7. COMBINED (IMPLICIT) MODEL

- Linearized functional model (m equations)

$$\mathbf{A}_{m,u}\,\widehat{\boldsymbol{\delta}}_{u,1} + \mathbf{B}_{m,n}\,\widehat{\boldsymbol{v}}_{n,1} + \mathbf{w}_{m,1} = \mathbf{0}_{m,1} \quad \textit{(u+n unknowns in m equations)}$$

- With stochastic model $\mathbf{C}_{l\,(n,n)}$

- Using Lagrange multipliers, the variation function is

$$\varphi = \hat{\mathbf{v}}^T\mathbf{P}\hat{\mathbf{v}} + 2\hat{\mathbf{k}}^T(\mathbf{A}\hat{\boldsymbol{\delta}} + \mathbf{B}\hat{\mathbf{v}} + \mathbf{w}) = \min$$

- To minimise the variation function, differentiate with respect to $\hat{\mathbf{v}}$, $\hat{\boldsymbol{\delta}}$, $\hat{\mathbf{k}}$ and set equal to zero:

$$\begin{aligned}
\frac{\partial\varphi}{\partial\hat{\mathbf{v}}} &= 2\hat{\mathbf{v}}^T\mathbf{P} + 2\hat{\mathbf{k}}^T\mathbf{B} = \mathbf{0}\\
\frac{\partial\varphi}{\partial\hat{\delta}} &= 2\hat{\mathbf{k}}^T\mathbf{A} = \mathbf{0}\\
\frac{\partial\varphi}{\partial\hat{\mathbf{k}}} &= 2\hat{\delta}^T\mathbf{A}^T + 2\hat{\mathbf{v}}^T\mathbf{B}^T + 2\mathbf{w}^T = \mathbf{0}
\end{aligned}$$

Dividing by 2 and transposing:

$$\begin{aligned}
\mathbf{P}\hat{\mathbf{v}} + \mathbf{B}^T\hat{\mathbf{k}} &= \mathbf{0}\\
\mathbf{A}^T\hat{\mathbf{k}} &= \mathbf{0}\\
\mathbf{A}\hat{\boldsymbol{\delta}} + \mathbf{B}\hat{\mathbf{v}} + \mathbf{w} &= \mathbf{0}
\end{aligned}$$

- The above equation can be written in hyper-matrix notation with the following conditions:
  1) The upper left matrix of the hyper-matrix must be invertible (the P matrix is invertible)
  2) The hyper-matrix should be symmetric (arrange the equations to achieve this condition)

$$\begin{pmatrix} \mathbf{P} & \mathbf{B}^T & \mathbf{0} \\ \mathbf{B} & \mathbf{0} & \mathbf{A} \\ \mathbf{0} & \mathbf{A}^T & \mathbf{0} \end{pmatrix}\begin{pmatrix} \hat{\mathbf{v}} \\ \hat{\mathbf{k}} \\ \hat{\boldsymbol{\delta}} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

- Partition into $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} + \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$ and eliminate **x** ($\hat{\text{v}}$ in this case) using

  $(\mathbf{D} - \mathbf{C}\,\mathbf{A}^{-1}\,\mathbf{B})\,\mathbf{y} + (\mathbf{v} - \mathbf{C}\,\mathbf{A}^{-1}\,\mathbf{u}) = \mathbf{0}$

- Substitute for **A, B, C, D, x, y, u,** and **v**

$$\left\{\begin{pmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{pmatrix} - \begin{pmatrix} \mathbf{B} \\ \mathbf{0} \end{pmatrix}\mathbf{P}^{-1}\begin{pmatrix} \mathbf{B}^T & \mathbf{0} \end{pmatrix}\right\}\begin{pmatrix} \hat{\mathbf{k}} \\ \hat{\delta} \end{pmatrix} + \left\{\begin{pmatrix} \mathbf{w} \\ \mathbf{0} \end{pmatrix} - \underbrace{\begin{pmatrix} \mathbf{B} \\ \mathbf{0} \end{pmatrix}\mathbf{P}^{-1}(\mathbf{0})}_{0}\right\} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\underbrace{\left\{\begin{pmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{pmatrix} - \begin{pmatrix} \mathbf{B}\mathbf{P}^{-1}\mathbf{B}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right\}}_{\Downarrow}\begin{pmatrix} \hat{\mathbf{k}} \\ \hat{\delta} \end{pmatrix} + \begin{pmatrix} \mathbf{w} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\left(\begin{array}{c:c} -\mathbf{B}\mathbf{P}^{-1}\mathbf{B}^T & \mathbf{A} \\ \hdashline \mathbf{A}^T & \mathbf{0} \end{array}\right)\left(\begin{array}{c} \hat{\mathbf{k}} \\ \hdashline \hat{\boldsymbol{\delta}} \end{array}\right) + \left(\begin{array}{c} \mathbf{w} \\ \hdashline \mathbf{0} \end{array}\right) = \left(\begin{array}{c} \mathbf{0} \\ \hdashline \mathbf{0} \end{array}\right)$$

