Chapter 5

Mathematical Models for GPS Positioning

Dilution of Precision (DOP) and Accuracy Measures
Pseudorange Point Positioning
Notions of Least-Squares Estimation and Kalman Filtering
Case Study: *Statistical Reliability Measures for GPS and Augmented GPS*

Performance Measures

- Accuracy: Ability of GPS to maintain the position within a total system error
- Availability: Percentage of time that GPS is usable
- Reliability: Ability to detect faults and to estimate the effects that undetected faults may have on solution
  - Internal reliability
    - Quantifies the smallest fault that can be detected on each observation through statistical testing of the least-squares residuals
  - External reliability
    - Quantifies the impact that an undetected fault can have on the estimated parameter
- Continuity: *ability* of a system to function within specified performance limits without interruption during a specified period (normally covering a particular manoeuvre, such as a turn)

Accuracy Measures

- **Accuracy**
  - Degree of closeness of an estimate to its true (but unknown) value
- **Precision**
  - Degree of closeness of observations to their means
  - In practice, accuracy and precision are often assumed to be the same
- **Predictable Accuracy**
  - Accuracy of position with respect to a reference coordinate system. Equivalent to absolute accuracy
- **Repeatable Accuracy**
  - Accuracy with which one can return to a position having coordinates which have been measured previously with same system
- **Relative Accuracy**
  - Accuracy of user's position to that of another user of the same navigation system; or accuracy of a user's position with respect to position in recent past
- **Resolution**
  - Measure of the degree of performance capability that a system can achieve

DRMS and CEP (2D)

- **DRMS (Distance Root Mean Squared):**
  - One number to express 2D accuracy
  - Convenient but not as rigorous as error ellipse or full covariance matrix
  - DRMS = $[\sigma_\phi^2 + \sigma_\lambda^2]^{1/2}$

    = Radial Error (Circle)

    = Mean Squared Position Error (MSPE)

    = Root Sum Square
  - Probability of circle with radius DRMS varies:

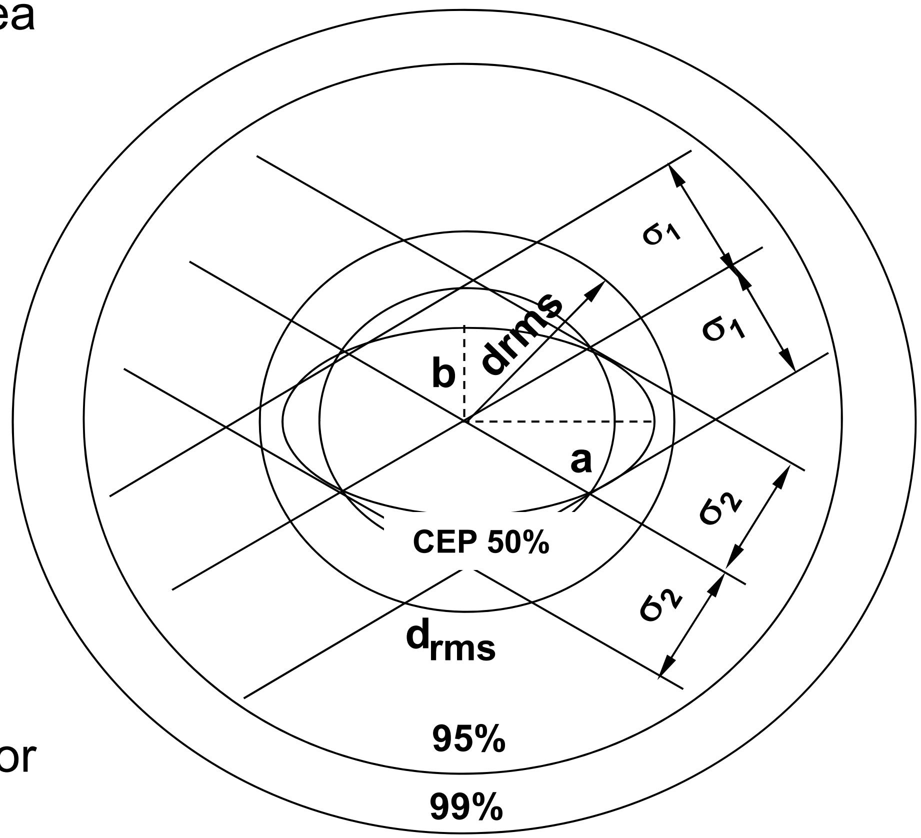
    $\sigma_\phi = \sigma_\lambda$ probability is 63%

    $\sigma_\phi = 10\ \sigma_\lambda$ probability is 68%
- **2DRMS:** 2 x DRMS: Probability between 95.4% & 98%
- **CEP (Circular Error Probable):**
  - Circle with 50% Probability {equivalent to 2D median error}
  - $CEP \approx 0.6[\sigma_\phi + \sigma_\lambda]$ {Approximation often used for GNSS when the 2 sigma values are similar in magnitude}
  - 95% Circle: CEP x 2.08 ≈ 2 x DRMS; 99% Circle: CEP x 2.58

- *Ref: Mikhail et al (1976) Observations and Least-Squares. IEP- A Dun- Donnelly Publisher, NY*

Comparison of Accuracy Measures (2D)

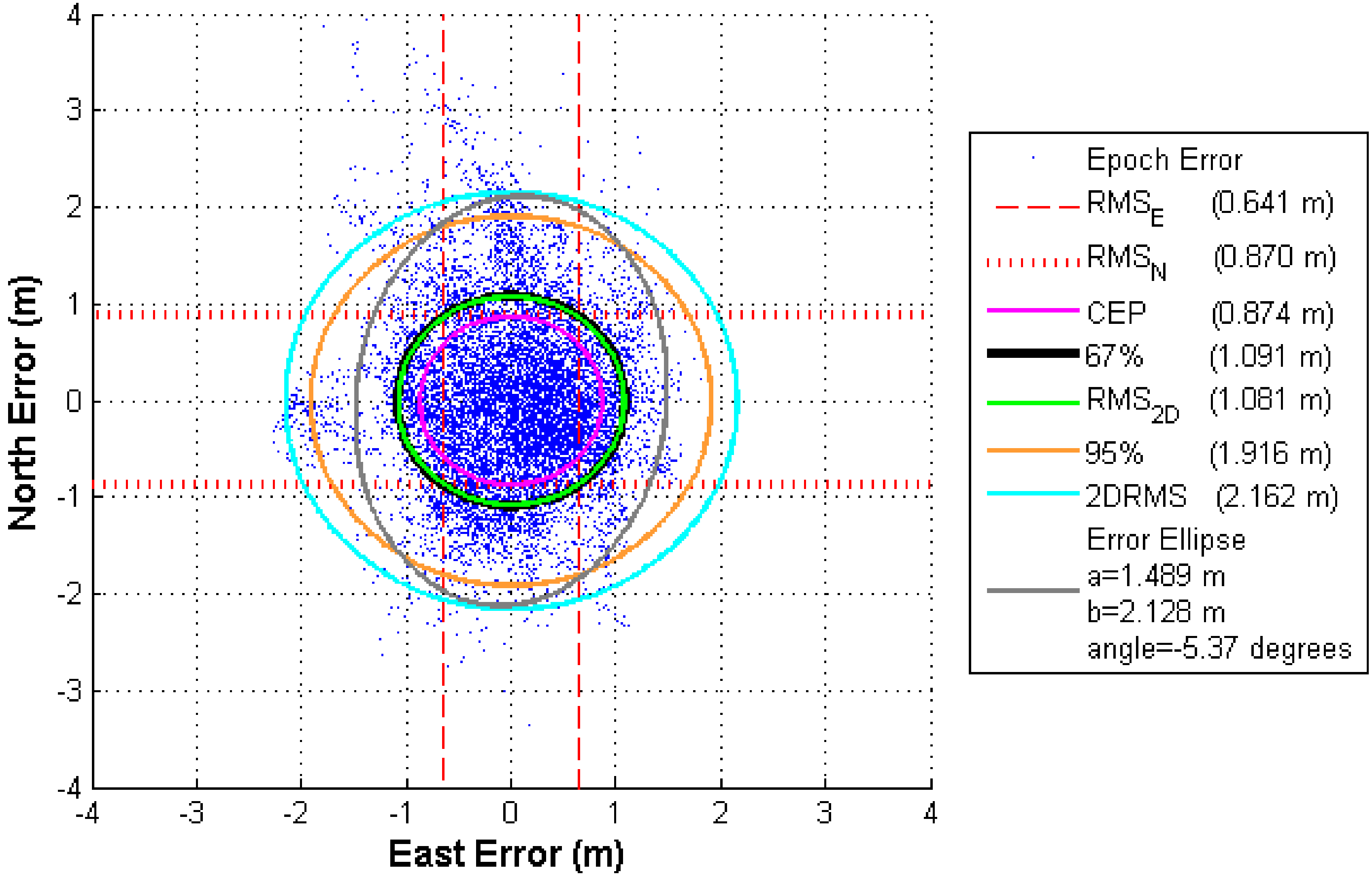
- DRMS
  - Probability of location within an area of constant radius
- Error ellipse
  - Constant probability, area varies
- **Error ellipse (2-D)**
  - Semi –major axes
  - Probability that measurement is within error ellipse: 39%
  - Axes x 2.45: Probability is 95%
- **Three dimensions:**
  - Error Ellipsoid: 19.9% probability
  - MRSE: Mean Radial Spherical Error



$\text{MRSE} = [\sigma_\phi^2 + \sigma_\lambda^2 + \sigma_h^2]^{1/2}$ probability of 61 %

SEP (Spherical error probable) $\approx 0.51\ [\sigma_\phi + \sigma_\lambda + \sigma_h]$ {sphere containing probability of 50%}

Various Accuracy Metrics Visualized



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Conversion Between Metrics

Given Accuracy Measurement \ Desired Accuracy Measurement	RMS_{1D}	RMS_{2D}	RMS_{3D}	RMS_V	CEP	2DRMS	67%	68%	95%	98%	SEP
RMS_{1D}	1	1.41	2.95	2.68	1.18	2.83	1.49	1.51	2.45	2.8	2.36
RMS_{2D}	0.71	1	2.10	1.89	0.84	2.00	1.06	1.07	1.74	1.99	1.70
RMS_{3D}	0.33	0.48	1	0.91	0.40	0.91	0.50	0.51	0.83	0.95	0.79
RMS_V	0.37	0.53	1.10	1	0.44	1.10	0.55	0.56	0.91	1.04	0.88
CEP	0.85	1.19	2.50	2.27	1	2.40	1.26	1.28	2.08	2.37	2.00
2DRMS	0.35	0.50	1.10	0.91	0.42	1	0.53	0.54	0.83	0.99	0.85
67%	0.67	0.95	1.98	1.80	0.79	1.89	1	1.01	1.64	1.88	1.58
68%	0.66	0.93	1.95	1.78	0.78	1.86	0.99	1	1.62	1.85	1.56
95%	0.41	0.58	1.20	1.10	0.41	1.20	0.61	0.62	1	1.14	0.96
98%	0.36	0.50	1.05	0.96	0.42	1.00	0.53	0.54	0.88	1	0.84
SEP	0.42	0.59	1.27	1.14	0.50	1.18	0.63	0.64	1.04	1.19	1

*Shaded areas are more susceptible to distribution assumptions

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Conversion Factor Assumptions

Three Assumptions When Converting Between Metrics

1. Errors are normally distributed about a zero mean
2. Geometry is equally balanced
   - Ratio of PDOP:HDOP is 2.1:1
   - Ratio between VDOP:HDOP is 1.9:1
3. The error distribution is circular

- Typically, these assumptions are not valid for all GPS data sets, but it is often acceptable to assume they are in order to compare various accuracy metrics
- Most conversions are within 10 cm, assuming nominal conditions
- Its common for a particular data sheet to provide one metric (e.g. CEP) but develop system accuracies with another metric (2DRMS)
- Manufacturers prefer CEP because the number is the lowest of the common statistical values
- Excellent References
  - Van Diggelen, F., "GPS Accuracy: Lies, Damn Lies, and Statistics." *GPS World*, pp. 41-44. Jan 1998.
  - Van Diggelen, F., "GNSS Accuracy: Lies, Damn Lies, and Statistics", *GPS World*, pp.17-21, Jan 2007

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Estimation Concept

What is the Problem?

- Estimation deals with estimating a set of parameters (states) using available measurements (observations)
- Estimating ‘N’ states using ‘N’ observations yields a unique solution
- What happens when more observations are available than is necessary to uniquely determine the states?
  - Example: What is the “true” length of a table using the following measurements?

Which is right?

Observations
1.48, 1.52, 1.55, 1.45,
1.55, 1.55, 1.40

Mean: 1.50 ?
Mode: 1.55 ?
Median: 1.52 ?

Estimation Concept

What is the Objective?

- The objective of estimation is to obtain a unique estimate of the state (unknowns) based on some **estimation criteria**
  - Unbiased estimate: The expected error of the estimated states is zero
  - Consistent estimates: As the number of observations tends to infinity, the estimates tends to the true value
  - Minimum variance: The state estimate has the smallest variance of all possible estimates

- Once the criteria are established, an appropriate estimator can be obtained
  - Least-squares and Kalman filtering are the two most common methods
  - ***These methods assume that measurement errors have a symmetrical distribution (which is not the case for multipath….. - Thus results are sub-optimal and can be biased)***

Math Model / Observation Equation

- The math model, or observation equation, forms the basis of least-squares estimation
- Assume the following linear relationship between the observations and the state vector

$$z = Hx + r$$

where

$\boldsymbol{z}$ is the observation vector, which has an associated covariance matrix C_z

$\boldsymbol{x}$ is the state vector (unknowns)

$\boldsymbol{H}$ is the design matrix

$\boldsymbol{r}$ is the measurement error vector (assumed to Gaussian white noise)

Ref: *Mikhail et al (1976) Observations and Least-Squares. IEP- A Dun- Donnelly Publisher, NY*

Koch, K. (1999), Parameter Estimation and Hypothesis Testing in Linear Models, Springer-Verlag, 2nd Edition

Leick, A. (1995) GPS Satellite Surveying, 2nd Edition, John Wiley & Sons.

