Math 451
Homework 11

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Due: April 18^{th}

Problem 1.

Simple L'Hospital: Let $f, g : (a,b) \to \mathbb{R}$ be continuously differentiable and $c \in (a,b)$. Assume that $f(c) = g(c) = 0$ and $g'(c) \neq 0$. Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}.$$

Remark: The more general L'Hospital theorem does not require f, g to be defined at c, and allows $c = \pm\infty$. The proof is more complicated. It can be found in Section 30 of our book.

Proof. Using the definition of the derivative we have

$$\frac{f'(c)}{g'(c)} = \frac{\lim_{x \to c} \frac{f(x)-f(c)}{x-c}}{\lim_{x \to c} \frac{g(x)-g(c)}{x-c}} = lim_{x \to c} \frac{f(x)-f(c)}{g(x)-g(c)}$$

Since $f(c) = g(c) = 0$ this is equivalent to

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

$\square$

Proof Difficulty: ★

Problem 2.

Is the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

defined on the set $[-1, 1]$ Riemann-integrable? Prove your answer.

Remark: We have seen that wild oscillations are a problem for continuity. Here you'll check if they are a problem for integrability.

Proof. We will show $f(x)$ is Riemann-integrable. Let $\varepsilon > 0$. First notice that $|f| \leq 1$. Now choose some $c \in \mathbb{R}$ such that $c < \frac{\varepsilon}{8}$. Now notice that $f|_{[-1,-c]}$ is continuous so there is a partition

$$P_1 := (-1 = a_0^{P_1}, a_1^{P_1}, ..., a_{|P_1|}^{P_1} = -c)$$

with step functions h_1 and h_2 adapted to P_1 such that $h_1 \leq f|_{[-1,-c]} \leq h_2$ where

$$\int_{-1}^{-c} h_2(x)\,dx - \int_{-1}^{-c} h_1(x)\,dx < \frac{\varepsilon}{4}$$

Similarily, $f|_{[c,1]}$ is continuous so there is a partition

$$P_2 := (c = a_0^{P_2}, a_1^{P_2}, ..., a_{|P_2|}^{P_2} = 1)$$

with step functions h_3 and h_4 adapted to P_2 such that $h_3 \leq f|_{[c,1]} \leq h_4$ where

$$\int_{c}^{1} h_4(x)\,dx - \int_{c}^{1} h_3(x)\,dx < \frac{\varepsilon}{4}$$

Now we will define

$$P := P_1 \cup P_2$$

Now we construct the step functions h_5 and h_6 adapted to P such that

$$h_5\Big|_{(a_i^P, a_{i+1}^P)} = \begin{cases} h_1|_{(a_i^P, a_{i+1}^P)}, & a_i^P < -c \\ \inf f([a_i^P, a_{i+1}^P]), & a_i^P = -c \\ h_3|_{(a_i^P, a_{i+1}^P)}, & a_i^P > -c \end{cases}$$

$$h_6\Big|_{(a_i^P, a_{i+1}^P)} = \begin{cases} h_2|_{(a_i^P, a_{i+1}^P)}, & a_i^P < -c \\ \sup f([a_i^P, a_{i+1}^P]), & a_i^P = -c \\ h_4|_{(a_i^P, a_{i+1}^P)}, & a_i^P > -c \end{cases}$$

By construction we have $h_5 \leq f \leq h_6$. Now we can take

$$\int_{-1}^{1} h_6(x)\,dx - \int_{-1}^{1} h_5(x)\,dx$$

$$= \int_{-1}^{-c} h_2(x)\,dx + \sup f([-c,c]) \cdot (c-(-c)) + \int_{c}^{1} h_4(x)\,dx - \left(\int_{-1}^{-c} h_1(x)\,dx + \inf f([-c,c]) \cdot (c-(-c)) + \int_{c}^{1} h_3(x)\,dx\right)$$

$$= \int_{-1}^{-c} h_2(x)\,dx - \int_{-1}^{-c} h_1(x)\,dx + \int_{c}^{1} h_4(x)\,dx - \int_{c}^{1} h_3(x)\,dx + 2c \cdot (\sup f([-c,c]) - \inf f([-c,c]))$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2 \cdot (\frac{\varepsilon}{8}) \cdot (1-(-1)) = \varepsilon$$

We can conclude that f is Riemann-integrable. □

Proof Difficulty: ★★★★

Problem 3.

