1/p}\Phi_n) = 0 \end{cases}$$

$$\overset{(1)-(2)}{\Rightarrow} \quad w^{1/p}\Phi_n = w^{1/p} z^n + \underbrace{\Big(w^{1/p} \overset{A}{P_{n-1}} w^{-1/p} - w^{-1/p'} \overset{B}{P_{n-1}} w^{1/p'}\Big)}_{Q_{w,p}} w^{1/p}\Phi_n$$

~~or~~

or $(I - Q_{w,p})\, w^{1/p}\Phi_n = w^{1/p} z^n$.

__Idea:__ for $[w]_{A_2} = 1+\tau$, $\tau \ll 1$, w is relatively flat, so $A, B \approx P_{n-1}$, i.e. $Q_{w,p} = A - B \approx 0$, or rather $Q_{w,p}$ is small, and can use geometric sum trick.

<u>Lemma</u> (A.-Aptekarev-Denisov; Lipschitz cty of weighted operators)

Spose H an operator s.t. $\|w^{1/p} H w^{-1/p}\|_{p,p} \leq F([w]_{A_p}, p)$,
cts in $[w]_{A_p}$

then $\|w^{1/p} H w^{-1/p} - H\|_{p,p} \lesssim \|\log w\|_{BMO}$.

(in particular, applies to H a CZO; Pattakos-Volberg before had just cty of norms)

↳ For us, get $\|w^{1/p} H w^{-1/p} - H\|_{p,p} \lesssim \|\log w\|_{BMO}$

~~So pick p large as you wa~~

So for $p \gg 1$, choose $\varepsilon \ll 1$ so that everything added-up $\leq 1-\delta < 1$ uniformly. □

In fact, can do more.

<u>Thm</u>: Spose $[w]_{A_2} \gg 1$. Then $\exists p > 2$ s.t. $I - Q_{w,p}$ invertible on $L^p(dx)$, with $\|(I - Q_{w,p})^{-1}\|_{p\to p} < C$.

In particular, get $\{\Phi_n\}$ bded in $L^p(w)$.

<u>Pf</u>: • $w \in A_2$ so $\exists p_0' < 2$ s.t. $w \in A_{p_0'}$; get

$\|Q_{w,p}\|_{p\to p} \leq \|A\|_{p\to p} + \|B\|_{p\to p} \leq C \quad \forall\, 2 \leq p \leq p_0$ (by remarks)

• Define $Q_{w,p(z)}$ analytic operator where $\dfrac{1}{p(z)} = \dfrac{z}{p_0} + \dfrac{1-z}{2}$.

<u>Key idea</u>: if $p(z) = 2$, $Q_{w,2}$ is anti-analytic, i.e. $Q_{w,2}^* = -Q_{w,2}$.

$\Rightarrow$ Spectrum $(Q_{w,2})$ is imaginary.

So $(I - Q_{w,2})^{-1}$ exists and $\|(I - Q_{w,2})^{-1}\| \leq 1$.

~~Start with~~

- Chop I to $I - Q_{w,p}$ into N pieces

so that $\left\| \frac{Q_{w,p}}{N} \right\|_{p \to p} \leq \frac{1}{2}$

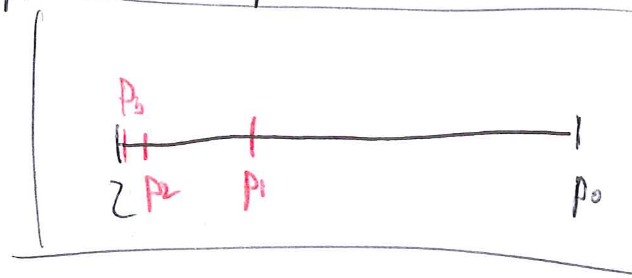
$\forall\, 2 \leq p \leq p_0$

$\hookrightarrow$ then $\left(I - \frac{Q_{w,p}}{N}\right)^{-1}$ exists in $L^p(dm)$, but all we can say is

$$\left\| \left(I - \frac{Q_{w,p}}{N}\right)^{-1} \right\|_{p,p} \leq 10^{10} \quad \text{for } 2 \leq p \leq p_0.$$

But have good L^2 bound, so complex interpolate to get

$$\left\| \left(I - \frac{Q_{w,p}}{N}\right)^{-1} \right\|_{p,p} \leq 2 \quad \text{for } 2 \leq p \leq p_1 \text{ for some } p_1.$$



Now add next piece

$$I - \frac{2Q_{w,p}}{N} = \underbrace{\left(I - \frac{Q_{w,p}}{N}\right)}_{\text{invertible}} - \underbrace{\frac{Q_{w,p}}{N}}_{\text{small}} \quad \text{in } L^p,\ 2 \leq p \leq p_1$$

By geometric sum,

get $\left\| \left(I - \frac{2Q_{w,p}}{N}\right)^{-1} \right\|_{p,p} \leq 10^{10}$ for $2 \leq p \leq p_1$

Interpolate w/ L^2 bd to get

$$\left\| \left(I - \frac{2Q_{w,p}}{N}\right)^{-1} \right\|_{p,p} \leq 2 \quad \text{for } 2 \leq p \leq p_2$$

N steps $\vdots$

$$\left\| \left(I - Q_{w,p}\right)^{-1} \right\|_{p,p} \leq 2 \quad \text{for } 2 \leq p \leq p_N. \quad \square$$

<u>What else can contractions do?</u>

What else can contractions do?

Consider $P^{w}_{[0,n]}$, projection onto $\{\varphi_0, \varphi_1, \ldots, \varphi_n\}$ wrt w.

Question: Can one show $\{P^{w}_{[0,n]}\}_n$ uniformly bdd $L^p(w) \to L^p(w)$?

Sergey & I first thought: nah, at least not using perturbative method.

A: Yes, using perturbative method!

Thm (A.-Aptekarev-Denisov): If $w \in A_2$, $\exists\, \varepsilon > 0$ s.t. if $|p-2| < \varepsilon$ then $\{P^{w}_{[0,n]}\}$ uniformly bdd in $\mathcal{L}(L^p(w))$.

weird though, since ~~...~~ bdding $\{P^{w}_{[0,n]}\}$ feels harder than $\{\varphi_n\}$...

Q: (a) What's the limitation of this technique? Is it that this is super-powerful, letting you prove lots of things?

(b) Or ~~...~~ are we actually doing very little, ~~...~~ not doing much with orthogonality of $\{\varphi_n\}$?

$\hookrightarrow$ Feels like (b) atm, though I often oscillate between both.