PHYS3115 - Particle Physics and the Early Universe
–particle physics–

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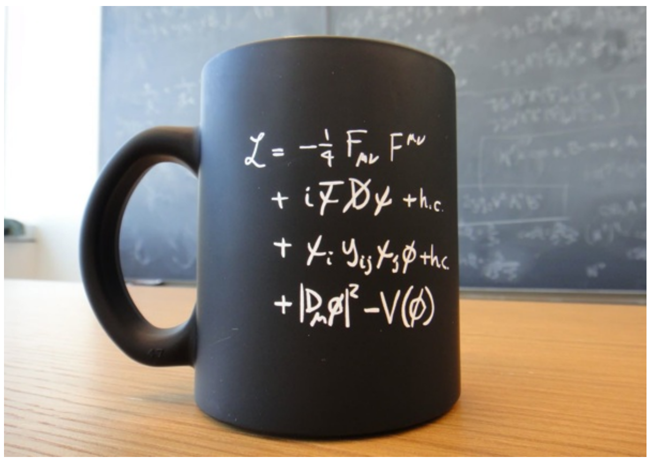
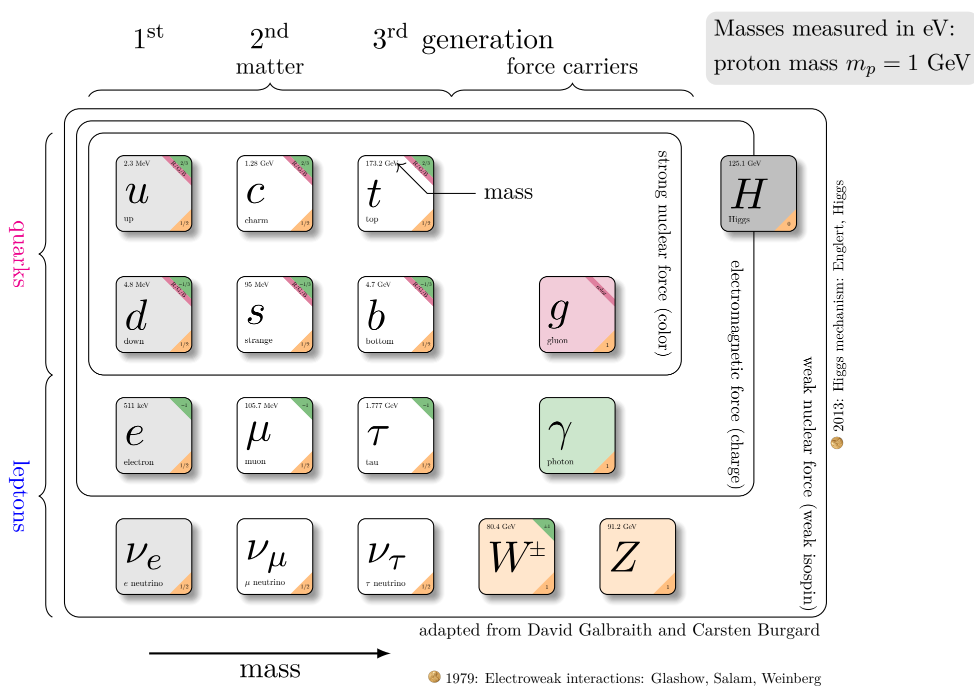
October 10, 2019





Particle physics is about the description of the fundamental building blocks of nature: its elementary particles and forces. It has close connections with the evolution of the universe at the earliest times, because the universe used to be much hotter and thus particles were more energetic and have been copiously produced. This is one of the reasons for a combined course on particle physics and the early universe.

The underlying theoretical framework of particle physics is quantum field theory. This course is not a course on quantum field theory which is the subject of the Honours level quantum field theory course, but I will touch on some aspects of quantum field theory. This course serves as an introduction to the bigger picture of particle physics. It will mostly focus on the theoretical side, but also contain some aspects of the experimental success of the Standard Model, mainly in the student presentations.

A good book to read while following this course is Mark Thompson, Modern Particle Physics which is based on a Part III (year 4) course taught by the author at Cambridge.

These notes are partly based on the A/Prof Yvonne Wong's lecture notes, who taught the course in previous years.

Conventions

Throughout the course we will use natural units

$$\hbar = c = k_B = 1$$

and the signature $(+---)$ for the metric. Thus length scales and time are measured with the same units

$$[\ell] = [t] = \frac{1}{[m]} = \frac{1}{[E]} = \frac{1}{[T]} = \mathrm{eV}^{-1}$$

which is the inverse of energy. Temperature is measured in the same units as energy. The relativistic energy-momentum relation reads

$$p^2 = p_\mu p^\mu = E^2 - \mathbf{p}^2 = m^2 \tag{0.1}$$

Contents

1 Introduction and preliminaries

Particle physics addresses the questions

- What are the elementary building blocks of nature?
- What is matter made of at the most fundamental level?
- How do the building blocks interact with each other?

In order to probe physics at the shortest distances, we have to use the highest energies. This concept is summarised in the de Broglie wavelength of a particle in quantum mechanics

$$\lambda = \frac{h}{|\mathbf{p}|} \ . \tag{1.1}$$

Fig. 1 illustrates the length scales which we are dealing with in particle physics. The energies are measured in eV, where 1eV equals $1.6 \times 10^{-19} J$, the energy gained/lost by an electron across a potential difference of $1V$. Molecules and solids are built up by atoms, which themselves have a nucleus and electrons. The nucleus is made of protons and neutrons, so-called nucleons, which are held together by the exchange of mesons. The nucleons belong to the larger family of baryons, bound states of three quarks, while mesons are bound states of a quark and an anti-quarks. Quarks interact via the strong force and electrons are leptons, which do not interact via the strong force.

According to today's knowledge up to energies of TeV= 10^{12} eV or length scales of 10^{-18}cm, quarks and leptons do not possess any substructure and are the elementary building blocks of nature.

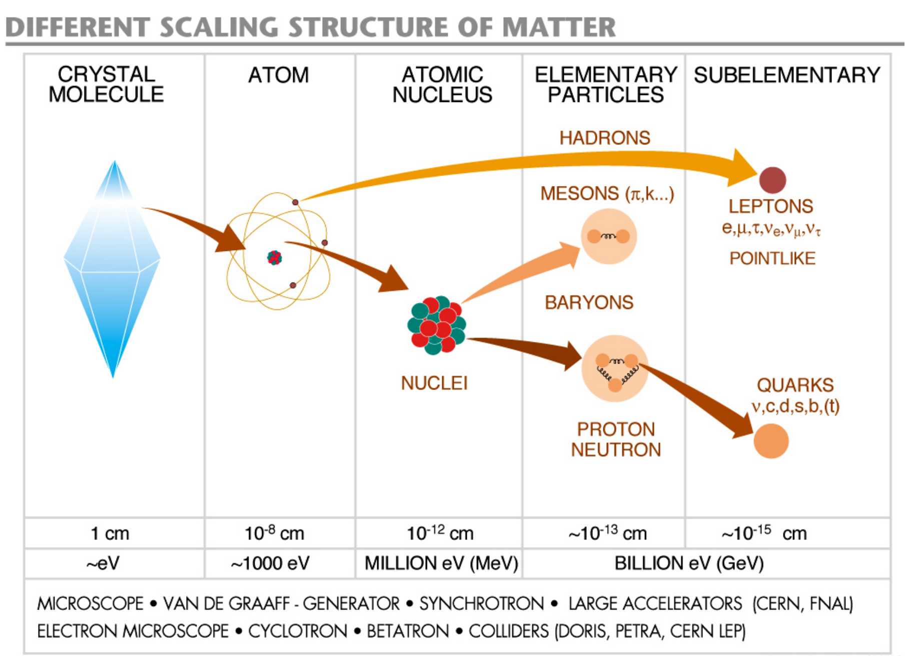


Figure 1: Comparison of different length scales. Taken from CERN

Apart from the elementary particles which constitute all ordinary matter (protons, neutrons, electrons), there are further heavier particles, which are generally unstable. They have been observed in cosmic rays, e.g. the muon, a heavy version of the electron, discovered in 1936 by Anderson and Neddermeyer while studying cosmic radiation (Nobel prize 1936) or the τ lepton, which has been predicted by Yung-su Tsai in 1971 and detected in experiments between 1974 and 1977 by Martin L. Perl (Nobel prize, Martin L. Perl 1995) and colleagues at the Stanford Linear Accelerator Center (SLAC) and the Lawrence Berkeley National Laboratory (LBL).

Experimental research in this field is often split in three different categories: the energy frontier, precision or intensity frontier and the cosmological frontier.

1. Typical experiments at the energy frontier are colliders, like the Large Hadron Collider (LHC), which accelerate particles to relativistic energies and then collide them to study physics at the highest energies. Typical analyses look at scattering of particles (deflection angle, when two particles collider) and decays of particles produced in collisions (debris left behind when a particle disintegrates). Cosmic radiation also probes the highest energies, even larger than the ones accessible at the LHC, but with much smaller fluxes, interaction rates.

