# The Steklov Problem for OPUC and Krein Systems

Michel Alexis

UW-Madison

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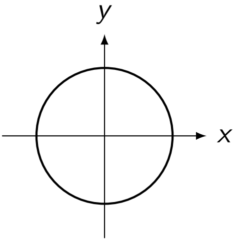

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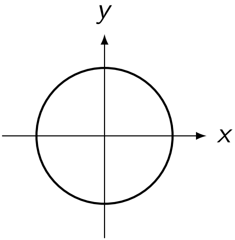


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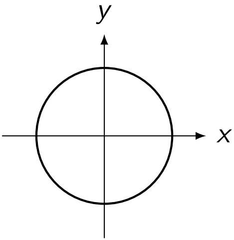


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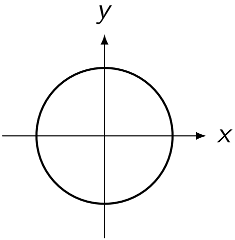


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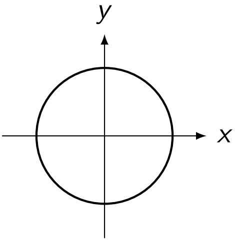


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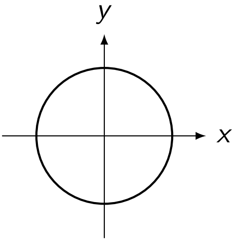


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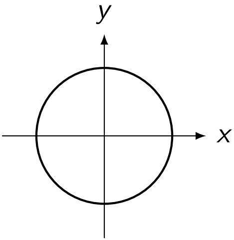


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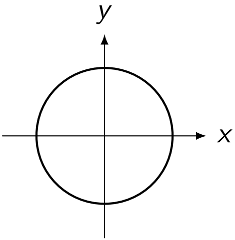


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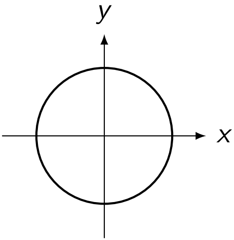


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- $\{\varphi_n(z)\}$ are called Orthogonal Polynomials on the Unit Circle.

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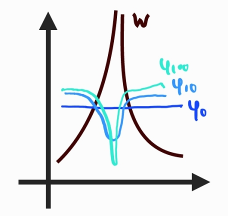

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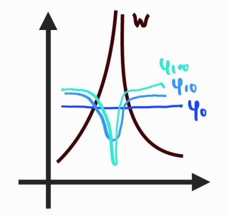


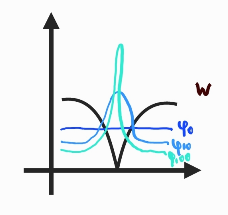

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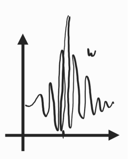

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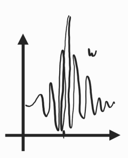


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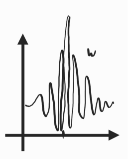


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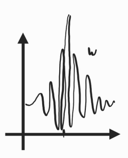


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**Remark:** If $\int_{\mathbb{T}} \log w > -\infty$, then $|\varphi_n(z)| \sim |\Phi_n(z)|$, where $\Phi_n(z)$ are the monic orthogonal polynomials of degree $n$.

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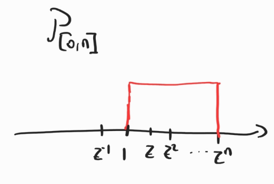

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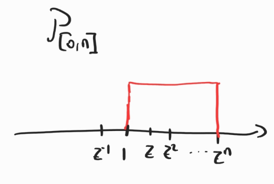


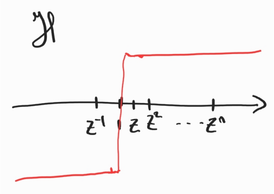

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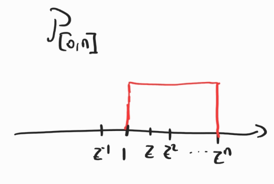


