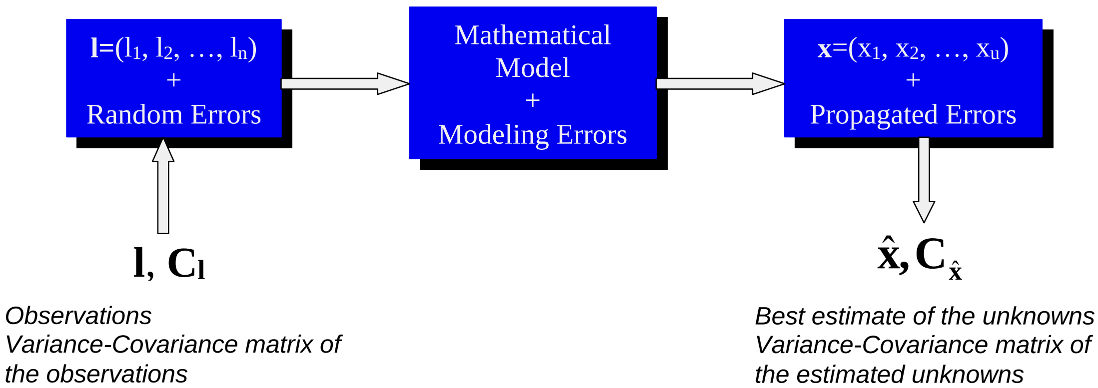
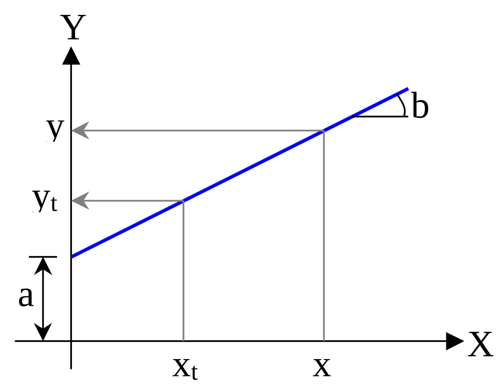
# 3. Error Propagation

- Error Propagation: is the process of evaluating the errors in estimated quantities (**x**) as functions of the errors in the measurements (**l**)
- Concept:




- Example: assume that a quantity (y) is estimated from a measured quantity (x) according to the following function (representing a straight line):

| | |
|---|---|
| $y = a + bx$ | (1) |
| And using the concept of true value, as introduced in chapter (1)<br>$y_t = a + bx_t$ | (2) |
| Defining the error of a measurement as the measured value minus the true value , (1) – (2)<br>$y - y_t = b\,(x - x_t)$<br>$dy = b \cdot dx$ | (3) |




  - From Equation (3), it is clear that any error in x of a value (dx) will introduce (i.e. propagate) an error of a value ($b \cdot dx$) in the y component
  - Any error in estimating b (i.e. any error in the math model) will introduce errors in y

- In general:
  - Given:

    *a) Observations:* $\mathbf{l} = \begin{bmatrix} l_1 & l_2 & . & . & . & l_n \end{bmatrix}^T$ *(recall $l_1$ is a uni-variable)*

    *b) Covariance matrix:* $\mathbf{C_l}$
  - Required

    *a) Estimated unknowns:* $\hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_1 & \hat{x}_2 & . & . & . & \hat{x}_u \end{bmatrix}^T$

    *b) Covariance matrix:* $\mathbf{C}_{\hat{\mathbf{x}}}$
  - Where

    $u$ = *number of unknowns*

    $n$ = *number of observations*

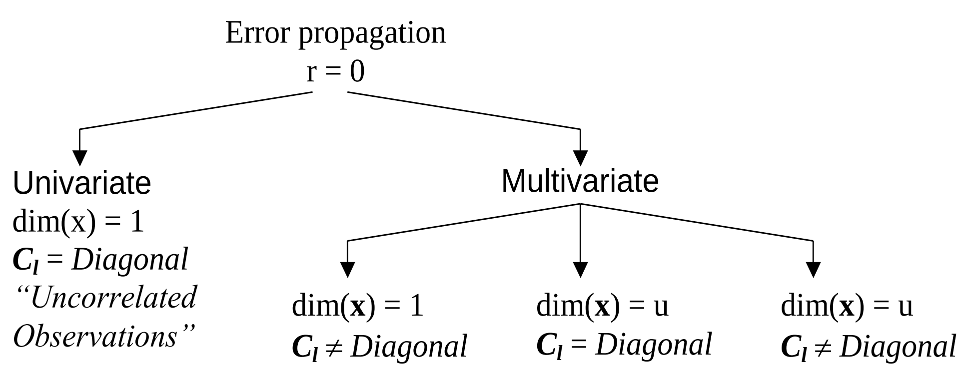
    $n_{necessary}$ = *the minimum number of observations required to estimate the unknowns*