2. At the precision/intensity frontier, experiments such as flavour physics experiments like Belle 2 in Japan, neutrino oscillation experiments, dark matter direction detection experiments, and others make high precision measurements at much lower energies. This allows to probe very weakly interacting physics like dark matter and neutrinos, probe for very rare processes such as $b \to s\gamma$. Through their high precision they are sensitive to small (quantum) corrections due to new physics.

3. Finally, the cosmology frontier uses our whole Universe to test the Standard Model (SM) and search for new physics using observations of big bang nucleosynthesis, the cosmic microwave background and other observables.

When combining quantum mechanics and special relativity, one naturally arrives at quantum field theory (QFT), which is the underlying framework of particle physics. Quantum field theory does not specify the particles and interactions itself. It only ensures that the introduced particles and interactions are consistent with special relativity and quantum physics. This is similar to Newton's second law in mechanics which does not specify the force. Particles are described by (quantum) excitations of a field, e.g. the photon is an excitation of the electromagnetic field. Our current understanding of particle physics is summarised in the SM of particle physics which describes nature (apart from a few deviations which we will comment on later). The Lagrangian which describes all interactions in the SM fits on a mug, which is shown on the title page. Before discussing the SM, we introduce natural units.

1.1 Natural units

In particle physics it is common to use natural units instead of SI units. This amounts to replacing basic SI units [kg,m,s,K] for mass, length, time and temperature with $[\hbar, c, \mathrm{GeV}, k_B]$. Thus we find

	SI	$[\hbar, c, \mathrm{GeV}]$	$\hbar = c = k_B = 1$
energy	$\mathrm{kg}\, m^2\, s^{-2}$	GeV	GeV
momentum	$\mathrm{kg}\, m\, s^{-1}$	GeV/c	GeV
mass	kg	GeV/c^2	GeV
time	s	$(\mathrm{GeV}/\hbar)^{-1}$	GeV^{-1}
length	m	$(\mathrm{GeV}/\hbar c)^{-1}$	GeV^{-1}
temperature	K	GeV/k_B	GeV

By setting $\hbar = c = k_B = 1$ we can express the units of all quantities in powers of GeV which is extremely convenient [1]. It is straightforward to convert between SI units and natural units using the expressions for $\hbar$, c, and k_B

$$1\,\mathrm{GeV} = 1.602 \times 10^{-10}\, J = 1.602 \times 10^{-10}\, \mathrm{kg}\, m^2\, s^{-2} \tag{1.2}$$

$$\hbar = 6.582 \times 10^{-25}\, \mathrm{GeV}\, s \tag{1.3}$$

$$c = 2.998 \times 10^{8}\, m\, s^{-1} \tag{1.4}$$

$$\hbar c = 0.197 \times 10^{-15}\, \mathrm{GeV}\, m \tag{1.5}$$

$$k_B = 8.617 \times 10^{-5}\, \mathrm{eV}\, K^{-1}\ . \tag{1.6}$$

For example for the root-mean-square charge radius of the proton

$$\langle r^2 \rangle^{1/2} = 4.1 \mathrm{GeV}^{-1} \tag{1.7}$$

can be converted to m by multiplying with the correct number of factors of $\hbar$ and c to obtain m

$$\langle r^2 \rangle^{1/2} = 4.1 \mathrm{GeV}^{-1} \hbar c = 0.81 \times 10^{-15}\, m\ . \tag{1.8}$$

Another common choice in cosmology is to introduce the reduced Planck mass

$$m_{\mathrm{Planck}} = \sqrt{\frac{\hbar c}{8\pi G}} = 2.435 \times 10^{18}\, \mathrm{GeV} \tag{1.9}$$

which will be discussed more in the second half of the course.

1.2 Standard Model of particle physics

The Standard Model (SM) of particle physics has been first formulated in the 1960's and finalised in its current form in the mid 1970's. It passed essentially all experimental tests and makes predictions

[1] The conversion from Heavyside-Lorentz units to natural units requires to set $\epsilon_0 = \mu_0 = 1$, where ϵ_0 and μ_0 are the permittivity and permeability in vacuum.

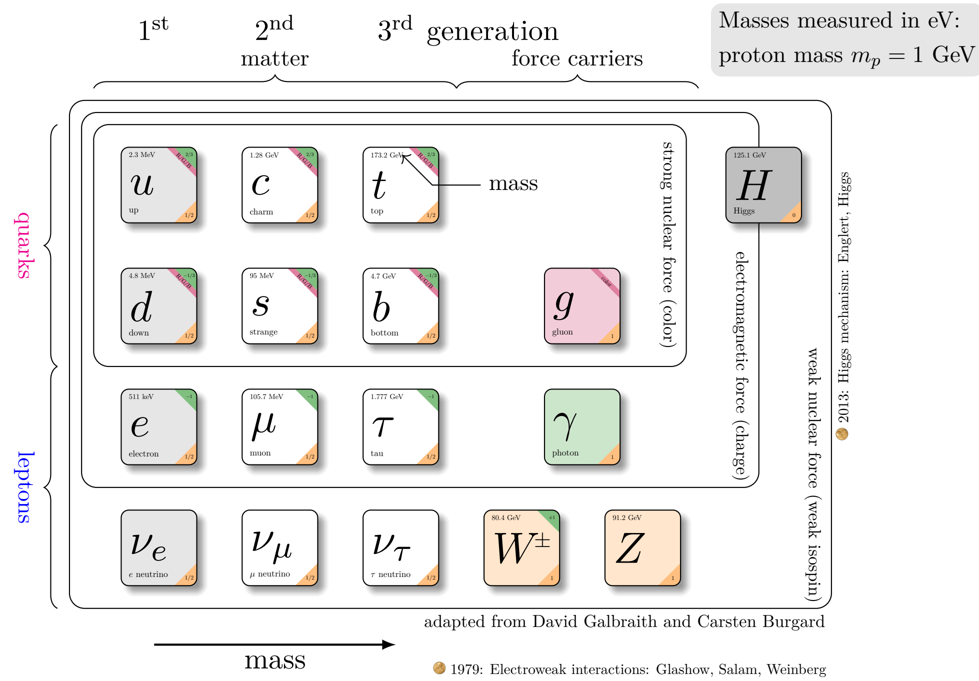


Figure 2: The Standard Model of particle physics.

with high accuracy. The anomalous magnetic moment of the electron agrees with the SM prediction to more than 12 significant figures precision and is one of the most precisely known quantities. Fig. 2 illustrates the particle content of the SM of particle physics. A good source for details about elementary particles is the website of the Particle Data Group http://pdg.lbl.gov.

The particles can be separated into spin $\frac{1}{2}$ fermions, the forces which are mediated by the spin 1 gauge bosons and the spin 0 Higgs boson. Gravity is mediated by a spin 2 graviton. It is not part of the SM of particle physics and one of the open questions in theoretical particle physics.

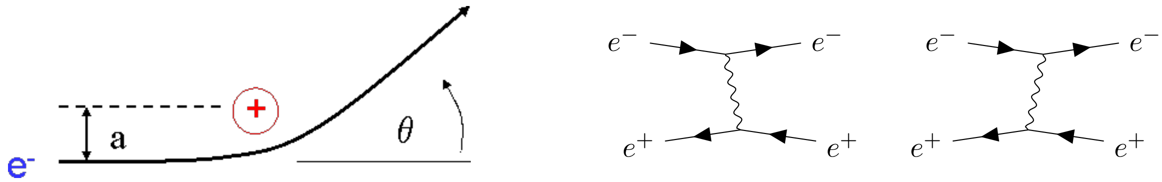
The SM describes three different forces:

1. the strong force, which is mediated by gluons and responsible for forming protons, neutrons, and more generally hadrons, composite states of multiple quarks and anti-quarks. We distinguish baryons which are composite states of three quarks and mesons which are composite states of a quark anti-quark pair;

2. the electromagnetic force, which is mediated by the photon;

3. and the weak nuclear force, which is responsible for beta decay

While in a classical theory (e.g. electromagnetism), a source charge will bend the trajectory of a test charge via the Coulomb interaction through an action at a distance (see Fig. 3a), in a quantum theory, the different particles interact by exchanging a photon (the force carrier of the electromagnetic interaction), see Fig. 3b. Either the electron or the positron in the shown figure emit a *virtual photon*



(a) Classical picture of scattering

(b) Quantum view of scattering

Figure 3: Scattering

with some momentum $\mathbf{\Delta p}$. This photon is absorbed by the other particle at a distance $\mathbf{\Delta x}$ and thus changes its momentum. The virtual photon mediates the interaction between the electron and positron and is the force carrier of electromagnetism. It is a virtual particle, because it does not respect Einstein's relation $E^2 = \mathbf{p}^2 + m^2$. As a photon is simply a quantized electromagnetic wave, and electromagnetic waves are oscillating $\mathbf{E}$ and $\mathbf{B}$ field solutions of Maxwell's equations. Electromagnetic fields are vector fields which corresponds to spin 1 in the quantum view of the photon. A similar discussion applies for the other force carriers (gluons and electroweak force).