- Partition this hyper-matrix to eliminate $\hat{\mathbf{k}}$, and isolate $\hat{\delta}$

$$[\mathbf{0} - (\mathbf{A}^T)\,(-\mathbf{B}\,\mathbf{P}^{-1}\,\mathbf{B}^T)^{-1}\,\mathbf{A}]\ \hat{\boldsymbol{\delta}} + [\mathbf{0} - (\mathbf{A}^T)\,(-\mathbf{B}\,\mathbf{P}^{-1}\,\mathbf{B}^T)^{-1}\,(\mathbf{w})] = \mathbf{0}$$

$$\therefore\ \mathbf{A}^T\,(\mathbf{B}\,\mathbf{P}^{-1}\,\mathbf{B}^T)^{-1}\,\mathbf{A}\ \hat{\boldsymbol{\delta}} + \mathbf{A}^T\,(\mathbf{B}\,\mathbf{P}^{-1}\,\mathbf{B}^T)^{-1}\,\mathbf{w} = \mathbf{0}$$

$$\therefore\ \mathbf{N}\,\hat{\boldsymbol{\delta}} + \mathbf{u} = 0$$

$$\hat{\boldsymbol{\delta}} = -\,\mathbf{N}^{-1}\,\mathbf{u} \qquad\qquad \textit{recall}\ \ \mathbf{P}^{-1} \propto \textbf{or} = \mathbf{C}_l$$

- Note: if **B = - I (i.e. parametric model)**

$$\mathbf{N} = \mathbf{A}^T\,[(-\mathbf{I})\,\mathbf{P}^{-1}\,(-\mathbf{I})^T)]^{-1}\,\mathbf{A} = \mathbf{A}^T\,(\mathbf{P}^{-1})^{-1}\,\mathbf{A} = \mathbf{A}^T\,\mathbf{P}\,\mathbf{A}$$

$$\mathbf{u} = \mathbf{A}^T\,[(-\mathbf{I})\,\mathbf{P}^{-1}\,(-\mathbf{I})^T)]^{-1}\,\mathbf{w} = \mathbf{A}^T\,(\mathbf{P}^{-1})^{-1}\,\mathbf{w} = \mathbf{A}^T\,\mathbf{P}\,\mathbf{w}$$

which is the same results as the parametric model

- Substitute for $\hat{\boldsymbol{\delta}}$ in the first group of equations of the last obtained hyper-matrix to estimate $\hat{\mathbf{k}}$

$$(-\mathbf{B}\,\mathbf{P}^{-1}\,\mathbf{B}^{T})\,\hat{\mathbf{k}} + \mathbf{A}\,\hat{\boldsymbol{\delta}} + \mathbf{w} = \mathbf{0}$$

$$\boxed{\begin{array}{l}\hat{\mathbf{k}} = (\mathbf{B}\,\mathbf{P}^{-1}\,\mathbf{B}^{T})^{-1}\,(\mathbf{A}\,\hat{\boldsymbol{\delta}} + \mathbf{w}) = (\mathbf{B}\,\mathbf{C_l}\,\mathbf{B}^{T})^{-1}\,(\mathbf{A}\,\hat{\boldsymbol{\delta}} + \mathbf{w}) \\ \hat{\mathbf{k}} = \mathbf{M}^{-1}\,(\mathbf{A}\,\hat{\boldsymbol{\delta}} + \mathbf{w}) \text{ .... } \textbf{where } \mathbf{M} = \mathbf{B}\,\mathbf{C_l}\,\mathbf{B}^{T}\end{array}}$$

