4. Adjustment of Observations

4.1. The concept of an adjustment

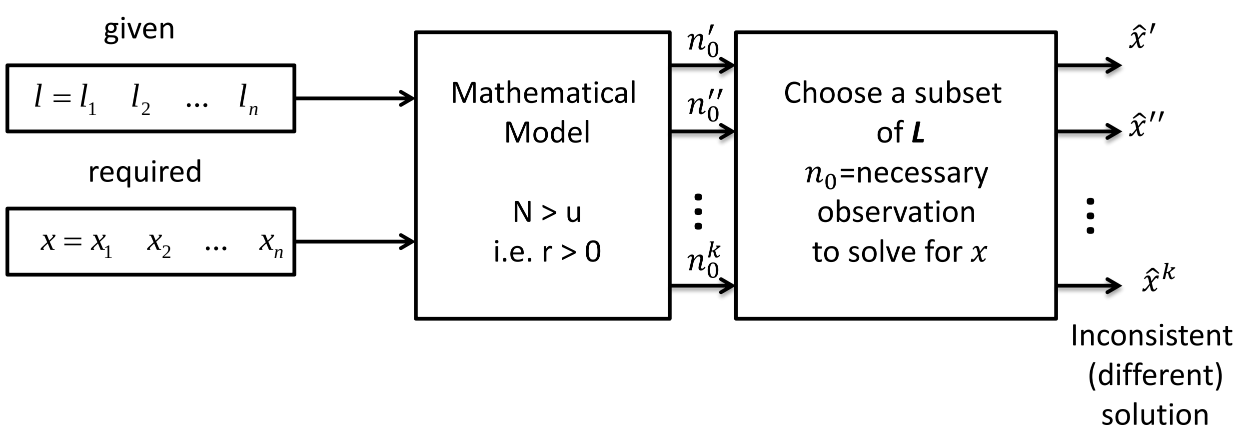
- In the previous chapters, we dealt with the case where the number of observations (n) is just enough to provide the necessary number of equations (m or $n_{necessary}$) to estimate (u) number of unknowns. This usually results in a *unique solution* for the unknowns.

- However, a unique solution is not practical, as an error in a single observation can radically affect the final solution of the unknowns. Therefore, in geomatics engineering applications we usually use *redundant observations*, i.e. having (taking) more observations than what is necessary for a unique solution.

  Number of redundant observations = $r = n - n_{necessary}$ (degrees of freedom)

- A mathematical model ($\mathbf{l} = \mathbf{f(x)}, \mathbf{g(l)} = \mathbf{0}, \mathbf{f(x,l)} = \mathbf{0}$) in which $r > 0$ is called ***over-determined mathematical model*** that can lead to more than one solution for the unknown parameters $\boldsymbol{x}$ (i.e. non-unique solution).

- This happens due to the discrepancies in the different equations of the math model caused by the effect of the random errors that still exist in some or all of the observables.



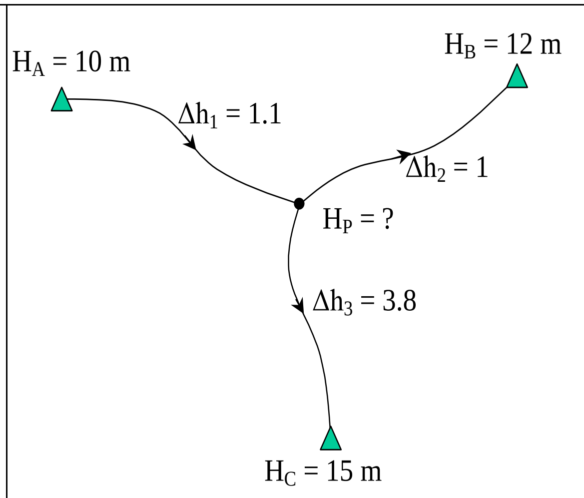
- The following examples will illustrate this fact:

1. Levelling networks

The following example shows three different routes (A→P, B→P, C→P) for estimating the elevation of point p.

$$
\begin{aligned}
H_p &= H_A + \Delta h_1 \\
&= 10 + 1.1 = 11.1m \\
\text{or } H_P &= H_B - \Delta h_2 \\
&= 12 - 1 = 11.0m \\
\text{or } H_P &= H_C - \Delta h_3 \\
&= 15 - 3.8 = 11.2\text{m}
\end{aligned}
$$

$u = 1$, $n = 3$, $r = n - u = 3 - 1 = 2$



Due to the different uncertainty (errors) in the observations Δh_i, the elevation of point P is not identical.