Let $f : [0, 1] \to \mathbb{R}$ be continuous.

(a) Assume $f(x) \geq 0$ for all x. Show that if $\int_0^1 f(x)\,dx = 0$ then $f(x) = 0$ for all x.

(b) Without assuming $f(x) \geq 0$ for all x, show that if $\int_a^b f(x)\,dx = 0$ for all $0 \leq a < b \leq 1$ then $f(x) = 0$ for all x.

Proof. (a) Let us assume, for the sake of contradiction, that there exists some $c \in [0, 1]$ such that $f(c) > 0$. Consider some $0 < \varepsilon < f(c)$. First consider the case where $c \in (0, 1)$. By the continuity of f we can find some $\delta > 0$ such that

$$x \in (c - \delta, c + \delta) \subset (0, 1) \Rightarrow |(f(x) - f(c))| < \varepsilon \Rightarrow f(x) > 0$$

We thus have

$$\int_{c-\delta}^{c+\delta} f(x)\,dx > 0$$

But since $f(x) \geq 0$ for all x this implies that

$$\int_0^1 f(x)\,dx > 0$$

This is a contradiction.

In the case where $c = 0$ we can just find some $\delta > 0$ such that

$$x \in [c, c + \delta) \subset (0, 1) \Rightarrow |(f(x) - f(c))| < \varepsilon \Rightarrow f(x) > 0$$

We can then proceed with same logic as above and find a contradiction.

In the case where $c = 1$ we can just find some $\delta > 0$ such that

$$x \in (c - \delta, c] \subset (0, 1) \Rightarrow |(f(x) - f(c))| < \varepsilon \Rightarrow f(x) > 0$$

We can then proceed with same logic as above and find a contradiction.

In every case we have found a contradiction. Thus we can conclude that $f(x) = 0$ for all x.

(b) Let us assume, for the sake of contradiction, that there exists some $c \in [0, 1]$ such that $f(c) \neq 0$. Without loss of generality, suppose that $f(c) > 0$; if $f(c) < 0$, we can consider the function $-f$, which would still satisfy the integral condition, and proceed analogously.

First consider the case where $c \in (0, 1)$. Since f is continuous, we can find some $\delta > 0$ such that

$$x \in (c - \delta, c + \delta) \subset (0, 1) \Rightarrow |f(x) - f(c)| < \frac{f(c)}{2} \Rightarrow f(x) > \frac{f(c)}{2}$$

Now we can take

$$\int_{c-\delta}^{c+\delta} f(x)\,dx > \int_{c-\delta}^{c+\delta} \frac{f(c)}{2}\,dx = (2\delta) \cdot \frac{f(c)}{2} = \delta f(c) > 0$$

This is a contradiction with our initial assumption.

Now in the case where $c = 0$ we can find some $\delta > 0$ such that

$$x \in [c, c + \delta) \subset (0, 1) \Rightarrow |f(x) - f(c)| < \frac{f(c)}{2} \Rightarrow f(x) > \frac{f(c)}{2}$$

We can then proceed with the same argument as above and find a contradiction.

When $c = 1$ we can find some $\delta > 0$ such that

$$x \in (c - \delta, c] \subset (0, 1) \Rightarrow |f(x) - f(c)| < \frac{f(c)}{2} \Rightarrow f(x) > \frac{f(c)}{2}$$

We can then proceed with the same argument as above and find a contradiction.

In every case we have found a contradiction. Thus, we can conclude $f(x) = 0$ for all x.