- Substitute $\hat{\mathbf{k}}$ in the first group of equations of the original hyper-matrix to estimate $\hat{\mathbf{v}}$

$$\mathbf{P}\,\hat{\mathbf{v}} + \mathbf{B}^{T}\,\hat{\mathbf{k}} = \mathbf{0}$$

$$\hat{\mathbf{v}} = -\,\mathbf{P}^{-1}\,\mathbf{B}^{T}\,\hat{\mathbf{k}} = -\,\mathbf{P}^{-1}\,\mathbf{B}^{T}\,(\mathbf{B}\,\mathbf{P}^{-1}\,\mathbf{B}^{T})^{-1}\,(\mathbf{A}\,\hat{\boldsymbol{\delta}} + \mathbf{w})$$

That is

$$\boxed{\begin{array}{l}\hat{\mathbf{v}} = -\,\mathbf{C_l}\,\mathbf{B}^{T}\,(\mathbf{B}\,\mathbf{C_l}\,\mathbf{B}^{T})^{-1}\,(\mathbf{A}\,\hat{\boldsymbol{\delta}} + \mathbf{w}) \\ \quad = -\,\mathbf{C_l}\,\mathbf{B}^{T}\,\hat{\mathbf{k}}\end{array}}$$

- Therefore, the adjusted quantities are:

$$\hat{\mathbf{x}} = \mathbf{x}^{0} + \hat{\boldsymbol{\delta}}$$

$$\hat{\mathbf{l}} = \mathbf{l} + \hat{\mathbf{v}}$$

- The covariance matrices are:

1. $C_{\hat{\delta}}$ (the covariance matrix of the solution vector)

- Functional model

$$\boxed{\hat{\delta} = -[A^{T}(BP^{-1}B^{T})^{-1}A]^{-1}A^{T}(BP^{-1}B^{T})^{-1}w}$$

$\hat{\boldsymbol{\delta}}$ = - constant $\mathbf{f}(\mathbf{x}^{0}, \mathbf{l}^{obs})$

$$\mathbf{C}_{\hat{\delta}} = \left(\frac{\partial\hat{\boldsymbol{\delta}}}{\partial\mathbf{l}}\right)\mathbf{C_l}\left(\frac{\partial\hat{\boldsymbol{\delta}}}{\partial\mathbf{l}}\right)^{\mathbf{T}}$$

where

$$\frac{\partial\hat{\boldsymbol{\delta}}}{\partial\mathbf{l}} = -\text{constant}\ \frac{\partial\mathbf{f}(\mathbf{l},\mathbf{x}^{0})}{\partial\mathbf{l}} = -\text{constant B}$$

- Therefore:

$$\mathbf{C}_{\hat{\delta}} = \left\{ [\mathbf{A}^T(\mathbf{B}\mathbf{C}_l\mathbf{B}^T)^{-1}\mathbf{A}]^{-1}\mathbf{A}^T(\mathbf{B}\mathbf{C}_l\mathbf{B}^T)^{-1}\mathbf{B} \right\}\mathbf{C}_l \left\{ \mathbf{B}^T(\mathbf{B}\mathbf{C}_l\mathbf{B}^T)^{-1}\mathbf{A}[\mathbf{A}^T(\mathbf{B}\mathbf{C}_l\mathbf{B}^T)^{-1}\mathbf{A}]^{-1} \right\}$$

$$\begin{aligned}
\mathbf{C}_{\hat{\delta}} &= \mathbf{N}^{-1}\mathbf{A}^T(\mathbf{B}\mathbf{C}_l\mathbf{B}^T)^{-1}\mathbf{B}\mathbf{C}_l\mathbf{B}^T(\mathbf{B}\mathbf{C}_l\mathbf{B}^T)^{-1}\mathbf{A}\mathbf{N}^{-1} \\
&= \mathbf{N}^{-1}\mathbf{A}^T(\mathbf{B}\mathbf{C}_l\mathbf{B}^T)^{-1}\mathbf{A}\mathbf{N}^{-1} \\
&= \mathbf{N}^{-1}\mathbf{N}\mathbf{N}^{-1} = \mathbf{N}^{-1} \\
&= [\mathbf{A}^T(\mathbf{B}\mathbf{C}_l\mathbf{B}^T)^{-1}\mathbf{A}]^{-1}
\end{aligned}$$

$$\boxed{\mathbf{C}_{\hat{\delta}} = [\mathbf{A}^T\mathbf{M}^{-1}\mathbf{A}]^{-1}}$$

