Some counterexamples in two-weight norm inequalities for Calderón-Zygmund operators

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Joint work with José Luis Luna-Garcia, Eric Sawyer and Ignacio Uriarte-Tuero

Bonn

Friday, November 3

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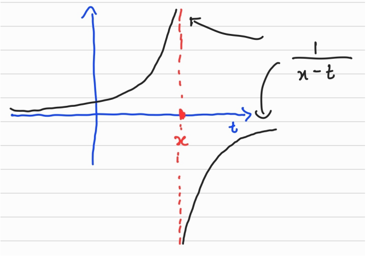

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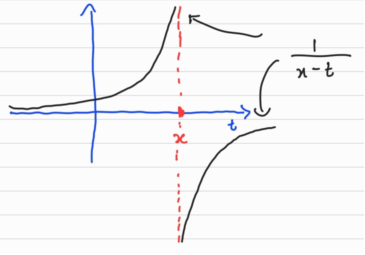

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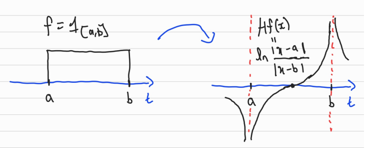

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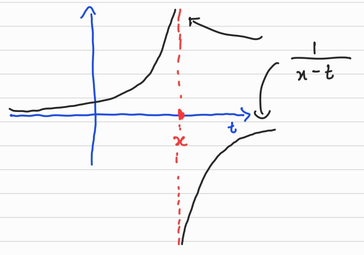

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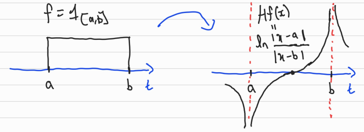

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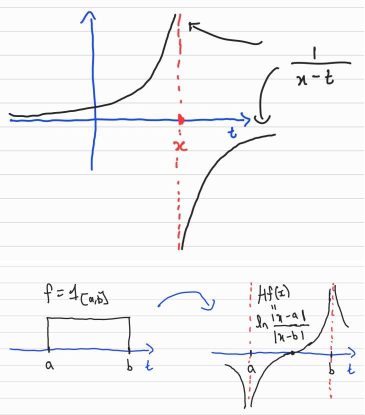

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The Hilbert transform is an example of a **Calderón-Zgymund Operator** (CZO).

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- The Riesz transforms pop-up in all sorts of PDEs and even some GMT problems.

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Can extend this characterisation to L^p by replacing the A_2 condition by an appropriate A_p condition.

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Theorem (Lacey-Sawyer-Shen-Uriarte-Tuero; Lacey, 2012)

Assume ω and σ don't have common point masses and a stronger A_2 condition holds where the averages are replaced by Poisson averages. Then

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if and only if the testing conditions hold, i.e.,

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Morally, to check the norm inequality, it suffices to *test* the norm inequality over indicators of cubes.

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Theorem (Sawyer-Wick, 2022)

If the measures σ, ω are doubling, then a more technical (but similar in spirit) L^p testing theorem holds.

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- Whereas A_2 is easy to check, since everything's positive and the averaging integrals are simple.

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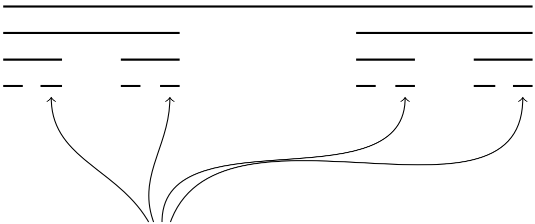

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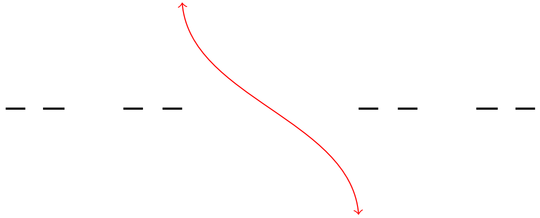

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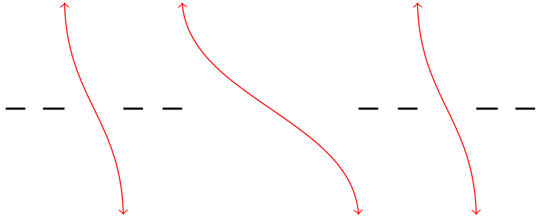

