5. Parametric Least-Squares Adjustment

- This chapter provides the detailed implementation of the least squares solution for the parametric method (also known as the observation equation least squares).

5.1. Estimated Parameters and Adjusted Observations

- Recall that the parametric math model is given by:

$$l_{n,1} = f_{n,1}(x_{u,1})$$

- Starting with the linearized functional model

$$\mathbf{A_{n,u}}\hat{\boldsymbol{\delta}}_{\mathbf{u,1}} + \mathbf{w_{n,1}} = \hat{\mathbf{v}}_{\mathbf{n,1}} \qquad \text{with POE}: \mathbf{x^o}$$

- The problem now on hand is to get the best estimate $\hat{x}$ where

$$\hat{\mathbf{x}} = \mathbf{x^o} + \hat{\boldsymbol{\delta}}$$

- As discussed before, the best estimate is the one that satisfies the least-squares condition:

$$\phi = \sum v^2 = \min \qquad or \qquad \sum pv^2 = \min$$

In matrix form, this is:

$$\phi = \hat{\mathbf{v}}^{\mathbf{T}}_{\mathbf{1,n}}\mathbf{P_{n,1}}\hat{\mathbf{v}}_{\mathbf{n,1}} = \min$$

- P is the weight matrix $\mathbf{P} \propto \mathbf{C_l^{-1}}$ ($\mathbf{C_l}$ is the variance-covariance matrix of the observations)

- Substituting for $\hat{\boldsymbol{v}} = \boldsymbol{A}\,\hat{\boldsymbol{\delta}} + \boldsymbol{w}$ in the variation function $\boldsymbol{\phi}$

$$\varphi = \left(\mathbf{A_{n,u}}\hat{\delta}_{\mathbf{u,1}} + \mathbf{w_{n,1}}\right)^{\mathbf{T}} \mathbf{P_{n,n}} \left(\mathbf{A_{n,u}}\hat{\delta}_{\mathbf{u,1}} + \mathbf{w_{n,1}}\right)$$

Transposing the 1st term (recall: $\boldsymbol{(A\,B)^T = B^T A^T}$)

$$= \left(\hat{\boldsymbol{\delta}}^{\mathbf{T}}_{\mathbf{1,u}}\mathbf{A}^{\mathbf{T}}_{\mathbf{u,n}} + \mathbf{w}^{\mathbf{T}}_{1,n}\right)\mathbf{P_{n,n}}\left(\mathbf{A_{n,u}}\hat{\boldsymbol{\delta}}_{\mathbf{u,1}} + \mathbf{w_{n,1}}\right)$$

Including P inside the 2nd term

$$= \left(\hat{\delta}^T_{1,u} A^T_{u,n} + w^T_{1,n}\right)\left(P_{n,n} A_{n,u} \hat{\delta}_{u,1} + P_{n,n} w_{n,1}\right)$$

$$= \hat{\delta}^T A^T P A \hat{\delta} + \underbrace{\hat{\delta}^T_{1,u} A^T_{u,n} P_{n,n} w_{n,1}}_{1x1} + \underbrace{w^T_{1,n} P_{n,n} A_{n,u} \hat{\delta}_{u,1}}_{1x1} + w^T P w$$

Note that: $\hat{\delta}^T_{1,u} A^T_{u,n} P_{n,n} w_{n,1} = w^T_{1,n} P_{n,n} A_{n,u} \hat{\delta}_{u,1}$ = scalar quantity, therefore:

$$\hat{\delta}^T_{1,u} A^T_{u,n} P_{n,n} w_{n,1} + w^T_{1,n} P_{n,n} A_{n,u} \hat{\delta}_{u,1} = 2(w^T_{1,n} P_{n,n} A_{n,u} \hat{\delta}_{u,1}) = 2(\hat{\delta}^T_{1,u} A^T_{u,n} P_{n,n} w_{n,1})$$

$$\boxed{\varphi = \hat{\delta}^T \left(A^T P A\right) \hat{\delta} + 2\left(w^T P A\right) \hat{\delta} + w^T P w = \min}$$

- The condition for ϕ to be a minimum quantity means its derivative with respect to all variables within the equation must be zero. The only variable in the minimization (variation) function ϕ is the vector $\hat{\delta}$.