    $r$ = *degrees of freedom* = $n - n_{necessary}$ *(normally, $n_{necessary} = u$)*

- Classification of problems

| **If r = 0** | **If r > 0** |
|---|---|
| ▪ we have $n_{necessary} = n$<br>▪ number of equations = number of unknowns<br>▪ unique solution | ▪ number of equations > number of unknowns<br>▪ requires adjustment of observations to reach a unique solution |
| <u>We will study:</u><br>▪ Univariate error propagation<br>▪ Multivariate error propagation | <u>We will study:</u><br>▪ The method of least squares |

- Zero-redundancy error propagation (r = 0)



## 3.1. Univariate Error Propagation

- Characteristics:

$$\mathbf{x} = [x]_{1x1}$$

- Steps of solution:

1. Construct the mathematical model (direct model)

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{l})$$

Where $\mathbf{l} = [l_1 \quad l_2 \quad \ldots \quad l_n]^{\mathrm{T}}$

2. Obtain the best estimate of $\boldsymbol{x}$

$$\hat{\boldsymbol{x}} = \boldsymbol{f}(\bar{\boldsymbol{l}})$$

3. Estimate the precision of $\hat{\boldsymbol{x}}$

$$\sigma_{\hat{x}}^2 = \left(\frac{\partial \boldsymbol{f}}{\partial l_1}\right)^2 \sigma_{l_1}^2 + \left(\frac{\partial \boldsymbol{f}}{\partial l_2}\right)^2 \sigma_{l_2}^2 + \ldots + \left(\frac{\partial \boldsymbol{f}}{\partial l_n}\right)^2 \sigma_{l_n}^2$$

$$\sigma_{\hat{x}}^2 = \sum_{i=1}^{n} \left(\frac{\partial \boldsymbol{f}}{\partial l_i}\right)^2 \sigma_{l_i}^2 \rightarrow \text{Observe the units}$$

- Example:

  - Given:

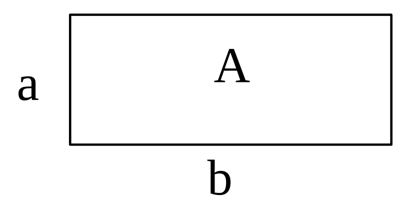
    x = A

    **l** = (a,b)

    $\bar{a} = 30m \;\; \sigma_{\bar{a}} = 0.1m$

    $\bar{b} = 40m \;\; \sigma_{\bar{b}} = 0.2m$

  - Required: $\hat{A} \; and \; \sigma_{\hat{A}}$




  - Solution:

1. Mathematical model

$$x = \boldsymbol{f}(\boldsymbol{l}) \rightarrow \mathrm{A} = \mathrm{a} \cdot \mathrm{b}$$

2. Best estimate

$$\hat{x} = \boldsymbol{f}(\bar{\boldsymbol{l}}) \rightarrow \hat{\mathrm{A}} = \bar{a} \cdot \bar{b} = 30 \text{ m} \cdot 40 \text{ m} = 1200 \text{ m}^2$$

3. Estimate the precision of $\hat{x} \rightarrow \sigma_{\hat{A}}(m^2)$

$$\sigma_{\hat{x}}^2 = \sum_{i=1}^{2}\left(\frac{\partial f}{\partial l_i}\right)^2 \sigma_{l_i}^2$$

$$\sigma_{\hat{A}}^2 = \left(\frac{\partial A}{\partial a}\right)^2 \sigma_{\bar{a}}^2 + \left(\frac{\partial A}{\partial b}\right)^2 \sigma_{\bar{b}}^2$$

$$\sigma_{\hat{A}}^2 = b^2 \cdot \sigma_{\bar{a}}^2 + a^2 \cdot \sigma_{\bar{b}}^2$$

$$\sigma_{\hat{A}}^2 = (40m)^2(0.1m)^2 + (30m)^2(0.2m)^2 \quad \textit{(Note: consistency of units} - m^2\textit{)}$$

$$\sigma_{\hat{A}}^2 = 1600\mathrm{m}^2 \cdot 0.01\mathrm{m}^2 + 900\mathrm{m}^2 \cdot 0.04\mathrm{m}^2 = 52\mathrm{m}^4$$

$$\sigma_{\hat{A}} = 7.211\mathrm{m}^2$$

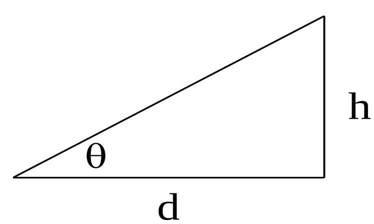
- Best estimate of $A = 1200\ m^2 \pm 7.211\ m^2$