The spin $\frac{1}{2}$ fermions constitute all matter and can be classified according to how they interact via the different forces: The quarks, the top-half in the figure, couple to gluons and interact via the strong interaction, while leptons do not couple to gluons. Each particle is accompanied by its anti-particle, which has the same mass, but opposite charges. See the discussion of the Dirac equation.

The quarks can be further separated into up-type quarks with electric charge $2/3e$ (first row) and down-type quarks with electric charge $-1/3e$ (2 second row). In nature we do not see elementary light quarks $[u, d, s, c, b]$, because they are always confined in composite states, the so-called hadrons. We distinguish between mesons, composite states of one quark and one anti-quark and baryons, composite states of three quarks. There are two different types of leptons: Charged leptons with electric charge $-1e$ (fourth row) and neutrinos which do not carry any electric charge (third row).

For each type of fermion, there are three copies, which are distinguished by their mass. These copies are commonly called generations, families or flavours. The first generation is made up of the lightest particles, the up (u) and down (d) quarks as well as the electron neutrino (ν_e) and the electron (e^-). The up and down quark form the building block for protons $p = (uud)$ and neutrons $n = (udd)$. Protons and neutrons form nuclei, which form atoms together with electrons. The fermion masses increase when going to the second and third generation.

The SM predicts neutrinos to be massless. However in 1998 Super-Kamiokande measured atmospheric neutrino oscillations and showed that neutrinos have a tiny mass. The absolute mass scale is not known, but restricted to be smaller than about $0.1 - 1$ eV. Neutrino masses are one evidence of physics beyond the SM.

Finally, the Higgs boson is a spin 0 particle, i.e. it remains the same under Lorentz transformations. It is required in the SM to give mass to all fermions [2] and gauge bosons. At high temperatures

[2]The mass mechanism for neutrinos is still unknown.

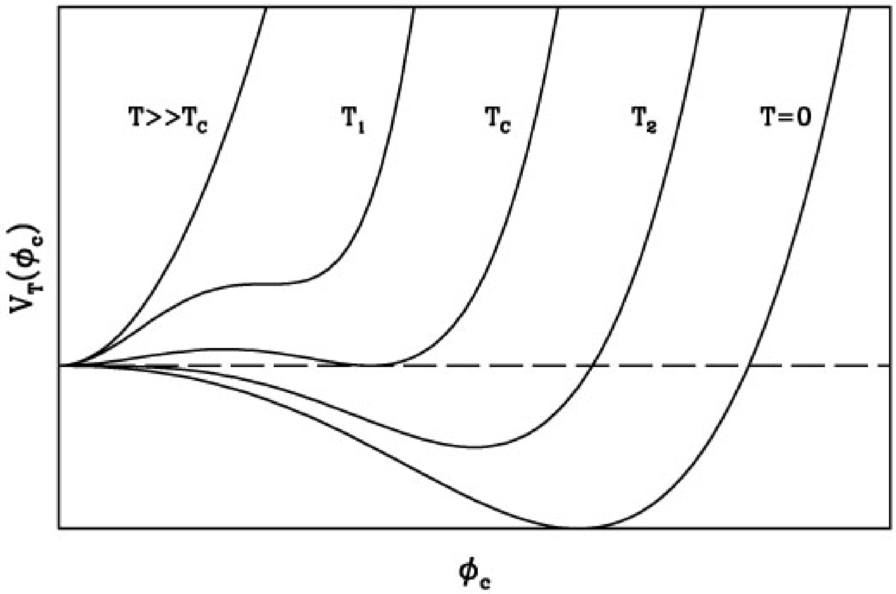


Figure 4: Electroweak phase transition and the Higgs potential $V_T(\phi_c)$ at finite temperature as a function of the Higgs field $|\phi_c|$.

and/or high energies [3] the symmetry of the SM is enhanced and the electromagnetic force and the weak nuclear force are unified in the electroweak force, i.e. they are effectively the same force. Below the critical temperature, there is a phase transition (the electroweak phase transition), this symmetry is broken and the W, Z boson and all the fermions obtain masses. See Fig. 4 for an illustration of the temperature dependence of the Higgs potential. The Higgs boson has postulated in 1964 and discovered in 2012 at the Large Hadron Collider (LHC).

1.3 Unsolved questions

There are several open questions in particle physics and in the intersection with cosmology[4]

1. What is the mass mechanism for neutrinos?
2. Why are there three generations?
3. Is there any explanation for the large hierarchy among the fermion masses?
4. What is the particle physics description of dark matter?

[3]At high energies $E \gg v = (\sqrt{2}G_F)^{-1/2} \simeq 246$ GeV (G_F is the Fermi constant), the effects of the symmetry breaking which are quantified by the order parameter v are small. In other words, the symmetry is enlarged in the limit $v \to 0$.

[4]Cosmology studies the evolution of the universe from very early times to today. According to the big bang model, as the universe was very hot and then slowly cooled down, the reactions in the early universe are described by particle physics. Thus particle physics gives insight into cosmology and particle physics is probed by cosmological measurements of early times. This leads to a close connection between the physics at the shortest distances with physics at the longest distances.

5. Why is there more matter than anti-matter? What is the mechanism of baryogenesis?

6. What is correct particle physics description of dark energy? How does it relate to the vacuum energy predicted by quantum field theory?

7. Which scalar field drives inflation?

1.4 Special relativity

The section provides a brief recap of special relativity. Special relativity is based on two postulates

1. The laws of physics are the same in all inertial reference frames.

2. The speed of light (in vacuum) is the same in all inertial reference frames.

Consider the emission and absorption of a light pulse in two different inertial reference frames. In the first inertial reference frame I, the light is emitted from a source S at position $\mathbf{x}_S$ time t_S and absorbed by a detector D at position $\mathbf{x}_D$ and time t_D. Similarly, in the second inertial reference frame the light is emitted at $(\mathbf{x}'_S, t'_S)$ and absorbed at $(\mathbf{x}'_D, t'_D)$. As the laws of physics are the same in both reference frames, the light is propagating along straight lines from the source to the detector in both frames and also the emission and absorption work in the same way. Moreover, light is propagating at the speed of light c and thus we find[5]

$$|\mathbf{x}_D - \mathbf{x}_S| = c(t_D - t_S) \qquad\qquad |\mathbf{x}'_D - \mathbf{x}'_S| = c(t'_D - t'_S)\ . \tag{1.10}$$

This motivates the definition of a distance between two points in spacetime $(t, \mathbf{x})$

$$\Delta s^2 \equiv c^2(t_D - t_S)^2 - (\mathbf{x}_D - \mathbf{x}_S)^2\ . \tag{1.11}$$

which is the same in all inertial reference frames $\Delta s^2 = \Delta s'^2$. This condition can be expressed very efficiently using 4-vector notation $x^\mu = (t^\mu, \mathbf{x}^\mu)$

$$\Delta s^2 = (x_D - x_S)^\mu (x_D - x_S)_\mu = \eta_{\mu\nu}(x_D - x_S)^\mu (x_D - x_S)^\nu \qquad (\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\ , \tag{1.12}$$

where we implicitly defined the 4-vector $x_\mu \equiv \eta_{\mu\nu} x^\nu = (t, -\mathbf{x})$ with a lower index and used the Einstein sum convention that repeated upper and lower indices are summed over. The quantity $\eta_{\mu\nu}$ is the *Minkowski metric* which can be used to lower indices. The inverse of the Minkowski metric $\eta^{\mu\nu}$ can be used to raise indices. Lorentz transformations from a frame I to a frame I'

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.13}$$

[5]Causality ensures that the time difference is positive in both cases.

leave Δs^2 invariant. For example a Lorentz boost along the x-direction is given by

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.14}$$

with $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. The upper index μ labels the rows and the lower index ν the columns of the matrix. The inverse Lorentz transformation is obtained by replacing $\beta \to -\beta$.

Any combination of 4-vectors x^μ and y^μ

$$\eta_{\mu\nu} x^\mu y^\nu \tag{1.15}$$

is invariant under Lorentz transformations and thus the same in all inertial reference frames, i.e. the metric $\eta_{\mu\nu}$ defines a scalar product. The set (more precisely vector space) of 4-vectors together with the Minkowski metric defines Minkowski space. 4-vectors with upper indices are called *contravariant* and 4-vectors with lower indices *covariant*.