Vanicek, P., and E.J. Krakiwsky (1986) Geodesy – The Concepts. Elsevier

Minimizing the Cost Function

- Observations errors are defined as follows (the "bar" represents estimates, namely the estimate of the state vector x in the following case)

$$z - H\bar{x}$$

- The *cost function* J, a quadratic form, can then be defined as

$$J = \left(z - H\bar{x}\right)^T P\left(z - H\bar{x}\right)$$

where P is a weighting function {defined in the next slide}

- The state vector estimate that minimizes the cost function is the least-squares estimate, given by

$$\bar{x} = \left(H^T PH\right)^{-1} H^T Pz$$

- *Note: In certain books, the above equation is negative. Whether a minus or plus sign should be used depends on how the misclosure vector is defined* ***in the non-linear case*** *as a function of initial and estimated states.*

Minimizing the Variance

- The variance of the least-squares estimate is given by

$$C_{\bar{x}} = \left(H^T P H\right)^{-1} H^T P \cdot E\left\{zz^T\right\} \cdot PH\left(H^T P H\right)^{-1}$$

where E{} is the expectation operator and $E\{zz^T\} = C_z$ (Covariance matrix of measurements)

- To minimize the variance, P should be equal to the inverse of the measurement covariance matrix

$$P = P_z = C_z^{-1}$$

such that

$$\bar{x} = \left(H^T C_z^{-1} H\right)^{-1} H^T C_z^{-1} z \qquad C_{\bar{x}} = \left(H^T C_z^{-1} H\right)^{-1}$$

Measurement Covariance Matrix

- C_z can be written as

$$C_z = \sigma_o^2 Q_z$$

  where Q_z is the cofactor matrix of measurements and σ_o^2 is the a priori variance of unit weight, which can be viewed as a scale factor
- If all measurements are uncorrelated and have the same variance, Q_z is the identity matrix and $C_z = \sigma_o^2 I$
- In the case of GPS pseudorange measurements, C_z could take the form

$$C_z = \sigma_o^2 \, \mathrm{Q}_z = \sigma_o^2 \begin{bmatrix} m(E,C/N_o)_1 & 0 & \ldots & 0 \\ 0 & m(E,C/N_o)_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & m(E,C/N_o)_i \end{bmatrix}$$

- where $m(E,C/N_o)_i$ could be a mapping function depending on the SV elevation E and/or C/N_o and σ_o^2 the variance of a zenith measurement with a nominal C/N_o value (typically 45 dB-Hz)

Least-Squares Solution

- Given that $C_z = \sigma_o^2 Q_z$, one can write

$$\bar{x} = \left(H^T C_z^{-1} H\right)^{-1} H^T C_z^{-1} z = -\left(H^T Q_z^{-1} H\right)^{-1} H^T Q_z^{-1} z$$

- which shows that the estimated parameters are independent of the *a priori* variance factor σ_o^2
- The covariance matrix can be written as

$$C_{\bar{x}} = \left(H^T C_z^{-1} H\right)^{-1} = \sigma_o^2 \left(H^T Q_z^{-1} H\right)^{-1}$$

- If $C_z = \sigma_o^2 I$ the above reduce to

$$\bar{x} = \left(H^T H\right)^{-1} H^T z \qquad C_{\bar{x}} = \sigma_o^2 \left(H^T H\right)^{-1}$$

$\left(H^T Q_z^{-1} H\right)^{-1}$ *or* $\left(H^T H\right)^{-1}$, *also* known as the cofactor matrices, are used to derive DOP

3D GPS Pseudorange Single-Point Positioning (1/2)

- Observation equation for measured pseudorange P:

$$P = \rho + d\rho + c(dt - dT) + d_{ion} + d_{trop} + \varepsilon_p$$

***Note:* Accuracy limited mainly by orbit, SV clock & atmospheric errors**

- Unknown parameters:
  - x, y, z coordinates of receiver contained in r = | $\mathbf{r}_s - \mathbf{r}_r$ |
  - receiver clock offset, cdT
- Partial derivatives of pseudorange to satellite i with respect to unknowns after linearization (Iterations are needed because the unknown rx coordinates are in the partial derivatives) are given below.

$$\frac{\partial P^i}{\partial x_r} = -\frac{(x_s^i - x_r)}{\rho^i} \qquad \frac{\partial P^i}{\partial y_r} = -\frac{(y_s^i - y_r)}{\rho^i}$$

$$\frac{\partial P^i}{\partial z_r} = -\frac{(z_s^i - z_r)}{\rho^i} \qquad \frac{\partial P^i}{\partial cdT} = -1$$

3D GPS Pseudorange Single-Point Positioning (2/2)

- For the case of four satellites, the design matrix at any one epoch is given below. The unknown coords of the user are in the partial derivatives and iterations are needed for convergence.

$$H = \begin{bmatrix} \frac{\partial P^i}{\partial x_r} & \frac{\partial P^i}{\partial y_r} & \frac{\partial P^i}{\partial z_r} & -1 \\ \frac{\partial P^j}{\partial x_r} & \frac{\partial P^j}{\partial y_r} & \frac{\partial P^j}{\partial z_r} & -1 \\ \frac{\partial P^k}{\partial x_r} & \frac{\partial P^k}{\partial y_r} & \frac{\partial P^k}{\partial z_r} & -1 \\ \frac{\partial P^l}{\partial x_r} & \frac{\partial P^l}{\partial y_r} & \frac{\partial P^l}{\partial z_r} & -1 \end{bmatrix} = \begin{bmatrix} -\frac{(x_s^i - x_r)}{\rho^i} & -\frac{(y_s^i - y_r)}{\rho^i} & -\frac{(z_s^i - z_r)}{\rho^i} & -1 \\ -\frac{(x_s^j - x_r)}{\rho^j} & -\frac{(y_s^j - y_r)}{\rho^j} & -\frac{(z_s^j - z_r)}{\rho^j} & -1 \\ -\frac{(x_s^k - x_r)}{\rho^k} & -\frac{(y_s^k - y_r)}{\rho^k} & -\frac{(z_s^k - z_r)}{\rho^k} & -1 \\ -\frac{(x_s^l - x_r)}{\rho^l} & -\frac{(y_s^l - y_r)}{\rho^l} & -\frac{(z_s^l - z_r)}{\rho^l} & -1 \end{bmatrix}$$

State Covariance Matrix – Geometrical Interpretation

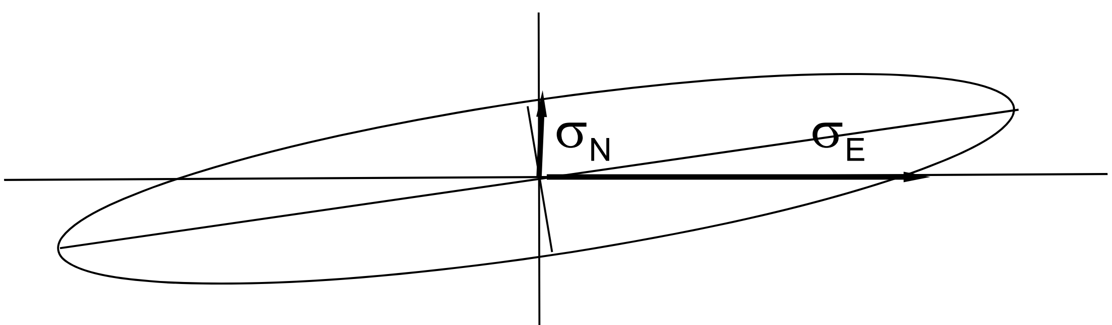
- Geometric interpretation of the covariance matrix in 2D:

$$C_{\bar{x}} = \begin{bmatrix} \sigma_N^2 & \sigma_{NE} \\ \sigma_{NE} & \sigma_E^2 \end{bmatrix}$$

N: Northing (φ)
E: Easting (λ)

- In 2D, errors can be represented by an ellipse which has an orientation that is a function of

$$\sigma_N^2, \sigma_E^2 \text{ and } \sigma_{NE}$$



$$DRMS = [\sigma_N^2 + \sigma_E^2]^{1/2}$$

$$HDOP = DRMS / \sigma_0$$

{The semi-axes of the error ellipse are equal to the eigenvalues of the covariance matrix and are oriented along the eigenvectors. Formulas to compute the error ellipse semi-axes can be found in Leick (1995)}

Residuals

- Once the final state estimate is obtained, the estimated measurement residuals can be computed as

$$\bar{r} = z - H\bar{x}$$

- Residuals are important for assessing the quality of the least-squares estimate
  - Large residuals imply the observations are not self-consistent, indicating a questionable solution
  - Small residuals imply the observations are self-consistent and a good solution
  - One large residual indicates a fault or blunder in the associated measurement
  - Residuals should be approximately normally (gaussian) distributed
  - If residuals have systematic variations (not gaussian), this indicates either biased measurements or a systematic effect not accounted for in the model
- The estimated or adjusted measurements are then given by

$$\bar{z} = z + \bar{r}$$

{See note on Slide #9 regarding signs in above equation}

A Posteriori Variance of Unit Weight

- Given “n” measurements and “m” unknowns (states), the a posteriori (after applying the least-squares algorithm) variance of unit weight is

$$\hat{\sigma}_o^2 = \frac{r^T P_z r}{n-m}$$

- where (n-m) in the number of degrees of freedom or redundancy. The expected value of $\hat{\sigma}_o^2$ is $E(\hat{\sigma}_o^2) = \sigma_o^2$

- Hence, if σ_o^2 was selected properly, $\hat{\sigma}_o^2 = \sigma_o^2$

- If $\hat{\sigma}_o^2 \neq \sigma_o^2$

$$C_{\bar{x}} = \frac{\hat{\sigma}_o^2}{\sigma_o^2}\left(H^T C_z^{-1} H\right)^{-1} = \hat{\sigma}_o^2\left(H^T Q_z^{-1} H\right)^{-1}$$

Other Covariance Matrices

- The cofactor and covariance matrices of the residuals and adjusted measurements are as follows

$$C_{\bar{r}} = C_z - H\left(H^T C_z^{-1} H\right)^{-1} H^T$$

$$C_{\bar{r}} = \hat{\sigma}_o^2 \, Q_{\bar{r}}$$

$$Q_{\bar{r}} = Q_z - H\left(H^T Q_z^{-1} H\right)^{-1} H^T$$

$$C_{\bar{z}} = C_z - C_{\bar{r}}$$

- $C_{\bar{r}}$ is used later in the calculation of the internal statistical reliability

Non-Linear Measurement Models

- The previous development assumed the measurement model was perfectly linear
- In practice, many math models are non-linear
- Linearization methods can be employed such that a least-squares estimate is still possible

$$z = f\left(x\right)$$

$$\downarrow$$

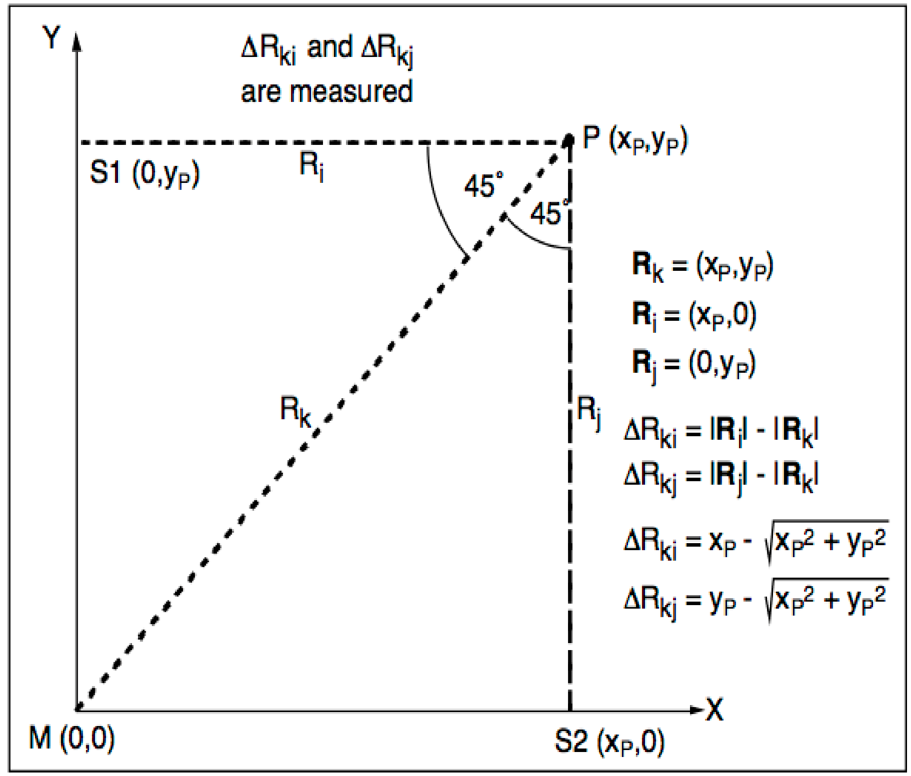
$$H_k = \left.\frac{f\left(x\right)}{\partial x}\right|_{x=\bar{x}_k} \qquad\qquad w_k = z - f\left(\bar{x}_k\right)$$

$$\downarrow$$

$$\bar{x}_{k+1} = \bar{x}_k + \left(H_k^T C_z^{-1} H_k\right)^{-1} H_k^T C_z^{-1} w \qquad\qquad C_{\bar{x}} = \left(H_k^T C_z^{-1} H_k\right)^{-1}$$

