How to represent a function in a quantum computer?

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University of Bonn

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Question: Given a function $f : [0,1] \to [-1,1]$, does there exist an infinite phase factor sequence $\Psi = \{\psi_k\}_{k=0}^{\infty}$ for which $f(x)$ is the imaginary part of the upper left entry of $U_\infty(x, \Psi)$?

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Why should we care? QSP is apparently a very simple and physically intuitive quantum algorithm.

Making sense of of the matrix product

Consider the truncated product

$$U_d(x,\Psi) := e^{i\psi_d Z} e^{i\theta X} e^{i\psi_{d-1} Z} e^{i\theta X} \dots e^{i\psi_0 Z} \dots e^{i\theta X} e^{i\psi_{d-1} Z} e^{i\theta X} e^{i\psi_d Z},$$

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Mutliplying two such matrices together yields another such matrix. Indeed

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So $U_d(x,\Psi) = \begin{pmatrix} poly(x) & poly(x)\sqrt{1-x^2} \\ poly(x)\sqrt{1-x^2} & poly(x) \end{pmatrix}$.

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Theorem (A.-Mnatsakanyan-Thiele, 2023)

For each f with $\|f\|_\infty < \frac{1}{\sqrt{2}}$, there exists a unique phase factor sequence Ψ such that the imaginary parts of the upper left entries of $U_d(x, \Psi)$ converge to f in $L^2\left(\frac{dx}{\sqrt{1-x^2}}\right)$, and we have the nonlinear Plancherel identity

$$\sum_k \log(1 + \tan^2 \psi_{|k|}) = -\frac{2}{\pi} \int_0^1 \log\left(1 - f(x)^2\right) \frac{dx}{\sqrt{1 - x^2}}.$$

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The key idea here is to use nonlinear Fourier analysis!

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linear Fourier transform	:	$(F_k)_k$	$\mapsto$	Laurent series $\sum_k F_k z^k$
nonlinear Fourier transform	:	$(F_k)_k$	$\mapsto$	a matrix of Laurent series.

The $SU(2)$-nonlinear Fourier transform (NLFT)

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The nonlinear Fourier transform of a sequence $(F_k)_k$ is the $SU(2)$-valued function

$$\begin{pmatrix} a(z) & b(z) \\ -b^*(z) & a^*(z) \end{pmatrix} = \prod_{j=-\infty}^{\infty} \frac{1}{\sqrt{1+|F_j|^2}} \begin{pmatrix} 1 & F_j z^j \\ -\overline{F_j} z^{-j} & 1 \end{pmatrix}$$

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The Laurent series a and b

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A more careful multilinear expansion shows that the support of F dictates the Fourier support of Laurent polynomials a, b

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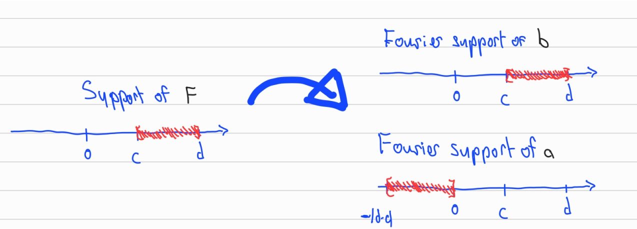

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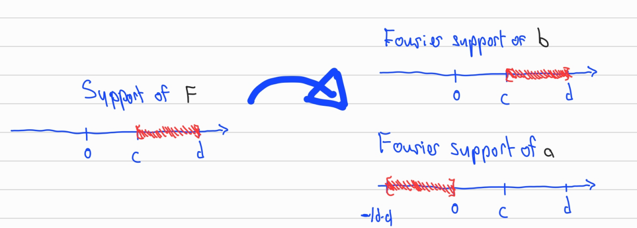

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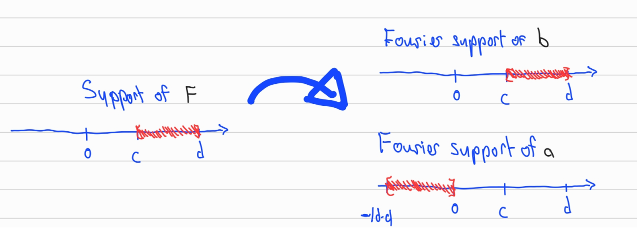