□

Proof Difficulty: ★★★★

Problem 4.

Prove that any continuous function $f : [a, b] \to \mathbb{R}$ is the uniform limit of a sequence of step functions.

Remark: This gives a different look on the definition of the integral as an approximation by integrals of step functions. It turns out that the function itself is approximated by step functions in a strong enough way.

Proof. We will construct (f_n) as follows. For each f_n consider $\frac{1}{n}$. Since f is continuous we can find some $\delta_n > 0$ such that for all $x, y \in [a, b]$ we have $|x - y| < \delta_n \Rightarrow |f(x) - f(y)| < \frac{1}{n}$.

Now let

$$s := \sup\{c \in \mathbb{N} : a + c\delta_n < b\}$$

For every $m \in \mathbb{N}_{\leq s}$ we define $x_m := a + m\delta_n$ and $x_{s+1} := b$. Now we set

$$f_n(x) = \begin{cases} f(x_m), & x_m \leq x < x_{m+1} \\ f(b), & x = b \end{cases}$$

We will now show that $f_n \to f$ uniformly. Let $\varepsilon > 0$ and by Arch Prop find some $N \in \mathbb{N}$ such that $\varepsilon > \frac{1}{N}$. By construction $\forall n > N$ we have,

$$|f_n(x) - f(x)| = |f(x - a) - f(x)| \text{ where } a < \delta_n$$

Furthermore, by construction we have

$$|f(x - a) - f(x)| < \frac{1}{n} < \frac{1}{N} < \varepsilon$$

We can conclude that $f_n \to f$ uniformly. Thus, any continuous function $f : [a, b] \to \mathbb{R}$ is the uniform limit of a sequence of step functions. □

Proof Difficulty: ★★★★

Problem 5.

Prove the integral test for convergence of series: Let $f : [1, \infty) \to \mathbb{R}$ be a non-negative monotonically decreasing function. Then $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

Proof. First assume that $\sum_{n=1}^{\infty} f(n)$ converges. Notice that since f is non-negative, we have $\sum_{n=1}^{\infty} f(n) \geq 0$. Since f is non-negative and monotonically decreasing we know for any $n \in \mathbb{N}$

$$f(n) = f(n) \cdot (n + 1 - n) \geq \int_n^{n+1} f(x)\,dx = \int_n^{n+1} f(x)\,dx$$

Therefore we have

$$\sum_{n=1}^{\infty} f(n) \geq \int_1^{\infty} f(x)\,dx$$

Since f is non-negative, we also know $\int_1^{\infty} f(x)\,dx \geq 0$. Combining this with the fact that $\int_1^{\infty} f(x)\,dx$ is bounded above by $\sum_{n=1}^{\infty} f(n) \in \mathbb{R}$, we can conclude that $\int_1^{\infty} f(x)\,dx$ converges.

Now assume that $\int_1^{\infty} f(x)\,dx$ converges. Notice that since f is non-negative, we have $\int_1^{\infty} f(n) \geq 0$. Since f is non-negative and monotonically decreasing we know for any $n \in \mathbb{N}$

$$\int_n^{n+1} f(x)\,dx \geq f(n+1) \cdot (n + 1 - n) = f(n+1)$$

Therefore we have

$$\int_1^{\infty} f(x)\,dx \geq \sum_{n=2}^{\infty} f(n) \Rightarrow \int_1^{\infty} f(x)\,dx + f(1) \geq \sum_{n=1}^{\infty} f(n)$$

Since f is non-negative, we also know $\sum_{n=1}^{\infty} f(n) \geq 0$. Combining this with the fact that $\sum_{n=1}^{\infty} f(n)$ is bounded above by $\int_1^{\infty} f(x)\,dx + f(1) \in \mathbb{R}$, we can conclude that $\sum_{n=1}^{\infty} f(n)$ converges.

We can conclude that $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges. $\square$

Proof Difficulty: ★★