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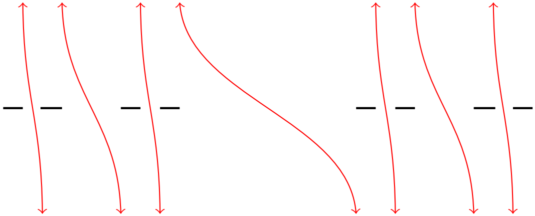

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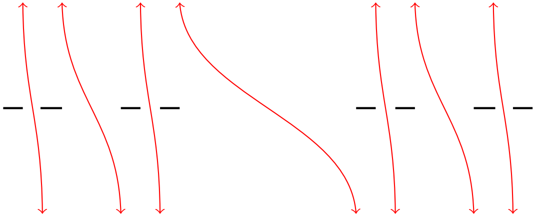

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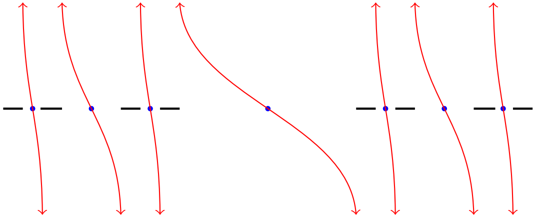

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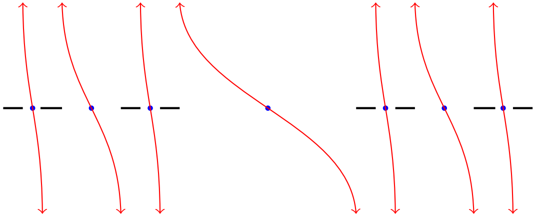

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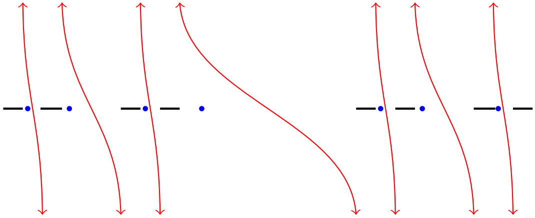

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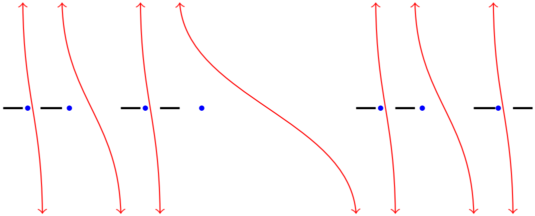

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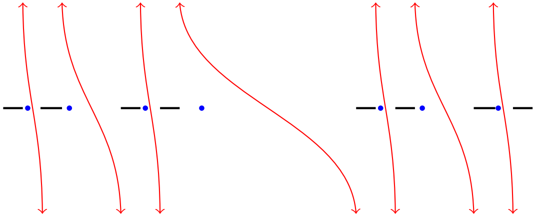

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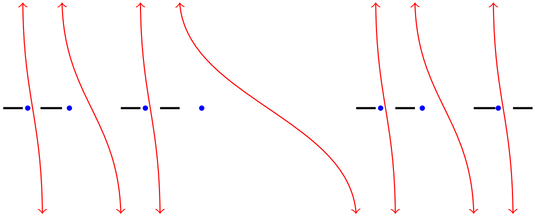

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Morally, this is why you need testing: one can only characterize the norm inequality with some biLipschitz unstable condition.

Rotational instability on doubling measures: big picture

Theorem (A.-Luna-Garcia-Sawyer-Uriarte-Tuero, 2022)

Consider the Riesz transform R_2 on $\mathbb{R}^2$. For every angle $\theta > 0$, there exist doubling measures σ, ω such that the norm inequality for R_2 holds w.r.t. (σ, ω), but does not w.r.t. the rotated measures $(\sigma_\theta, \omega_\theta)$.

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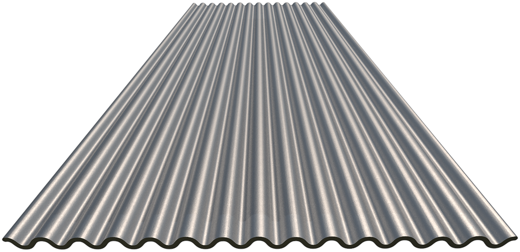

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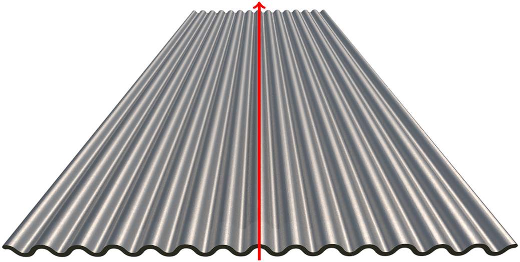