- Therefore, for $\phi = \min \rightarrow \frac{\partial \phi}{\partial \delta} = 0$ (recall: $\frac{\partial (x^T C x)}{\partial x} = 2x^T C$)

$$\frac{\partial \varphi}{\partial \delta} = 2\hat{\delta}^T \left(A^T P A\right) + 2\left(w^T P A\right) + 0 = 0$$

- Transpose the whole equation (note because P is symmetric matrix, then: $P^T = P$)

$$\boxed{\underbrace{\left(A^T_{u,n} P_{n,n} A_{n,u}\right)}_{u,u} \hat{\delta}_{u,1} + \underbrace{A^T_{u,n} P_{n,n} w_{n,1}}_{u,1} = 0}$$

$$\boxed{N_{u,u} \hat{\delta}_{u,1} + u_{u,1} = 0}$$

- This expression is known as the normal equation

  N = normal equations matrix

  u = normal equations vector (also known as the vector of constant terms)

- The solution of the normal equations for δ is:

$$\boxed{\begin{array}{l}\hat{\boldsymbol{\delta}} = -\mathbf{N}^{-1}\mathbf{u} \\ \hat{\boldsymbol{\delta}} = -\left(\mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{w} \\ \hat{\mathbf{x}} = \mathbf{x}^{\mathbf{o}} + \hat{\boldsymbol{\delta}} \\ \downarrow \end{array}}$$

adjusted parameters

- It is interesting to note here that the normal equations system can be derived directly from the variation function in a more direct way:

$$\varphi = \hat{\mathbf{v}}^{\mathbf{T}}\mathbf{P}\hat{\mathbf{v}} = \min$$

where

$$\hat{\mathbf{v}} = \mathbf{A}\hat{\boldsymbol{\delta}} + \mathbf{w}$$

Therefore:

$$\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{\delta}} = \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{v}} \cdot \frac{\partial \mathbf{v}}{\partial \boldsymbol{\delta}} = \left(2\hat{\mathbf{v}}^{\mathbf{T}}\mathbf{P}\right)\cdot\left(\mathbf{A}\right) = \mathbf{0}$$

which, after transposing, yields the following:

$$\begin{aligned} \mathbf{A}^{\mathbf{T}}\mathbf{P}\hat{\mathbf{v}} &= \mathbf{0} \\ \mathbf{A}^{\mathbf{T}}\mathbf{P}\left(\mathbf{A}\hat{\boldsymbol{\delta}} + \mathbf{w}\right) &= \mathbf{0} \\ \left(\mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{A}\right)\hat{\boldsymbol{\delta}} + \mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{w} &= \mathbf{0} \\ \mathbf{N} \quad \hat{\boldsymbol{\delta}} \quad + \quad \mathbf{u} \quad &= \mathbf{0} \end{aligned}$$

- Solution for the residuals vector (the correction to the observations):

$$\begin{aligned} \hat{\mathbf{v}} &= \mathbf{A}\boldsymbol{\delta} + \mathbf{w} \\ &= -\mathbf{A}\left(\mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{w} + \mathbf{w} \\ &= \left(-\mathbf{A}\left(\mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{P} + \mathbf{I}\right)\mathbf{w} \\ \hat{\mathbf{v}} &= \left(\mathbf{I} - \mathbf{A}\left(\mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{P}\right)\mathbf{w} \end{aligned}$$

- The adjusted observations vector:

$$\boxed{\hat{\mathbf{l}}_{\mathbf{n,1}} = \mathbf{l}^{\mathbf{obs}}_{\mathbf{n,1}} + \hat{\mathbf{v}}_{\mathbf{n,1}}}$$

5.2. Estimated Variance-Covariance Matrices for the Adjusted Parameters and the Adjusted Observables

- Since least squares only estimates the values of $\hat{\mathbf{x}}$ ($\hat{\mathbf{x}} = \mathbf{x}^o + \hat{\boldsymbol{\delta}}$) and $\hat{\mathbf{l}}$ ($\hat{\mathbf{l}} = \mathbf{l} + \hat{\mathbf{v}}$), it is always necessary to compute a measure of precision for these estimated quantities ($\mathbf{C}_{\hat{\mathbf{x}}}$ and $\mathbf{C}_{\hat{\mathbf{l}}}$).
- The only general way to compute $\mathbf{C}_{\hat{\mathbf{x}}}$ and $\mathbf{C}_{\hat{\mathbf{l}}}$ is through the use of the covariance law.