Numerical Example – TOA mode (1/3)

- Figure shows 3 distances from 3 transmitters to point P – 2D positioning
- TOA mode (3 ranges measured with an accuracy of 21 m)
- Use of least-squares:



  - $x = (H^T Q_z^{-1} H)^{-1} H^T Q_z^{-1} z$
  - $C_{\bar{x}} = \sigma_o^2 (H^T Q_z^{-1} H)^{-1}$
  - H**:** design matrix (3,2)
  - Q_z**:**(3,3)
  - z: measurements (3,1)
  - σ_o^2: measurement variance
  - x: state vector (unknowns) (2,1)
  - $R_i = R_j = 1000$ m
  - $R_k = 1400$ m
  - Approximate position of P: 1000 m, 1000 m

Numerical Example – TOA mode (2/3)

$$R_i = \sqrt{(x_P - x_{S1})^2 + (y_P - y_{S1})^2}$$

$$H = \begin{bmatrix} \frac{\partial R_i}{\partial x} & \frac{\partial R_i}{\partial y} \\ \frac{\partial R_k}{\partial x} & \frac{\partial R_k}{\partial y} \\ \frac{\partial R_j}{\partial x} & \frac{\partial R_j}{\partial y} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -0.707 & -0.707 \\ 0 & -1 \end{bmatrix}$$

$$\frac{\partial R_i}{\partial x} = -\frac{(x_P - x_{S1})}{R_i}$$

$$H^T H = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \qquad \left(H^T H\right)^{-1} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

$$C_x = \sigma_o^2 \left(H^T H\right)^{-1} = (21\ m)^2 \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix} = \begin{bmatrix} (18\ m)^2 & 110\ m^2 \\ 110\ m^2 & (18\ m)^2 \end{bmatrix}$$

$$HDOP = \sqrt{0.75 + 0.75} = 1.2$$

$$DRMS = \sqrt{\sigma_x^2 + \sigma_y^2} = \sigma_o\ HDOP \approx 25\ m$$

Assume that $Q_z = I$ and $\sigma_o = 21\,m$ for a range measurement

Numerical Example – TOA mode (3/3)

Now try with only the two ranges perpendicular to the two axes. The correlation is 0. The DRMS accuracy is lower as only two ranges are used.

$$H = \begin{bmatrix} \frac{\partial R_i}{\partial x} & \frac{\partial R_i}{\partial y} \\ \frac{\partial R_j}{\partial x} & \frac{\partial R_j}{\partial y} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad H^T H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C_x = \sigma_o^2 \left(H^T H\right)^{-1} = (21\ m)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (21\ m)^2 & 0 \\ 0 & (21\ m)^2 \end{bmatrix}$$

$$HDOP = 1.4$$

$$DRMS = \sqrt{\sigma_x^2 + \sigma_y^2} = \sigma_o\, HDOP = 29\ m$$

Numerical Example – Pseudoranging mode (1/1)

In this case, a time bias common to all three measurements is added to each observation equation:

$$R_i = \sqrt{(x_P - x_{S1})^2 + (y_P - y_{S1})^2} + cdt$$

$$H^T H = \begin{bmatrix} 1.5 & 0.5 & 1.707 \\ 0.5 & 1.5 & 1.707 \\ 1.707 & 1.707 & 3 \end{bmatrix}$$

$$H = \begin{bmatrix} \frac{\partial R_i}{\partial x} & \frac{\partial R_i}{\partial y} & \frac{\partial R_i}{\partial cdt} \\ \frac{\partial R_k}{\partial x} & \frac{\partial R_k}{\partial y} & \frac{\partial R_k}{\partial cdt} \\ \frac{\partial R_j}{\partial x} & \frac{\partial R_j}{\partial y} & \frac{\partial R_j}{\partial cdt} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ -0.707 & -0.707 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\left(H^T H\right)^{-1} = \begin{bmatrix} 9.206 & 8.206 & -9.907 \\ 8.206 & 9.206 & -9.907 \\ -9.907 & -9.907 & 11.608 \end{bmatrix}$$

$$C_x = \sigma_o^2 \left(H^T H\right)^{-1} = (21\ m)^2 \left(H^T H\right)^{-1} = \begin{bmatrix} (64\ m)^2 & 3618\ m^2 & -4369\ m^2 \\ 3618\ m^2 & (64\ m)^2 & -4369\ m^2 \\ -4369\ m^2 & -4369\ m^2 & (72\ m)^2 \end{bmatrix}$$

$$HDOP = \sqrt{9.206 + 9.206} = 4.3$$

$$DRMS = \sqrt{\sigma_x^2 + \sigma_y^2} = \sigma_o\ HDOP = 90\ m$$

Numerical Example – TDOA mode (1/2)

Only two range differences are available, namely

$$\Delta R_{ki} = R_k - R_i$$
$$\Delta R_{kj} = R_k - R_j$$

$$H = \begin{bmatrix} \frac{\partial \Delta R_{ki}}{\partial x} & \frac{\partial \Delta R_{ki}}{\partial y} \\ \frac{\partial \Delta R_{kj}}{\partial x} & \frac{\partial \Delta R_{kj}}{\partial y} \end{bmatrix} = \begin{bmatrix} -0.707 & 0.293 \\ 0.293 & -0.707 \end{bmatrix}$$

Assume that $Q_z = I$ (correlations between ranges are incorrectly neglected) and $\sigma_o = \sqrt{(21\,m)^2 + (21\,m)^2} = 30\,m \; \{\textit{for a range difference}\}$

$$C_x = \sigma_o^2 \left(H^T H\right)^{-1} = (30\,m)^2 \begin{bmatrix} 3.414 & 2.414 \\ 2.414 & 3.414 \end{bmatrix} = \begin{bmatrix} (55.4\,m)^2 & 2172.4\,m^2 \\ 2172.4\,m^2 & (55.4\,m)^2 \end{bmatrix}$$

$$HDOP = \sqrt{3.414 + 3.414} = 2.6$$

$$DRMS = \sqrt{(55.4\,m)^2 + (55.4\,m)^2} = \sigma_o\, HDOP = 78\,m$$

Numerical Example – TDOA mode (2/2)

Let's now look at the serial correlation introduced by differencing measurements. One can write

$$\Delta R = BR = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} R_k \\ R_i \\ R_j \end{bmatrix} \qquad Q_{\Delta R} = \sigma_o^2 Q_{\Delta R} = \sigma_o^2 BB^T = \sigma_o^2 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$C_x = \sigma_o^2 \left(H^T Q_{\Delta R}^{-1} H \right)^{-1} = (21\, m)^2 \begin{bmatrix} 9.2 & 8.2 \\ 8.2 & 9.2 \end{bmatrix} = \begin{bmatrix} (64\, m)^2 & 3709\, m^2 \\ 3709\, m^2 & (64\, m)^2 \end{bmatrix}$$

21 m is the standard deviation for a singe range equivalent measurement as opposed to 30 m for that of a measurements difference $\left\{ 30\, m = 21\, m\sqrt{2} \right\}$

$$HDOP = 4.3$$

$$DRMS = \sqrt{\sigma_x^2 + \sigma_y^2} = \sigma_o HDOP = 90\ m$$ (Same result as pseudorange mode)

Comments on Numerical Examples

- TOA mode yields the best results (best geometry – lowest DOP - and smallest position covariance matrix)
- TDOA and pseudoranging modes yield identical results
- All results obtained without assuming actual measurements. The results are valid for any scale of the configuration
- In all cases, except the 1st case (TOA with 3 ranges), the number of measurements is the same as the number of estimates. Thus no least-squares criteria is used in such a case. The actual estimation process would produce residuals with 0 values and no $\sigma_o^{2'}$ could be computed as the measurements would fit the model perfectly. Faults would not be detected.

Cofactor Matrices

- Assuming observations have the same standard deviation, then the covariance matrix of the estimated parameters can be written as follows

$$C_{\bar{x}} = \left(H^T C_z^{-1} H\right)^{-1} = \sigma_o^2 \left(H^T Q_z^{-1} H\right)^{-1} = \sigma_o^2 Q_x$$

where Q_x is the cofactor matrix as follows

$$Q_x = (H^T Q_z^{-1} H)^{-1} = \begin{pmatrix} q_{xx} & q_{xy} & q_{xz} & q_{xt} \\ q_{xy} & q_{yy} & q_{yz} & q_{yt} \\ q_{xz} & q_{yz} & q_{zz} & q_{zt} \\ q_{xt} & q_{yt} & q_{zt} & q_{tt} \end{pmatrix}$$

- In this way, the differences in the accuracy of the measurements can be accounted for

DOP Computation

- DOP can be computed from the cofactor matrix Q_x

$$GDOP = \sqrt{q_{xx} + q_{yy} + q_{zz} + q_{tt}} \text{ Geometric DOP (3D position + time)}$$

$$PDOP = \sqrt{q_{xx} + q_{yy} + q_{zz}} \text{ 3D position (x, y, z or } \varphi, \lambda, \text{h) DOP}$$

$$TDOP = \sqrt{q_{tt}} \text{ time DOP}$$

- DOP are often computed using the following cofactor matrix Q_x to measure strictly the satellite geometry

$$Q_x = (H^T H)^{-1}$$

- However the previous method is more realistic, especially in the indoors

Computation of HDOP and VDOP (1/2)

- Two other DOPs commonly used in GPS are HDOP and VDOP

  HDOP : Horizontal DOP (2D position – horizontal plane)
  VDOP : Vertical DOP (1D position – height)

- In order to compute HDOP and VDOP, the cofactor matrix Q_x which is expressed in the equatorial system must be transformed into the topocentric local (L) coordinate system using covariance propagation (i.e. ignoring the time parameter) {See next slide}

- x, y and z in the topocentric local are Northing, Easting and height

$$Q_{x_L} = R Q_x R^T = \begin{bmatrix} q_{xLxL} & q_{xLyL} & q_{xLzL} \\ q_{xLyL} & q_{yLyL} & q_{yLzL} \\ q_{xLzL} & q_{yLzL} & q_{zLzL} \end{bmatrix}$$

Computation of HDOP and VDOP (2/2)

- HDOP and VDOP can then be defined as

$$HDOP = \sqrt{q_{xLxL} + q_{yLyL}}$$

$$VDOP = \sqrt{q_{hLhL}}$$

- PDOP is the same in both the local and global coordinate systems

- Some useful DOP relationships:

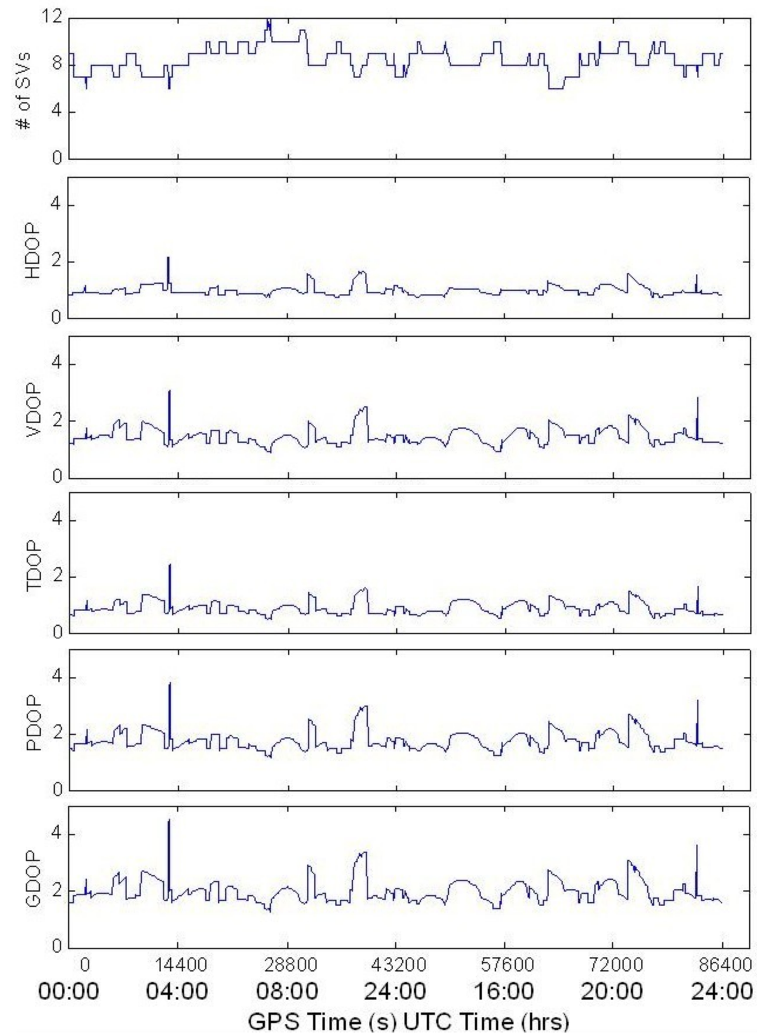
$$GDOP = \sqrt{HDOP^2 + VDOP^2 + TDOP^2}$$

$$PDOP = \sqrt{GDOP^2 - TDOP^2}$$

$$= \sqrt{HDOP^2 + VDOP^2}$$

Sample DOP Computation

Calgary April 12, 1998 (5° cutoff)



Statistical Method

- Knowledge of height in GNSS solution estimation is common for many applications because height can be obtained from maps for ground users or from atmospheric pressure changes (1 mbar change =1 hPa ≈ 10.5 m)
- The easiest method to implement this knowledge is to use the a priori information as a quasi-observation through the a priori covariance matrix C_{x0} of the state vector as

$$X = -\sigma_o^2 (H^T Q_z^{-1} H + C_{x0}^{-1})^{-1} H^T Q_z^{-1} z$$

- The appropriate height “constraint” is assigned through a “small” height variance in C_{x0}
- This method is simple to implement but not as effective as the inequality constraint method