♦ Example on univariate error propagation

- Given:

| | *Mean* | $\sigma$ | *n* |
|---|---|---|---|
| $d$ | 56.78*m* | 2*cm* | 4 |
| $\theta$ | 9°12'7" | 30" | 9 |

- Required: $\hat{h}$ *and* $\sigma_{\hat{h}}$




- Solution:
  1. Mathematical model

     $$x = \boldsymbol{f}(\boldsymbol{l}) \rightarrow h = d \cdot tan\theta$$

  2. Best estimate

     $\hat{\boldsymbol{x}} = \boldsymbol{f}(\bar{\boldsymbol{l}})$, where each $l_i$ is an average value, i.e., $\bar{\mathrm{d}}$ or $\bar{\theta}$

     $$\hat{\mathrm{h}} = \bar{\mathrm{d}}\tan\bar{\theta} = 56.78\mathrm{m}\ \tan 9°12'7" = 9.198\mathrm{m}$$

  3. Estimate the precision of $\hat{h}$

     $$\underbrace{\sigma_{\hat{h}}^2}_{\mathrm{cm}^2} = \underbrace{\left(\frac{\partial h}{\partial d}\right)^2 \sigma_{\bar{d}}^2}_{\mathrm{cm}^2} + \underbrace{\left(\frac{\partial h}{\partial \theta}\right)^2 \sigma_{\bar{\theta}}^2}_{\mathrm{cm}^2}$$

$$\sigma_{\bar{d}} = \frac{\sigma_d}{\sqrt{4}} = \frac{2}{\sqrt{4}} = \frac{2}{2} = 1\mathrm{cm} \quad \text{and} \quad \sigma_{\bar{\theta}} = \frac{\sigma_\theta}{\sqrt{9}} = \frac{30}{\sqrt{9}} = \frac{30''}{3} = 10''$$

$$\frac{\partial h}{\partial d} = \tan\theta \text{ (unit less)} \rightarrow \left(\frac{\partial h}{\partial d}\right)^2 \sigma_{\bar{d}}^2 \rightarrow cm^2$$

$$\frac{\partial h}{\partial \theta} = d \cdot \sec^2\theta \left(\text{units } m^2\right) \rightarrow \left(\frac{\partial h}{\partial \theta}\right)^2 \sigma_{\bar{\theta}}^2 \rightarrow m^2["]^2$$

$$\sigma_{\hat{h}}^2 = \left(tan9^0 12'17''\right)^2 (1)^2 + \left(56.78x100sec^2 9^0 12'17''\right)^2 \left(\frac{10''}{206265}\right)^2 = 0.1061 \quad cm^2$$

$$\hat{h} = 9.198m \pm 0.003m$$

## 3.2. Multivariate Error Propagation

- In the previous section we discussed error propagation in the univariate case. That is single unknown quantity derived from uncorrelated observables and thus we applied the law of propagation of variances.
- In this chapter, we will discuss error propagation in the multivariate case in which we will apply the law of propagation of variance-covariances (also known as the covariance law)
- To derive the covariance law, let us start with the special case of a univariate variable (single unknown $\boldsymbol{x}$ as a function of uncorrelated observables)

$$\boldsymbol{x} = \boldsymbol{f}\left(l_1, l_2, \ldots l_n\right) = \boldsymbol{f}\left(\boldsymbol{l}\right)$$

with the variance of $\hat{\boldsymbol{x}}$, $\sigma_{\hat{x}}^2$, derived from the law of propagation of variances as:

$$\sigma_{\hat{x}}^2 = \left(\frac{\partial x}{\partial l_1}\right)^2 \sigma_{l_1}^2 + \left(\frac{\partial x}{\partial l_2}\right)^2 \sigma_{l_2}^2 + \ldots\ldots + \left(\frac{\partial x}{\partial l_n}\right)^2 \sigma_{l_n}^2$$