4-momentum: Apart from 4-vectors in coordinate space, we can define other 4-vectors. Energy and 3-momentum can be combined in a 4-vector in special relativity

$$p^\mu = (E, \mathbf{p}) \ . \tag{1.16}$$

The scalar product $p_\mu p^\mu = E^2 - \mathbf{p}^2$ is invariant under Lorentz transformations and for a single particle it is simply given by in terms of the (rest) mass m of the particle, $p_\mu p^\mu = m^2$.

4-derivative: We can define the four-derivative

$$\frac{\partial}{\partial x^\mu} \equiv \left(\frac{\partial}{\partial t}, \nabla\right) \tag{1.17}$$

How does it transform under Lorentz transformations? Using the explicit definition of the 4-derivative we obtain

$$\sum_\mu \frac{\partial}{\partial x^\mu} x^\mu = 4 \ , \tag{1.18}$$

which is the same constant independent of the inertial reference frame and thus a Lorentz scalar. As x^μ is a contravariant 4-vector, $\partial/\partial x^\mu$ has to be a covariant vector with a lower index. We thus define

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \ . \tag{1.19}$$

Using the inverse metric, we can define the contravariant derivative

$$\partial^\mu = \eta^{\mu\nu}\partial_\nu = \left(\frac{\partial}{\partial t}, -\nabla\right) \ . \tag{1.20}$$

2 A first attempt at relativistic quantum mechanics

In quantum mechanics nature is described by the wave function $\psi(x,t)$, whose time evolution is governed by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi\ . \tag{2.1}$$

For a non-relativistic one-particle system, the Hamiltonian is generally given by

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\ . \tag{2.2}$$

In the absence of a potential $V(x)$, the Schrödinger equation can be solved in terms of plane waves

$$Ne^{i\mathbf{p}\cdot\mathbf{x}-i\omega_k t/\hbar} \qquad \text{with} \qquad \omega_k = \frac{\hbar^2\mathbf{p}^2}{2m} \tag{2.3}$$

with some normalisation constant N. ω_p is the energy of the non-relativistic particle moving with momentum $\hbar\mathbf{p}$. The general solution is given by a superposition of the different solutions

$$\psi(t,\mathbf{x}) = \int d^3p f(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}-i\omega_p t/\hbar} \tag{2.4}$$

in terms of a function f subject to the normalisation $\int|\psi|^2 d^3x = 1$. The probability density of a free particle satisfies a *continuity equation* ($\hbar = 1$)

$$\frac{\partial}{\partial t}\psi^*\psi = \psi^*\frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial t}\psi \tag{2.5}$$

$$\stackrel{(2.1)}{=} -i\psi^*\left(-\frac{\nabla^2}{2m}\psi\right) + i\left(-\frac{\nabla^2}{2m}\psi^*\right)\psi \tag{2.6}$$

$$= \frac{i}{2m}\left(\psi^*\nabla^2\psi - (\nabla^2\psi^*)\psi\right) \tag{2.7}$$

$$= \frac{i}{2m}\nabla\cdot\left(\psi^*\nabla\psi - (\nabla\psi^*)\psi\right)\ . \tag{2.8}$$

It is of the form of a continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla\cdot\mathbf{j} = 0 \tag{2.9}$$

with the probability density $\rho = \psi^*(\mathbf{x},t)\psi(\mathbf{x},t)$ and the probability current

$$\mathbf{j} = \frac{1}{2im}\left(\psi^*\nabla\psi - (\nabla\psi^*)\psi\right)\ . \tag{2.10}$$

The continuity equation relates the probability to find the particle in a volume element ρdV with the flow of the probability out of the volume element $\mathbf{j}\cdot d\mathbf{S}$ via a surface $d\mathbf{S}$. The continuity equation ensures that the probability to find the particle anywhere in spacetime remains the same and it is thus an important ingredient in quantum mechanics.

For a single wave as defined in Eq. (2.3), the probability density is simply given by $\rho = |N|^2$, which can be interpreted as representing as the number density of particles, while the probability current is

$$\mathbf{j} = \frac{1}{2im}\left(\psi^* i\mathbf{p}\psi - (-i\mathbf{p}\psi^*)\psi\right) = \frac{\mathbf{p}}{2m}2\psi^*\psi = \frac{\mathbf{p}}{m}|N|^2\ , \tag{2.11}$$

which is proportional to the non-relativistic velocity $\mathbf{p}/m$ of the particles. Thus the plane wave $\psi(\mathbf{x}, t)$ represents a region of space with number density $|N|^2$ particles per unit volume moving with average velocity $\mathbf{p}/m$.

The Schrödinger equation is inherently non-relativistic, since it treats time and position differently: while there is a second derivative with respect to x, there is only one derivative with respect to time. In order to obtain a Lorentz-invariant version of the Schrödinger equation we have to treat time and position on an equal footing, i.e. either use one derivative or a second derivative.

2.1 Klein-Gordon equation

Let us start with second derivatives.

In special relativity a particle with 4-momentum p^μ has to satisfy the relativistic dispersion relation

$$p^\mu p_\mu = E^2 - \mathbf{p}^2 = m^2 \ . \tag{2.12}$$

Following the standard practice in quantum mechanics we replace the energy and momentum by operators (setting $\hbar = 1$)

$$E \to i\partial_t \qquad\qquad \mathbf{p} \to -i\nabla \tag{2.13}$$

and postulate the wave equation for a relativistic spin-0 particle

$$m^2\phi = (i\partial_t)^2 - (-i\vec{\nabla})^2\phi = (-\partial_t^2 + \nabla^2)\phi \qquad\qquad (\partial_\mu\partial^\mu + m^2)\phi = 0 \tag{2.14}$$

which is the so-called *Klein-Gordon equation*. By construction it is explicitly Lorentz-invariant.

The Klein-Gordon equation is a second-order partial differential equation and is solved in terms of plane waves

$$N\exp(i\mathbf{k}\cdot\mathbf{x} \pm i\omega_k t) \qquad \text{with} \qquad \omega_k = (\mathbf{k}^2 + m^2)^{1/2} \ . \tag{2.15}$$

Note that there are solutions with negative energy! This is a problem, because the Hilbert space is formed by all possible solutions.

A bigger problem is related to the probability density. Similarly to the non-relativistic Schrödinger equation, we can define a continuity equation. We start by considering the following construct

$$\phi^*\frac{\partial^2\phi}{\partial t^2} - \phi\frac{\partial^2\phi^*}{\partial t^2} \stackrel{(2.14)}{=} \phi^*\left(\nabla^2\phi - m^2\phi\right) - \phi\left(\nabla^2\phi^* - m^2\phi^*\right) = \phi^*\nabla^2\phi - \phi\nabla^2\phi^* \tag{2.16}$$

This can be rewritten in form of a continuity equation with

$$\rho = -i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right) \qquad\qquad \mathbf{j} = -i\left(\phi^*\nabla\phi - \phi\nabla\phi^*\right) \ , \tag{2.17}$$

where we added $-i$ to make the quantities real. Although the function ϕ satisfies a continuity equation, we can not interpret it as a continuity equation for the probability density: For a plane wave we find that the density is given by

$$\rho = \pm\omega_k\phi^*\phi - \phi(\mp\omega_k)\phi^* = \pm 2\omega_k|N|^2 \tag{2.18}$$

and thus proportional to the energy of the plane wave. As the density can become negative, it can not be interpreted as probability density.

2.2 Dirac equation

As relativistic quantum mechanics with second derivatives has problems, let us try to use first derivatives. Dirac's strategy was to factor the energy-momentum relation into two factors linear in the 4-momentum

$$0 = p^\mu p_\mu - m^2 = (\beta^\alpha p_\alpha + m)(\gamma^\lambda p_\lambda - m) \; , \tag{2.19}$$

where β^α and γ^λ, $\alpha, \lambda = 0,1,2,3$ are eight coefficients which have to be determined. Eq. (2.19) is solved if either of the two factors vanishes, i.e.

$$\beta^\alpha p_\alpha + m = 0 \qquad \text{or} \qquad \gamma^\lambda p_\lambda - m = 0 \; . \tag{2.20}$$

We can use either of the two linear equations to define a relativistic equivalent of the Schödinger equation after the canonical substitution $p_\mu \to i\partial_\mu$. For example the second possibility with $-m$ in Eq. (2.20) leads to

$$i\gamma^\mu \partial_\mu \psi = m\psi \tag{2.21}$$

$$\Rightarrow i\gamma^0 \partial_t \psi = \left[-i\gamma^i \frac{\partial}{\partial x^i} + m \right] \psi \tag{2.22}$$

$$\Rightarrow i\partial_t \psi = \left[-i(\gamma^0)^{-1}\gamma^i \frac{\partial}{\partial x^i} + m(\gamma^0)^{-1} \right] \psi \equiv \hat{H}\psi \; , \tag{2.23}$$

where we used the definition of ∂_μ in the second line and defined the Hamiltonian for the Dirac equation in the third line. So far we did not yet determine the coefficients β^α and γ^λ. What are those coefficients? In order to answer this question we multiply out the right-hand side of Eq. (2.19)

$$(\beta^\alpha p_\alpha + m)(\gamma^\lambda p_\lambda - m) = \beta^\alpha p_\alpha \gamma^\lambda p_\lambda - m(\beta^\lambda - \gamma^\lambda)p_\lambda - m^2 \; . \tag{2.24}$$