Use of Known Height in Least-Squares Solution

Least-Squares Inequality (LSI) Constraint

- Problem: Constraint the height within known error bounds
- LS with such a constraint can be written
- Minimize || **Hx** – **z** || subject to **Gx** ≥ **h**
  - **H** is the design matrix
  - **x** is the vector of unknowns
  - **z** is the misclosure vector
  - **G** is the constraint matrix
  - **h** is the constant vector of inequality constraints
- For instance, assume that δh is within c = +/- 2 m, the constraint is written as follows and the least-squares solution is given in (Lu et al 1993):

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \delta\varphi \\ \delta\lambda \\ \delta h \\ \delta t \end{bmatrix} \geq \begin{bmatrix} -c \\ -c \end{bmatrix}$$

Reference: G.Lu et al (1993) Application of inequality constraint least-squares to GPS navigation under selective availability. Manuscripta Geodaetica, 18, 124-130

Impact of Height Constraint on HDOP

- The advantage of a height constraint is the gain of one degree of freedom. This improves robustness/reliability
- The HDOP with a fixed or quasi-fixed height improves substantially
- A position solution can be obtained with three satellites instead of four
- If the receiver clock can be constrained over a short time period, a minimum of two satellites in an appropriate geometry would be sufficient
- Because of the correlation between the clock state and the height, a height constraint significantly improves the clock estimate and a clock constraint will significantly improve the height estimate

Curvilinear Position Calculations

- Most textbooks show how to compute the Earth-Centred Earth-Fixed (ECEF) Cartesian coordinates of the receiver

- However, it is often more interesting/intuitive to have solutions in a horizontal and vertical frame
  - This can be accomplished through the use of latitude, longitude and height (curvilinear coordinates)
  - Similarly, if the user is considered the origin, the north, east and vertical directions define a topocentric coordinate frame

- There are two ways to achieve this
  - The first approach is to compute the navigation solution using Cartesian coordinates and then convert the result to latitude, longitude and altitude
  - The second approach is to compute the navigation solution directly in terms of latitude, longitude and altitude

Transforming from Cartesian to Curvilinear Coordinates

- For the first case, we assume we have a Cartesian solution and we wish to transform it to curvilinear coordinates
  - There are well known equations for doing this with the coordinates
  - For the covariance matrix, we need to apply the following transformation

$$C_x^{Curv} = BC_x^{Cart} B^T$$

where B is a transformation matrix from Cartesian to curvilinear (NEU) coordinates, and is given by

$$B = P_2 R_2 \left(\phi - \pi/2\right) R_3 \left(\lambda - \pi\right)$$

Computing Curvilinear Position Directly (1/4)

- The second approach is to compute the latitude, longitude and height directly in the least-squares solution
- In this case, the misclosure vector is computed in the same way as before (after converting the current estimate of the curvilinear coordinates to Cartesian coordinates to compute the range)
- The design matrix must be updated to take the derivatives with respect to latitude, longitude and height directly
  - Using the chain rule, the derivative of the pseudorange with respect to latitude can be written as (similar formulations for longitude and height)

$$\frac{\partial P}{\partial \phi} = \frac{\partial P}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial \phi}$$

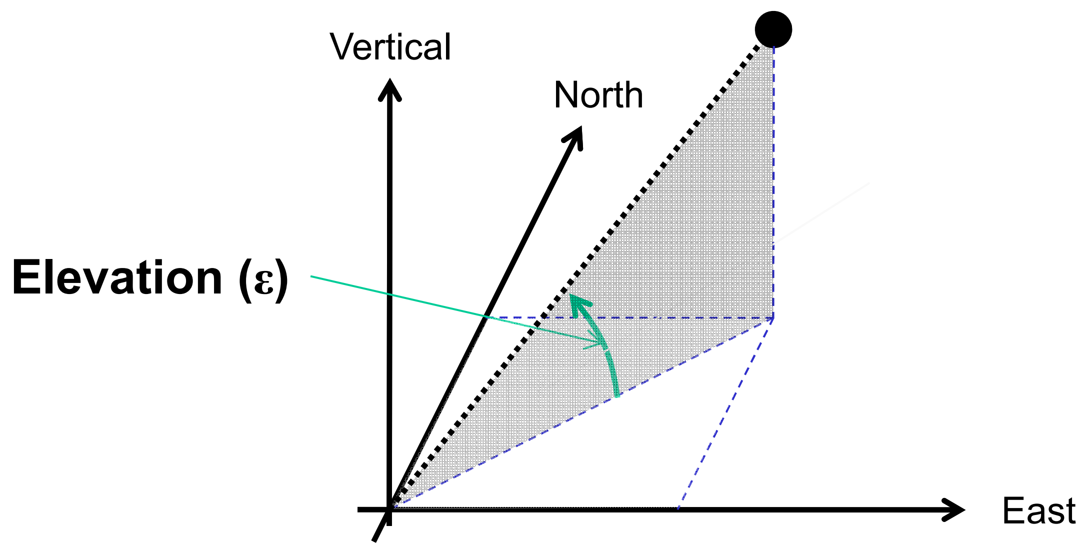
Computing Curvilinear Position Directly (2/4)

- The derivatives on the previous page are very tedious
- Fortunately, the problem can be simplified by considering the design matrix for the Cartesian case

$$H_i = \begin{bmatrix} \frac{\Delta x}{\rho} & \frac{\Delta y}{\rho} & \frac{\Delta z}{\rho} & 1 \end{bmatrix}$$

- We see that the first three elements of each row of the design matrix forms a unit vector that points to the satellite that is parameterized in Cartesian coordinates
- For the curvilinear case, we instead compute the unit vector in topocentric terms, namely in north (latitude), east (longitude) and up (height) directions

Computing Curvilinear Position Directly (3/4)



- From the above, we can easily define the (NEU) unit vector in terms of the azimuth and elevation of the satellite such that

$$H_i = \begin{bmatrix} \cos\varepsilon\cos\alpha & \cos\varepsilon\sin\alpha & \sin\varepsilon & 1 \end{bmatrix}$$

Computing Curvilinear Position Directly (4/4)

- Using the design matrix on the previous slide, the corrections to the current position estimate are computed in units of length (i.e., metres)
- To correct the curvilinear position we apply the following equations

$$\phi = \phi + \frac{\Delta N}{\left(R_E + h\right)}$$

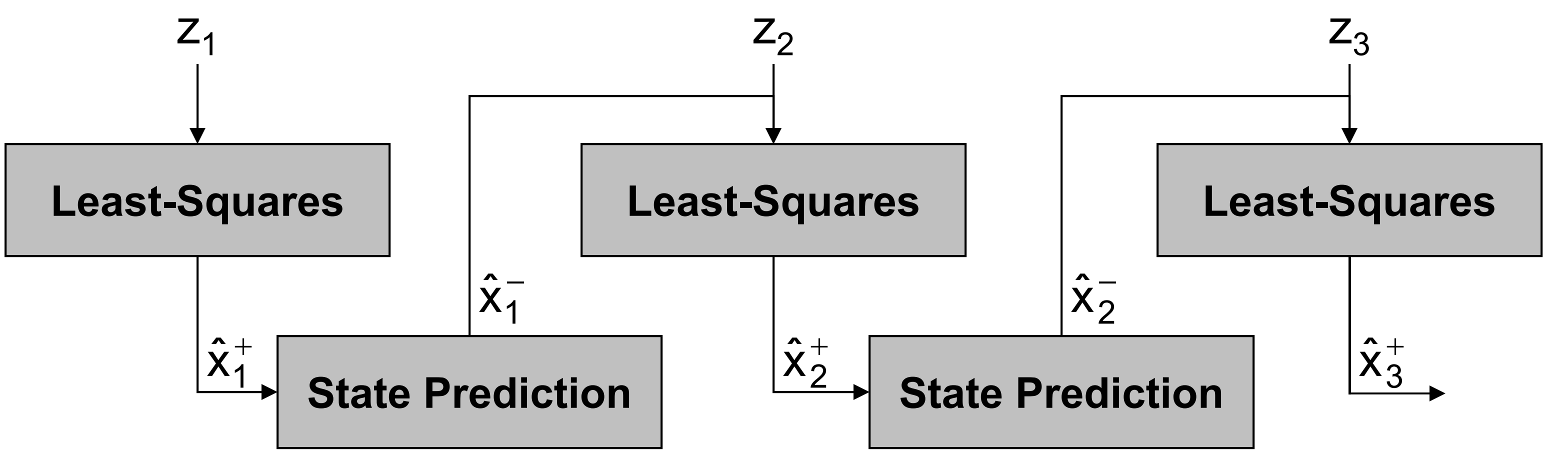
$$\lambda = \lambda + \frac{\Delta E}{\left(R_E + h\right)\cos\phi}$$

$$h = h + \Delta U$$

- By estimating the curvilinear coordinates directly, we are able to separate motion in the horizontal and vertical directions, which can be very helpful for many applications (i.e., using a height constraint)

Kalman Filtering Concept

- Kalman filtering is conceptually very similar to least-squares
- The difference is that in a Kalman filter, an assumption about how the state vector changes with time is used to (hopefully) improve the final state estimate
  - Conceptually, this is equivalent to adding extra observations to the least-squares approach, but the “observations” are based on predictions of the state vector since the last least-squares update



System (Dynamics) Model

- In addition to the observation (math) model in least-squares, Kalman filtering also uses a system (dynamics) model that describes how the state vector behaves over time

$$\dot{x} = Fx + Gw$$

where

x is the state vector

F is the dynamics matrix (usually based on physical properties)

G is a shaping matrix

w is a vector of Gaussian white noise noise processes (driving noise)

System (Dynamics) Model

- Once the dynamics model is formed, the state prediction can be written as

$$x_{k+1}^{-} = \Phi_{k,k+1} x_k \qquad C_{x_{k+1}}^{-} = \Phi_{k,k+1}^{T} C_{x_k} \Phi_{k,k+1} + Q_k$$

where

Φ is the state transition matrix, which is a function of the dynamics matrix (F)

Q is the process noise matrix, which is a function of the driving noise term (Gw)

Kalman Filter Update Equations

- As with least-squares, it is assumed that measurements are available that are linearly related to the state vector as follows

$$z = Hx + r$$

- How does a Kalman filter incorporate these measurements, especially while still taking into account the information from the dynamics model?

- To answer this question, we begin by assuming that the measurements will be incorporated using the following linear, recursive model

$$x^{+} = x^{-} + K\left(z - H\overline{x}\right)$$

where “super minus” represents a value immediately before an update, and “super plus” a value immediately after an update

Equation Summary

Prediction Equations

$$x_{k+1}^{-} = \Phi_{k,k+1} \cdot x_k$$

$$C_{x_{k+1}}^{-} = \Phi_{k,k+1}^{T} \cdot C_{x_k} \cdot \Phi_{k,k+1} + Q_k$$

Update Equations

$$K_k = C_{x_k}^{-} \cdot H_k^T \left(H_k \cdot C_{x_k}^{-} \cdot H_k^T + C_z \right)^{-1}$$

$$x_k^{+} = x_k^{-} + K_k \cdot \left(z - H_k \cdot x_k^{-} \right)$$

$$C_{x_k}^{+} = \left(I - K_k \cdot H_k \right) \cdot C_{x_k}^{-}$$

Refs: Gelb, A. (1974), Applied Optimal Estimation, The M.I.T. Press.
Grewal, M. and A. Andrews (2001) Kalman Filtering - Theory and Practice. Wiley.

Concept of Reliability

- Statistical reliability is a theoretical extension of blunder or fault detection
- Used in pre-analysis or post-analysis mode to detect the reliability of measurements, given the observation accuracy
- Statistical reliability is completely reliant on the probability of committing a type I (α) and type II (β) error
  - Type I: probability of rejecting a good measurement
  - Type II: probability of accepting a bad measurement
- The above probabilities are used to define the non-centrality parameter (δ_o), which is the smallest bias in the standardized residuals that can be detected (at the α and β probability levels)
- Some references on statistical reliability related to GNSS:

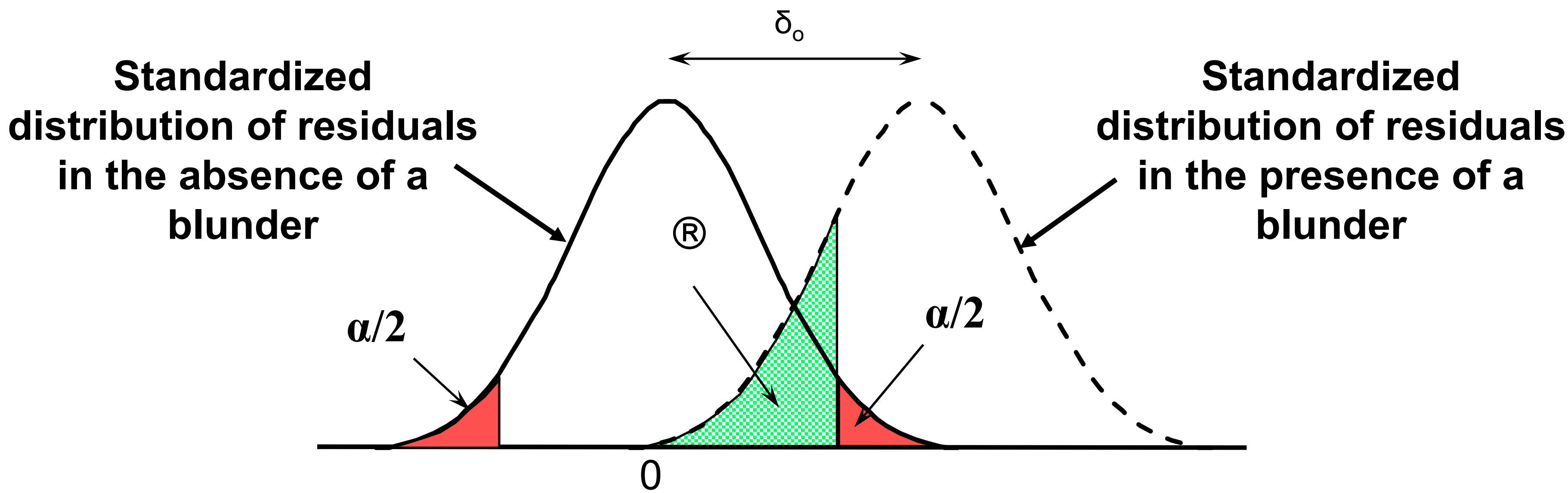
Baarda W. (1968) A testing procedure for use in geodetic networks. Netherlands Geodetic Commission, Publication on Geodesy, New Series 2, 5, Delft, Netherlands, pp 1–97

Lu, G. (1991) Quality Control for Differential Kinematic GPS Positioning. MSc. Thesis, published as Report No. 20042, Department of Surveying Engineering, The University of Calgary.