$$= \left(\frac{\partial x}{\partial l_1}\right)\sigma_{l_1}^2\left(\frac{\partial x}{\partial l_1}\right) + \left(\frac{\partial x}{\partial l_2}\right)\sigma_{l_2}^2\left(\frac{\partial x}{\partial l_2}\right) + \ldots\ldots + \left(\frac{\partial x}{\partial l_n}\right)\sigma_{l_n}^2\left(\frac{\partial x}{\partial l_n}\right)$$

$$\sigma_{\hat{x}}^2 = \begin{bmatrix} \frac{\partial x}{\partial l_1} & \frac{\partial x}{\partial l_2} \cdots \frac{\partial x}{\partial l_n} \end{bmatrix} \begin{bmatrix} \sigma_{l_1}^2 & 0 & \cdots & 0 \\ & \sigma_{l_2}^2 & \cdots & 0 \\ \text{symmmetric} & & \ddots & \vdots \\ & & & \sigma_{l_n}^2 \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial l_1} \\ \frac{\partial x}{\partial l_2} \\ \vdots \\ \frac{\partial x}{\partial l_n} \end{bmatrix}$$

$$\sigma_{\hat{x}_{1,1}}^2 = \boldsymbol{J}_{\boldsymbol{l}_{1,n}} \boldsymbol{C}_{\boldsymbol{l}_{n,n}} \boldsymbol{J}_{\boldsymbol{l}_{n,1}}^T$$

- Now if we have more than one unknown, say a vector ***x*** of u unknowns, that are related to the n observations as follows:

$$\mathbf{x}_{ux1} = \mathbf{f}_{ux1}(\mathbf{l}_{nx1})$$

Or,

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{f}_1(l_1, l_2, ..., l_n) \\ \mathbf{x}_2 &= \mathbf{f}_2(l_1, l_2, ..., l_n) \\ &\vdots \\ \mathbf{x}_u &= \mathbf{f}_u(l_1, l_2, ..., l_n) \end{aligned}$$

- In this case, $\sigma^2_{\hat{x}}$ will be a matrix $\mathbf{C}_{\hat{\mathbf{x}}}$ with dimensions (*u x u)* that takes the following form:

$$\mathbf{C}_{\hat{\mathbf{x}}} = \begin{bmatrix} \sigma^2_{x_1} & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_u} \\ & \sigma^2_{x_2} & \cdots & \sigma_{x_2x_u} \\ \text{symmetric} & & \ddots & \vdots \\ & & & \sigma^2_{x_u} \end{bmatrix}$$

- Moreover, the ***J<sub></sub>l*** matrix will contain all the partial derivatives of the functions w.r.t the observations: $\dfrac{\partial x_j}{\partial l_i}$ $j = 1,2,..u$ $i = 1,2,..n$

$$J_{l_{u,n}} = \begin{bmatrix} \dfrac{\partial x_1}{\partial l_1} & \dfrac{\partial x_1}{\partial l_2} & \cdots & \dfrac{\partial x_1}{\partial l_n} \\ \dfrac{\partial x_2}{\partial l_1} & \dfrac{\partial x_2}{\partial l_2} & \cdots & \dfrac{\partial x_2}{\partial l_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial x_u}{\partial l_1} & \dfrac{\partial x_u}{\partial l_2} & \cdots & \dfrac{\partial x_u}{\partial l_n} \end{bmatrix}$$

- $\boldsymbol{J_l}$ is usually called the *Jacobian matrix*. It is also sometimes called the coefficient matrix (since, in the case of a linear mathematical model, the partial derivatives are just the coefficients of the observations).

- When some observations have correlation, then the variance-covariance matrix $\boldsymbol{C_l}$ will not be a diagonal matrix (i.e. it will be fully populated).

$$\mathbf{C}_{\mathbf{l}} = \begin{bmatrix} \sigma^2_{l_1} & \sigma_{l_1l_2} & \cdots & \sigma_{l_1l_n} \\ & \sigma^2_{l_2} & \cdots & \sigma_{l_2l_n} \\ \text{symmetric} & & \ddots & \vdots \\ & & & \sigma^2_{l_n} \end{bmatrix}$$

- Taking all the above characteristics of the multivariate case into account, the final covariance law will be: $\mathbf{C}_{\hat{\mathbf{x}}} = \mathbf{J}_{\mathbf{l}} \cdot \mathbf{C}_{\mathbf{l}} \cdot \mathbf{J}_{\mathbf{l}}^{\mathbf{T}}$

