The Steklov Problem for OPUC and Krein Systems

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McMaster University

JMM, January 6 2023

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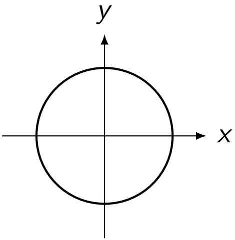

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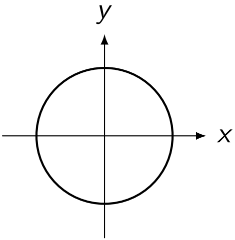

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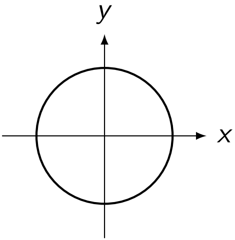

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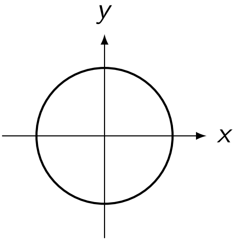

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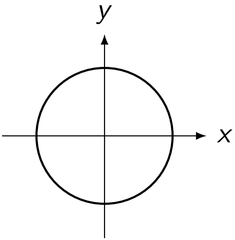

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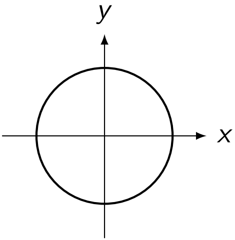

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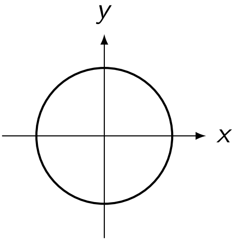

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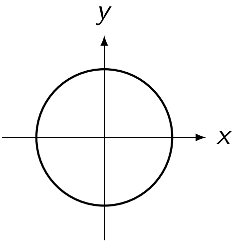

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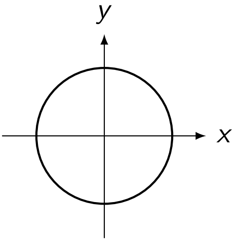

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- $\{\varphi_n(z)\}$ are called Orthogonal Polynomials on the Unit Circle.

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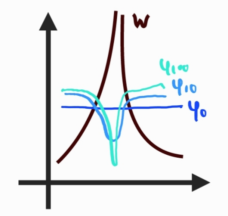

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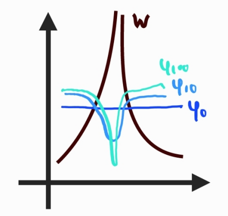

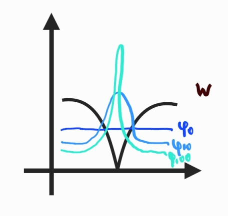

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Conjecture (Steklov's Conjecture)

If $w \geqslant \delta > 0$*, then* $\{\varphi_n\}$ *are bounded above, i.e.* $\sup_n \|\varphi_n\|_{L^\infty(w)} < \infty$.

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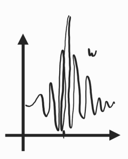

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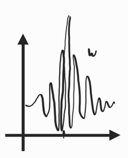

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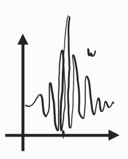

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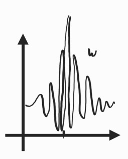

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Remark: If $\int_{\mathbb{T}} \log w > -\infty$, then $|\varphi_n(z)| \sim |\Phi_n(z)|$, where $\Phi_n(z)$ are the monic orthogonal polynomials of degree n.

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Proof:

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Suppose $\epsilon \leqslant w \leqslant \cancel{C} 1$. Then there exists $p > 2$ for which $\sup_n \|\Phi_n\|_{L^p} < \infty$.

Proof: $\mathcal{P}_{[0,n]} \overset{\text{def}}{=}$ Projection onto $\mathrm{Span}\{1, z, z^2, z^3, \dots, z^n\}$.