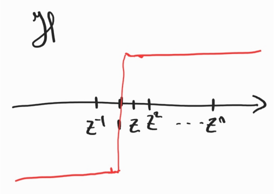


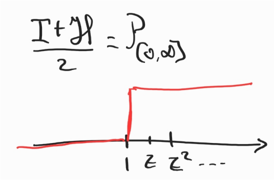

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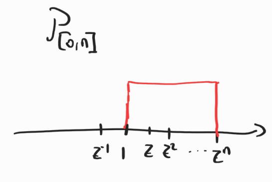


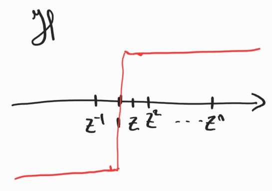


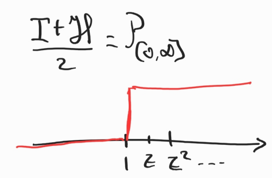


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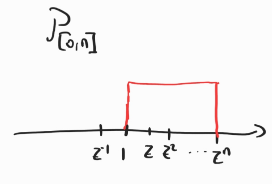


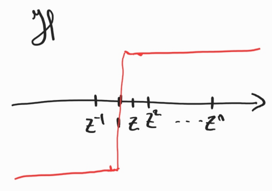


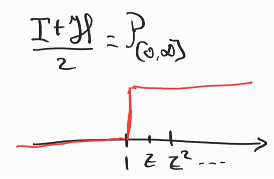


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□

# $A_p$ weights

- We say $w \in A_2$ if

$$[w]_{A_2} \stackrel{\text{def}}{=} \sup_{I\text{: arc in } \mathbb{T}} \left( \frac{1}{|I|} \int_I w \right) \left( \frac{1}{|I|} \int_I w^{-1} \right) < \infty \,.$$

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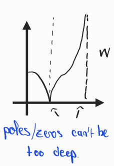

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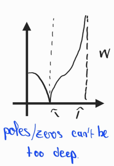


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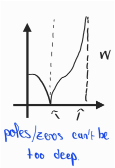


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$A_p$ plays nicely with the Hilbert transform $\mathcal{H}$, so makes sense to adapt previous proof to $A_p$ weights.

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**Theorem (A.-Aptekarev-Denisov, '20)**

*If $w \in A_2$, then $\sup_n \|w^{1/p}\Phi_n\|_{L^p} = \sup_n \|\Phi_n\|_{L^p(w)} < \infty$ for some $p > 2$.*

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**Proof.**

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- And as it turns out we can invert $I - Q_{w,p}$ for some $p$ near 2 using spectral theory and analytic interpolation. (Deferred till end of presentation, if time permits.)

$\square$

# Krein Systems, the continuous analogue of OPUC

$(d\mu, \mathbb{T})$

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OPUC

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- $\lambda$ is the variable denoting points in the space $\mathbb{R}$.

We will be interested in the case when $d\sigma = w(\lambda)\frac{d\lambda}{2\pi}$, where $w - 1 \in L^1(\mathbb{R}) + L^2(\mathbb{R})$ to ensure $P(r,\lambda)$ exists.

# A Steklov problem for Krein Systems

**Problem (Steklov problem for Krein Systems)**

*Does there exist* $p > 2$ *for which* $\{P(r,\lambda)\}_{r\geqslant 0}$ *is bounded in* $L^p_{loc}(w,\mathbb{R})$*?*

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**Remark:** In fact, this is nontrivial for $p = 2$ and even $p < 2$.

# Generalizing the OPUC result to Krein systems

**Theorem**

*Suppose $w \in A_2(\mathbb{R})$ and $w - 1 \in L^1(\mathbb{R})$. Then there exists $\epsilon > 0$ for which*

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Some algebra and the same process as before shows that

$$(I - Q_{w,p})X = -w^{-1/p'}\mathcal{P}_{[0,r]}w^{1/p'}(w^{1/p} - w^{-1/p'})e^{i\lambda r}$$

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where $X = w^{1/p}(P(r,\lambda) - e^{i\lambda r})$. Since $I - Q_{w,p}$ is invertible, we're done. $\square$

# An orthogonality property unique to $\mathbb{R}$

Recall that for OPUC we have

$$\mathcal{P}_{[0,n-1]} w\Phi_n = 0\,.$$

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Under certain conditions on $w$ (see next slide), $\alpha_2(r) \in L^2_{dr}(\mathbb{R}^+)$, $\alpha_\infty(r) \in L^\infty_{dr}(\mathbb{R}^+)$ and $\lim_{r\to\infty} \alpha_\infty(r)$ exists.