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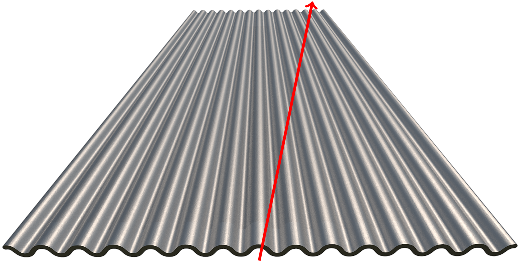

Start with R_2: it points vertically along the level curves of the sheet metal.
Rotate R_2 slightly: it sees all the ripples all of a sudden.

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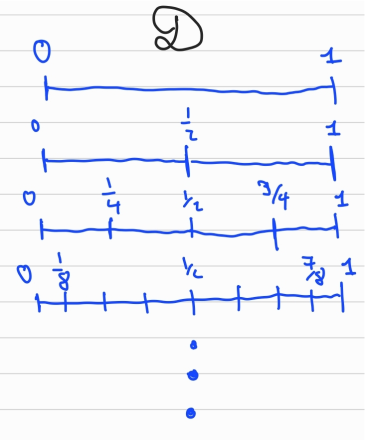

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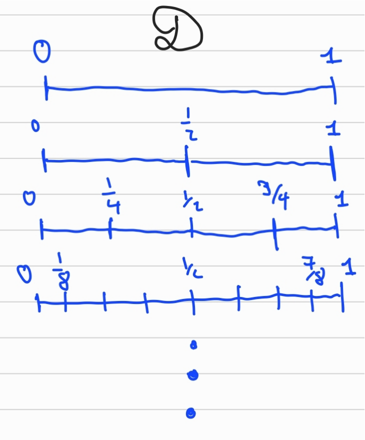

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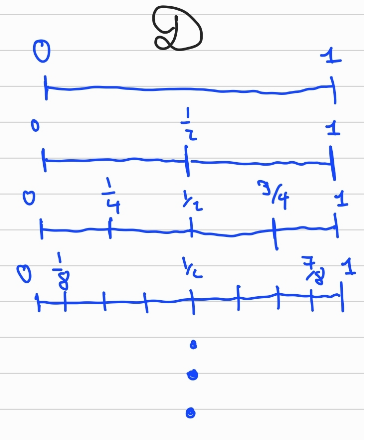

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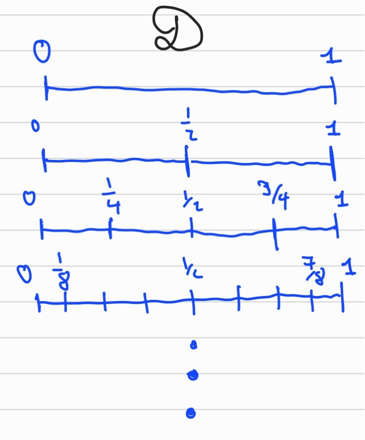

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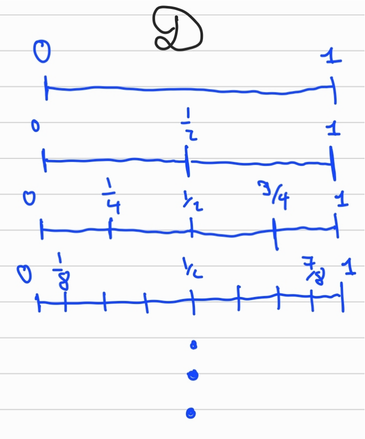

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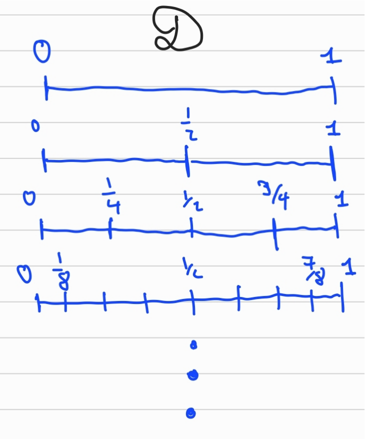

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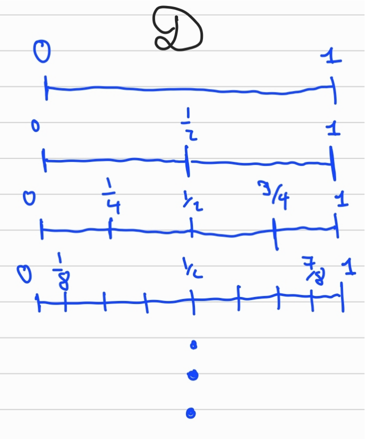