2. Area of triangle

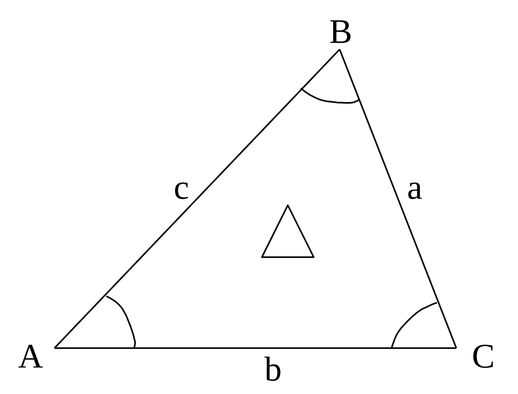
$u = 1$, $n_{necessary} = 3$, $n = 6$

$$\Delta_1 = \frac{1}{2} ab \sin C$$

$$\Delta_2 = \frac{1}{2} cb \sin A$$

$$\Delta_3 = \frac{1}{2} ac \sin B$$

Again, identical results may not (most probably) be obtained, due to the errors in the observation vector ***l***.

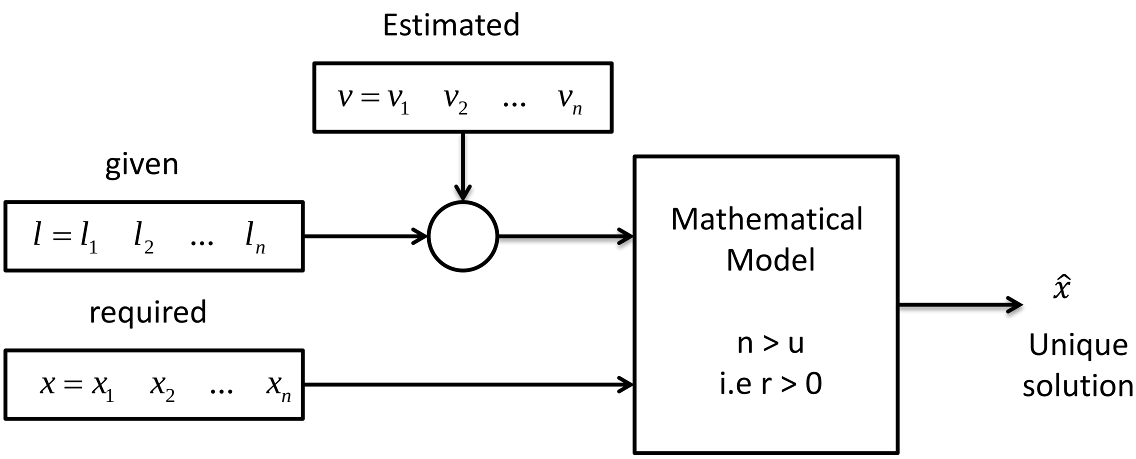


The problems associated with the over-determined math model can be overcome by *adjusting the observations*.

- The apparent inconsistency, due to measurement errors, with the math model can be resolved through the replacement of the given observations **l** by another set of the so-called best estimates of the observations $\hat{\mathbf{l}}$ such that the new set $\hat{\mathbf{l}}$ fits the model exactly.

  Where $\hat{\mathbf{l}}_{\mathbf{n,1}} = \mathbf{l}_{\mathbf{n,1}} + \hat{\mathbf{v}}_{\mathbf{n,1}}$

Note: The estimated residuals $\hat{\mathbf{v}}$ are unknown and must be determined before the observations can be estimated.



- How we choose $\boldsymbol{v}$: There are essentially an infinite number of possible sets of residuals that provide estimated observations $\hat{\mathbf{l}}$ that fits the math model. However, there is only one set of residuals that yield the optimal least squares solution: $\sum v_i^2 = \mathbf{v}^{\mathbf{T}}\mathbf{v} = \min$

4.2. The Least Squares Method

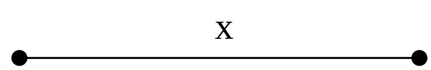
- Principle:

In addition to the fact that the adjusted observations must satisfy the mathematical model exactly, the corresponding residuals must satisfy the least squares criterion.

$$\varphi = \sum_{i=1}^{n} v_i^2 = v_1^2 + v_2^2 + \ldots\ldots\ldots + v_n^2 = min$$

- Simple example on the concept of least squares:

Required:



x ———— *u (unknowns)* = 1

Given:

$$\mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 15.12 \\ 15.14 \end{bmatrix} m$$ ———— *n (observations)* = 2

r (redundancy/extra observations) = *n* – *u* = 2 – 1 = 1

The best estimate value of $\hat{\mathbf{x}}$ can be obtained from the following observation equations:

$$\left.\begin{aligned}\hat{x}_1 &= l_1 + \hat{v}_1 = \hat{l}_1 \\ \hat{x}_2 &= l_2 + \hat{v}_2 = \hat{l}_2\end{aligned}\right\} \text{ Find } \hat{v}_1 \text{ and } \hat{v}_2 \text{ such that } \hat{l}_1 = \hat{l}_2$$

	For l_1	For l_2	$\hat{l}_1$	$\hat{l}_2$	$\sum v^2$
Possible values for v	0	-0.02	15.12	15.12	$\varphi_1 = (0)^2 + (0.02)^2 = 4x10^{-4}$
	0.01	-0.01	15.13	15.13	$\varphi_2 = (0.01)^2 + (-0.01)^2 = 2x10^{-4}$
	0.015	-0.005	15.135	15.135	$\varphi_3 = (0.015)^2 + (-0.005)^2 = 2.5x10^{-4}$

Note that ϕ_2 is the smallest, but is it the minimum value when all possible combinations of corrections are considered?