A comparison with the left-hand side of Eq. (2.19) shows that $\beta^\lambda = \gamma^\lambda$ and thus

$$p^\mu p_\mu = \gamma^\alpha p_\alpha \gamma^\lambda p_\lambda \; . \tag{2.25}$$

Writing out Eq. (2.25) in components we find

$$(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = (\gamma^0 p_0 - \gamma^1 p_1 - \gamma^2 p_2 - \gamma^3 p_3)^2 \tag{2.26}$$

$$\begin{aligned} &= (\gamma^0)^2(p^0)^2 + (\gamma^1)^2(p^1)^2 + (\gamma^2)^2(p^2)^2 + (\gamma^3)^2(p^3)^2 \\ &+ (\gamma^0\gamma^1 + \gamma^1\gamma^0)p^0p^1 + (\gamma^0\gamma^2 + \gamma^2\gamma^0)p^0p^2 + (\gamma^0\gamma^3 + \gamma^3\gamma^0)p^0p^3 \\ &+ (\gamma^1\gamma^2 + \gamma^2\gamma^1)p^1p^2 + (\gamma^2\gamma^3 + \gamma^3\gamma^2)p^2p^3 + (\gamma^1\gamma^3 + \gamma^3\gamma^1)p^1p^3 \end{aligned} \tag{2.27}$$

Comparing the left-hand side with the right-hand side, we find the following relations for the coefficients γ^λ for $i = 1,2,3$

$$(\gamma^0)^2 = 1 \qquad\qquad (\gamma^i)^2 = -1 \qquad\qquad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \tag{2.28}$$

The last condition can not be satisfied by ordinary commuting numbers (c-numbers), unless at least three coefficients vanish which is in contradiction to the other two conditions. The simplest way out is to assume that the coefficients are matrices. The conditions in Eq. (2.28) can be rewritten in a compact form

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} , \tag{2.29}$$

where $\eta^{\mu\nu}$ is the (inverse) Minkowski metric and the curly braces are the *anticommutator*,

$$\{A, B\} \equiv AB + BA \tag{2.30}$$

In addition to the defining relation (2.29) for the γ matrices, we have to require that γ^0 and $\gamma^0\gamma^i$ are hermitian matrices[6], in order to have a hermitian Hamiltonian. Thus the matrices must be square matrices. This implies the relation

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{2.31}$$

which is obvious for $\mu = 0$ and it follows from the hermiticity of $\gamma^0\gamma^i$ for $\mu = i$

$$\gamma^0\gamma^i = (\gamma^0\gamma^i)^\dagger = \gamma^{i\dagger}\gamma^{0\dagger} = \gamma^{i\dagger}\gamma^0 \Rightarrow (\gamma^i)^\dagger = \gamma^0\gamma^i\gamma^0 . \tag{2.32}$$

It turns out that the smallest matrices, which satisfy the above conditions are 4×4 matrices. The set of matrices is not unique. One specific representation of the γ matrices is given by the *Dirac-Pauli representation*

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{2.33}$$

where 1 denotes a 2×2 identity matrix, 0 a 2×2 matrix of zeros and σ^i, $i = 1, 2, 3$ are the *Pauli matrices*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.34}$$

Another common representation for the γ matrices is the *Weyl* or *chiral representation*

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \tag{2.35}$$

where σ^0 is the 2×2 identity matrix, σ^i the Pauli matrices, and $\bar\sigma^0 = \sigma^0$ and $\bar\sigma^i = -\sigma^i$. In the following we will use the Pauli-Dirac representation, unless explicitly specified.

Summarising the above, we find that

$$p^\mu p_\mu - m^2 = (\gamma^\alpha p_\alpha + m) \left(\gamma^\lambda p_\lambda - m\right) = 0 . \tag{2.36}$$

[6] Note that $(\gamma^0)^{-1} = \gamma^0$ from Eq. (2.28).

Either factor can be used as relativistic wave equation. Conventionally the second factor (with $-m$) is picked. After the substitution $p_\mu \to i\partial_\mu$ we obtain the *Dirac equation*

$$(i\partial\!\!\!/ - m)\,\psi = (i\gamma^\mu\partial_\mu - m)\,\psi = 0\;, \tag{2.37}$$

where we introduced the commonly-used Feynman-slash notation $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$. Note that the wave function ψ is not a number, but has four components

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}\;. \tag{2.38}$$

It is called a *Dirac spinor* and should *not* be confused with a Lorentz 4−vector, since it does not transform as a 4−vector under Lorentz transformations, but as a spinor. The individual components are c-numbers. You find details in any book on relativistic quantum mechanics (e.g. Schwabl: "Advanced Quantum Mechanics"). Furthermore any solution to the Dirac equation is also a solution to the Klein-Gordon equation (2.14). If we multiply the Dirac equation with $(i\gamma^\nu\partial_\nu + m)$ from the left we obtain

$$\begin{aligned}
0 &= (i\gamma^\nu\partial_\nu + m)(i\gamma^\mu\partial_\mu - m)\psi &(2.39)\\
&= \left(-\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + m(i\gamma^\mu\partial_\mu - i\gamma^\nu\partial_\nu) - m^2\right)\psi &(2.40)\\
&= \left(-\frac{1}{2}\left\{\gamma^\nu,\gamma^\mu\right\}\partial_\nu\partial_\mu - m^2\right)\psi &(2.41)\\
&\overset{(2.29)}{=} \left(-\partial^\mu\partial_\mu - m^2\right)\psi &(2.42)\\
&= -\left(\partial^\mu\partial_\mu + m^2\right)\psi\;. &(2.43)
\end{aligned}$$

Similarly to the Klein-Gordon equation, we can define a continuity equation. For this we need the Dirac equation for the (complex) conjugate Dirac spinor

$$0 = (i\gamma^\mu\partial_\mu\psi - m\psi)^\dagger = -i\partial_\mu\psi^\dagger\gamma^{\mu\dagger} - m\psi^\dagger \overset{(2.31)}{=} -i\partial_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^0 - m\psi^\dagger\;. \tag{2.44}$$

This motivates the definition of the *adjoint spinor*

$$\bar\psi = \psi^\dagger\gamma^0 \tag{2.45}$$

which satisfies the equation

$$0 = -i\partial_\mu\bar\psi\gamma^\mu - m\bar\psi \tag{2.46}$$

Using the above equation and the Dirac equation we find

$$\partial_\mu\left(\bar\psi\gamma^\mu\psi\right) = (\partial_\mu\bar\psi)\gamma^\mu\psi + \bar\psi\gamma^\mu\partial_\mu\psi \underset{(2.37)}{\overset{(2.46)}{=}} im\bar\psi\psi - i\bar\psi m\psi = 0 \tag{2.47}$$

and thus we find the density $\rho \equiv \bar{\psi}\gamma^0\psi$ and the current $\mathbf{j} = \bar{\psi}\gamma^i\psi$ of the continuity equation. The density ρ is positive definite

$$\rho = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi = \sum_i |\psi_i|^2 \geq 0 \tag{2.48}$$

and thus can be interpreted as a probability density in contrast to the Klein-Gordon equation. It can be shown that $j^\mu = \bar{\psi}\gamma^\mu\psi$ transforms as a 4-vector under Lorentz transformations.