- **Summary of steps in multivariate error propagation:**
  - Given: $\bar{\mathbf{l}}_{nx1}, \mathbf{C}_{\bar{\mathbf{l}}\,nxn}$
  - Required: $\hat{\mathbf{x}}_{ux1}, \mathbf{C}_{\hat{\mathbf{x}}\,uxu}$

1. Form the direct (explicit) mathematical model

   $$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{l})$$

2. Establish the variance-covariance matrix of the observations $\mathbf{C}_{\mathbf{l}}$

3. Evaluate the elements of the Jacobian matrix:

   $$J_l = \frac{\partial x_j}{\partial l_i} \quad j = 1,2,...\text{u} \qquad i = 1,2,...\text{n}$$

4. Adjust the physical units in the covariance law

5. Apply the covariance law to get $\boldsymbol{C}_{\hat{\boldsymbol{x}}}$

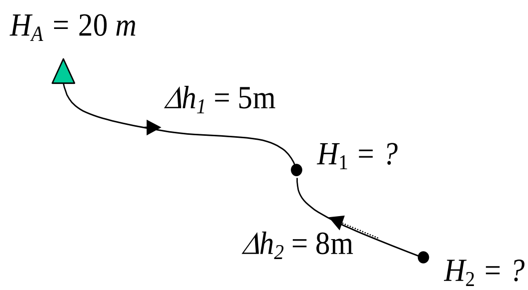
- Example (levelling network):

  - Given:

$$H_A = 20\ m \quad (BM)$$

$$\mathbf{l} = \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} m$$

$$\mathbf{C_l} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} mm^2$$




  - Required:

$$\hat{x} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \text{ and } \boldsymbol{C}_{\hat{x}}$$

    - Discuss the degree of correlation between $H_1$ and $H_2$.

  - Solution:

1. Mathematical model (direct model)

$$\begin{aligned} H_1 &= H_A + \Delta h_1 && = 20m + 5m && = 25m \\ H_2 &= H_A + \Delta h_1 - \Delta h_2 && = 20m + 5m - 8m && = 17m \end{aligned}$$

2. Variance-covariance matrix of the observations:

   $\mathbf{C_l}$ is given, note that the observations are uncorrelated (however, this does not mean that the best estimates of x will be uncorrelated as well)

3. Construct the Jacobian matrix

$$\boldsymbol{J}_{\boldsymbol{l}_{2,2}} = \begin{bmatrix} \dfrac{\partial H_1}{\partial \Delta h_1} & \dfrac{\partial H_1}{\partial \Delta h_2} \\ \dfrac{\partial H_2}{\partial \Delta h_1} & \dfrac{\partial H_2}{\partial \Delta h_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \ (\textit{no units})$$

   (Note: A linear model will result in the coefficients of the observations as the elements of $\mathbf{J_l}$)

4. Adjust the physical units in the covariance law.

   Since all the partial derivatives are unit-less, and the variances of the observations are in $mm^2$ (i.e. the same units as in $\boldsymbol{C}_{\hat{x}}$ ) no scaling is needed.

5. Apply the covariance law to get $\boldsymbol{C}_{\hat{x}}$

$$\boldsymbol{C}_{\hat{x}} = \mathbf{J_l} \cdot \mathbf{C_l} \cdot \mathbf{J_l^T}$$

$$\begin{bmatrix} \sigma^2_{H_1} & \sigma_{H_1H_2} \\ \sigma_{H_2H_1} & \sigma^2_{H_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 4 & -9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 13 \end{bmatrix} \mathrm{mm}^2$$

From which we get

$$\sigma_{H_1} = \sqrt{4\,mm^2} = 2\,mm \qquad \sigma_{H_2} = \sqrt{13\,mm^2} = 3.6\,mm$$

Note:

$$\begin{aligned} \sigma^2_{H_1} &= \sigma^2_{H_A} + \sigma^2_{\Delta h_1} \\ &= 0 + \sigma^2_{\Delta h_1} = 4\ \mathrm{mm}^2 \end{aligned} \quad \text{and} \quad \begin{aligned} \sigma^2_{H_2} &= \sigma^2_{H\,A} + \sigma^2_{\Delta h_1} + \sigma^2_{\Delta h_2} \\ &= 0 + 4 + 9 = 13\ \mathrm{mm}^2 \end{aligned}$$

- Correlation:

$$\rho_{H_1H_2} = \frac{\sigma_{H_1H_2}}{\sigma_{H_1}\sigma_{H_2}} = \frac{4}{2 \cdot 3.6} = +0.55\ \text{(signifcant )}$$