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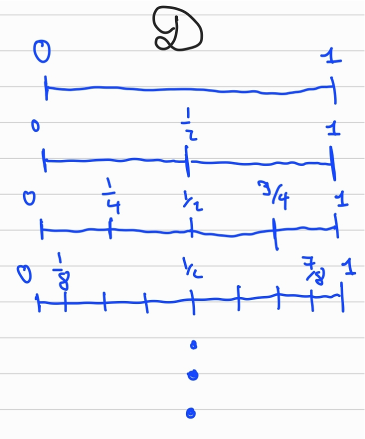

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Kakaroumpas-Treil got explicit weights for an essentially equivalent pb.

Rotational instability on doubling measures: the key ideas

We really just need to find doubling measures (σ, ω) which are well-behaved with respect to R_2, but badly behaved with respect to R_1.

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Find doubling measures (σ, ω) on $[0,1] \subset \mathbb{R}$ which satisfy $A_2(\sigma, \omega) < \infty$, but fail the testing condition for H.

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Find dyadically doubling measures (σ, ω) on $[0,1] \subset \mathbb{R}$ which satisfy the dyadic $A_2^{\mathcal{D}}(\sigma, \omega) < \infty$, but fail the testing condition for the dyadic square function

$$Sf(x) \equiv \sqrt{\sum_{J \in \mathcal{D}} \left(E_{J_-} f - E_{J_+} f\right)^2 |J| \mathbf{1}_J(x)}\,.$$

A transferrence principle: from dyadic square function to H

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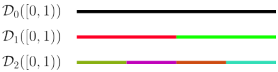

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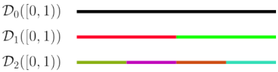

σ, ω are dyadic step functions s.t.
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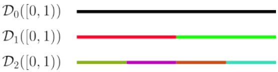

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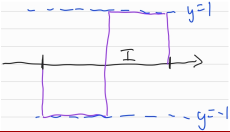

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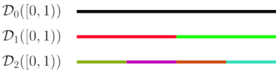

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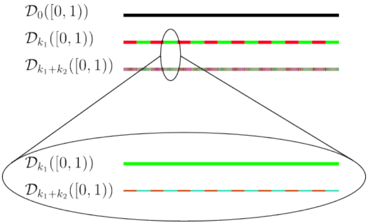

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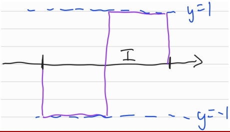

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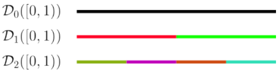

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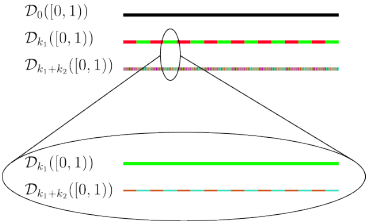

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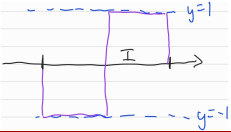

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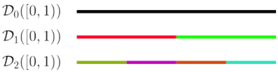

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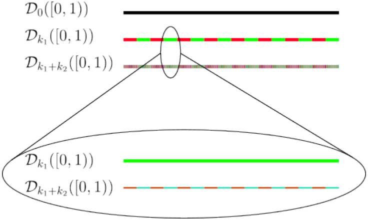
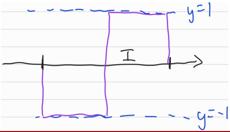


σ, ω are dyadic step functions s.t. testing fails for S. $\Rightarrow$ Spacing out Haar frequencies of σ and ω makes testing fail for H

$\sigma = E_{[0,1]}\sigma + \sum_{I \in \mathcal{D}} c_I h_I$, where h_I is a Haar function $\Rightarrow$ $\sigma = E_{[0,1]}\sigma + \sum_{I \in \mathcal{S}} c_I s^I_{k_I}$, where $\mathcal{S}$ is a sparser set of cubes



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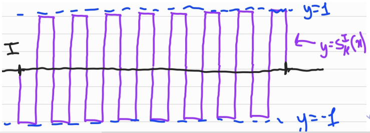

How the s_k's give us estimates for the Hilbert transform

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By complex function theory, $(s_k + iHs_k)^2 \to 0$ weakly. Imaginary part gives $(Hs_k)\, s_k \to 0$ weakly. Real part gives $(Hs_k)^2 \to (s_k)^2 = \mathbf{1}_{[0,1]}^2$ weakly.
Looking at $(s_k + iHs_k)^3 \to 0$ weakly yields last term $\to 0$.