2.2.1 Solutions of the Dirac equation

We expect that the solutions of the Dirac equation are also given by plane waves, i.e. they are of the form

$$\psi(x^\mu) = u(p^\mu)e^{-ip_\mu x^\mu} \tag{2.49}$$

where $u(p^\mu)$ is a time-independent 4-component spinor. Substitution into the Dirac equation (2.37) shows

$$0 = (i\gamma^\mu\partial_\mu - m)\psi = (\gamma^0 p_0 - \sum_i \gamma^i p_i - m)u(p^\mu)e^{-ip_\mu x^\mu} \ . \tag{2.50}$$

Thus Dirac equation simplifies to an algebraic problem and the solutions to the Dirac equation have to satisfy

$$0 = (\gamma^\mu p_\mu - m)u \ . \tag{2.51}$$

We first solve it for a particle at rest with $\mathbf{p} = 0$ and energy $E = \pm m$. In this case Eq. (2.51) simplifies to

$$0 = (\gamma^0 E - m)u = \begin{pmatrix} E-m & & & \\ & E-m & & \\ & & -E-m & \\ & & & -E-m \end{pmatrix} u \tag{2.52}$$

which has four independent solutions

$$u_1(E=m, \mathbf{p}=0) = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_2(E=m, \mathbf{p}=0) = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \tag{2.53}$$

$$u_3(E=-m, \mathbf{p}=0) = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad u_4(E=-m, \mathbf{p}=0) = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{2.54}$$

with a normalisation constant N, which we will determine later. The first two ($u_{1,2}$) are for positive energies $E = +m$ and the last two ($u_{3,4}$) for negative energies $E = -m$. The corresponding wave functions for a particle at rest are given by

$$\psi_1 = u_1 e^{-imt} \qquad \psi_2 = u_2 e^{-imt} \qquad \psi_3 = u_3 e^{+imt} \qquad \psi_4 = u_4 e^{+imt} \tag{2.55}$$

For general $\mathbf{p} \neq 0$, the equation for u becomes

$$\left[\gamma^0 E - \begin{pmatrix} 0 & \sigma \cdot \mathbf{p} \\ -\sigma \cdot \mathbf{p} & 0 \end{pmatrix} - m\right] u = 0 \tag{2.56}$$

The solutions to this equation are given by

$$u_1(E > 0) = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \qquad u_2(E > 0) = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \tag{2.57}$$

$$u_3(E < 0) = N \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \qquad u_4(E < 0) = N \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix} \tag{2.58}$$

Like for the Klein-Gordon equation, there are negative energy solutions which we have to be interpreted.

2.2.2 Antiparticles

The presence of negative energy solutions seems unavoidable and can not be simply discarded. If negative energy solutions represent negative energy particle states, positive energy particle states should fall into the lower negative energy particle states. Dirac proposed that the vacuum corresponds to the case, where all negative energy eigenstates are filled, the so-called *Dirac sea*, and thus due to the Pauli exclusion principle the positive energy particle states won't fall into the occupied negative energy states.

The modern interpretation is due to Feynman and Stückelberg. The $E < 0$ states are interpreted as negative energy particle which propagate backwards in time. These negative energy particles correspond to physical positive energy antiparticle states with opposite charge, which propagate forwards in time. As the time dependence of the wave function e^{-iEt} is unchanged under the simultaneous transformation $(E, t) \to (-E, -t)$, the two pictures are mathematically equivalent.

As u_3 and u_4 are interpreted as travelling backwards in time, the momentum in the spinor is the negative of the physical momentum. It is easier to work with antiparticle spinors which we define by reversing the signs of $(E, \mathbf{p})$

$$v_1(E, \mathbf{p}) e^{-i(\mathbf{p}\cdot\mathbf{x} - Et)} = u_4(-E, -\mathbf{p}) e^{i(-\mathbf{p}\cdot\mathbf{x} - (-E)t)} \tag{2.59}$$

$$v_2(E, \mathbf{p}) e^{-i(\mathbf{p}\cdot\mathbf{x} - Et)} = u_3(-E, -\mathbf{p}) e^{i(-\mathbf{p}\cdot\mathbf{x} - (-E)t)} \tag{2.60}$$

and thus

$$v_1 = N \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad\qquad v_2 = N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \tag{2.61}$$

and the wave functions are given by

$$\psi_i = v_i e^{-i(\mathbf{p}\cdot\mathbf{x} - Et)} \ . \tag{2.62}$$

If we substitute it into the Dirac equation we obtain

$$(\gamma^\mu p_\mu + m)v = 0 \tag{2.63}$$

We did not fix the normalisation yet. There are different conventions. A common one is to normalise the density to

$$\rho = \psi^\dagger \psi = 2E \ . \tag{2.64}$$

For example for the ψ_1 solution we obtain explicitly

$$\psi_1^\dagger \psi_1 = u_1^\dagger u_1 = |N|^2 \left(1 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2 + p_y^2}{(E+m)^2} \right) = |N|^2 \frac{2E}{E+m} \qquad \Rightarrow \qquad N = \sqrt{E+m} \tag{2.65}$$

The same normalisation is obtained for all u and v spinors.

The full solution is a linear combination of the 4 linearly-independent ψ_i, summed over waves of all possible momenta

$$\psi(x^\mu) = \sum_{i=1,2} \int \frac{d^3p}{(2\pi)^3} \left[A_i(\mathbf{p}) u_i(E, \mathbf{p}) e^{-ip_\mu x^\mu} + B_i(\mathbf{p}) v_i(E, \mathbf{p}) e^{ip_\mu x^\mu} \right] \ . \tag{2.66}$$

where A_i and B_i are c-numbers determined by the initial conditions.

2.2.3 Spin, helicity, and chirality

For a particle at rest, particles are described by the top two components of the Dirac spinor and antiparticles by the lower two components of the Dirac spinor. In order to interpret the different components of the spinors, we have to introduce one more concept: the spin operator

$$\hat{S} = \frac{1}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \tag{2.67}$$

with the Pauli spin matrix σ_3. The spinors u_1 and u_2 are eigenstates of the spin operator with eigenvalues $+1/2$ and $-1/2$ respectively. Thus at rest, u_1 describes a particle with spin up and u_2 describes a particle with spin down. We have to be more careful, when considering the antiparticle spinors. Applying the Hamiltonian and momentum operator to $\psi = v(E,\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x}-Et)}$ does *not* give the physical momentum

$$\hat{H}\psi = i\frac{\partial\psi}{\partial t} = -E\psi \qquad\qquad \hat{\mathbf{p}}\psi = -\mathbf{p}\psi \tag{2.68}$$

The physical energy and momentum are obtained via the operators

$$\hat{H}^{(v)} = -i\frac{\partial}{\partial t} \qquad\qquad \hat{\mathbf{p}}^{(v)} = i\nabla \tag{2.69}$$

Also the operator for the orbital angular momentum has to be adjusted $\mathbf{L} = \mathbf{r}\times\mathbf{p} \to \mathbf{L}^{(v)} = \mathbf{r}\times\mathbf{p}^{(v)}$. Finally, in order for the commutator

$$[\hat{H}, \hat{L} + \hat{S}] = 0 \tag{2.70}$$

to vanish, i.e. the sum of orbital angular momentum and spin is conserved, also the spin operator has to change sign for antiparticles

$$\hat{S}^{(v)} = -\hat{S}\ . \tag{2.71}$$

Thus at rest the spinor v_1 describes an antiparticle with spin up and the spinor v_2 describes an antiparticle with spin down.

The spin operator $\hat{S}$ commutes with the Hamiltonian $\hat{H}$ with $\mathbf{p} = 0$, but not for $\mathbf{p} \neq 0$ and thus is not a good quantum number for $\mathbf{p} \neq 0$. However we can define an operator which commutes for $\mathbf{p} \neq 0$, the *helicity* operator

$$h \equiv \frac{1}{2p}\begin{pmatrix} \sigma\cdot\mathbf{p} & 0 \\ 0 & \sigma\cdot\mathbf{p}\end{pmatrix}\ . \tag{2.72}$$

It is the projection of spin onto the momentum. For $\mathbf{p} = (0, 0, p_z)$, it exactly agrees with the spin operator for a particle at rest. It can be shown that it commutes with the Hamiltonian $[\hat{H}, h] = 0$ and thus is a conserved quantity. For $p_{x,y} = 0$, the u_1 (u_2) spinor describes a particle with positive (negative) helicity and v_1 (v_2) describes an antiparticle with negative (positive) helicity. Similarly to spin, we have to change the sign for antiparticles. A general helicity eigenstate can be constructed by searching for simultaneous eigenstates of $\hat{H}$ and h. See Thompson for details.

Helicity commutes with the Hamiltonian and is conserved, but it is not invariant under Lorentz transformations for massive particles, because the direction (sign) of the 4-momentum $\mathbf{p}$ is not Lorentz invariant.