5.2.1. $\mathbf{C}_{\delta}$ - The V-C matrix of the solution vector $\hat{\delta}$:

Functional model:

$$
\begin{aligned}
\hat{\delta} &= -\mathbf{N}^{-1}\mathbf{u} \\
&= -\left(\mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{w} \\
&= -\left(\mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{P}\left(\mathbf{f}\left(\mathbf{x}^{\mathbf{0}}\right)-\mathbf{l}^{\text{obs}}\right) \\
&= \underbrace{-\underbrace{\left(\mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{P}}_{\text{CONSTANT}}\ \underbrace{\mathbf{f}\left(\mathbf{x}^{\mathbf{0}}\right)}_{\text{CONSTANT}}}_{K_2} + \underbrace{\underbrace{\left(\mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{P}}_{\text{CONSTANT}}}_{\mathbf{K}_1}\ \underbrace{\mathrm{l}^{\text{obs}}}_{\text{VARIABLE}}
\end{aligned}
$$

$$
\therefore \hat{\delta}_{\mathbf{u},\mathbf{1}} = K_{2_{u,1}} + K_{\mathbf{1}_{\mathbf{u},\mathbf{n}}}\mathbf{l}^{\text{obs}}{}_{\mathbf{n},\mathbf{1}} \qquad \left(\mathbf{K}_{\mathbf{1}} = \mathbf{N}^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{P}\right)
$$

Using the law of propagation of V-C (covariance law)

$$
\mathbf{C}_{\boldsymbol{\delta}\,\mathbf{u},\mathbf{u}} = \mathbf{J}_{\mathrm{l}\,\mathbf{u},\mathbf{n}}\mathbf{C}_{\mathbf{l}\,\mathbf{n},\mathbf{n}}\mathbf{J}^{\mathbf{T}}_{\mathrm{l}\,\mathbf{n},\mathbf{u}}
$$

$$
\mathbf{C}_{\boldsymbol{\delta}} = \mathbf{K}_{\mathbf{1}}\mathbf{C}_{\mathbf{l}}\mathbf{K}_{\mathbf{1}}^{\mathbf{T}}
$$

Recall that

$\mathbf{P} \propto \mathbf{C}_{\mathbf{l}}^{-1}$	C_l = V-C matrix of the observations
$\mathbf{P} = \sigma_{\mathbf{0}}^{2} C_l^{-1}$ $C_l = \sigma_{\mathbf{0}}^{\mathbf{2}}\mathbf{P}^{-1}$	σ_0^2 is called the variance factor (or variance of unit weight, or a-priori variance factor) It is usually chosen = 1 (one)

Using the covariance law,

$$
\mathbf{C}_{\delta} = \mathbf{K}_{\mathbf{1}}\mathbf{C}_{\mathbf{l}}\mathbf{K}_{\mathbf{1}}^{\mathbf{T}} \qquad \textit{Note}: \mathbf{K}_{\mathbf{1}} = \mathbf{N}^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{P} \text{ and } C_l = \sigma_{\mathbf{0}}^{\mathbf{2}}\mathbf{P}^{-1}
$$

Substituting for K_1 and $C_l = \sigma_{\mathbf{0}}^{\mathbf{2}}\mathbf{P}^{-1}$

$$
\begin{aligned}
&= (N^{-1}A^T\ P)\ \sigma_0^2\ P^{-1}\left(N^{-1}A^T P\right)^T \\
&= N^{-1}A^T\sigma_0^2\ \underbrace{P\ P^{-1}}_{I}\ \left(N^{-1}A^T P\right)^T \\
&= \sigma_0^2\ N^{-1}A^T\ \ I\ \left(N^{-1}A^T P\right)^T \\
&= \sigma_0^2\ N^{-1}A^T\ (P\ A\ \ N^{-1}) \\
&= \sigma_0^2\ N^{-1}\ \underbrace{A^T\ P\ A}_{N}\ N^{-1} \\
&= \sigma_0^2\ N^{-1}NN^{-1} \\
C_\delta &= \sigma_0^2\ N^{-1}
\end{aligned}
$$