2. $\mathbf{C}_{\hat{\mathbf{x}}}$ (the covariance matrix of the adjusted parameters)

$$\hat{\mathbf{x}} = \mathbf{x}^o + \hat{\boldsymbol{\delta}}$$

$$\mathbf{C}_{\hat{\mathbf{x}}} = \hat{\sigma}^2\, \mathbf{C}_{\hat{\boldsymbol{\delta}}}$$

3. $\mathbf{C}_{\hat{\mathbf{v}}}$ (the covariance matrix of the residuals) without proof

$$\boxed{\mathbf{C}_{\hat{\mathbf{v}}} = \mathbf{C}_l\mathbf{B}^T\mathbf{M}^{-1}\mathbf{B}\mathbf{C}_l - \mathbf{C}_l\mathbf{B}^T\mathbf{M}^{-1}\mathbf{A}\mathbf{N}^{-1}\mathbf{A}^T\mathbf{M}^{-1}\mathbf{B}\mathbf{C}_l}$$

- where

$$\mathbf{M} = \mathbf{B}\mathbf{C}_l\mathbf{B}^T$$

4. $\mathbf{C}_{\hat{\mathbf{l}}}$ (the covariance matrix of the adjusted observations)

$$\hat{\mathbf{l}} = \mathbf{l} + \hat{\mathbf{v}}$$

$$\mathbf{C}_{\hat{\mathbf{l}}} = \mathbf{C}_l - \mathbf{C}_{\hat{\mathbf{v}}}$$

7.1. Iterative Solution of the Combined Model

- Linearized model

$$\mathbf{A}\hat{\delta} + \mathbf{B}\hat{v} + \mathbf{w} = \mathbf{0}$$

- At iteration (i), calculate

$$\mathbf{A}_{(i)} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{X}^{o}_{(i)},\mathbf{l}} \quad \mathbf{B}_{(i)} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{l}}\right|_{\mathbf{X}^{o}_{(i)},\mathbf{l}}$$

At the current point of expansion (POE)

$$\mathbf{X}^{\mathbf{0}}_{(i)} = \hat{\mathbf{X}}_{(i-1)}$$

$$\mathbf{l} = \mathbf{l}^{obs}$$

- calculate $\hat{\boldsymbol{\delta}}_{(i)}$ and $\hat{\mathbf{v}}_{(i)}$

$$\hat{\boldsymbol{\delta}}_{(i)} = -\,\mathbf{N}_{(i)}^{-1}\,\mathbf{u}_{(i)}$$

$$\hat{\mathbf{v}}_{(i)} = -\,\mathbf{C}_l\,\mathbf{B}_{(i)}^{T}\,\hat{\boldsymbol{k}}_{(i)}$$

- Where

$$\mathbf{N}_{(i)} = \mathbf{A}_{(i)}^{T}\,\mathbf{M}^{-1}_{(i)}\,\mathbf{A}_{(i)}$$
$$\mathbf{u}_{(i)} = \mathbf{A}_{(i)}^{T}\,\mathbf{M}^{-1}_{(i)}\,\mathbf{w}_{(i)}$$

and

$$M_{(i)} = \mathbf{B}_{(i)}\,\mathbf{P}^{-1}\,\mathbf{B}_{(i)}^{T}$$

$$\mathbf{w}_{(i)} = \mathbf{f}(\mathbf{x}_{(i)},\mathbf{l}_{(i)}) + \mathbf{B}_{(i)}(\mathbf{l}^{obs} - \hat{\mathbf{l}}_{(i-1)})$$ ……. [Note for i =1 the second term will be zero]

$$\hat{k}_{(i)} = \mathbf{M}^{-1}_{(i)}\,(\mathbf{A}_{(i)}\,\hat{\boldsymbol{\delta}}_{(i)} + \mathbf{w}_{(i)})$$

- where

$$\hat{\mathbf{l}}_{(i)} = \mathbf{l} + \hat{\mathbf{v}}_{(i)}$$

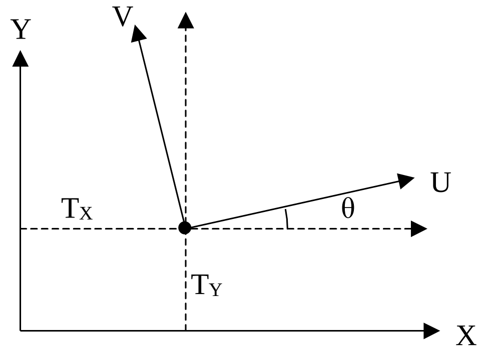
$$\hat{\mathbf{x}}_{(i)} = \mathbf{x}^{\mathbf{0}}_{(i)} + \hat{\delta}_{(i)} = \hat{\mathbf{x}}_{(i-1)} + \hat{\delta}_{(i)}$$