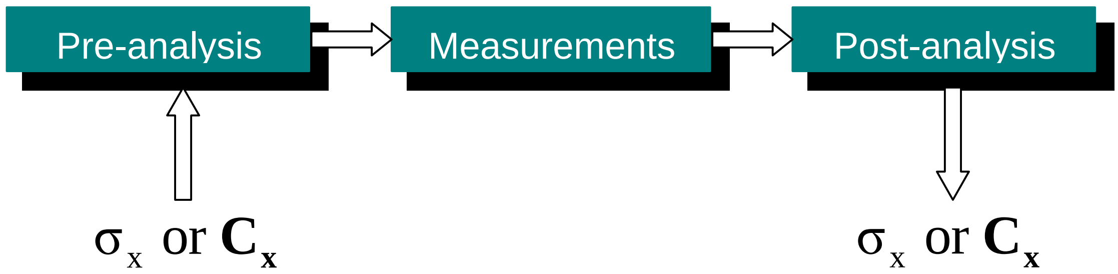
*(Positive errors in $H_1 \Rightarrow$ positive errors in $H_2$)*

- Homework:

Consider $\rho_{\Delta h_1 \Delta h_2} = 0.25$, estimate $\rho_{H_1H_2}$ ?

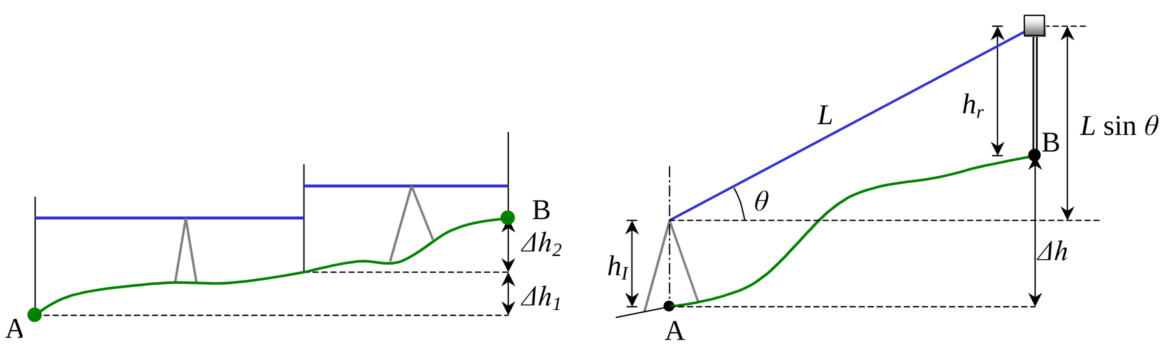
## 3.3. Pre-analysis of Survey Measurements

- Types of analysis




*Pre-specified accuracy* *Actual accuracy*

- Pre-analysis is the analysis of the measurement components before the project is actually undertaken.
- Assumptions: all components of a set of survey measurements are free of bias caused by systematic errors. This means that variances, or standard deviations, can be used as measures of accuracy as well as measures of precision.
- Main items to consider in pre-analysis of a certain survey project are:
  1. Possible survey techniques (and thus the corresponding mathematical model)
     - Example:




$H_B = H_A + \Sigma\Delta h_i$
*(Spirit leveling)*

***OR***

$H_B = H_A + h_I + L \sin\theta - h_r$
*(Trigonometric levelling)*

  2. Available surveying instruments (cost, simplicity, and the precision of a single measurement).

## 3.4. Principles of Pre-analysis

- Consider the simple case of having a single unknown and uncorrelated observations – the process of pre-analysis is performed through the application of the law of propagation of variances:

Recall:

$$\underbrace{\sigma_{\hat{x}}^2 = \left(\frac{\partial x}{\partial l_1}\right)^2 \frac{\sigma_{l_1}^2}{n_1} + \left(\frac{\partial x}{\partial l_2}\right)^2 \left(\frac{\sigma_{l_2}^2}{n_2}\right) + \ldots\ldots + \left(\frac{\partial x}{\partial l_N}\right)^2 \left(\frac{\sigma_{l_N}^2}{n_N}\right)}_{\text{N-terms}}$$

$$\sigma_{\hat{x}}^2 = \sum_{i=i}^{N} \left(\frac{\partial x}{\partial l_i}\right)^2 \left(\frac{\sigma_{l_i}^2}{n_i}\right)$$

$\sigma_{\hat{x}}^2$ ... *Final required accuracy/precision*

$\left(\frac{\partial x}{\partial l_i}\right)$ ...*Effect of the math model*

$\left(\frac{\sigma_{l_i}^2}{n_i}\right)$ ... *Effect of the instrument and the number of observations*

Note:

$\sigma_{\hat{x}}$ ... *is usually given in this case (as a pre-specified value)*

$\left(\frac{\partial x}{\partial l_i}\right)$ ... *will depend on the mathematical model*

$\sigma_{l_i}$ ... *depends on the precision of the used equipment*

$n_i$... *is the number of observations*

- Usually $\left(\frac{\partial x}{\partial l_i}\right)$ and $\sigma_{l_i}$ are related and straight forward to decide upon, and therefore the number of observations $n_i$ are the main quantities we are interested in.