That is, the variance-covariance matrix of the solution vector $\hat{\boldsymbol{\delta}}$ is

$$
\boxed{\mathbf{C}_\delta = \sigma_0^2\ \mathbf{N}^{-1}}
$$

5.2.2. $\mathbf{C}_{\hat{\mathbf{x}}}$ - The V-C matrix of the adjusted parameters $\hat{x}$:

$$
\hat{\mathbf{x}} = \underbrace{\mathbf{x}^{\mathbf{o}}}_{constant} + \hat{\boldsymbol{\delta}}
$$

$$
\begin{aligned}
\mathbf{C}_{\hat{\mathbf{x}}} &= \mathbf{J}_\delta \mathbf{C}_{\boldsymbol{\delta}} \mathbf{J}_\delta^{\mathbf{T}} \\
&= \left(\frac{\partial \hat{\mathbf{x}}}{\partial \boldsymbol{\delta}}\right)\mathbf{C}_{\boldsymbol{\delta}}\left(\frac{\partial \hat{\mathbf{x}}}{\partial \boldsymbol{\delta}}\right)^{\mathbf{T}}
\end{aligned}
$$

$\therefore \mathbf{C}_{\hat{\mathbf{x}}} = \mathbf{C}_{\boldsymbol{\delta}} = \mathbf{N}^{-1}$, note that if $\boxed{\begin{array}{l}\hat{\sigma}_0^2 \neq 1, \quad then \\ C_{\hat{x}} = \hat{\sigma}_0^2 N^{-1}\end{array}}$

Note: we have introduced a new quantity $\hat{\sigma}_{\mathbf{0}}^{\mathbf{2}}$ which is called the a-posteriori variance factor, where $\hat{\sigma}_{\mathbf{0}}^{\mathbf{2}} = \dfrac{\hat{\mathbf{v}}^{\mathbf{T}}\mathbf{P}\hat{\mathbf{v}}}{\mathbf{d.o.f}}$ (degree of freedom (d.o.f.) = n – u)

5.2.3. $\mathbf{C}_{\hat{\mathbf{l}}}$ - The V-C matrix of the adjusted observations $\hat{l}$:

Function model:

$$\hat{\mathbf{l}} = \mathbf{f}(\hat{\mathbf{x}}) = \mathbf{A}\hat{\mathbf{x}}$$

To get $\mathbf{C}_{\hat{\mathbf{l}}}$, apply the covariance law

$$\mathbf{C}_{\hat{\mathbf{l}}} = \left(\frac{\partial \hat{\mathbf{l}}}{\partial \hat{\mathbf{x}}}\right) \mathbf{C}_{\hat{\mathbf{x}}} \left(\frac{\partial \hat{\mathbf{l}}}{\partial \hat{\mathbf{x}}}\right)^{\mathbf{T}} = \mathbf{J}_{\hat{\mathbf{x}}} \mathbf{C}_{\hat{\mathbf{x}}} \mathbf{J}_{\hat{\mathbf{x}}}^{\mathbf{T}}$$

$$\mathbf{C}_{\hat{\mathbf{l}}\,\mathbf{n,n}} = \mathbf{A}_{\mathbf{n,u}} \mathbf{C}_{\hat{\mathbf{x}}\,\mathbf{u,u}} \mathbf{A}^{\mathbf{T}}_{\mathbf{u,n}}$$

We can also derive an expression for the variance-covariance matrix for $\hat{\boldsymbol{v}}$ (the residuals vector)

$$\mathbf{C}_{\hat{\mathbf{v}}} = \mathbf{C}_{\mathbf{l}} - \mathbf{C}_{\hat{\mathbf{l}}}$$

5.3. Iterative Solution to the Parametric Model

- Steps:

1. Identify the elements of $\mathbf{x}_{\mathbf{u,1}}$ and $\mathbf{l}_{\mathbf{n,1}}$
2. Form the observation equations $\mathbf{l} = \mathbf{f}(\mathbf{x})$
3. Find approximate values for $\mathbf{x}^{\mathbf{o}}$ (using u observation equations) and use them to evaluate the design linearized equations:

$$\hat{\mathbf{v}}_{\mathbf{n,1}} = \mathbf{A}_{\mathbf{n,u}} \hat{\boldsymbol{\delta}}_{\mathbf{u,1}} + \mathbf{w}_{\mathbf{n,1}}$$

Where

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^{\mathbf{o}}} \qquad \mathbf{w}_{\mathbf{n,1}} = \mathbf{f}_{\mathbf{n,1}}\left(\mathbf{x}^{\mathbf{o}}\right) - \mathbf{l}^{\mathrm{obs}}{}_{\mathbf{n,1}}$$

4. Establish the C_l matrix (check the units)
5. Establish the weight matrix

$\mathbf{P} = \sigma_0^2 \mathbf{C}_{\mathbf{l}}^{-1}$ (choose the desired variance factor to simplify the computations)

Usually σ_0^2 can be assumed = 1 (one).

Note: If $\mathbf{C}_{\mathbf{l}}$ is diagonal matrix

$$\mathbf{C}_l = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 4 & 0 & 0 \\ sym & & 6 & 0 \\ & & & 9 \end{bmatrix} \Rightarrow \mathbf{C}_l^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1/4 & 0 & 0 \\ sym & & 1/6 & 0 \\ & & & 1/9 \end{bmatrix}$$

6. Solve for $\hat{\delta}_1 = -N_1^{-1}u_1$, where $N_1 = A_1^T P A_1$ and $u_1 = A_1^T P w_1$
7. Update the approximate values

   $$\hat{\mathbf{x}}_1 = \mathbf{x}_1^0 + \hat{\boldsymbol{\delta}}_1$$

   Note: If model is linear go to step 11

8. Check the magnitude of each element of $\hat{\boldsymbol{\delta}}_1$ for significance
9. If $\hat{\boldsymbol{\delta}}_1$ had significant element (will be discussed later) choose the new POE

   $$\mathbf{x}_2^0 = \hat{\mathbf{x}}_1$$

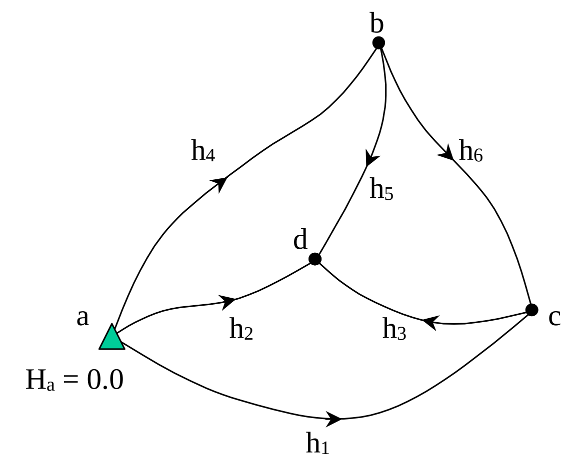
   - Calculate $\mathbf{A}_2$, $\mathbf{w}_2$ with $\mathbf{x}_2^0$ and $\boldsymbol{l}^{obs}$
   - Solve for $\hat{\delta}_2 = -\mathbf{N}_2^{-1}\,\mathbf{u}_2$ (step 6)
   - Update the approximate values (step 7)
   - Check the elements of $\hat{\boldsymbol{\delta}}_2$ (step 8)

10. Repeat step 9 until there is no significant elements in $\hat{\boldsymbol{\delta}}$
11. Calculate $\mathbf{C}_{\hat{\mathbf{x}}}$, $\mathbf{C}_{\hat{\mathbf{l}}}$, $\mathbf{C}_{\hat{\mathbf{v}}}$ and $\hat{\boldsymbol{\sigma}}_0^2 = \dfrac{\hat{\mathbf{v}}^T\mathbf{P}\hat{\mathbf{v}}}{\mathbf{d.o.f}}$

5.4. Numerical Example: Parametric Least Squares

- The opposite sketch shows a levelling network *abcd*, in which point (a) is assumed to be fixed with zero elevation.