# A remainder estimate

Estimating $\|R(r,\lambda)\|_{L^p(w)}$ quantifies the decay of $P(r,\lambda) - e^{i\lambda r}$ and provides an asymptotic expansion of sorts, in both $\lambda$ and $r$.

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**Theorem**

*Suppose $w \in A_2(\mathbb{R})$, $\langle\lambda\rangle^q(w-1) \in L^1(\mathbb{R})$ for some $q > 2$, and*

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**Examples:** If $-1 < \beta < 1$, then $w(\lambda) = |\lambda|^\beta$ in $[-1,1]$, and equals 1 outside $[-1,1]$ satisfies all the conditions of the above.

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- Estimating $\|P(r,\lambda) - e^{i\lambda r}\|_{L^p(w)}$ is kind of overkill. Seems much more natural to try and estimate

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I have no clue how to come up with such an estimate beyond the content of this presentation.

*Thank you for Listening!*

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Just kidding!

# Inverting $I - Q_{w,p}$

A few facts:

- Recall $Q_{w,p} = w^{1/p}\mathcal{P}_{[0,n-1]}w^{-1/p} - w^{-1/p'}\mathcal{P}_{[0,n-1]}w^{1/p'}$.

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  - In fact, an inner-product computation shows $\|(I - \kappa Q_{w,2})f\|_2^2 \geqslant \|f\|_2^2$.

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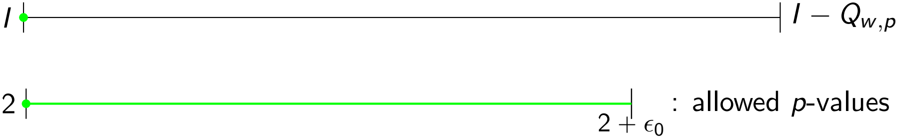


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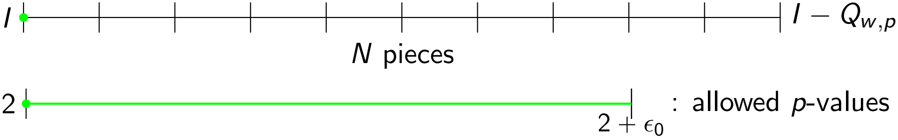


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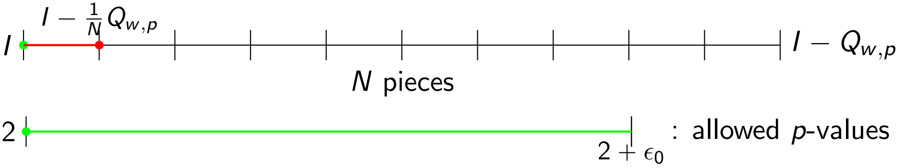


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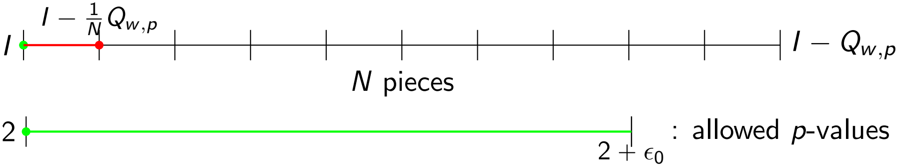


Algorithm:

1. By geometric sum,
$\|(I - \frac{1}{N}Q_{w,p})^{-1}\|_{p,p} = |(I - \frac{0}{N}Q_{w,p} - \frac{Q_{w,p}}{N})^{-1}\|_{p,p} \leqslant 10^{10}$ for $p$ green.