Another commonly used concept is *chirality* which is Lorentz invariant, but not conserved: We can form another γ-matrix out of the 4 γ^μ, the so-called γ_5 matrix:

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{2.73}$$

In the Pauli-Dirac representation it is given by

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\ . \tag{2.74}$$

The γ_5 matrix satisfies the following properties

$$(\gamma_5)^2 = 1 \qquad \gamma_5^\dagger = \gamma_5 \qquad \{\gamma_5, \gamma^\mu\} = 0 \tag{2.75}$$

which can be demonstrated by a straightforward calculation. In the ultrarelativistic limit $E \gg m$, the helicity eigenstates are also eigenstates of the γ_5 matrix (chirality operator), in particular for $p_{x,y} = 0$ they also agree with the spin eigenstates and thus

$$\gamma_5 u_1 = u_1 \qquad \gamma_5 u_2 = -u_2 \tag{2.76}$$

$$\gamma_5 v_1 = -v_1 \qquad \gamma_5 v_2 = v_2 \; . \tag{2.77}$$

However in general the spin, helicity and chirality do not coincide. We can define projection operators onto the right-chiral $P_R = (1+\gamma_5)/2$ and left-chiral $P_L = (1-\gamma_5)/2$ particle states. They satisfy the usual properties of a projection operator $P_{R,L}^2 = P_{R,L}$ and divide the space in two chiralities, i.e. $P_L + P_R = 1$ and $P_L P_R = 0$. In particular we find for the scalar $\bar\psi\psi$ and vector bilinears $\bar\psi\gamma^\mu\psi$

$$\bar\psi\psi = \bar\psi(P_L + P_R)\psi = \bar\psi(P_L^2 + P_R^2)\psi = (P_R\psi)^\dagger\gamma^0 P_L\psi + (P_L\psi)^\dagger\gamma^0 P_R\psi \tag{2.78}$$

$$\bar\psi\gamma^\mu\psi = \bar\psi\gamma^\mu(P_L + P_R)\psi = \bar\psi\gamma^\mu(P_L^2 + P_R^2)\psi = (P_L\psi)^\dagger\gamma^0\gamma^\mu P_L\psi + (P_R\psi)^\dagger\gamma^0\gamma^\mu P_R\psi \tag{2.79}$$

The mass term connects the two chiralities and hence for massless particles the Dirac equation splits into one equation for left-chiral particles and one for right-chiral particles. The chiral or Weyl representation is specially suited to discuss *chiral theories* like the Standard Model of particle physics, where left-chiral and right-chiral particles have different quantum numbers. In the Weyl representation γ_5 is given by

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \; . \tag{2.80}$$

3 The Standard Model Lagrangian in a nutshell

3.1 Classical field theory

The Klein-Gordon and Dirac equations can also be derived from the stationary action principle. For definiteness we will consider the Klein-Gordon equation in the following which can be derived from the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{3.1}$$

and thus the action $S = \int dt \int d^3x\mathcal{L}$. Consider a variation of the action with respect to the field ϕ and the coordinates x^μ

$$x'^\mu = x^\mu + \delta x^\mu \qquad\qquad \phi'(x) = \phi(x) + \delta\phi(x)\ . \tag{3.2}$$

In the absence of an explicit x-dependence, the variation of the action yields

$$\begin{aligned}
\delta S &= \int_R d^4x \left[\mathcal{L}(\phi', \partial_\mu\phi') - \mathcal{L}(\phi, \partial_\mu\phi)\right] && (3.3)\\
&= \int_R d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right) && (3.4)\\
&= \int_R d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \int_R d^4x\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi && (3.5)\\
&= \int_R d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \int_{\partial R} d\sigma_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi && (3.6)
\end{aligned}$$

Using Gauss theorem the second term can be rewritten as a surface integral. If we demand that there is no variation, $\delta x^\mu = 0$ and $\delta\phi = 0$, on the boundary ∂R, the second term vanishes and we obtain the Euler-Lagrange equations

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \tag{3.7}$$

For a real scalar field the Euler-Lagrange equation is

$$0 = \frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} = -\partial_\mu\partial^\mu\phi - m^2\phi \tag{3.8}$$

which is exactly the Klein-Gordon equation. The conjugate momentum to the field ϕ is

$$\pi = \frac{\partial\mathcal{L}}{\partial\partial_0\phi} \tag{3.9}$$

and the Hamiltonian density is obtained by a Legendre transformation

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}\ . \tag{3.10}$$

3.2 Symmetries

Similarly to the classical mechanics of point particles, Noether's theorem establishes a connection between continuous symmetries and conserved quantities. We will only consider transformations of the fields ϕ_i and not spacetime transformations. One example of such a transformation is $\phi \to e^{-i\epsilon}\phi$ or more generally for $i = 1, \ldots, N$ fields

$$\phi_i(x) \to \phi_i'(x) = \phi_i(x) + \delta\phi_i(x) = \phi_i(x) - i\epsilon_a F_i^a[\phi_j(x)] \ . \tag{3.11}$$

ϵ_a are real (x-independent) parameters, $F_i^a = \frac{\partial \delta\phi_i}{\partial \epsilon_a}|_{\epsilon_a=0}$ are functions of the fields and we neglected terms of order ϵ^2 and higher. From Eq. (3.6) we find that the symmetry of the action, implies that the second term vanishes for an arbitrary surface ∂R and thus the integrand has to vanish itself

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi = \epsilon_a \partial_\mu J^\mu \ , \tag{3.12}$$

where we defined the 4-current density

$$J^{\mu,a} = -i\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} F_i^a \ . \tag{3.13}$$

The divergence of the 4-current density is

$$\partial_\mu J^{\mu,a} = -i\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\right) F_i^a - i\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\partial_\mu F_i^a = -i\left(\frac{\partial \mathcal{L}}{\partial \phi_i}\right) F_i^a - i\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}\partial_\mu F_i^a \ , \tag{3.14}$$

where we used the product rule in the first step and the equations of motion in the second step. Thus we find a version of Noether's theorem: *For the global (spacetime-independent) symmetry $\phi_i \to \phi_i' = \phi_i + \delta\phi_i$, the 4-current $J^{\mu,a}$ is conserved, $\partial_\mu J^{\mu,a} = 0$. For a conserved current the charge*

$$Q^a(t) = \int d^3x J_0^a(x) \tag{3.15}$$

is time independent, i.e. a constant of motion.

3.3 Dirac Lagrangian

Let us consider one example. The Dirac equation is the equation of motion of a field theory. The Lagrangian corresponding to the Dirac equation is given by

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \ , \tag{3.16}$$

where ψ_i and $\bar{\psi}_i$ can be considered as independent fields (Complex fields have two independent real components.). The equation of motion for $\bar{\psi}$ leads to the Dirac equation

$$0 = \frac{\partial \mathcal{L}}{\partial \bar{\psi}_i} = \left[(i\partial\!\!\!/ - m)\,\psi\right]_i \tag{3.17}$$

Exercise:

- Show that the Lagrangian (3.16) is invariant under a global (x independent) symmetry $\psi \to e^{-i\alpha}\psi$.
- Show that the associated conserved current is

$$j^\mu = \bar{\psi}\gamma^\mu\psi \; . \tag{3.18}$$

- The conserved charge is

$$Q = \int d^3x \psi^\dagger \psi \; . \tag{3.19}$$

This motivates the ad-hoc definition of the continuity equation. What do we have to change in order to obtain a Lagrangian which is invariant under local (x-dependent) transformations, $\psi \to e^{-i\alpha(x)}\psi$? A local transformation leads to the following variation of the Lagrangian

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = \bar{\psi}e^{i\alpha(x)}(i\partial\!\!\!/ - m)e^{-i\alpha(x)}\psi - \bar{\psi}(i\partial\!\!\!/ - m)\psi = j^\mu \partial_\mu \alpha \tag{3.20}$$

The kinetic term (the term with the derivative) led to the non-trivial variation of the Lagrangian. We can make the Lagrangian invariant under a local (or gauge) transformation by replacing the partial derivative by a covariant derivative $D_\mu = \partial_\mu - ieA_\mu$ where A_μ is a field which transforms as a Lorentz 4-vector. The covariant derivative should satisfy

$$D_\mu\psi \to e^{-i\alpha} D_\mu \psi \; . \tag{3.21}$$

How does A_μ transform if $\psi \to e^{-i\alpha}\psi$? Consider a general transformation $A_\mu \to A'_\mu$

$$D_\mu\psi \to (\partial_\mu - ieA'_\mu)e^{-i\alpha}\psi = e^{-i\alpha\psi}(\partial_\mu - ieA'_\mu - i\partial_\mu\alpha)\psi \tag{3.22}$$

Hence A_μ transforms as follows

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \alpha \; . \tag{3.23}$$

which is exactly the gauge transformtion in electrodynamics. The vector field A_μ is a gauge field. It could for example describe the electromagnetic field. In electromagnetism the time component corresponds to the charge density $A^0 = \phi$ and the spatial components form the vector potential $(A^i) = \mathbf{A}$. The corresponding field strength tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \; . \tag{3.24}$$

For electrodynamics it is given in terms of the electric $\mathbf{E}$ and magnetic fields $\mathbf{B}$

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \; . \tag{3.25}$$

3.4 A first look at the Standard Model Lagrangian

We are now in the position to understand the basic properties of the Standard Model Lagrangian. Without going into details, the relevant parts of the Lagrangian are given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R + |D_\mu \phi|^2 - V(\phi) + \left(y \bar{\psi}_L \psi_R \phi + \text{h.c.} \right) \ . \tag{3.26}$$

where $F_{\mu\nu}$ stands for the field strength tensors, D denotes the covariant derivatives and $\psi_L \equiv P_L \psi$ ($\psi_R \equiv P_R \psi$) denotes a left-chiral (right-chiral) fermion, ϕ describes the Higgs field, a spin-0 scalar field. Time-permitting, we will discuss the underlying symmetry in more detail in the last lecture.