- repeat until $\hat{\boldsymbol{\delta}}_{(i+1)} - \hat{\boldsymbol{\delta}}_{(i)}$ approaches 0

7.2. Example

- The 2-D coordinate transformation (shift, rotation, and scale) between two coordinate systems (X, Y) and (U, V) is given by

$$\begin{pmatrix} X \\ Y \end{pmatrix}_i = S \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_i + \begin{pmatrix} T_X \\ T_Y \end{pmatrix}$$



- For simplicity we will assume that $T_X = T_Y = 0$, therefore we can write the 2-D model as

$$\begin{pmatrix} X \\ Y \end{pmatrix}_i = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_i, \text{where} \quad a = S\cdot\cos\theta, b = S\cdot\sin\theta$$

- Therefore, the unknowns are a and b
- In order to estimate a and b, observations are required. In this case, the following table gives three points of known co-ordinates in both systems.

i	U	V	X	Y
1	0.0	1.0	-2.1	1.1
2	1.0	0.0	1.0	2.0
3	1.0	1.0	-0.9	2.8

- For all three given points, the $\mathbf{C_l}$ matrix of the (U, V) co-ordinates is

$$\mathbf{C}_{\mathbf{l}_i} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} = 0.01\ \mathbf{I}_{2,2}$$

$$\therefore \mathbf{C_l} = 0.01\ \mathrm{I}_{6,6}$$

- All (X, Y) coordinates are to be considered constants.

- Solution:

1. **l** and **x**

$$\mathbf{l} = \begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{pmatrix} \quad n = 6 \qquad\qquad \mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix} \quad u = 2$$

2. Mathematical model

f_1: a U_i - b V_i - X_i = 0

f_2: b U_i + a V_i - Y_i = 0 $\Rightarrow$ a V_i + b U_i - Y_i = 0

3. Linearized equations

$$\mathbf{A}_{6,2}\hat{\boldsymbol{\delta}}_{2,1} + \mathbf{B}_{6,6}\hat{\mathbf{v}}_{6,1} + \mathbf{w}_{6,1} = \mathbf{0}$$

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^0, \mathbf{l}=\mathbf{l}^{obs}} = \begin{bmatrix} U_1 & -V_1 \\ V_1 & U_1 \\ U_2 & -V_2 \\ V_2 & U_2 \\ U_3 & -V_3 \\ V_3 & U_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Note that **A** is not a function of **x** ~~or~~, but it is a function of **l**.

$$\mathbf{B} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{l}}\right|_{\mathbf{x}=\mathbf{x}^0, \mathbf{l}=\mathbf{l}^{obs}} = \begin{array}{c} \begin{matrix} U_1 & V_1 & U_2 & V_2 & U_3 & V_3 \end{matrix} \\ \begin{bmatrix} a^0 & -b^0 & 0 & 0 & 0 & 0 \\ b^0 & a^0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^0 & -b^0 & 0 & 0 \\ 0 & 0 & b^0 & a^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a^0 & -b^0 \\ 0 & 0 & 0 & 0 & b^0 & a^0 \end{bmatrix} \end{array}$$

Therefore, a^o and b^o are needed to evaluate **B**. They (a^o and b^o) can be evaluated by simple computation through the use of 2 equations of the math model.

$a^0 = 1$, $b^0 = 2$ using the equations of point 2

$$\mathbf{w} = \mathbf{f}(\mathbf{x^0}, \mathbf{l}^{obs}) = \begin{bmatrix} a^0U_1 - b^0V_1 - X_1 \\ a^0V_1 + b^0U_1 - Y_1 \\ a^0U_2 - b^0V_2 - X_2 \\ a^0V_2 + b^0U_2 - Y_2 \\ a^0U_3 - b^0V_3 - X_3 \\ a^0V_3 + b^0U_3 - Y_3 \end{bmatrix} = \begin{bmatrix} 0.1 \\ -0.1 \\ 0.0 \\ 0.0 \\ -0.1 \\ 0.2 \end{bmatrix}$$