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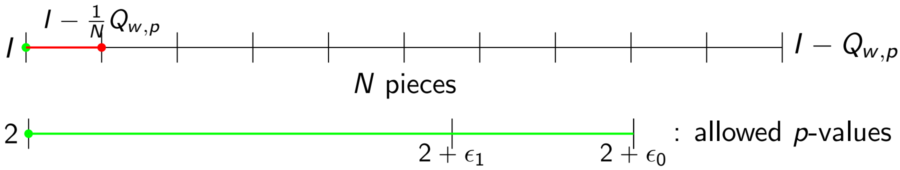
- Choose $N$ large enough so that $\|\frac{Q_{w,p}}{N}\|_{p,p} \leqslant \frac{1}{10}$.




Algorithm:

1. By geometric sum,
$\|(I - \frac{1}{N}Q_{w,p})^{-1}\|_{p,p} = |(I - \frac{0}{N}Q_{w,p} - \frac{Q_{w,p}}{N})^{-1}\|_{p,p} \leqslant 10^{10}$ for $p$ green.
2. Analytically interpolate with good $p = 2$ estimate to get
$\|(I - \frac{1}{N}Q_{w,p})^{-1}\|_{p,p} \leqslant 5$ for $2 \leqslant p \leqslant 2 + \epsilon_1$, where $\epsilon_1 < \epsilon_0$.

# Inverting $I - Q_{w,p}$

A few facts:

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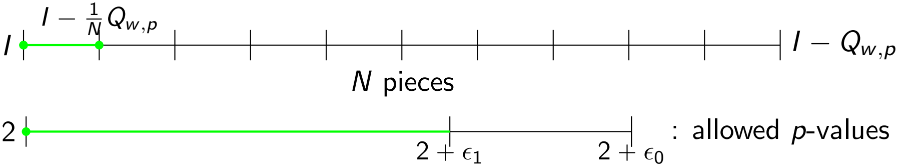


Algorithm:

1. By geometric sum,
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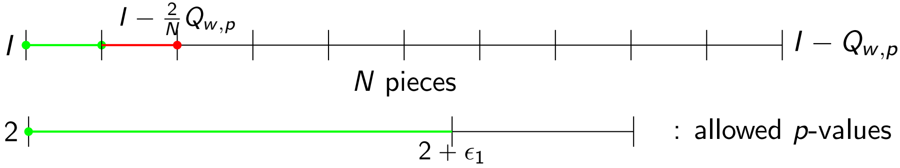


Algorithm:

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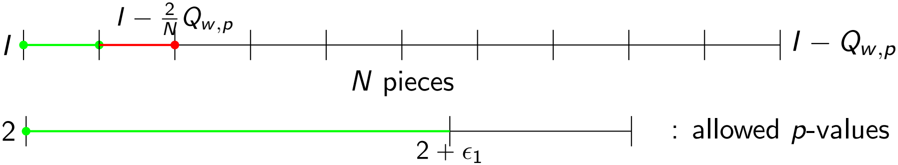


Algorithm:

1. By geometric sum,
$\|(I - \frac{2}{N}Q_{w,p})^{-1}\|_{p,p} = |(I - \frac{1}{N}Q_{w,p} - \frac{Q_{w,p}}{N})^{-1}\|_{p,p} \leqslant 10^{10}$ for $p$ green.

# Inverting $I - Q_{w,p}$

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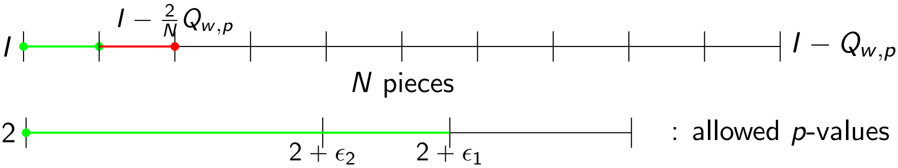


Algorithm:

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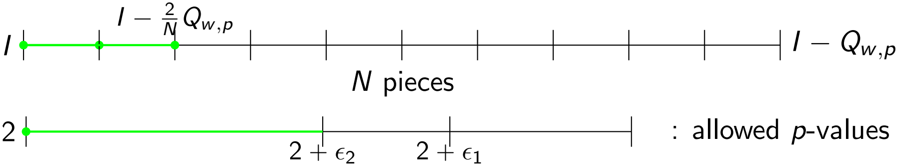