- Geometric interpretation of $\sum v^2 = \min$

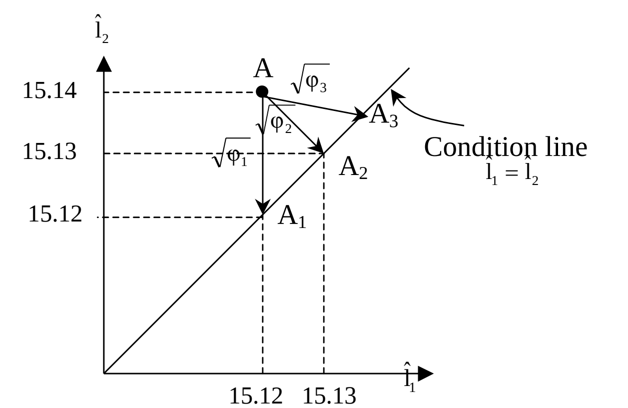
The adjusted observations $\hat{l}_1$ and $\hat{l}_2$ are related to each other by $\hat{l}_1$ - $\hat{l}_2$ = 0 or $\hat{l}_1$ = $\hat{l}_2$ which is a line with 45° inclination (condition line)

The observation

$$\mathbf{l} = \begin{bmatrix} 15.12 \\ 15.14 \end{bmatrix}$$

can be represented by point (A)

The projection of point (A) into the condition line have 3 possibilities



A_1, A_2, and A_3 correspond to φ_1, φ_2 and φ_3 respectively, and all the 3 points satisfy the condition line that is $\hat{l}_1 = \hat{l}_2$ where,

$$\hat{\mathbf{l}} = \mathbf{A_1} = \begin{bmatrix} 15.12 \\ 15.12 \end{bmatrix} \; or \; \mathbf{A_2} = \begin{bmatrix} 15.13 \\ 15.13 \end{bmatrix} \; or \; \mathbf{A_3} = \begin{bmatrix} 15.135 \\ 15.135 \end{bmatrix}$$

However, A_2 is the only solution that satisfies

$$\mathbf{v_2^T} \cdot \hat{\mathbf{l}} = 0 \quad \text{(dot product).}$$

That is,

$$\begin{bmatrix} 0.01 & -0.01 \end{bmatrix} \begin{bmatrix} 15.13 \\ 15.13 \end{bmatrix} = 0$$

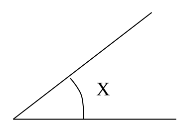
This means that the vector $\mathbf{v}_2$ is perpendicular to the condition line and therefore it is the minimum function.

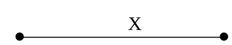
Note as well that $\hat{\mathbf{x}} = \hat{\mathbf{l}} = \frac{\sum l_i}{n} = \frac{15.12 + 15.14}{2} = 15.13$ – that is, the mean satisfies the least squares condition.

- Proof that the mean is the least squares estimate for a group of measurements of a certain parameter:

Given: $\begin{bmatrix} l_1 & l_2 & \dots & l_n \end{bmatrix}$ Required: $\mathbf{x}$

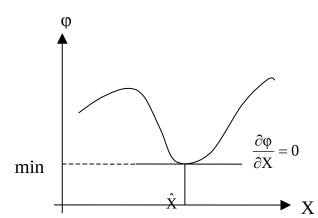
$$\begin{aligned} l_1 + v_1 &= \hat{x} \\ l_2 + v_2 &= \hat{x} \\ l_n + v_n &= \hat{x} \end{aligned}$$





Using the least squares condition:

$$\begin{aligned} \varphi &= \sum_{i=1}^{n} v_i^2 \\ &= v_1^2 + v_2^2 + \dots\dots + v_n^2 = \min \end{aligned}$$



Or,

$$\varphi = \left(\hat{x} - l_1\right)^2 + \left(\hat{x} - l_2\right)^2 + \dots\dots\dots + \left(\hat{x} - l_n\right)^2 = \min$$

For φ to be minimum,

$$\frac{\partial \varphi}{\partial \hat{x}} = 0$$

$$\frac{\partial \varphi}{\partial \hat{x}} = 2\left(\hat{x} - l_1\right) + 2\left(\hat{x} - l_2\right) + \dots. + 2\left(\hat{x} - l_n\right) = 0$$

$$0 = 2\,n\,\hat{x} - 2\left(l_1 + l_2 + \dots\dots\dots + l_n\right)$$

$$0 = n\hat{x} - \sum_{i=1}^{n} l_i$$

$$\hat{x} = \frac{\sum l_i}{n}$$

If the observations are unequal in precision, then we have to consider the weight of the observations, that is

$$\hat{x} = \frac{\sum_{i=1}^{n} P_i l_i}{\sum_{i=1}^{n} P_i} \Rightarrow \text{weighted mean}$$