Ryan, S. (2002) Augmentation of DGPS for Marine Navigation. PhD Thesis, published as Report No. 20164, Department of Geomatics Engineering, The University of Calgary

Statistical Reliability

Graphical Representation of Concept



α (Type I Error)	β (Type II Error)	™
0.050	0.20	2.80
0.025	0.20	3.10
0.001	0.20	4.12
0.050	0.10	3.24
0.025	0.10	3.52
0.001	0.10	4.57

Internal Reliability

- **Internal Reliability:** It is a measure of the capability of the system (group of measurements) to detect and localize a blunder. The marginally detectable blunder (MDB) is the smallest blunder which can be detected, should it occur. If one assumes uncorrelated measurements and the presence of a single blunder at a specific epoch, the least-squares MDB for the i^th observation (there is a different formulation for a Kalman filter) can be calculated as follows:

$$\nabla_{o_i} = \frac{\delta_o \left(C_z\right)_{ii}}{\sqrt{\left(C_{\bar{r}}\right)_{ii}}}$$

where the covariance matrix of the residuals $C_{\bar{r}}$ is given by:

$$C_{\bar{r}} = C_z - H\left(H^T C_z^{-1} H\right)^{-1} H^T$$

External Reliability

- **External Reliability**: It is the effect (Δx) that a hypothetical blunder of magnitude "MDB" has on the estimated state vector

$$\Delta x = \left(H^T C_z^{-1} H\right)^{-1} H^T C_z^{-1} e_i \nabla_{o_i}$$

where

$$e_i = \begin{bmatrix} 0 & \cdots & 0 & \underbrace{\boxed{1}}_{i^{th}\text{ element}} & 0 & \cdots & 0 \end{bmatrix}^T$$

- The effect of a blunder on a particular parameter can be determined by looking at the corresponding element of the Δx vector

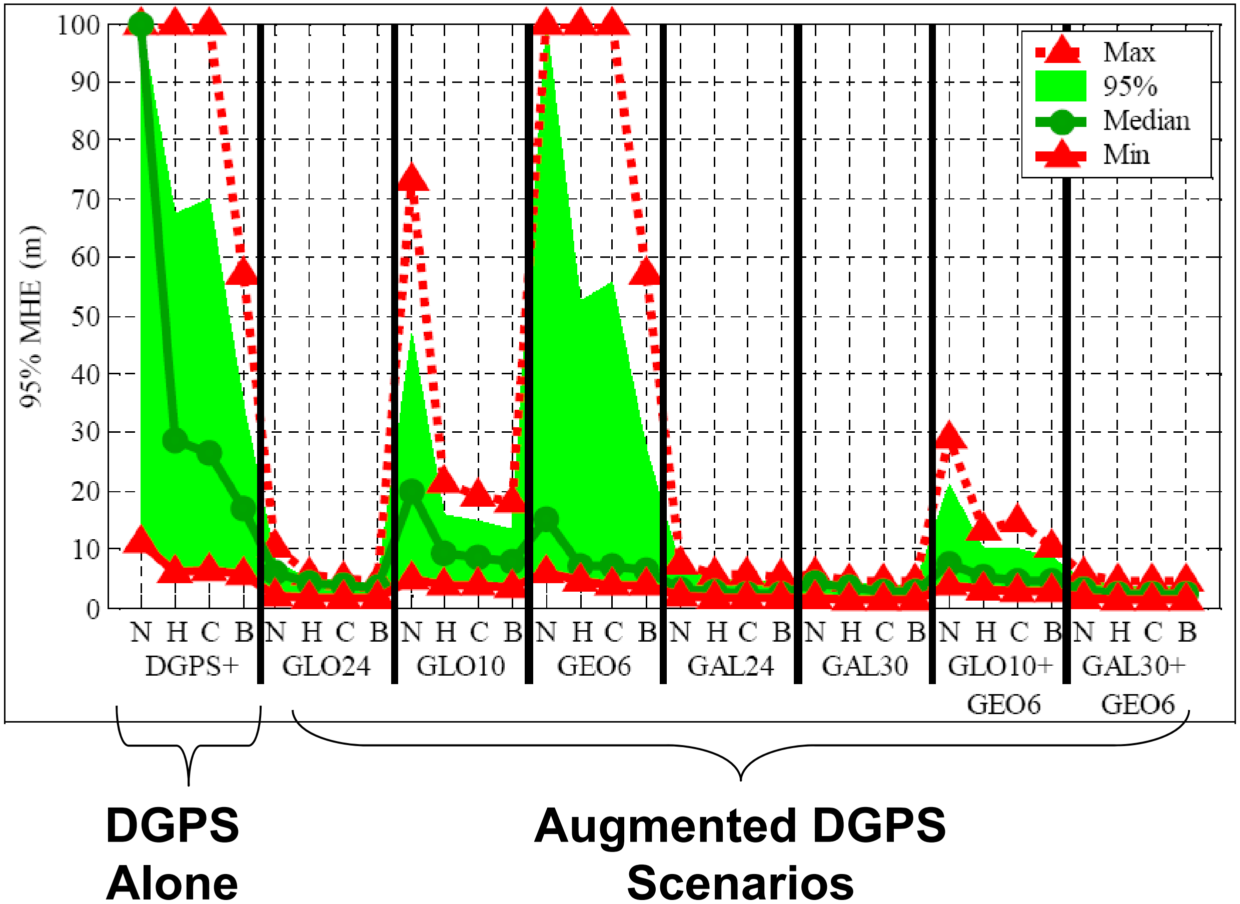
- Statistical reliablity is challenging in the indoors due to multiple range errors (noise and multipath) – See Kuusniemi et al

Kuusniemi, H., A. Wieser, G. Lachapelle, and J. Takala (2007) User-level Reliability Monitoring in Urban Personal Satellite Navigation. IEEE Transactions on Aerospace and Electronic Systems, 43, 4, 1305-1318.

Statistical Reliability

Sample Results

- Maximum Horizontal Error (MHE) (95th percentile across the globe) using different satellite constellations $\left\{ MHE = \sqrt{\Delta\varphi^2 + \Delta\lambda^2} \right\}$



GLO …GLONASS
GAL …GALILEO
GEO …Geostationary

N… no constraint
H… Height constraint
C… Clock constraint
B… Both H & C

Case Study

Statistical Reliability Measures for GPS and Augmented GPS

REFERENCE:

RYAN, S. (2002) Augmentation of DGPS for Marine Navigation. PhD Thesis, Report No. 20164, Department of Geomatics Engineering, University of Calgary.

Available on http://PLAN.geomatics.ucalgary.ca

Note: Notation in this case study is different from the notation used earlier in this section

Reliability Theory

External Reliability on Position

- Calculate the Marginally Detectable Blunder (MDB) for each observation:

$$\nabla_{o_i} = \frac{\delta_o \cdot \left(C_z\right)_{ii}}{\sqrt{\left(C_{\bar{r}}\right)_{ii}}}$$

- Calculate the impact of each MDB on the parameters.
  - Assume only 1 blunder occurs at any time
  - Calculate the effect that each MDB could have on the parameters:

$$\Delta X = -\left(A^T C_l^{-1} A\right)^{-1} A^T C_l^{-1} \nabla$$

  - For each blunder determine the Horizontal Error:

$$\text{Horizontal Error} = \sqrt{\Delta\varphi^2 + \Delta\lambda^2}$$

  - The MDB that produces the Maximum Horizontal Position Error (HPE) represents the **External Reliability**.

Reliability Theory

Least Squares Blunder Detection

- Residual Testing:

$$\bar{r}_i^* = \left| \frac{\bar{r}_i}{\sqrt{C_{r\,ii}}} \right| < n_{1-\frac{\alpha}{2}}$$

  - If σ_o unknown, the student distribution must be used.
- If a blunder is detected, perform sub-set testing:
  - Reject one (1) observation at a time and test the resulting subset of residuals.
  - If only one (1) sub-set passes, the blunder has been isolated
  - Otherwise the blunder has been detected, but cannot be isolated.

Reliability Theory

Least Squares Multiple Blunder Detection

- Assume that blunders are present on satellites "i" and "j". The k^{th} satellite's normalized residual must be < δ.

$$\frac{\bar{r}_k}{\sqrt{C_{\bar{r}_{kk}}}} = \frac{\left|R_{ki}\nabla_i + R_{kj}\nabla_j\right|}{\sqrt{C_{\bar{r}_{kk}}}} \leq \delta$$

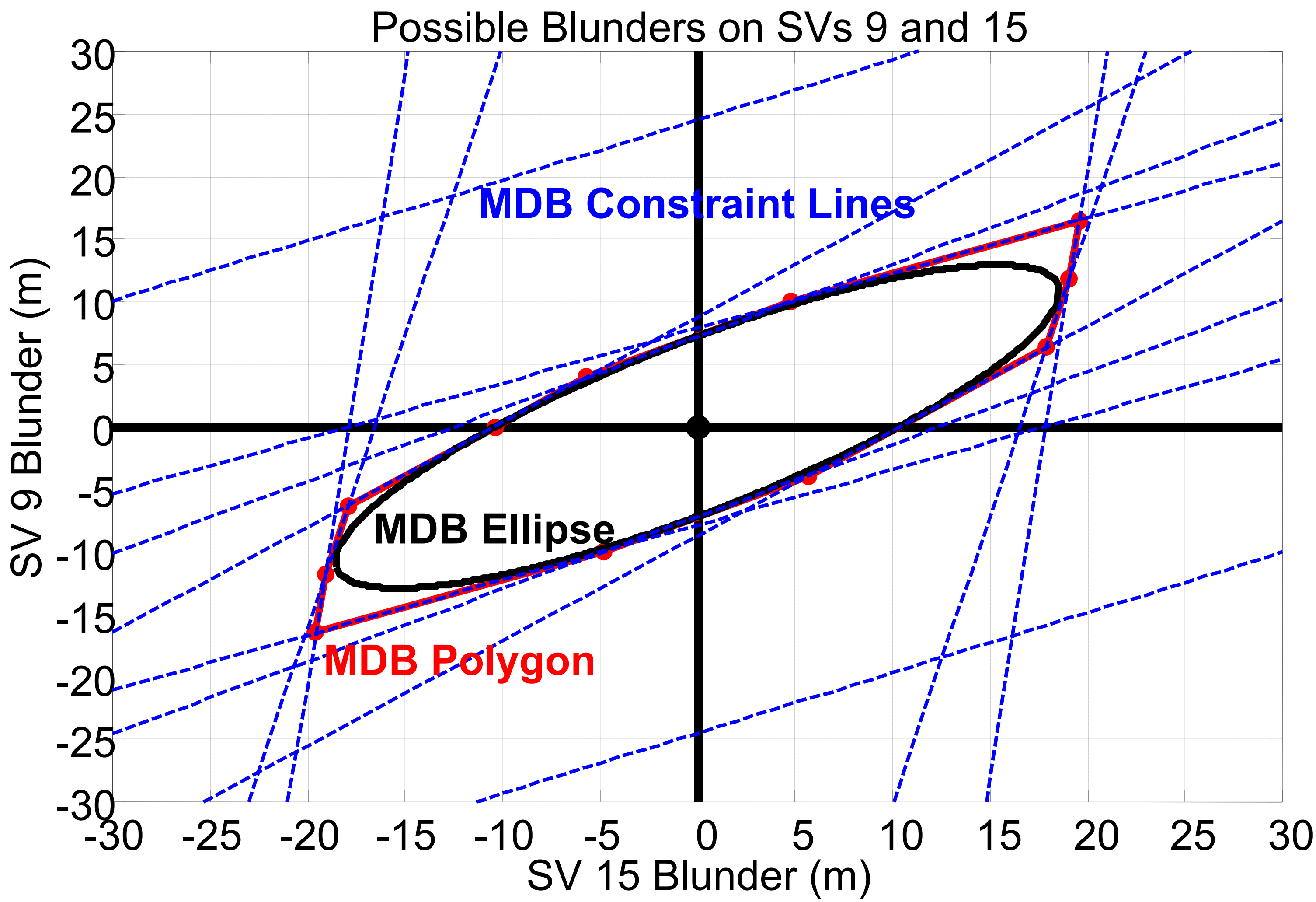
- With n observations there will be 2n of these constraints on the blunders, which define a MDB polygon in "i" and "j" blunder space. Substituting into the SSR results in the MDB ellipse.

$$\bar{r}^T C_l^{-1} \bar{r} \leq \delta^2$$

$$\sum_{k=1}^{k=n} \left(R_{ki}^2 \nabla_i^2 + 2R_{ki}R_{kj}\nabla_i\nabla_j + R_{kj}^2\nabla_j^2\right) C_{l_{kk}}^{-1} \leq \delta^2$$

Reliability Theory

Least Squares Multiple Blunder Detection Example



Kalman Filtering
Introduction

- Use all previous data to detect failures in the current epoch.
- Model:

$$x_k = \Phi_k x_{k-1} + w_k, w_k \sim n(0, Q_k)$$

$$z_k = H_k x_k + e_k, e_k \sim n(0, R_k)$$

- Based on our vehicle, we select a Dynamics Model and use it to estimate our parameters.
  - Constant Velocity (P and V)
  - Constant Acceleration (P, V, and A)
  - Time Correlation (ie Gauss Markov) (P, V, and A)

Kalman Filtering

Propagation and Updating

- Use the Dynamics Model to propagate the parameters to the next epoch.

$$\bar{x}_k^- = \Phi_k \bar{x}_{k-1}^+ \quad P_k^- = \Phi_k P_{k-1}^+ \Phi_k^T + Q_k$$

- Update our Parameters, using the current measurements and the propagated parameters.

$$\bar{x}_k^+ = \bar{x}_k^- + K_k (z_k - H_k \bar{x}_k^-) \quad P_k^+ = P_k^- - K_k H_k P_k^-$$

- Use the innovation sequence to detect blunders, similar to least-squares.