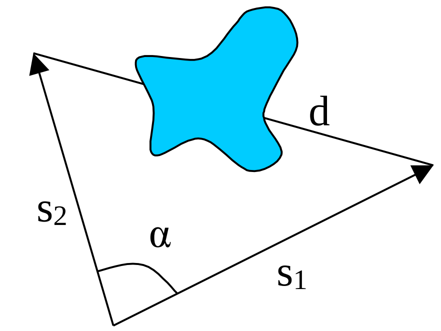
- Since we have only a single equation and $N$ unknowns, we cannot estimate $n_i$ unless we have some additional information or make some assumptions (e.g., the $N$ terms equally contribute to the total error budget).

♦ Example:

- The (d) distance cannot be measured directly, however, we can measure $s_1$, $s_2$ and $\alpha$.
- We know the following:

| *Measurement* | *Standard deviation* |
|---|---|
| $s_1 = 136\ m$ | $\sigma_{s_1} = 1.5\ cm$ |
| $s_2 = 115\ m$ | $\sigma_{s_2} = 1.5\ cm$ |
| $\alpha = 50°$ | $\sigma_\alpha = 10''$ |

Note: the value of the measurements can be obtained from a map (i.e. approximate values)




- Required

$$n_{s_1}, n_{s_2}, n_\alpha \text{ such that } \sigma_{\hat{d}} \le 0.5\ \text{cm}$$

- Solution:

1. Mathematical model

$$d = \left(s_1^2 + s_2^2 - 2s_1 s_2 \cos\alpha\right)^{1/2} = 107.77\ m \quad \textit{(cosine law)}$$

2. Error model

$$\sigma_{\hat{d}}^2 = \left(\frac{\partial d}{\partial s_1}\right)^2 \sigma_{\bar{s}_1}^2 + \left(\frac{\partial d}{\partial s_2}\right)^2 \sigma_{\bar{s}_2}^2 + \left(\frac{\partial d}{\partial \alpha}\right)^2 \sigma_{\bar{\alpha}}^2$$

$$\frac{\partial d}{\partial s_1} = \frac{2s_1 - 2s_2\cos\alpha}{2d} = 0.576 \ \textit{(unitless)}$$

$$\frac{\partial d}{\partial s_2} = \frac{2s_2 - 2s_1\cos\alpha}{2d} = 0.255 \ \textit{(unitless)}$$

$$\frac{\partial d}{\partial \alpha} = \frac{2s_1 s_2 \sin\alpha}{2d} = 111.17\ m = 11117\ cm$$

$$(0.5\ \text{cm})^2 \ge (0.576)^2 \sigma_{\bar{s}_1}^2 + (0.255)^2 \sigma_{\bar{s}_2}^2 + (11117\ \text{cm})^2 \frac{\sigma_{\bar{\alpha}}^2}{(206265)^2}$$

$$0.25\text{cm}^2 \ge 0.332\sigma_{\bar{s}_1}^2 + 0.065\sigma_{\bar{s}_2}^2 + 0.003\sigma_{\bar{\alpha}}^2$$

$$0.25\text{cm}^2 \ge 0.332\frac{\sigma_{s_1}^2}{n_{s_1}} + 0.065\frac{\sigma_{s_2}^2}{n_{s_2}} + 0.003\frac{\sigma_\alpha^2}{n_\alpha}$$

- We have 3 unknowns in one equation. Therefore, to solve this equation, we must impose some conditions:

1. 1st trial

Assume $s_1$, $s_2$ and $\alpha$ contribute equally to $\sigma^2_{\hat{d}}$.

$$\frac{0.25}{3} = 0.332\frac{(1.5)^2}{n_{s_1}} \quad \rightarrow n_{s_1} = 9$$

$$\frac{0.25}{3} = 0.065\frac{(1.5)^2}{n_{s_2}} \quad \rightarrow n_{s_2} = 2$$

$$\frac{0.25}{3} = 0.003\frac{(10)^2}{n_{\alpha}} \quad \rightarrow n_{\alpha} = 4$$