- Proof that the weighted mean is the least squares estimate for a group of observations, with different precision, of the same parameters:

$$\varphi = \sum P_i v_i^2 = min \quad where \begin{matrix} l_i + v_i = \hat{x} \\ v_i = \hat{x} - l_i \end{matrix}$$

$$\begin{aligned} &subsitute\ the\ expression\ for\ v_i \\ &\varphi = \sum P_i \left(\hat{x} - l_i\right)^2 = min \\ &\varphi = \sum P_i \left(\hat{x}^2 - 2\hat{x} l_i + l_i^{\,2}\right) = min \\ &\varphi = \hat{x}^2 \sum P_i - 2\hat{x} \sum P_i l_i + \sum P_i l_i^2 = min \end{aligned}$$

For φ to be minimum

$$\frac{\partial \varphi}{\partial \hat{x}} = 0$$

$$\frac{\partial \varphi}{\partial \hat{x}} = 2\hat{x} \sum P_i - 2 \sum P_i l_i = 0$$

$$\hat{x} = \frac{\sum P_i l_i}{\sum P_i}$$

It is important to note that for

$$\varphi = \sum v_i^2 = min \qquad \frac{\partial \varphi}{\partial v} = 2 \sum v_i = 0 \rightarrow \sum v_i = 0$$

$$and$$

$$\varphi = \sum P_i v_i^2 = min \qquad \frac{\partial \varphi}{\partial v} = 2 \sum P_i v_i = 0 \rightarrow \sum P_i v_i = 0$$

(Check Chapter 1 and Lab 1)

- Based on the above discussion, the basic concept of adjustment is, therefore, to allow the observables **l** to change slightly while solving for **x**.

- This means that in the over determined model **f(x,l) = 0**, we consider (**l**) as approximate values for the observables which need to be corrected by certain small amount, denoted by **v,** so as to yield a unique solution. (**v** is the vector of residuals)

- The mathematical model becomes:

$$\mathbf{f}\left(\hat{\mathbf{x}}_{\mathbf{u},\mathbf{1}},\hat{\mathbf{l}}_{\mathbf{n},\mathbf{1}}\right)=\mathbf{f}\left(\hat{\mathbf{x}}_{\mathbf{u},\mathbf{1}},\mathbf{l}_{\mathbf{n},\mathbf{1}}+\mathbf{v}_{\mathbf{n},\mathbf{1}}\right)=\mathbf{0}$$

(unknowns…) $\hat{\mathbf{x}}$ is the best estimate (adjusted) values of the parameters
$\hat{\mathbf{l}}$ is the adjusted vector of the observables
$\mathbf{l}$ is the original observation vector
(unknowns…) v is the residuals vector

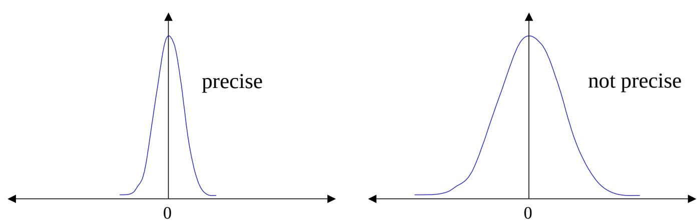
- The above model cannot be solved for **x** and **v** simultaneously because we have more unknowns (*u* "for x" + *n* "for v") in *n* equations (i.e. we need extra *u* equations)

- Recall, the residuals v are small and will behave according to the Gauss law of random errors ($\Sigma v = 0$). As a result we can find several conditions that can be used to provide us with the required *u* extra equations which will enable us to solve for both **x** and **v** simultaneously.

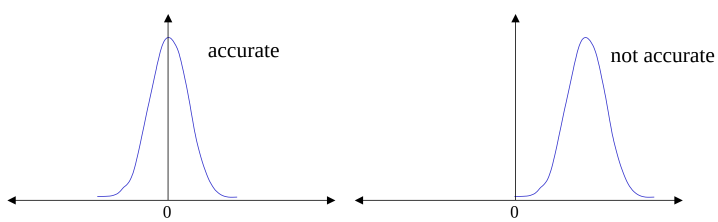
- The condition of least squares on the residuals (Σv^2 = min) was found to satisfy the properties of the best estimate:

1) Maximum likelihood (most probable)

2) Minimum variance (most precise)



3) Unbiased (most accurate)



4.3. Least Squares Techniques

General (or implicit) model

$$\mathbf{f(x_{u,1}, l_{n,1}) = 0}$$

x and l cannot be written as an explicit function compared to the parametric or conditional models

Parametric Adjustment $l_{n,1} = f(x_{u,1})$	Conditional Adjustment $f_{r,1}(l_{n,1}) = 0$
• Each observed quantity provides one observation equation • One stage adjustment in which $\hat{\mathbf{x}}$ and $\hat{\mathbf{l}}$ are estimated simultaneously	• $r = n - u$; the degree of freedom • After adjusting the observations, the unknown parameters can be estimated using the direct math model: $\hat{\mathbf{x}}_{u,1} = f_{u,1}(\hat{\mathbf{l}}_{n,1})$ • Two stage adjustment, in which $\hat{\mathbf{l}}$ is estimated first, then $\hat{\mathbf{x}}$ as a function of $\hat{\mathbf{l}}$

- These math models can be either linear or non-linear. Non-linear models should be linearized first before conducting the least squares adjustment.