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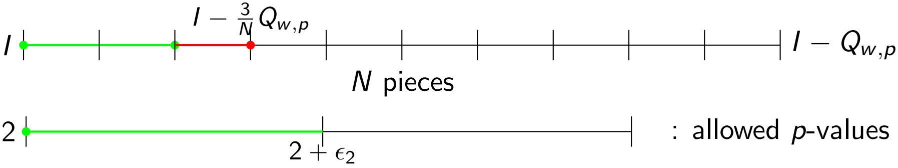


Algorithm:

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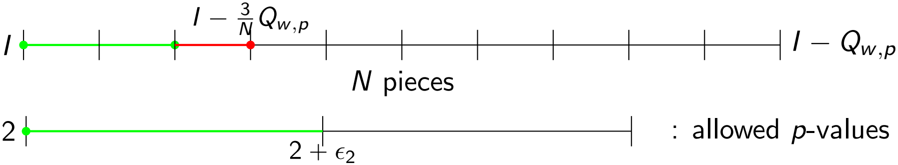


Algorithm:

1. By geometric sum,
   $\|(I - \frac{3}{N}Q_{w,p})^{-1}\|_{p,p} = |(I - \frac{2}{N}Q_{w,p} - \frac{Q_{w,p}}{N})^{-1}\|_{p,p} \leqslant 10^{10}$ for $p$ green.

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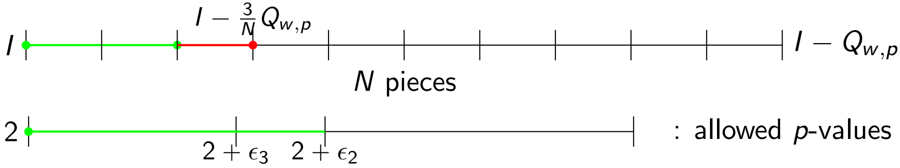


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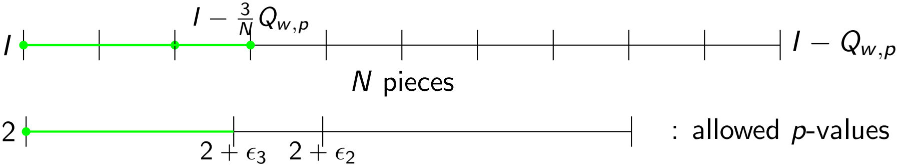


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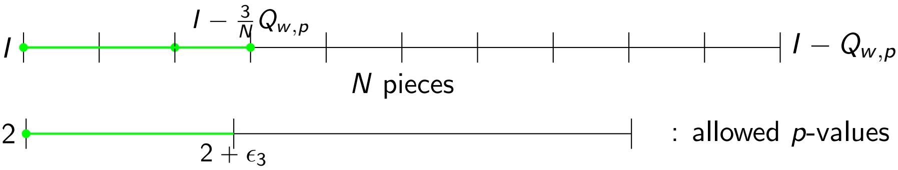


Algorithm: Repeat $N$ times ...

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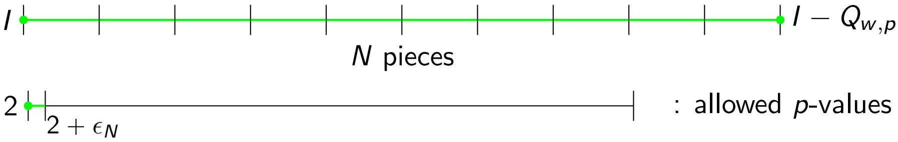


Algorithm:

# Inverting $I - Q_{w,p}$

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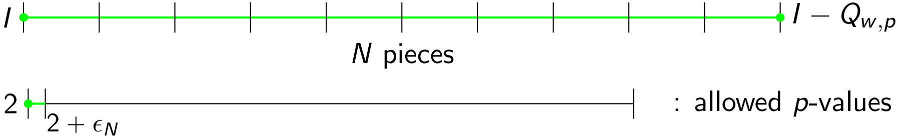


Algorithm: Done!

*Thank you for Listening (for real this time)!*