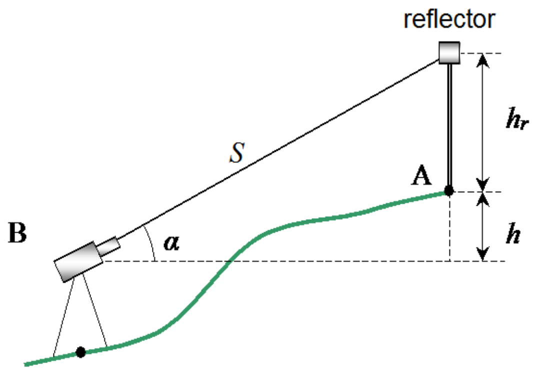
Note: since $\sigma_{s_1} = \sigma_{s_2}$ but $n_{s_1} >> n_{s_2}$ $\therefore$ equal contribution is not a proper assumption.

2. 2nd trial

$$\begin{array}{cccc}
0.25 = & 0.15 + & 0.05 + & 0.05 \\
 & \updownarrow & \updownarrow & \updownarrow \\
0.25 = & 0.332\frac{(1.5^2)}{n_{s_2}} + & 0.065\frac{(1.5)^2}{n_{s_2}} + & 0.003\frac{(10)^2}{n_{\alpha}} \\
 & \downarrow & \downarrow & \downarrow \\
 & n_{s_1} = 5 & n_{s_2} = 3 & n_{\alpha} = 6
\end{array}$$

Which is a more realistic than the first assumption especially for $n_{s_1}$ and $n_{s_2}$

- Example:

| | |
|---|---|
| ▪ The height h of a survey station (A) above the instrument at (B) is required with a precision/accuracy of 0.01m.<br><br>$h = s \cdot \sin(\alpha) - h_r$<br><br>$s = 400m, \alpha = 30^o$<br><br>$h_r$ = ? |  |



1. Estimate $\sigma_{\bar{s}}, \sigma_{\bar{\alpha}}, \sigma_{\bar{h}_r}$ assuming balanced (equal) precisions:

$$\sigma_{\hat{h}}^2 = \left(\frac{\partial h}{\partial s}\right)^2 \sigma_{\bar{s}}^2 + \left(\frac{\partial h}{\partial \alpha}\right)^2 \sigma_{\bar{\alpha}}^2 + \left(\frac{\partial h}{\partial h_r}\right)^2 \sigma_{\bar{h}_r}^2$$

$$\sigma_{\bar{s}} = \frac{\sigma_{\hat{h}}/\sqrt{3}}{\partial h/\partial s} = \frac{0.01/\sqrt{3}}{\sin(30)} = \frac{0.01/\sqrt{3}}{0.50} \qquad = 0.0115\ m$$

$$\sigma_{\bar{\alpha}} = \frac{\sigma_{\hat{h}}/\sqrt{3}}{\partial h/\partial \alpha} = \frac{0.01/\sqrt{3}}{s \cdot \cos(30)} = \frac{0.01/\sqrt{3}}{346.4} \qquad \begin{aligned} &= 1.67x10^{-5}\ rad \\ &= 1.67x10^{-5} * 206265'' \\ &= 3.4'' \end{aligned}$$

$$\sigma_{\bar{h}_r} = \frac{\sigma_{\hat{h}}/\sqrt{3}}{\partial h/\partial h_r} = \frac{0.01/\sqrt{3}}{(1)} \qquad = 0.006\ m$$

1. If $\sigma_{\bar{\alpha}}$ is limited by the instrument (5″ for example), re-evaluate $\sigma_{\bar{s}}$ and $\sigma_{\bar{h}_r}$ to accommodate for this limitation in $\sigma_{\bar{\alpha}}$. From Step 1:

$$(0.01)^2 = (0.5)^2 \sigma_s^2 + (346.4)^2 \left(\frac{5}{206265}\right)^2 + (-1)^2 \sigma_{h_r}^2$$
$$(0.0054)^2 m^2 = (0.5)^2 \sigma_s^2 + \sigma_{h_r}^2$$

Balancing the precisions of the two terms:

$$\sigma_{\bar{s}} = \frac{0.0054/\sqrt{2}}{0.5} = 0.008m\ (8mm) \rightarrow choose\ an\ EDM$$

$$\sigma_{\bar{h}_r} = \frac{0.0054/\sqrt{2}}{1} = 0.004m\ (4mm)$$