Kalman Filtering

Innovation Sequence Testing

- Test the Normalized Sum Square of the Innovations

$$Test = i^T C_i^{-1} i \sim \chi^2(m,0), \ \ where \, i = z_k - H_k \bar{x}_k^-$$

- Assume only one Blunder Occurs at any time and calculate the MDB.

$$\lambda_o = \nabla_k^2 \left(C_i^{-1}\right)_{kk} \Rightarrow \nabla_k = \sqrt{\frac{\lambda_o}{\left(C_i^{-1}\right)_{kk}}}$$

- This gives the same MDB as east-squares if we include a priori information on the parameters.
- Calculate the impact of each MDB on the parameters and generate the HPE similar to least-squares.

Kalman Filtering

Spectral Densities

- The filter is only as good as the model assumed
- In the simulations to follow, the Dynamics Model and Spectral Densities were extrapolated from an actual Canadian Coast Guard Survey Launch.

First Order Gauss Markov Process

Direction	σ^2 (10^{-3})	Time Constant
North & East	300	10 s
Up	10	1 s

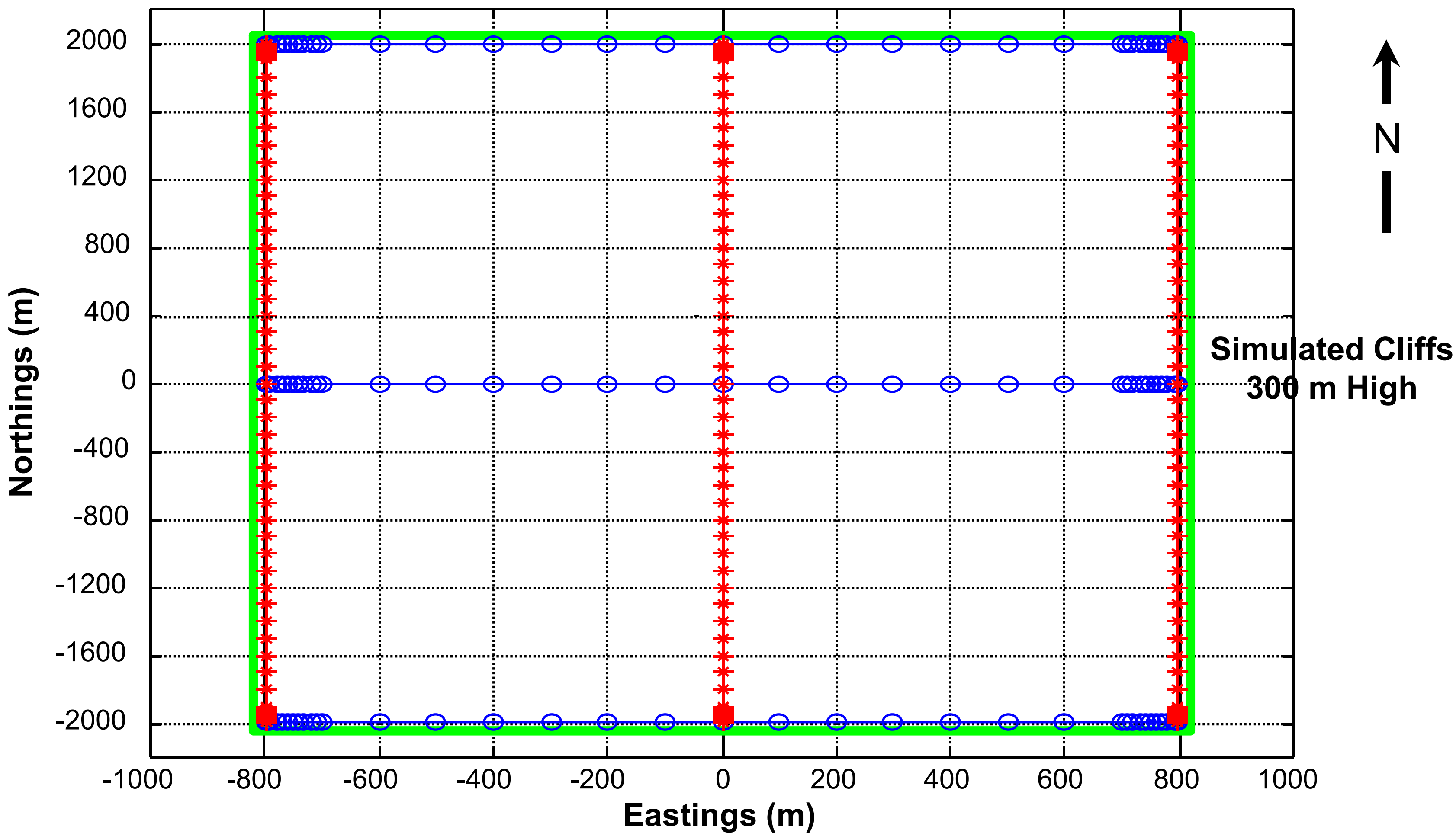
Simulation Description

Test Parameters

- Kalman Filter:
  - Survey Launch
- Constellations:
  - DGPS (1 m^2)
  - DGPS+DGEO (1 m^2)
  - DGPS+DGLO (1 m^2)
  - DGPS+DGEO+DGLO (1 m^2)
- Constraints:
  - Height Constraint (4 m^2)
  - Clock Constraint (1 m^2)
- Reliability Parameters:
  - α = 0.1%, β = 10%, δ = 4.57
- Simulation Data:
  - Date: July 25, 1997
  - Time: 24 Hours
  - Location: IOS Victoria, BC
    48^o N 123^o W
  - 25 GPS SV
  - 15 GLONASS SV
  - 6 Geostationary SV

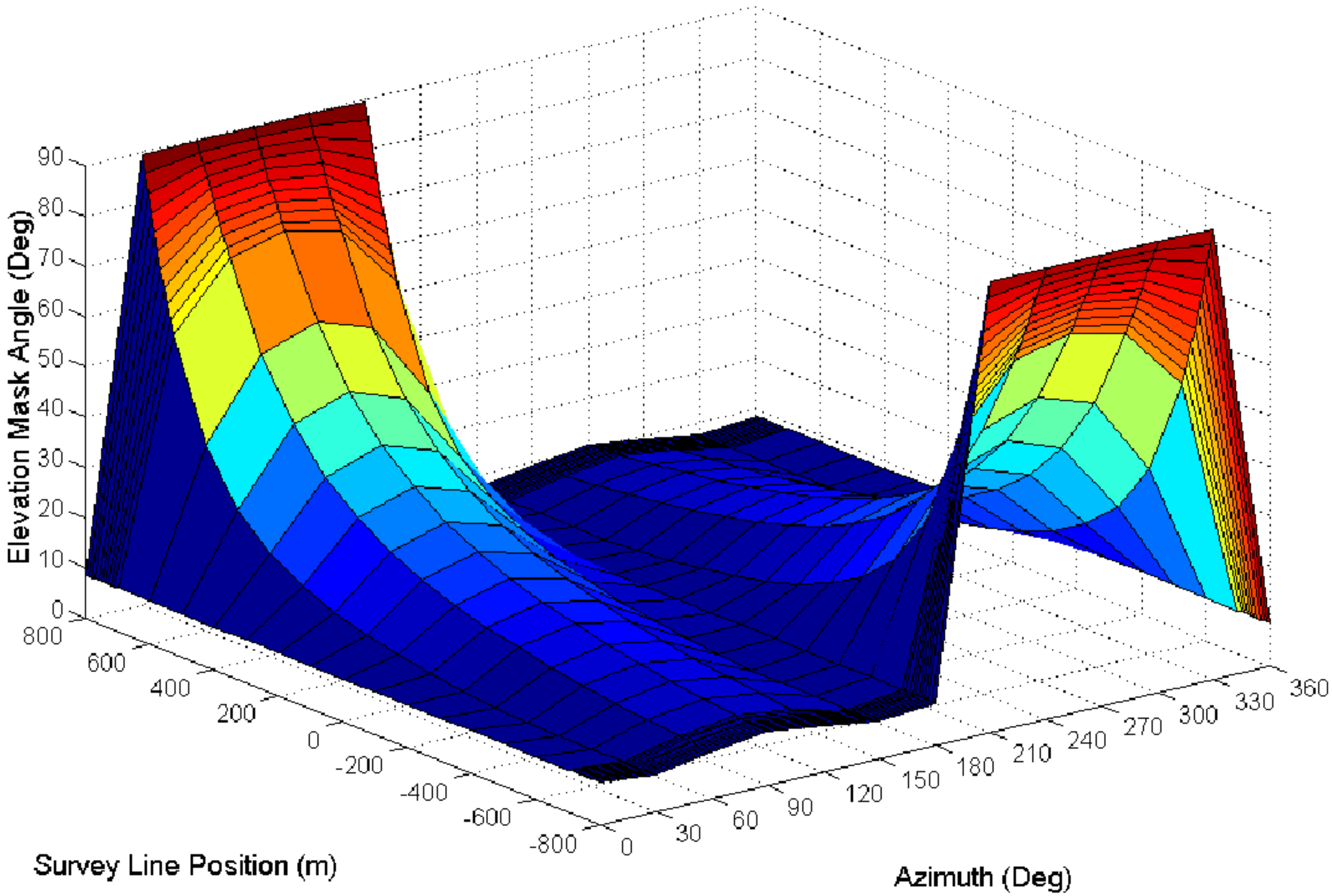
Simulation Description

Trajectory and Terrain



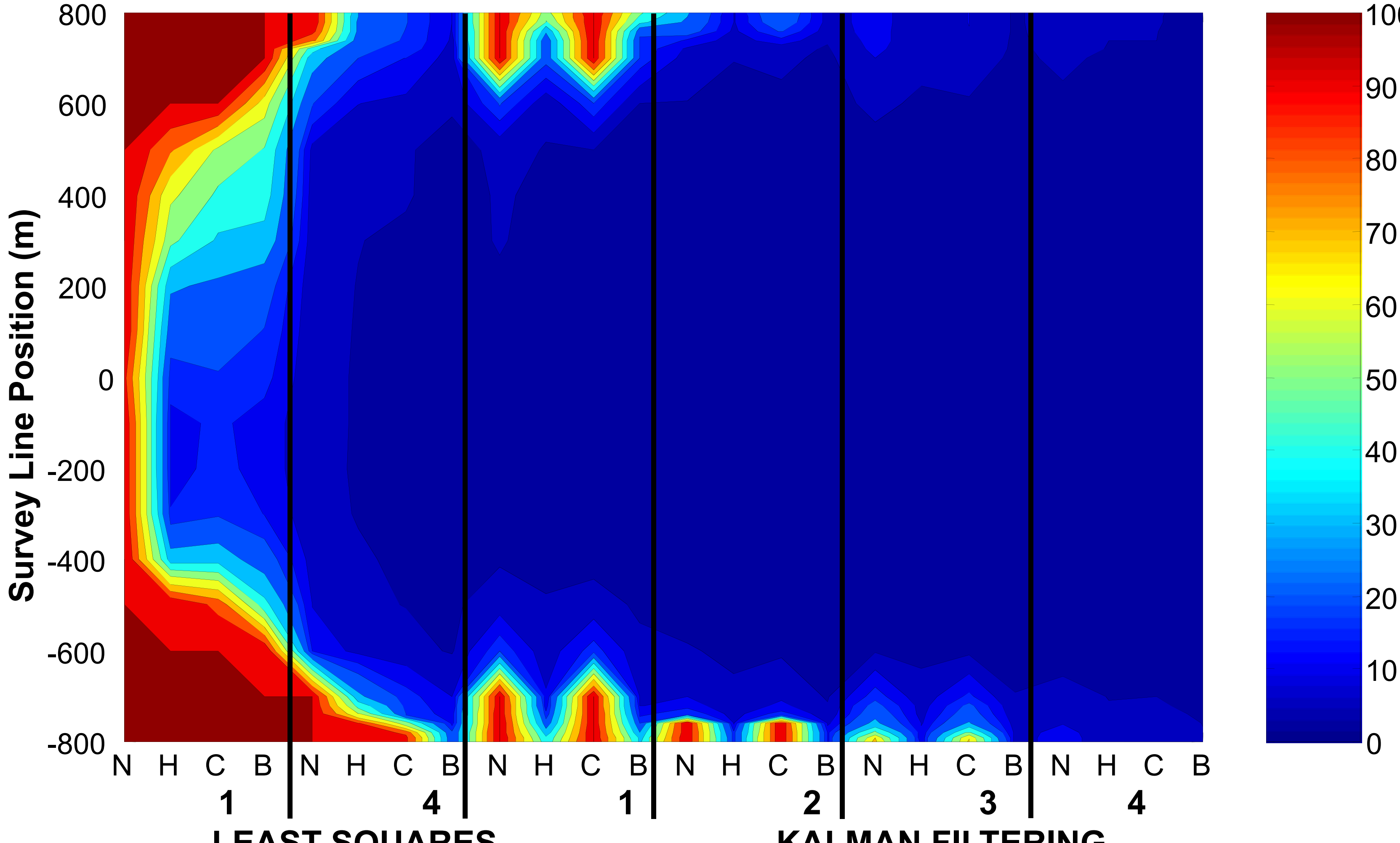
Simulation Description

East/West Line in the Middle - Mask Profile



Simulation Results (24 Hours)

