# Steklov's Problem for OPUC

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UW-Madison

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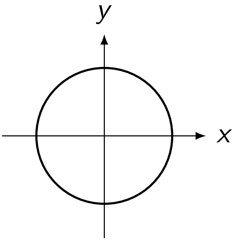

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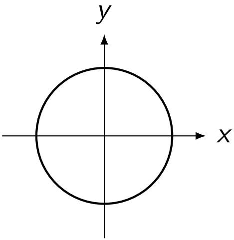


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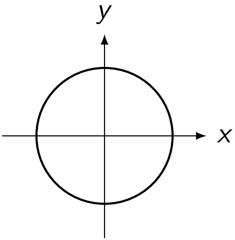


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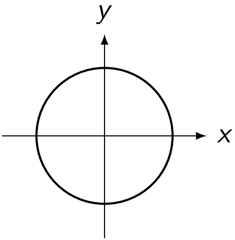


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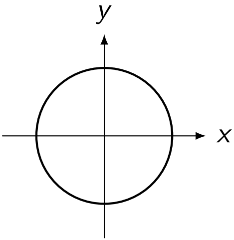


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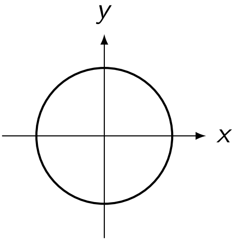


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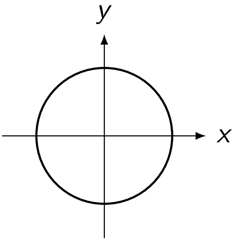


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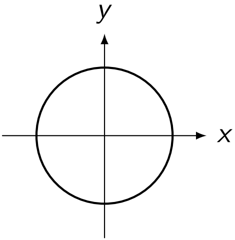


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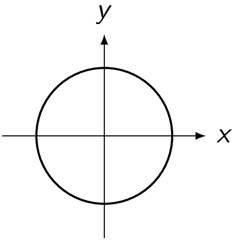


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- $\{\varphi_n(z)\}$ are called Orthogonal Polynomials on the Unit Circle.

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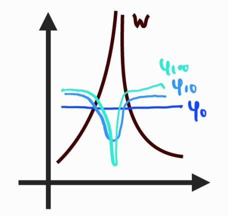

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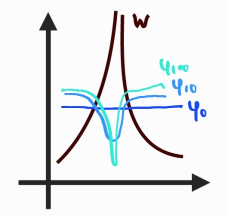


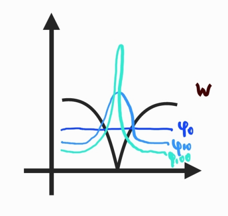

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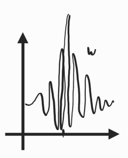

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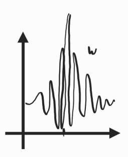


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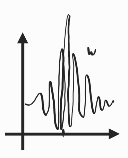


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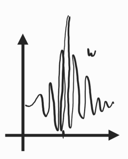


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**Remark:** If $\int_{\mathbb{T}} \log w > -\infty$, then $|\varphi_n(z)| \sim |\Phi_n(z)|$, where $\Phi_n(z)$ are the monic orthogonal polynomials of degree $n$.

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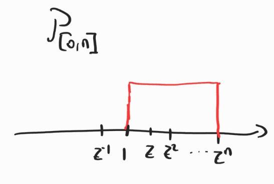

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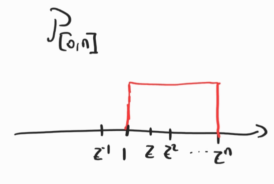


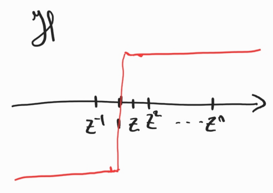

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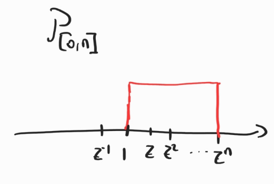


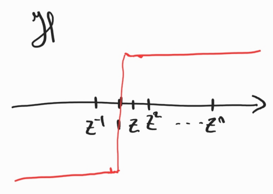


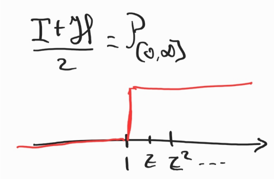

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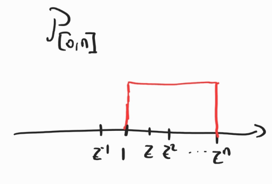


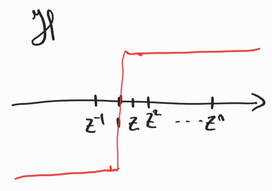


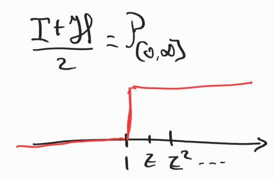


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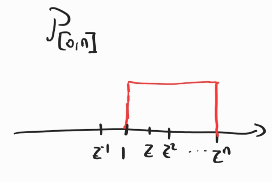


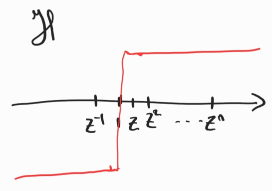


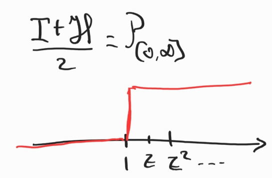


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□

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- We say $w \in A_2$ if $\sup_{I\text{: arc in } \mathbb{T}} \left( \frac{1}{|I|} \int_I w \right) \left( \frac{1}{|I|} \int_I w^{-1} \right) < \infty.$

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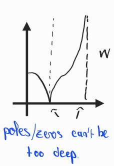

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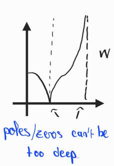

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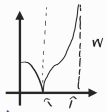


poles/zeros can't be too deep.

**Theorem (Hunt-Muckenhoupt-Wheeden)**

*If $w \in A_p$, then* $\|w^{1/p}\mathcal{H}w^{-1/p}\|_{p,p} = \|\mathcal{H}\|_{L^p(w)\to L^p(w)} < \infty$.

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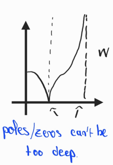


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$A_p$ plays nicely with the Hilbert transform $\mathcal{H}$, so makes sense to adapt previous proof to $A_p$ weights.

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**Theorem (A.-Aptekarev-Denisov, '20)**

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- Can invert $I - Q_{w,p}$ for some $p$ near 2 using spectral theory and analytic interpolation. (Skipped because not main goal of talk).

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**A:** Yes!

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*If $w \in A_2$, then there exists $\delta > 0$ such that whenever $|\frac{1}{p} - \frac{1}{2}| < \delta$, then*

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Now we can proceed exactly as before! □

*Thank you for Listening!*