	Section length
Observations: $\mathbf{l}_{6,1} = \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta h_3 \\ \Delta h_4 \\ \Delta h_5 \\ \Delta h_6 \end{bmatrix} = \begin{bmatrix} 6.16 \\ 12.57 \\ 6.41 \\ 1.09 \\ 11.58 \\ 5.07 \end{bmatrix} \text{m}$	4 km 2 km 2 km 4 km 2 km 4 km
Parameters (unknowns): $\mathbf{x}_{3,1} = \begin{bmatrix} h_b \\ h_c \\ h_d \end{bmatrix}$	



- Required

Given that the variance of the elevation differences = 1 cm^2/km estimate the elevation of points b, c, d and their variance-covariance matrix $\mathbf{C}_{\hat{\mathbf{x}}}$.

Note: n = 6, u = 3

- Solution

1. The six observation equations – $\hat{\mathbf{l}} = \mathbf{f}(\hat{\mathbf{x}})$

$$\underbrace{\begin{matrix} \Delta\hat{h}_1 = \\ \Delta\hat{h}_2 = \\ \Delta\hat{h}_3 = \\ \Delta\hat{h}_4 = \\ \Delta\hat{h}_5 = \\ \Delta\hat{h}_6 = \end{matrix}}_{\mathbf{l}} \underbrace{\begin{matrix} & \hat{h}_c & & -h_a \\ & & \hat{h}_d & -h_a \\ & -\hat{h}_c & +\hat{h}_d & \\ \hat{h}_b & & & -h_a \\ -\hat{h}_b & & +h_d & \\ -\hat{h}_b & +\hat{h}_c & & \end{matrix}}_{\mathbf{f}(\hat{\mathbf{x}})}$$

2. Estimate an approximate value (POE) of the parameters $\mathbf{x}^{\mathbf{o}}$ using any 3 equations of the 6 observation equations.

   Using equations 4, 1, and 2 (use the simplest equations):

$$h_b^0 = ha + \Delta h_4 = 0.0 + 1.09 = 1.09m$$

$$h_c^0 = ha + \Delta h_1 = 0.0 + 6.16 = 6.16m$$

$$h_d^0 = ha + \Delta h_2 = 0.0 + 12.57 = 12.57m$$

Hence

$$\mathbf{x}^{\mathbf{o}}_{\mathbf{3,1}} = \begin{bmatrix} 1.09 \\ 6.16 \\ 12.57 \end{bmatrix} m$$

3. Calculate the misclosure vector: $\mathbf{w} = \mathbf{f}\left(\mathbf{x}^{\mathbf{o}}\right) - \mathbf{l}^{\text{obs}}$

   (Remember that our model **l = f(x)** will not provide unique value for **x** because **l** contains random errors and this is the main reason for vector **w** **not being zero.** We use an approximate value for **x** which we call $\mathbf{x}^{\mathbf{0}}$ and while a different $\mathbf{x}^{\mathbf{0}}$ will lead to a different **w, we should end up with the same solution**)

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \underbrace{\begin{bmatrix} h_c^0 & -h_a \\ h_d^0 & -h_a \\ -h_c^0 & +h_d^0 \\ h_b^0 & -h_a \\ -h_b^0 & +h_d^0 \\ -h_b^0 & +h_c^0 \end{bmatrix}}_{\mathbf{f}\left(\mathbf{x}^{\mathbf{0}}\right)} - \underbrace{\begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta h_3 \\ \Delta h_4 \\ \Delta h_5 \\ \Delta h_6 \end{bmatrix}}_{\mathbf{l}} = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ -0.1 \\ 0.0 \end{bmatrix} m$$

Note: equations 4, 1, and 2 will, definitely, have zero elements for the **w** vector

4. Write down the linearized equations: $\hat{\mathbf{v}}_{\mathbf{6,1}} = \mathbf{A}_{\mathbf{6,3}}\hat{\boldsymbol{\delta}}_{\mathbf{3,1}} + \mathbf{w}_{\mathbf{6,1}}$

$$\mathbf{A}_{6,3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \text{ unitless}$$

Note: the units of $\hat{\boldsymbol{v}}$, **w**, and $\hat{\boldsymbol{\delta}}$ are all in metres.