East/West Line - HPE 95% - DGPS & KF



1 = DGPS , 2 = DGPS + DGLO, 3 = DGPS + DGEO, 4 = DGPS + DGLO + DGEO

Double Blunder Simulations

Test Parameters

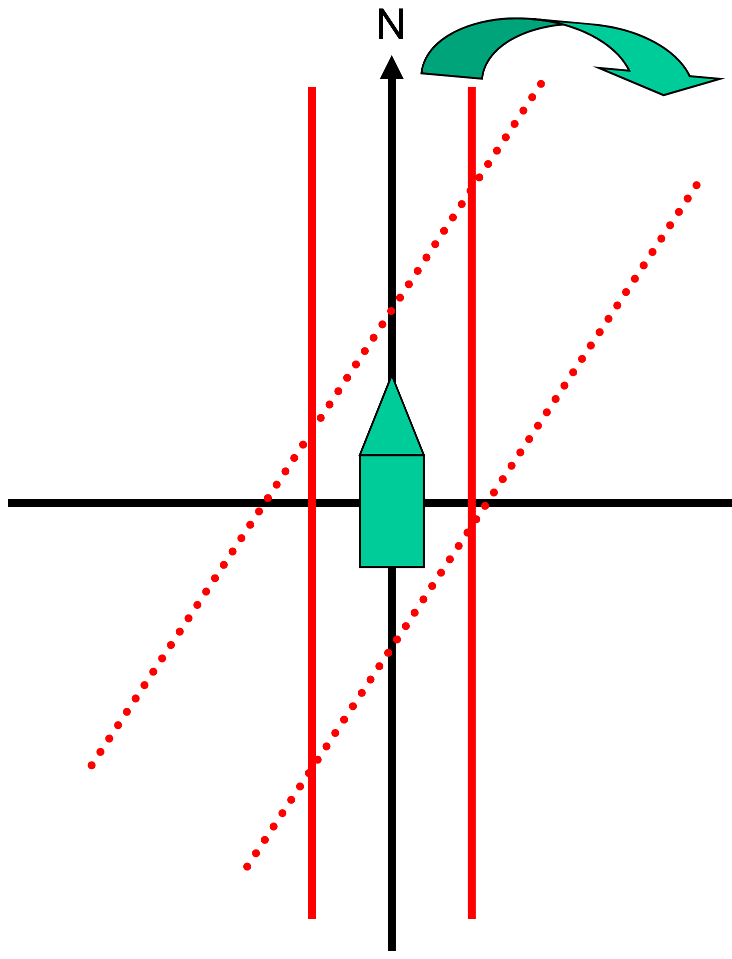
- Constellations:
  - DGPS (Obs m^2)
  - DGPS+DGLO a(Obs m^2)
  - DGPS+DGEO (Obs m^2)
  - DGPS+DGEO+DGLO (Obs m^2)
- Constraints:
  - None
  - Height Constraint (4 m^2)
  - Clock Constraint (1 m^2)
  - Height+Clock (4/1 m^2)
- Observation Variance
  - Narrow Correlator 1 m^2
  - Wide Correlator 9 m^2
- Mask Profile
  - Channel Rotated 180^o
- Reliability Parameters:
  - α = 0.1%, β = 10%, δ_o = 4.57
  - Single and Double Blunders
- Simulation Data:
  - Date: March 22, 2000
  - Time: 24 Hours
  - Location: 50^o N 114^o W
  - 27 GPS SVs
  - 8 GLONASS SVs
  - 6 GEO SVs

Simulation Description

Constricted Channels

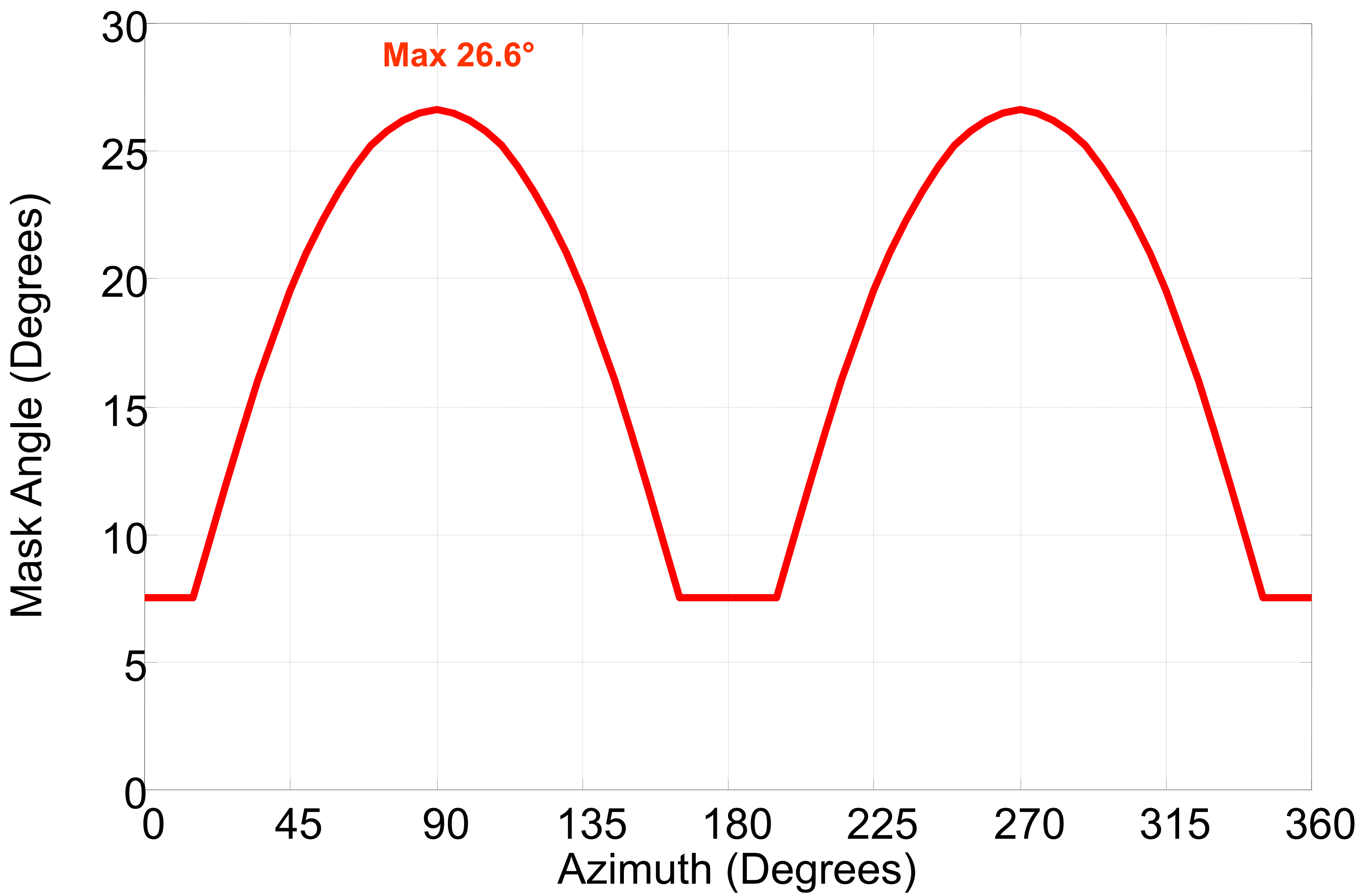
- Channel rotated 180° in 30° increments
- Reliability Analysis
  - Single Blunder
  - Double Blunder

- Note:
  - A pedestrian system could not use such constraints
  - A vehicle could use some trajectory constraints