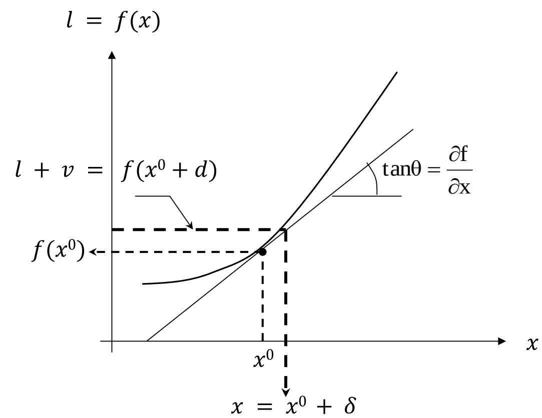
4.4. Linearization of Non-linear Models

- Linearization: is the process of approximating non-linear functions by a linear one. The most convenient and efficient approach of linearization used in geomatics engineering is the Taylor series expansion.

- The expansion of a function is performed about a point of expansion (POE) (i.e. good approximate values for the involved variables).

    1) Univariate function (parametric model): $\mathbf{l = f(x)}$

$$l = f(x)$$
$$l = f(x^o) + \frac{\partial f}{\partial x}\Big|_{x^0}\left(x - x^0\right)$$
$$+ \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}\Big|_{x^0}\left(x - x^0\right)^2$$
$$+ \textit{higher order terms}$$



$$\mathbf{l} = \mathbf{f}\left(\mathbf{x^0}\right) + \frac{\partial f}{\partial x}\Big|_{\mathbf{x^0}} . \boldsymbol{\delta}$$
$$\mathbf{0} = \left(\mathbf{f}\left(\mathbf{x^0}\right) - \mathbf{l}\right) + \frac{\partial f}{\partial x}\Big|_{\mathbf{x^0}} . \boldsymbol{\delta}$$

$$\underset{n\times 1}{w} + \underset{n\times u}{A}\ \underset{u\times 1}{\delta} = 0$$

Misclosure vector ← → Correction to the unknown parameters

2) Multivariate functions (combined model)

$\mathbf{f}_{m,1}\ (\mathbf{x}_{u,1}\ ,\ \mathbf{l}_{n,1}) = \mathbf{0}$

POE:

$\mathbf{x^0}$ – approximate values of the unknowns can be estimated from u group of observations (use the number of observations necessary to solve for $\mathbf{x}$)
$\mathbf{l^{obs}}$ – observed values

$$\mathbf{f}\left(\mathbf{x},\mathbf{l}\right) \approx \mathbf{f}\left(\mathbf{x^0},\mathbf{l^{obs}}\right) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\Big|_{\mathbf{x^0},l^{\mathbf{obs}}} . \left(\hat{\mathbf{x}} - \mathbf{x^0}\right) + \frac{\partial \mathbf{f}}{\partial \mathbf{l}}\Big|_{\mathbf{x^0},\mathbf{l}^{obs}} . \left(\hat{\mathbf{l}} - \mathbf{l^{obs}}\right)$$

In matrix form

$$\underset{m\times 1}{w} + \underset{m\times u}{A}\ \underset{u\times 1}{\delta} + \underset{m\times n}{B}\ \underset{n\times 1}{v} = 0$$

w_{mx1}	misclosure vector
A_{mxu}	1st design matrix (partial derivatives of the function w.r.t. the unknown parameters)
$\delta_{u,1}$	Corrections to the approximate values of the unknown parameters x^0

$B_{m,n}$	2nd design matrix (partial derivatives of the functions w.r.t. the observables)
$v_{n,1}$	corrections to the observations (residuals)
Note:	$\hat{\mathbf{l}}_{n,1} = \mathbf{l}^{obs}_{n,1} + \mathbf{v}_{n,1}$ adjusted observations
	$\hat{\mathbf{x}}_{u,1} = \mathbf{x}^{o}_{u,1} + \boldsymbol{\delta}_{u,1}$ adjusted parameters

- From the multivariate linearization, we can derive the two general adjustments

  1) Parametric model

$$l_{n,1} = f_{n,1}(x_{u,1}) \ or \ \ f_{n,1}(x_{u,1}) - l_{n,1} = 0_{n,1}$$

$$POE: \ \ x^0, \ l^{obs}$$

$$\underbrace{f(x^0) - l^{obs}}_{w} + \underbrace{\frac{\partial f}{\partial x}}_{A}\delta - \underbrace{\frac{\partial l}{\partial l}}_{I}v = 0$$

$$w_{n,1} + A_{n,u}\,\delta_{u,1} - v_{n,1} = 0_{n,1}$$

  Can be derived from the implicit model by substituting B = -I

  2) Conditional model

$$f_{r,1}(l_{n,1}) = 0_{r,1}$$

$$POE: \ \ l^{obs}$$

$$f(l^{obs}) + \frac{\partial f}{\partial l}v = 0$$

$$w_{r,1} + B_{r,n}v_{n,1} = 0_{r,1}$$

  Can be derived from the implicit model by substituting A = 0

- Note: The solution to linearized models is <u>iterative</u> (will be discussed later).

