A Two-Weight T1 Theorem for Singular Integral Operators with respect to Doubling Measures

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and some continuity condition. Example: $K(x, y) = \frac{1}{x-y}$ in dimension 1.

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Assume ω, σ are doubling measures and T is a Calderón-Zygmund operator. Then for all functions $f \in L^2(\sigma)$, we have

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Note that the statement is only meaningful if $A_2(\omega,\sigma), \mathfrak{T}, \mathfrak{T}^*$ are all finite.

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Then we want to show $\left|\sum_{I,J\in\mathcal{D}} \langle T_\sigma \Delta_I^\sigma f, \Delta_J^\omega g\rangle_\omega\right| \lesssim \|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}$.

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This type of estimate pervades most of the proof. Note that for each $\mathrm{P}_1(J, 1_{(2J)^c}\nu)$ we have only $\sqrt{\omega(J)}$.

Base case: "flat" measures and the pivotal condition

Using the Haar cancellation condition leads to lots of terms like

$$\sum_{r=1}^{\infty} \mathrm{P}_1(Q_r, 1_Q\sigma)^2\omega(Q_r)\,, \text{ where } Q = \sqcup Q_r\,.$$

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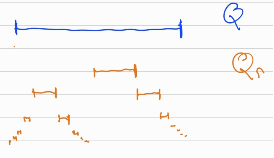

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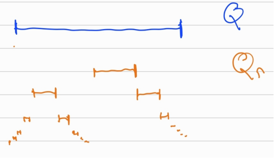

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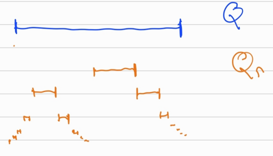

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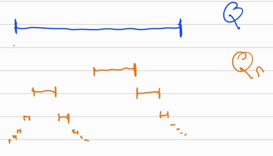

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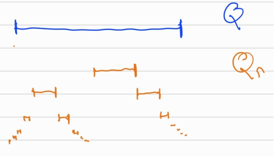

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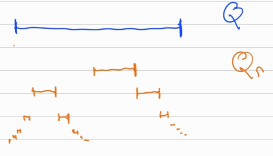

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When ω, σ are “flat,” we can control the above terms by the A_2 condition.

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So left with estimating terms where I, J are overlapping or close.

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Q: How do we handle cubes which are near each other?

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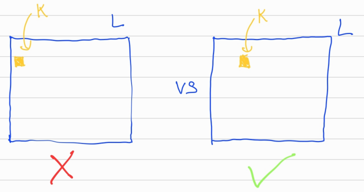

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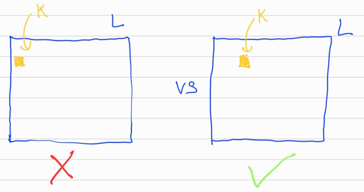

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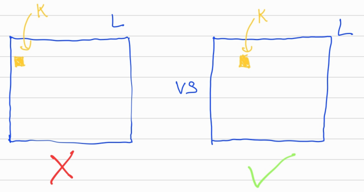

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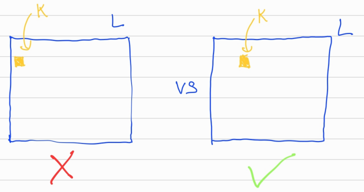

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So only have to deal with $\sum_{I,J\,:\,J \Subset I} + \sum_{I,J\,:\,\ell(I) \sim \ell(J) \text{ and } J \subset 3I}$ (and the same terms with I, J swapped, which can be handled similarly by duality).

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So now we only have to bound

$$|\sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{D} \\ J\Subset I}} \langle T_\sigma \Delta_J^\sigma f, \Delta_J^\omega g\rangle_\omega|$$

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We get

$$\sum_{r=1}^{\infty} \mathrm{P}_\kappa(Q_r, 1_Q \sigma)^2 \omega(Q_r) \lesssim A_2(\omega, \sigma)\sigma(Q)\,.$$

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To do this, you need to consider functions b_Q to not just have mean 0, but rather $\int y^d b_Q(y) dy = 0$ for $d = 0, \ldots, \kappa - 1$. Need the same properties of $\Delta_I^\omega g$.

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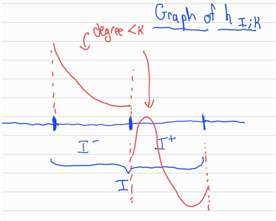

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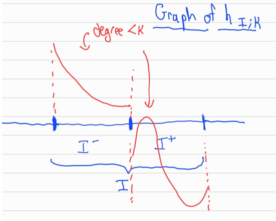

So to accomodate arbitrary doubling constants and the P_κ, we decompose f into $L^2(\sigma)$-Alpert wavelets: $f = \sum_{I \in \mathcal{D}} \Delta^\sigma_{I;\kappa} f$ (and similarly for g).

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Thank you for Listening!