4. The $\hat{\boldsymbol{\delta}}$ vector

$$\hat{\boldsymbol{\delta}} = -\mathbf{N}^{-1}\mathbf{u} = -[\mathbf{A}^{\mathbf{T}}(\mathbf{BP}^{-1}\mathbf{B}^{\mathbf{T}})^{-1}\mathbf{A}]^{-1}\mathbf{A}^{\mathbf{T}}(\mathbf{BP}^{-1}\mathbf{B}^{\mathbf{T}})^{-1}\mathbf{w}$$

$$\hat{\boldsymbol{\delta}} = \begin{bmatrix} 0.0 \\ -0.05 \end{bmatrix}$$

$$\therefore \hat{\mathbf{x}} = \mathbf{x^0} + \hat{\boldsymbol{\delta}} = \begin{bmatrix} 1.0 \\ 2.0 \end{bmatrix} + \begin{bmatrix} 0.0 \\ -0.05 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.95 \end{bmatrix}$$

5. The $\hat{\mathbf{v}}$ vector

$$\hat{\mathbf{v}} = -\mathbf{C_l B^T}(\mathbf{B C_l B^T})^{-1}(\mathbf{A}\hat{\boldsymbol{\delta}} + \mathbf{w})$$

$\hat{\mathbf{v}}^{\mathbf{T}} = [0.01 \;\; 0.08 \;\; 0.02 \;\; 0.01 \;\; -0.05 \;\; -0.05]$

$$\hat{\mathbf{l}} = \mathbf{l} + \hat{\mathbf{v}} = \begin{bmatrix} 0.0 \\ 1.0 \\ 1.0 \\ 0.0 \\ 1.0 \\ 1.0 \end{bmatrix} + \begin{bmatrix} 0.01 \\ 0.08 \\ 0.02 \\ 0.01 \\ -0.05 \\ -0.05 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 1.08 \\ 1.02 \\ 0.01 \\ 0.95 \\ 0.95 \end{bmatrix}$$

Check: make use of $\hat{\mathbf{x}} = \begin{bmatrix} 1.0 \\ 1.95 \end{bmatrix}$ to calculate the values of the (X,Y) coordinates using the (U, V) coordinates and the math model:

$$\begin{pmatrix} X \\ Y \end{pmatrix}_i = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix}_i$$

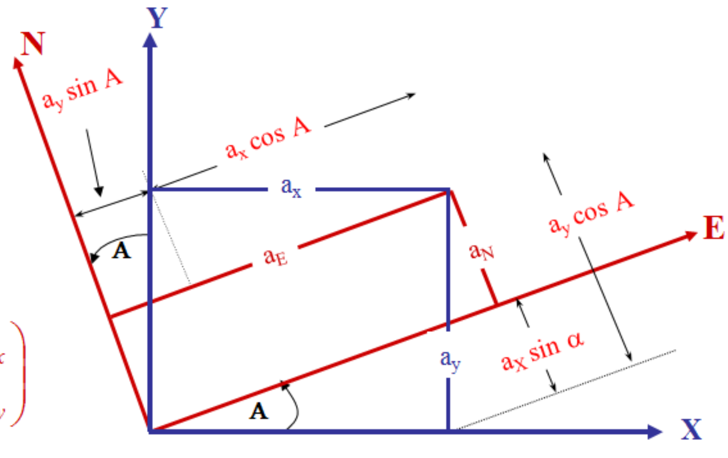
7.3. Side Note: Rotation Matrices in 2D

$$\mathbf{a_E} = a_y \sin A + a_x \cos A$$

$$\mathbf{a_N} = a_y \cos A - a_x \sin A$$

In Matrix Form

$$\begin{pmatrix} a_E \\ a_N \end{pmatrix} = \begin{pmatrix} \cos A & \sin A \\ -\sin A & \cos A \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$



If there is a difference in a scale between the two coordinate systems, then a scale factor should be included:

$$\begin{pmatrix} a_E \\ a_N \end{pmatrix} = S \begin{pmatrix} \cos A & \sin A \\ -\sin A & \cos A \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

If there is a shift between the two coordinate systems, then two shift components should be included as well:

$$\begin{pmatrix} a_E \\ a_N \end{pmatrix} = S \begin{pmatrix} \cos A & \sin A \\ -\sin A & \cos A \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix} + \begin{bmatrix} T_E \\ T_N \end{bmatrix}$$