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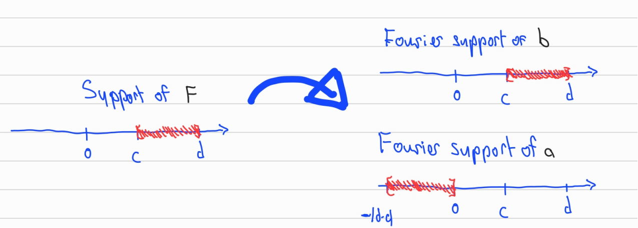

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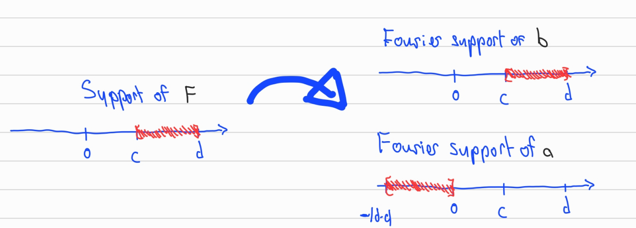

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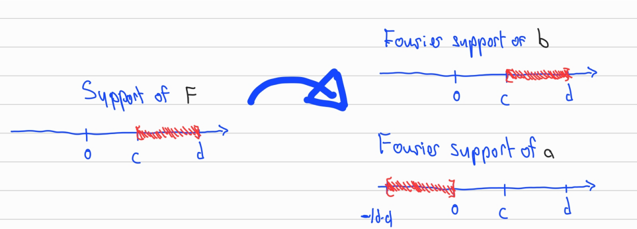

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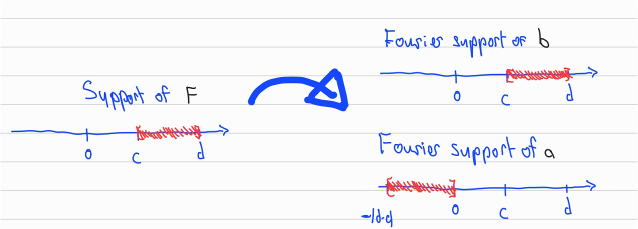

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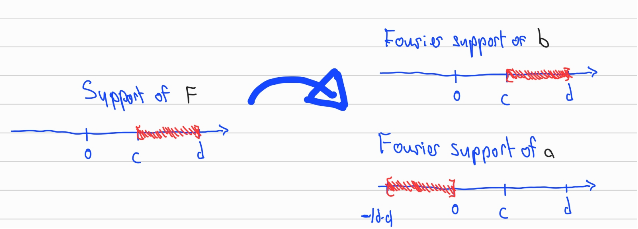

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Thus $a^*(0) = a^*_-(0)a^*_+(0)$.

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By the determinant identity $|a|^2 + |b|^2 = 1$ on $\mathbb{T}$ and the assumption $\|b\|_\infty < \frac{1}{\sqrt{2}}$, we get $\|\frac{b}{a}\|_\infty < 1$.

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By the determinant identity $|a|^2 + |b|^2 = 1$ on $\mathbb{T}$ and the assumption $\|b\|_\infty < \frac{1}{\sqrt{2}}$, we get $\|\frac{b}{a}\|_\infty < 1$.
So have the operator norm estimate $\|M\| \leqslant \|\frac{b}{a}\|_\infty < 1$ (viewing M as an operator on $H^2(\mathbb{D}^*) \times H^2(\mathbb{D})$).

Very Reduced problem: Given $(a, b) \in H^2(\mathbb{D}^*) \times L^\infty(\mathbb{T})$ such that a^* outer, $\|b\|_\infty < \frac{1}{\sqrt{2}}$, factor $(a, b) = (a_-, b_-)(a_+, b_+)$.
If we can solve for $(A, B) = a_+^*(0)(a_+, b_+)$ in

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Thus by geometric sum, we have $(I + M)^{-1} = \sum_{k=0}^{\infty}(-M)^k$ on $H^2(\mathbb{D}^*) \times H^2(\mathbb{D})$. So $\begin{pmatrix} A \\ B \end{pmatrix} = \sum_{k=0}^{\infty}(-M)^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

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Thank you for Listening!