5. Construct the $\boldsymbol{C_l}$ matrix – in a levelling network the variance of the observed elevation-differences are proportional to the length of the corresponding section and are usually uncorrelated.

$$\therefore \sigma^2_{\Delta h} \propto L \quad \text{in our example } \sigma^2_{\Delta h_i} = 1\text{cm}^2/\text{km}$$

$$\mathbf{C}_{\mathbf{l}\,6,6} = \text{diag}\begin{bmatrix}\sigma^2_{\Delta h_1} & \sigma^2_{\Delta h_2} & \sigma^2_{\Delta h_3} & \sigma^2_{\Delta h_4} & \sigma^2_{\Delta h_5} & \sigma^2_{\Delta h_6}\end{bmatrix}$$

$$\mathbf{C}_\mathbf{l} = \text{diag}\begin{bmatrix}4 & 2 & 2 & 4 & 2 & 4\end{bmatrix}\text{cm}^2$$

Scale $\mathbf{C_l}$ to (metre)^2 to be consistent with **v**, **w**, and δ

$$\mathbf{C}_\mathbf{l} = 10^{-4}\text{diag}\begin{bmatrix}4 & 2 & 2 & 4 & 2 & 4\end{bmatrix}\text{m}^2$$

And recalling that $\mathbf{P}$ matrix $\propto \mathbf{C}_\mathbf{l}^{-1}$, $\mathbf{P} = \sigma_0^2\mathbf{C}_\mathbf{l}^{-1}$

$$\mathbf{P} = \sigma_0^2 \cdot \underbrace{10^4 \cdot \text{diag}\begin{bmatrix}\tfrac{1}{4} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{4} & \tfrac{1}{2} & \tfrac{1}{4}\end{bmatrix}}_{\mathbf{C}_\mathbf{l}^{-1}} \tfrac{1}{\text{m}^2}$$

If we assume $\sigma_0^2 = 10^{-4}$

$$\therefore \mathbf{P} = 10^{-4}\cdot 10^{-4}\text{diag}\begin{bmatrix}\tfrac{1}{4} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{4} & \tfrac{1}{2} & \tfrac{1}{4}\end{bmatrix}\tfrac{1}{\text{m}^2}$$

$$\therefore \mathbf{P} = \text{diag}\begin{bmatrix}\tfrac{1}{4} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{4} & \tfrac{1}{2} & \tfrac{1}{4}\end{bmatrix}\tfrac{1}{\text{m}^2}$$

Note: we choose σ_0^2 to simplify the computations.

6. Solve for the estimated parameters

$$\hat{\boldsymbol{\delta}}_{\mathbf{3,1}} = -\mathbf{N}^{-1}_{\mathbf{3,3}}\mathbf{u}_{\mathbf{3,1}}$$

$$\mathbf{N}_{3,3} = \mathbf{A}^\mathbf{T}_{3,6}\mathbf{P}_{6,6}\mathbf{A}_{6,3} = \begin{bmatrix}1.0 & -0.25 & -0.5\\ -0.25 & 1.0 & -0.5\\ -0.5 & 0.5 & 1.5\end{bmatrix}$$

(Note: N is always symmetric)

$$\mathbf{N}^{-1} = \begin{bmatrix}1.6 & 0.8 & 0.8\\ 0.8 & 1.6 & 0.8\\ 0.8 & 0.8 & 1.2\end{bmatrix} \qquad \mathbf{u}_{3,1} = \mathbf{A}^\mathbf{T}\mathbf{P}\mathbf{w} = \begin{bmatrix}0.05\\ 0.00\\ -0.05\end{bmatrix}$$

$$\therefore \hat{\boldsymbol{\delta}} = -\mathbf{N}^{-1}\mathbf{u} = \begin{bmatrix}-0.04\\ 0.00\\ 0.02\end{bmatrix}\text{m} \rightarrow \hat{\mathbf{x}} = \mathbf{x}^\mathbf{o} + \hat{\boldsymbol{\delta}} = \begin{bmatrix}1.09\\ 6.16\\ 12.57\end{bmatrix} + \begin{bmatrix}-0.04\\ 0.00\\ 0.02\end{bmatrix} = \begin{bmatrix}1.05\\ 6.16\\ 12.59\end{bmatrix}\text{m}$$

$$\text{Adjusted parameters: } \hat{\mathbf{x}} = \begin{bmatrix} 1.05 \\ 6.16 \\ 12.59 \end{bmatrix} m$$