4.5. Linearization Examples

- Linear Parametric Math Model Example: Levelling Network

<table>
<tr>
<td>
Constants: H_A
Observations (n = 3):
$\mathbf{l} = [\Delta h_1 \quad \Delta h_2 \quad \Delta h_3]^T$
Unknowns (u = 2):
$\mathbf{x} = [h_B \quad h_C]^T$
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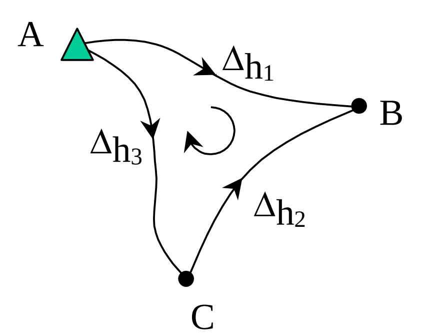
<td>
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Observation equation for a single observation: $\hat{\mathbf{l}} = \mathbf{f}(\hat{\mathbf{x}})$: $\Delta\hat{h}_1 = \hat{h}_B - h_A$

Expand into differential form: $\boldsymbol{l} + \boldsymbol{v} = \boldsymbol{f(x^o)} + \partial \boldsymbol{f(x)} / \partial \mathbf{x} \, .\boldsymbol{\delta}$:

$$\begin{aligned} \Delta h_1 + v_1 &= h_B^o - h_A + (1)\delta_{h_B} \\ v_1 &= \underbrace{\left(h_B^o - h_A - \Delta h_1\right)}_{w} + \delta_{h_B} \end{aligned}$$

Similarly for Δh_2:

$$\begin{aligned} \Delta h_2 + v_2 &= \hat{h}_B - \hat{h}_C \\ &= \left(h_B^o - h_C^o\right) + \delta_{h_B} - \delta_{h_C} \\ v_2 &= \underbrace{\left(h_B^o - h_C^o - \Delta h_2\right)}_{w} + \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \delta_{h_B} \\ \delta_{h_C} \end{bmatrix} \end{aligned}$$

A similar equation can be derived for Δh_3, and together the three equations can be written in matrix form:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_{h_B} \\ \delta_{h_C} \end{bmatrix} + \begin{bmatrix} h_B^o - h_A - \Delta h_1 \\ h_B^o - h_C^o - \Delta h_2 \\ h_C^o - h_A - \Delta h_3 \end{bmatrix}$$

$$\boldsymbol{v}_{n,1} = \boldsymbol{A}_{n,u}\boldsymbol{\delta}_{u,1} + \boldsymbol{w}_{n,1}$$

- Non-linear Parametric Math Model Example: 2D Distance Network

Constants: x_A, y_A, x_B, y_B, x_C, y_C

Observations (n = 3):

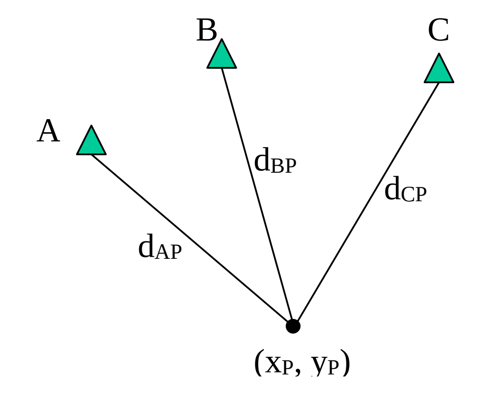
$$\mathbf{l} = \begin{bmatrix} d_{AP} & d_{BP} & d_{CP} \end{bmatrix}^T$$

Unknowns (u = 2):

$$\mathbf{x} = \begin{bmatrix} x_P & y_P \end{bmatrix}^T$$

Observation Equation: $\hat{\mathbf{l}} = \mathbf{f}(\hat{\mathbf{x}})$

$$\hat{d}_{AP} = \sqrt{(\hat{x}_P - x_A)^2 + (\hat{y}_P - y_A)^2}$$



Expand into differential form

$$v_{dAP} = w + A\ \delta$$

$$v_{dAP} = \left(f\left(x^0\right) - l^{obs}\right) + \begin{bmatrix} \frac{\partial d_{AP}}{\partial x_P} & \frac{\partial d_{AP}}{\partial y_P} \end{bmatrix} \begin{bmatrix} \delta x_P \\ \delta y_P \end{bmatrix}$$

$$v_{dAP} = \left(d^0_{AP} - d^{obs}_{AP}\right) + \begin{bmatrix} \frac{\Delta x^0}{d^0_{AP}} & \frac{\Delta y^0}{d^0_{AP}} \end{bmatrix} \begin{bmatrix} \delta x_P \\ \delta y_P \end{bmatrix}$$