Rotational instability on doubling measures: the key ideas

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By (1), the testing condition for R_1 fails, like for the Hilbert transform.
By (2), R_2 doesn't interact with the horizontal oscillation, so testing for R_2 holds!

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Here the *quadratic, or vector-valued* $A_p^{\ell^2}(\sigma,\omega)$ condition is

$$\left\| \left\{ \sum_Q a_Q^2 \left(E_Q\sigma\right)^2 \mathbf{1}_Q \right\}^{\frac{1}{2}} \right\|_{L^p(\omega)} \lesssim \left\| \left\{ \sum_Q a_Q^2 \mathbf{1}_Q \right\}^{\frac{1}{2}} \right\|_{L^p(\sigma)},$$

which must hold over all sequences of cubes $\{Q\}$, sequences of reals $\{a_Q\}$.

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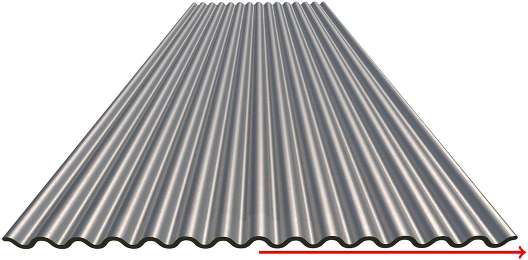

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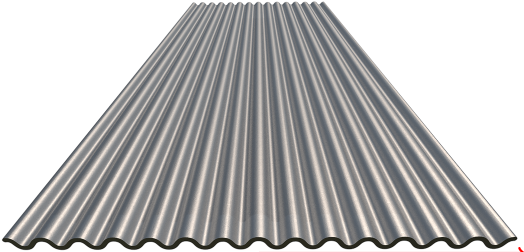

$\longrightarrow$ $A_p^{\ell^2}(\sigma,\omega)$ fails "along" $\mathbf{e}_1$

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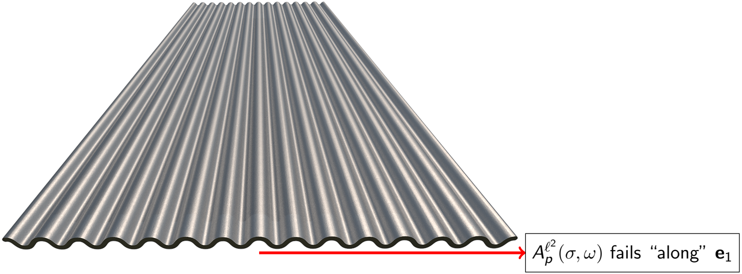

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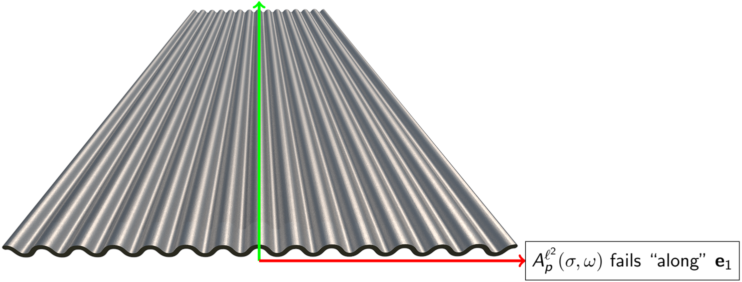

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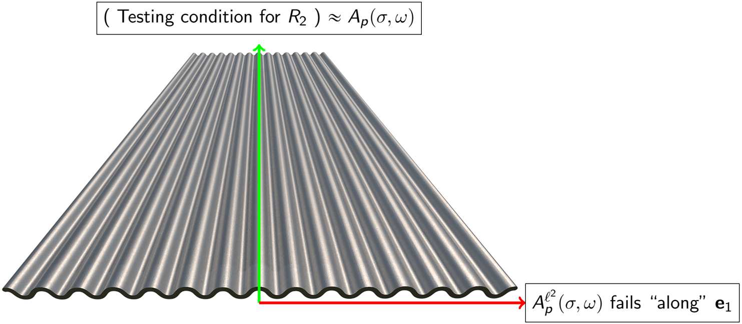

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