7. Adjusted observations $\hat{\text{l}} = \text{l} + \hat{\text{v}}$

$$\hat{\mathbf{v}} = \mathbf{A}\hat{\boldsymbol{\delta}} + \mathbf{w} = \begin{bmatrix} 0.00 \\ 0.02 \\ 0.02 \\ -0.04 \\ -0.04 \\ 0.04 \end{bmatrix} m \quad \hat{\mathbf{l}} = \mathbf{l} + \hat{\mathbf{v}} = \begin{bmatrix} 6.16 \\ 12.57 \\ 6.41 \\ 1.09 \\ 11.58 \\ 5.07 \end{bmatrix} + \begin{bmatrix} 0.00 \\ 0.02 \\ 0.02 \\ -0.04 \\ -0.04 \\ 0.04 \end{bmatrix} = \begin{bmatrix} 6.16 \\ 12.59 \\ 6.43 \\ 1.05 \\ 11.54 \\ 5.11 \end{bmatrix} \text{m}$$

Note: If we use any 3 equations of the 6 observation equations and we use $\hat{\mathbf{l}}$ instead of **l**, we should estimate the same $\hat{\mathbf{x}}$.

8. Now, we estimate some parameters which express the accuracy of $\hat{\mathbf{x}}$ and $\hat{\mathbf{l}}$

$$\mathbf{C}_{\hat{\mathbf{x}}3,3} = \hat{\sigma}_0^2 \mathbf{N}^{-1}$$

$$\hat{\sigma}_0^2 = \text{a-posteriori variance factor} = \frac{\hat{\mathbf{v}}^{\mathbf{T}}_{1,6} \mathbf{P}_{6,6} \hat{\mathbf{v}}_{6,1}}{\underbrace{\text{n} - \text{u}}_{\text{degrees of freedom}}} = \frac{0.002}{6-3} = 6.7\text{x}10^{-4}$$

Note: v ~ metre, P ~ 1/metre^2 $\therefore \hat{\sigma}_0^2$ ~ unitless

$$\mathbf{C}_{\hat{\mathbf{x}}} = \hat{\sigma}_0^2 \mathbf{N}^{-1} = 10^{-4} \begin{bmatrix} 10.67 & 5.33 & 5.33 \\ 5.33 & 10.67 & 5.33 \\ 5.33 & 5.33 & 8.00 \end{bmatrix} m^2$$

$$\sigma_{h_b} = \sqrt{10^{-4} x 10.67} = 3.27 cm$$

$$\sigma_{h_c} = \sqrt{10^{-4} x 10.67} = 3.27 cm$$

$$\sigma_{h_d} = \sqrt{10^{-4} x 8} = 2.83 cm$$

Finally (if needed), the V-C matrix of $\hat{\mathbf{l}}$ is computed

$$\mathbf{C}_{\hat{\mathbf{l}}} = \mathbf{A}\mathbf{C}_{\hat{\mathbf{x}}}\mathbf{A}^{\mathrm{T}} = 10^{-4}\begin{bmatrix} 10.67 & 5.33 & -5.33 & 5.33 & 0.00 & 5.33 \\ & 8.00 & 2.67 & 5.33 & 2.67 & 0.00 \\ & & 8.00 & 0.00 & 2.67 & -5.33 \\ \text{symmetrical} & & & 10.67 & -5.33 & -5.33 \\ & & & & 8.00 & 5.33 \\ & & & & & 10.67 \end{bmatrix}$$

Homework: Try to solve this problem with $\mathbf{x^o} = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}$, you should end up with the same results. In general, for any linear model, you can always assume $\mathbf{x^o} = 0$.