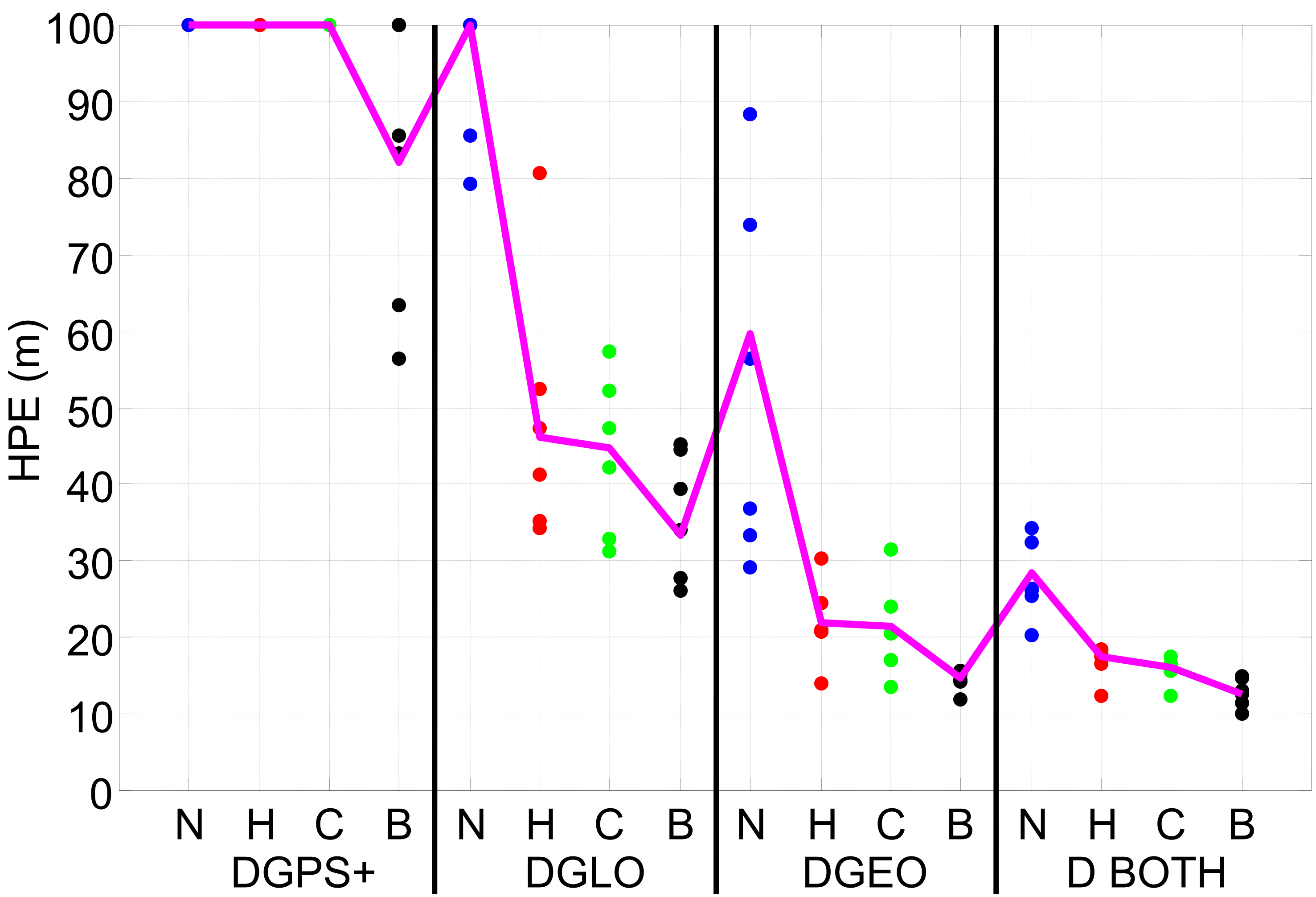
Simulation Description

Masking Profiles for the Constricted Channel



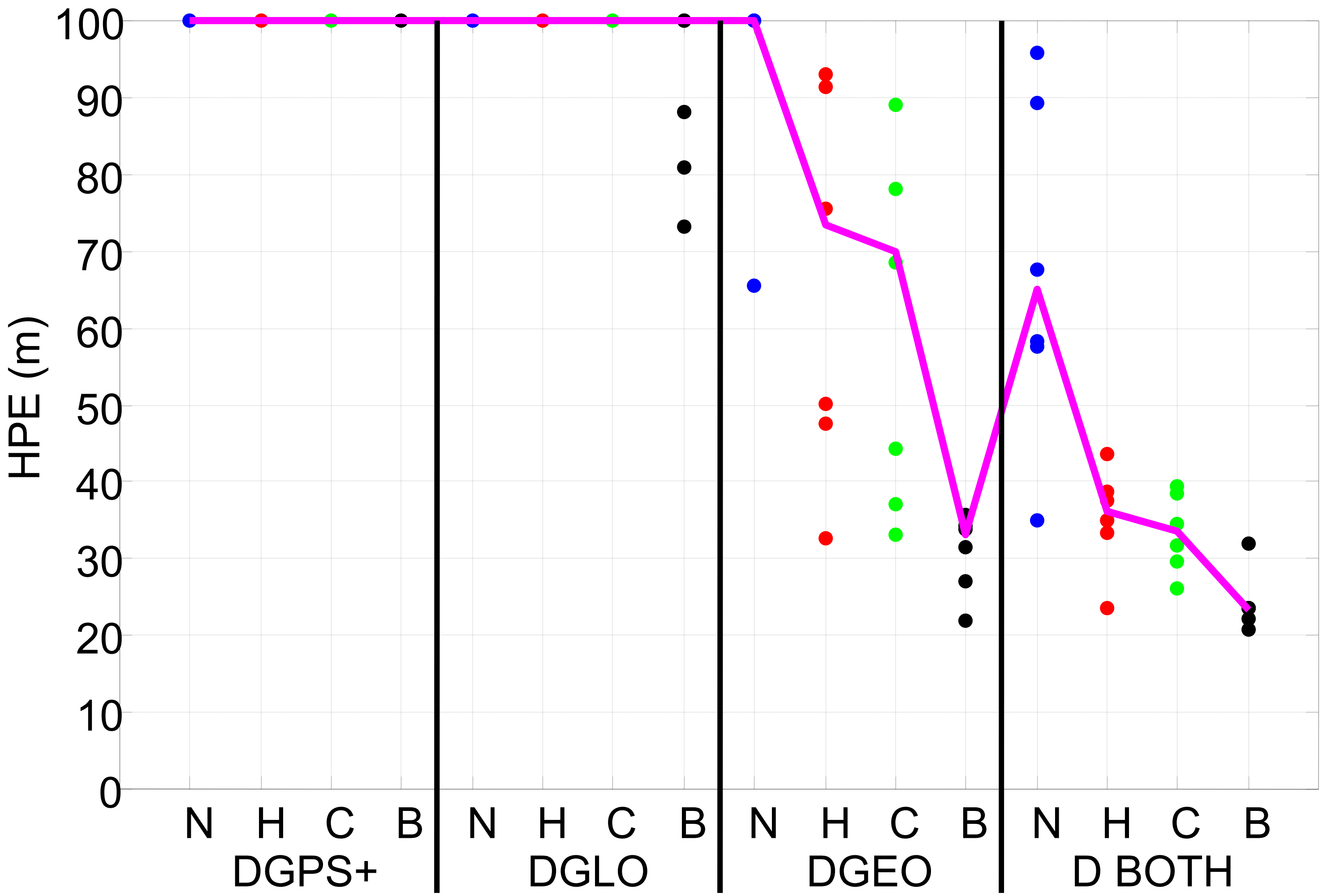
Simulation Results

95% HPE - Single Blunder - Wide Correlator



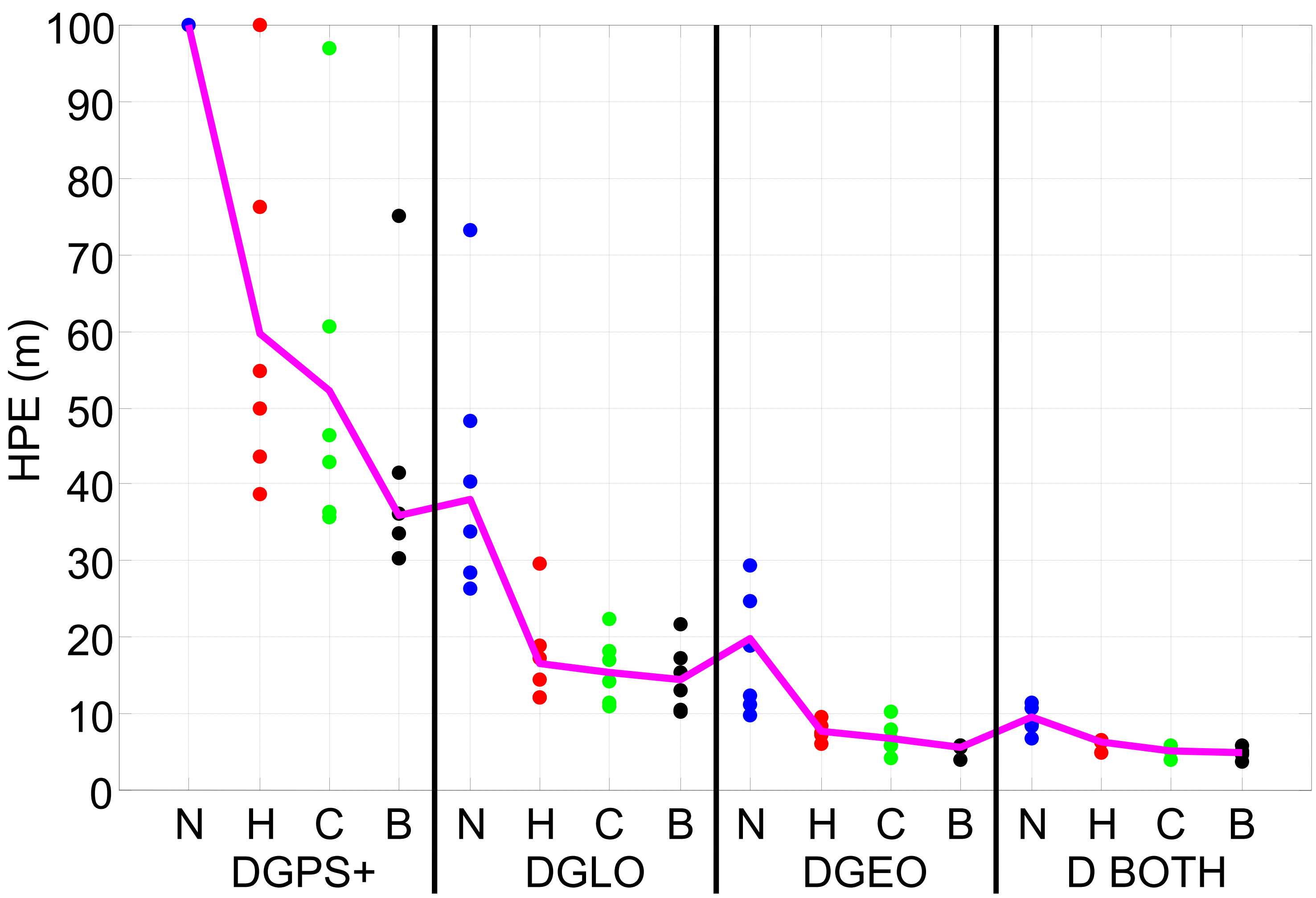
Simulation Results

95% HPE - Double Blunder - Wide Correlator



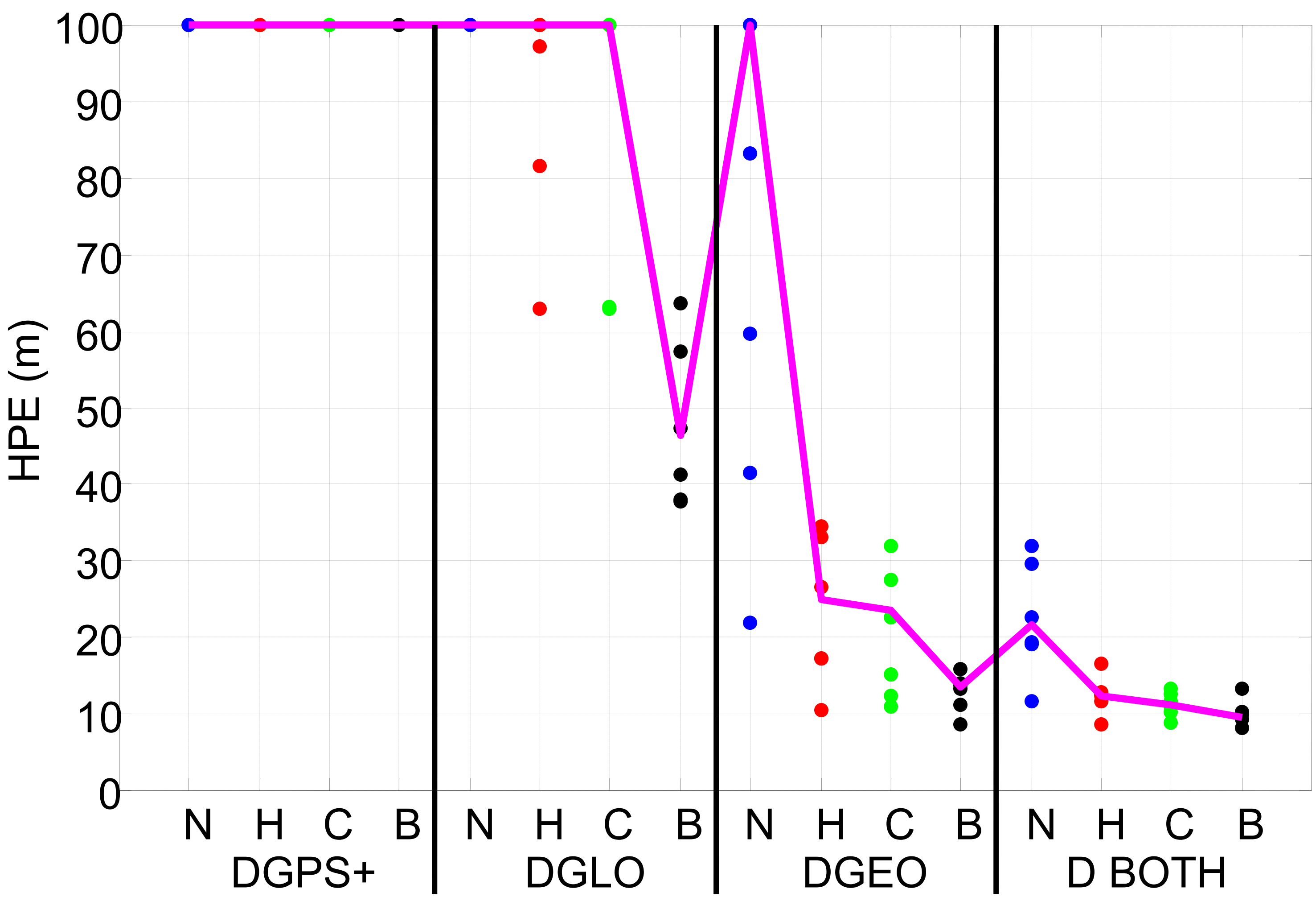
Simulation Results

95% HPE - Single Blunder - Narrow Correlator



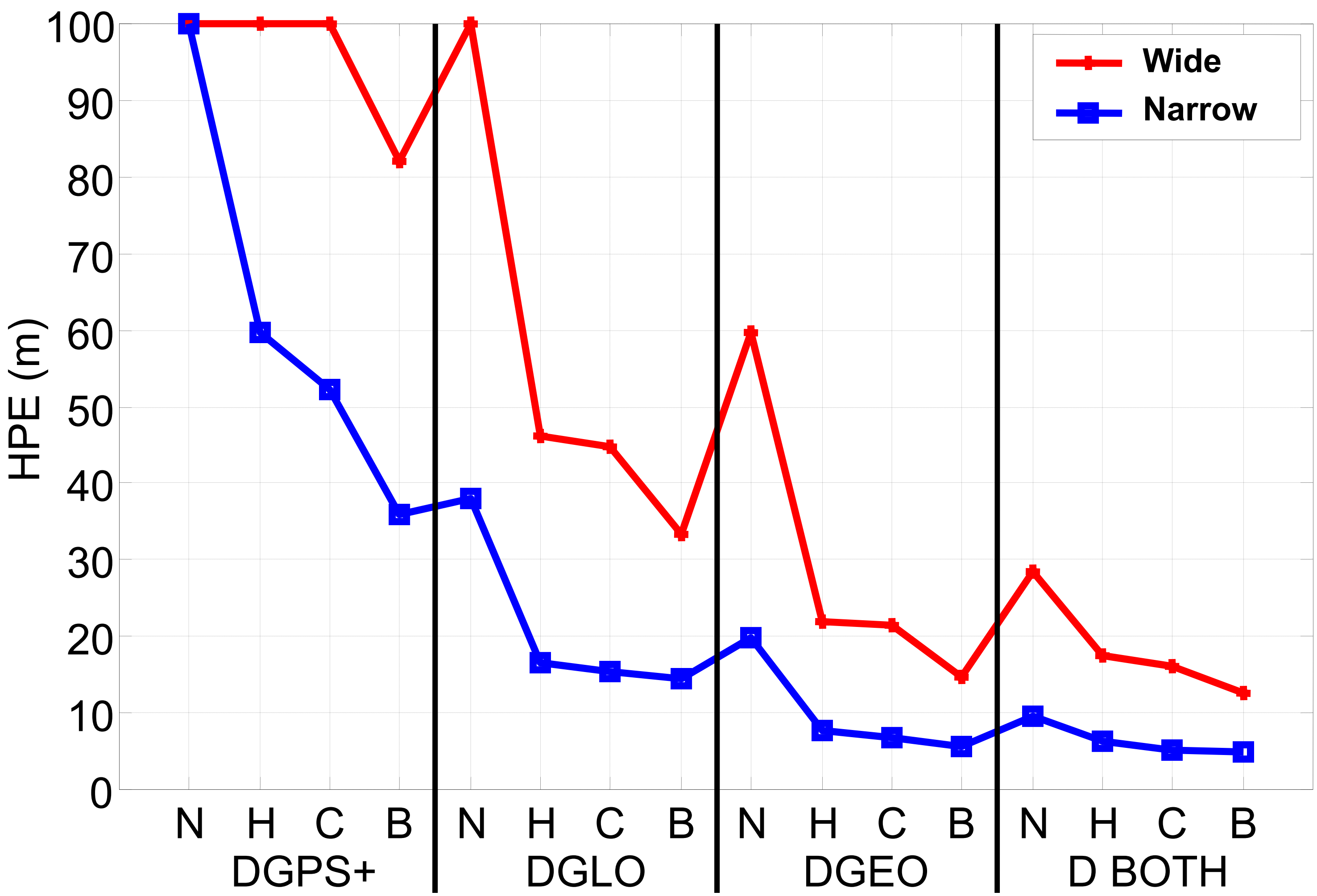
Simulation Results

95% HPE - Double Blunder - Narrow Correlator



Simulation Results

95% HPE - Single Blunder



Simulation Results

95% HPE - Double Blunder