$$\frac{\partial d_{AP}}{\partial x_P} = \frac{2\left(x^0_P - x_A\right)(1)}{2\sqrt{\left(x^0_P - x_A\right)^2 + \left(y^0_P - y_A\right)^2}} = \frac{\Delta x^0_{AP}}{d^0_{AP}}$$

$$\frac{\partial d_{AP}}{\partial y_P} = \frac{2\left(y^0_P - y_A\right)(1)}{2\sqrt{\left(x^0_P - y_A\right)^2 + \left(y^0_P - y_A\right)^2}} = \frac{\Delta y^0_{AP}}{d^0_{AP}}$$

$$\begin{bmatrix} v_{dAP} \\ v_{dBP} \\ v_{dCP} \end{bmatrix} = \begin{bmatrix} \frac{\Delta x^0_{AP}}{d^0_{AP}} & \frac{\Delta y^0_{AP}}{d^0_{AP}} \\ \frac{\Delta x^0_{BP}}{d^0_{BP}} & \frac{\Delta y^0_{BP}}{d^0_{BP}} \\ \frac{\Delta x^0_{CP}}{d^0_{CP}} & \frac{\Delta y^0_{CP}}{d^0_{CP}} \end{bmatrix} \begin{bmatrix} \delta x_P \\ \delta y_P \end{bmatrix} + \begin{bmatrix} d^0_{AP} - & d^{obs}_{AP} \\ d^0_{BP} - & d^{obs}_{BP} \\ d^0_{CP} - & d^{obs}_{CP} \end{bmatrix}$$

$$\boldsymbol{v}_{n,1} = \boldsymbol{A}_{n,u}\boldsymbol{\delta}_{u,1} + \boldsymbol{w}_{n,1}$$

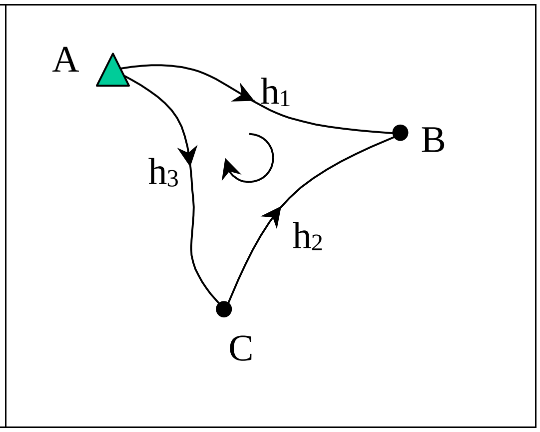
- <u>Conditional Model Example</u>

Levelling network: same network as in the parametric example

$$\boldsymbol{l} = \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta h_3 \end{bmatrix},\ n = 3$$

$$\boldsymbol{x} = \begin{bmatrix} h_B \\ h_c \end{bmatrix},\quad u = 2$$

$$r = n - u = 3 - 2 = 1$$



Note: The condition equation is a geometric or physical condition that must be satisfied by the adjusted observation $\hat{\mathbf{l}}$

$$\mathbf{g}\left(\hat{\mathbf{l}}\right) = 0$$

$$\hat{h}_1 - \hat{h}_2 - \hat{h}_3 = 0$$

Expand into differential form:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{l}} \qquad \mathbf{f}(\mathbf{l}^0) = \mathbf{f}(\mathbf{l}^{obs})$$

$$\underset{r\times n}{\mathbf{B}}\ \underset{n\times 1}{\mathbf{v}} + \underset{r\times 1}{\mathbf{w}} = \underset{r\times 1}{\mathbf{0}}$$

$$\begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + [h_1 - h_2 - h_3] = 0$$

- The number of equations is fewer than in the parametric model, but the conditional model is more difficult to program
- If there is more than one condition, they must be independent (i.e. we cannot derive one equation from the other condition equations)
- Following the estimation of the $\mathbf{v}_{\mathbf{n,1}}$ vector, we can estimate the best estimate value of the unknown parameters.

$$\hat{\mathbf{l}}_{\mathbf{n,1}} = \mathbf{l}_{\mathbf{n,1}} + \mathbf{v}_{\mathbf{n,1}}$$

- Since there are more observations than unknowns, there are several possibilities for the direct model equations.
- All possibilities will be equivalent if the <u>adjusted observations</u> are used, i.e

$$\hat{\mathbf{x}} = \mathbf{f}(\hat{\mathbf{l}})$$

$$\begin{bmatrix} \hat{h}_B \\ \hat{h}_c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \end{bmatrix} + \begin{bmatrix} h_A \\ h_A \end{bmatrix}$$

Another possibility is:

$$\hat{h}_B = h_A + h_1$$

$$\hat{h}_c = h_A + h_1 - h_2$$

$$\begin{bmatrix} \hat{h}_B \\ \hat{h}_c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \end{bmatrix} + \begin{bmatrix} h_A \\ h_A \end{bmatrix}$$

Note: That this is a direct model with redundant adjusted observations.

