How to represent a function in a quantum computer

Michel Alexis (Bonn)

Joint work with Lin Lin (Berkeley), Gevorg Mnatsakanyan (UW-Madison), Christoph Thiele (Bonn), and Jiasu Wang (Industry)

El Escorial 2025

June 11, 2025

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$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} = \left(\prod_j (1+|F_j|^2)^{-\frac{1}{2}}\right) \prod_j \begin{pmatrix} 1 & F_j z^j \\ -\overline{F_j} z^{-j} & 1 \end{pmatrix}$$

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$a =$	$1+$ higher order terms	$b =$	

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$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \approx \prod_j \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & F_j z^j \\ -\overline{F_j} z^{-j} & 0 \end{pmatrix} \right)$$

$$= \sum_{n \geqslant 0} \sum_{j_1 < \ldots < j_n} \begin{pmatrix} 0 & F_{j_1} z^{j_1} \\ -\overline{F_{j_1}} z^{-j_1} & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & F_{j_n} z^{j_n} \\ -\overline{F_{j_n}} z^{-j_n} & 0 \end{pmatrix}.$$

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$a =$	$1+$ higher order terms	$b =$	$\sum\limits_j F_j z^j$ + higher order terms

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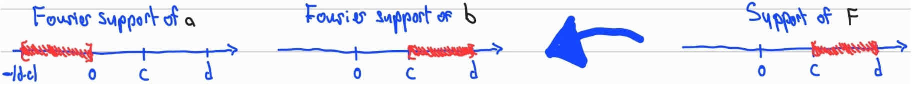
Expansion also shows the support of F dictates the frequency supports of a, b.



Representing a function in a quantum computer

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Theorem (A.-Lin-Mnatsakanyan-Thiele-Wang, 2024)

If $b : \mathbb{T} \to \mathbb{C}$ satisfies $\|b\|_\infty \leqslant 1$ and $\int_{\mathbb{T}} \log(1 - |b(z)|^2) > -\infty$, then there exists a unique sequence $F \in \ell^2(\mathbb{Z})$ such that $(*, b) = NLFT(F)$, and we have the nonlinear Plancherel identity

$$\sum_k \log(1 + |F_k|^2) = -\int_{\mathbb{T}} \log\left(1 - |b(z)|^2\right) .$$

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Question: How do define $NLFT(F)$ for $F \in \ell^2(\mathbb{Z})$?

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$$(a,b) = (a_-, b_-)(a_+, b_+)$$

where $(a_-, b_-) \in \mathbf{H}_{<0}$ is the NLFT of $F_{<0}$, and $(a_+, b_+) \in \mathbf{H}_{\geqslant 0}$ is the NLFT of $F_{\geqslant 0}$.

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$$\left[\mathrm{Id} + \begin{pmatrix} 0 & P_- \frac{\overline{b}}{\overline{a}} P_+ \\ -P_+ \frac{b}{a} P_- & 0 \end{pmatrix} \right] \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

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$$\left[\mathrm{Id} + \begin{pmatrix} 0 & P_- \frac{\overline{b}}{\overline{a}} P_+ \\ -P_+ \frac{b}{a} P_- & 0 \end{pmatrix}\right] \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

But our operator looks like Id $+$antisymmetric operator on $L^2(\mathbb{T}) \times L^2(\mathbb{T})$.

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But our operator looks like Id +antisymmetric operator on $L^2(\mathbb{T}) \times L^2(\mathbb{T})$.
So (A, B) exists and is always unique! Uniqueness of (a_+, b_+) then follows!

Thank you for Listening!