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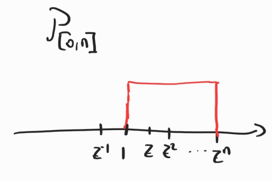

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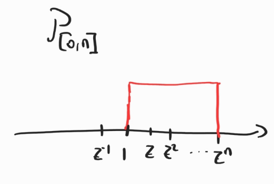

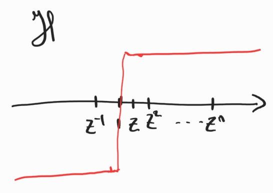

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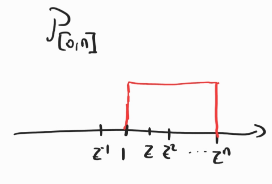

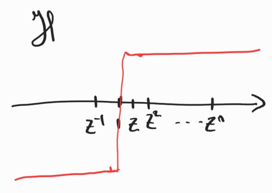

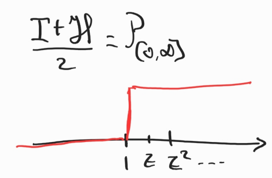

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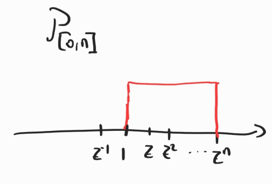

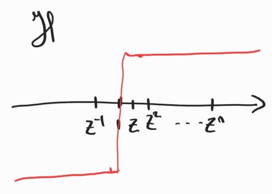

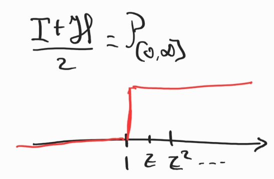

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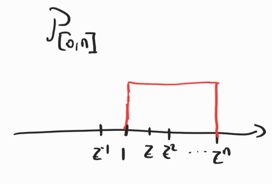

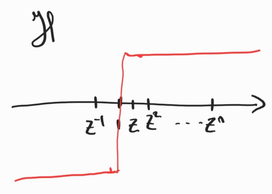

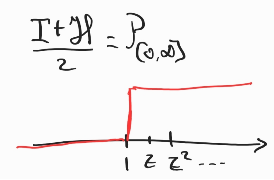

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A_p weights

- We say $w \in A_2$ if

$$[w]_{A_2} \stackrel{\text{def}}{=} \sup_{I:\text{ arc in } \mathbb{T}} \left(\frac{1}{|I|} \int_I w \right) \left(\frac{1}{|I|} \int_I w^{-1} \right) < \infty \,.$$

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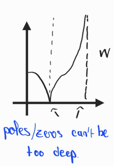

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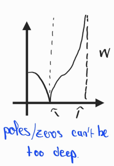

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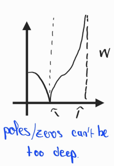

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Theorem (A.-Aptekarev-Denisov, '20)

If $w \in A_2$, *then* $\sup_n \|w^{1/p}\Phi_n\|_{L^p} = \sup_n \|\Phi_n\|_{L^p(w)} < \infty$ *for some* $p = p([w]_{A_2}) > 2$.

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- We can then invert $I - Q_{w,p}$ when $|\frac{1}{2} - \frac{1}{p}| < \epsilon([w]_{A_2})$ using spectral theory and analytic interpolation.

$\square$

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We will be interested in the case when $d\sigma = w(\lambda)\frac{d\lambda}{2\pi}$, where $w - 1 \in L^1(\mathbb{R}) + L^2(\mathbb{R})$ to ensure $P(r,\lambda)$ exists.

Steklov for Krein systems and an explicit formula

Theorem (A. '21)

Suppose $w \in A_2(\mathbb{R})$. *Then there exists* $\epsilon = \epsilon([w]_{A_2}) > 0$ *for which*

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whenever $2 - \epsilon \leqslant p \leqslant 2 + \epsilon$.

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$$\sup_{r \geqslant 0} \| P(r, \lambda) - e^{i\lambda r} \|_{L^p(w)} < \infty$$

whenever $\max\{2 - \epsilon, p_0\} \leqslant p \leqslant 2 + \epsilon$.

The proof is similar to the previous one, using the useful explicit identity

$$X_p = (I - Q_{w,p})^{-1} w^{-1/p'} \mathcal{P}_{[0,r]}(w-1) e^{i\lambda r} ,$$

where $X_p \stackrel{\text{def}}{=} w^{1/p} \left(P(r, \lambda) - e^{i\lambda r} \right)$.

Steklov for Krein systems and an explicit formula

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Corollary

Above theorem is sharp: one cannot take $p < p_0$.

Thank you for Listening!