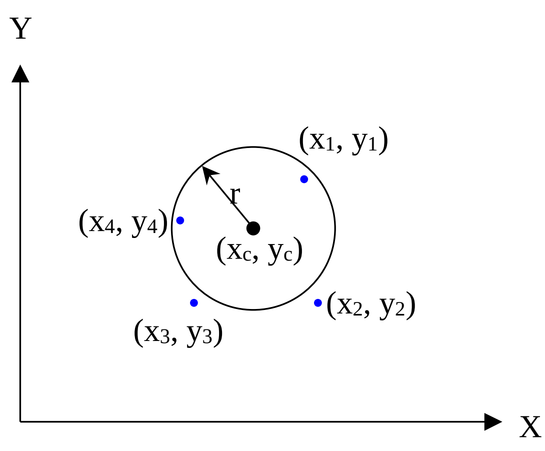
♦ <u>Combined Model Example: Circle Fit Problem</u>

Observations: (x_i, y_i) on the circumference of the circle

Unknowns: Centre point of circle and radius

$$\boldsymbol{l}_{8,1} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_c \\ y_c \\ r \end{bmatrix}$$

$n = 8$ and $u = 3$



Math model: **f(x,l) = 0**

$$(\hat{x}_i - \hat{x}_c)^2 + (\hat{y}_i - \hat{y}_c)^2 - \hat{r}^2 = 0 \qquad i = 1,2,3,4,$$

(note that the observables and unknown parameters are not separable)

$$\mathbf{A_{m,u}}\boldsymbol{\delta}_{\mathbf{u,1}} + \mathbf{B_{m,n}}\mathbf{v_{n,1}} + \mathbf{w_{m,1}} = \underline{\mathbf{0}_{\mathbf{m,1}}}$$

$$\mathbf{A_{4,3}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_c} & \frac{\partial f_1}{\partial y_c} & \frac{\partial f_1}{\partial r} \\ \frac{\partial f_2}{\partial x_c} & \frac{\partial f_2}{\partial y_c} & \frac{\partial f_2}{\partial r} \\ \frac{\partial f_3}{\partial x_c} & \frac{\partial f_3}{\partial y_c} & \frac{\partial f_3}{\partial r} \\ \frac{\partial f_4}{\partial x_c} & \frac{\partial f_4}{\partial y_c} & \frac{\partial f_4}{\partial r} \end{bmatrix} \qquad \boldsymbol{\delta}_{\mathbf{3,1}} = \begin{bmatrix} \delta x_c \\ \delta y_c \\ \delta r \end{bmatrix}$$

$$\boldsymbol{w}_{4,1} = \boldsymbol{f}\left(\boldsymbol{x}^0, \boldsymbol{l}^0\right) = \boldsymbol{f}\left(\boldsymbol{x}^0, \boldsymbol{l}^{obs}\right) = \begin{bmatrix} \left(x_1 - x_c^0\right)^2 + & \left(y_1 - y_c^0\right)^2 & -r_0^2 \\ \left(x_2 - x_c^0\right)^2 + & \left(y_2 - y_c^0\right)^2 & -r_0^2 \\ \left(x_3 - x_c^0\right)^2 + & \left(y_3 - y_c^0\right)^2 & -r_0^2 \\ \left(x_4 - x_c^0\right)^2 + & \left(y_4 - y_c^0\right)^2 & -r_0^2 \end{bmatrix}$$

$$\mathbf{B_{4,8}} = \frac{\partial \mathbf{f}}{\partial \mathbf{l}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial y_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial y_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial y_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial f_4}{\partial x_4} & \frac{\partial f_4}{\partial y_4} \end{bmatrix}$$

e.g. $(\mathrm{x}_1 - \mathrm{x}_c)^2 + (\mathrm{y}_1 - \mathrm{y}_c)^2 - r_0^2 = 0$

$$\left.\begin{aligned} \frac{\partial \mathrm{f}_1}{\partial \mathrm{x}_1} &= 2\left(\mathrm{x}_1 - \mathrm{x}_c^0\right) \\ \frac{\partial \mathrm{f}_1}{\partial \mathrm{y}_1} &= 2\left(\mathrm{y}_1 - \mathrm{y}_c^o\right) \end{aligned}\right\} \text{1st row of the B matrix}$$

$$\left.\begin{aligned} \frac{\partial \mathrm{f}_1}{\partial \mathrm{x}_c} &= -2\left(\mathrm{x}_1 - \mathrm{x}_c^0\right) \\ \frac{\partial \mathrm{f}_1}{\partial \mathrm{y}_c} &= -2\left(\mathrm{y}_1 - \mathrm{y}_c^0\right) \\ \frac{\partial \mathrm{f}_1}{\partial \mathrm{r}} &= -2r_0 \end{aligned}\right\} \text{1st row of the A matrix}$$

$$\mathbf{v}_{\mathbf{1,n}}^{\mathbf{T}} = \begin{bmatrix} v_{x1} & v_{y1} & v_{x2} & v_{y2} & v_{x3} & v_{y3} & v_{x4} & v_{y4} \end{bmatrix}$$