4 A taste of quantum field theory

4.1 Quantum harmonic oscillator

Before moving to quantizing a scalar field let us review the quantization of the harmonic oscillator. The quantum harmonic oscillator is defined by the Lagrangian

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2 \tag{4.1}$$

We can directly determine the conjugate momentum, Hamiltonian and Poisson brackets using standard techniques

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{4.2}$$

$$H = p\dot{x} - L = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2 \tag{4.3}$$

$$\{x, p\} = 1 \qquad \{p, p\} = \{x, x\} = 0 \tag{4.4}$$

We obtain the quantum harmonic oscillator by replacing x and p by operators $\hat{x}$ and $\hat{p} = -i\frac{d}{dx}$ and the Poisson bracket by the commutator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{x}^2 \tag{4.5}$$

$$[\hat{x}, \hat{p}] = i \qquad [\hat{p}, \hat{p}] = [\hat{x}, \hat{x}] = 0\ . \tag{4.6}$$

In order to solve the quantum harmonic oscillator we want to factorise the Hamiltonian in analogy to the identity

$$u^2 + v^2 = (u - iv)(u + iv)\ . \tag{4.7}$$

Hence we form two new operators as linear combination of the old ones

$$a = \sqrt{\frac{m\omega}{2}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right) \qquad\qquad a^\dagger = \sqrt{\frac{m\omega}{2}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)\ . \tag{4.8}$$

The operators satisfy the following commutation relation

$$[a, a^\dagger] = \left[\sqrt{\frac{m\omega}{2}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right), \sqrt{\frac{m\omega}{2}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)\right] = \frac{i}{2}\left([\hat{x}, -\hat{p}] + [\hat{p}, \hat{x}]\right) = 1 \tag{4.9}$$

The Hamiltonian can be rewritten as

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right)\ . \tag{4.10}$$

Exercise: Show by substituting the definitions of a and $a^\dagger$ that it is indeed the Hamiltonian.

Before interpreting the operators $\hat{a}$ and $\hat{a}^\dagger$ physically, we have to show another commutation relation

$$[H, a] = \left[\omega\left(a^\dagger a + \frac{1}{2}\right), a\right] = \omega[a^\dagger a, a] = \omega a^\dagger[a, a] + \omega[a^\dagger, a]a = -\omega a \tag{4.11}$$

or more explicitly

$$
\begin{aligned}
[H,a] &= Ha - aH && (4.12)\\
&= \omega\left(a^\dagger a + \frac{1}{2}\right)a - a\hbar\omega\left(a^\dagger a + \frac{1}{2}\right) && (4.13)\\
&= \omega\left(a^\dagger a a - a a^\dagger a\right) && (4.14)\\
&= \omega\left(a^\dagger a a - \left(a^\dagger a + [a,a^\dagger]\right)a\right) && (4.15)\\
&= -\omega a\ . && (4.16)
\end{aligned}
$$

Now given an energy eigenstate $|E\rangle$ with a given energy E, we can calculate the energy eigenvalue of the states $a\,|E\rangle$ as follows

$$
\begin{aligned}
H(a\,|E\rangle) &= Ha\,|E\rangle && (4.17)\\
&= (aH + [H,a])\,|E\rangle && (4.18)\\
&= (aE - \omega a)\,|E\rangle && (4.19)\\
&= (E-\omega)(a\,|E\rangle) && (4.20)
\end{aligned}
$$

Similarly for the operator $a^\dagger$

$$
[H,a^\dagger] = +\omega a^\dagger \qquad\qquad H(a^\dagger\,|E\rangle) = (E+\omega)(a^\dagger\,|E\rangle)\ . \tag{4.21}
$$

Hence the states $a\,|E\rangle$, $a^\dagger\,|E\rangle$ are also energy eigenstates with energies $E\pm\omega$, respectively. The operators a and $a^\dagger$ transform a state with energy E into a state with energy $E\pm\omega$. They are denoted *ladder operators*, more specifically $a^\dagger$ is denoted *raising operator* and a *lowering operator*.

Next we have to find the lowest energy eigenstate or ground state $|0\rangle$. Classically we observe that there is a minimum energy of the harmonic oscillator. Hence there has to be a lowest energy eigenstate

$$
a\,|0\rangle = 0\ . \tag{4.22}
$$

This is called the *ladder termination condition*. The energy of this lowest energy eigenstate is given by

$$
H\,|0\rangle = \omega\left(a^\dagger a + \frac{1}{2}\right)|0\rangle = \frac{1}{2}\omega\,|0\rangle\ . \tag{4.23}
$$

Note that lowest energy level is not zero as it would be for a classical harmonic oscillator, but $\frac{1}{2}\omega$. It is known as *zero point energy* and ultimately due to the non-vanishing commutator of the ladder operators $[a,a^\dagger]=1$. The energy of the n^{th} state $|n\rangle$ is given by

$$
E_n = \omega\left(n + \frac{1}{2}\right)\ , \tag{4.24}
$$

because applying the raising operator $a^\dagger$ n times increases the energy with respect to the lowest energy eigenstate by $n\times\omega$. In addition to the ladder operators it is convenient to introduce the *number operator*,

$$
\hat{N} = \hat{a}^\dagger\hat{a}\ , \tag{4.25}
$$

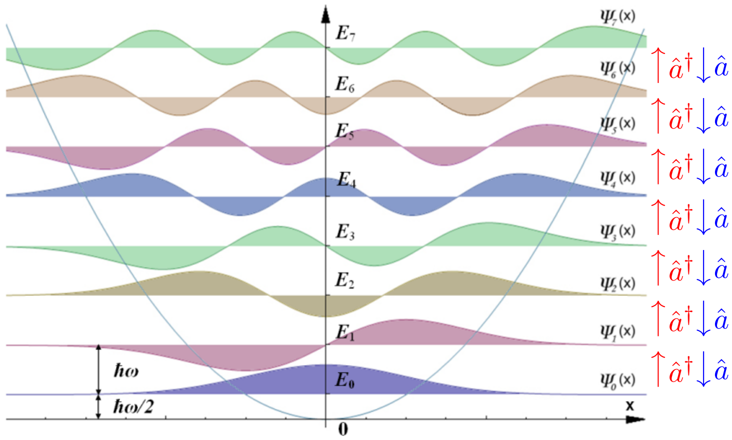


Figure 5: Energy levels of harmonic oscillator. Raising operator $a^\dagger$ increases energy by ω and lowering operator a lowers it. Figure taken from Wikipedia.

which counts the energy quanta. It fulfils the following eigenvalue equation

$$\hat{N}\,|n\rangle = n\,|n\rangle \ , \tag{4.26}$$

where the n in $|n\rangle$ denotes the number of energy quanta. We can rewrite the Hamiltonian as

$$\hat{H} = \omega\left(\hat{N} + \frac{1}{2}\right) \ . \tag{4.27}$$

See Fig. 5 for an illustration of the action of the ladder operator on the energy eigenstates. All other energy eigenstates can be constructed from the lowest energy eigenstate using the raising operator. By demanding that all states $|n\rangle$ are properly normalised,

$$\langle n|n\rangle = 1 \ , \tag{4.28}$$

it is possible to show[7] that the raising and lowering operators act on a state $|n\rangle$

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad\qquad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \ . \tag{4.29}$$

Thus we can write the state $|n\rangle$ as follows

$$|n\rangle \equiv \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n |0\rangle \ , \tag{4.30}$$

where we denoted the lowest energy eigenstate by $|0\rangle$. The factor $1/\sqrt{n!}$ ensures that the states are correctly normalised.

The wave function of the lowest energy eigenstate $\phi_0(\xi)$ can be determined from the ladder termination condition in Eq. (4.22)

$$0 = a\phi_0(x) = \sqrt{\frac{m\omega}{2}}\left(x + \frac{1}{m\omega}\frac{d}{dx}\right)\phi_0(x) \ . \tag{4.31}$$

[7]See the discussion in McIntyre Chap.9.

It is an ODE, which can be solved using standard techniques

$$\phi_0(x) = \left(\frac{m\omega}{\pi}\right)^{1/4} e^{-m\omega x^2/2} \,. \tag{4.32}$$

4.2 Heisenberg picture

In quantum mechanics we typically consider time-independent operators like the Hamiltonian and time-dependent quantum states (wave functions). This is commonly denoted as *Schrödinger picture.* If we however consider the states[8]

$$|\psi\rangle_H \equiv e^{i\hat{H}t} |\psi(t)\rangle = \left(1 + i\hat{H}t + \frac{1}{2}\left[i\hat{H}t\right]^2\right)|\psi(t)\rangle \tag{4.33}$$

we find that

$$i\frac{\partial}{\partial t}|\psi\rangle_H = i\frac{\partial}{\partial t}e^{i\hat{H}t}|\psi(t)\rangle = e^{i\hat{H}t}\left(-\hat{H}|\psi(t)\rangle + i\frac{\partial}{\partial t}|\psi(t)\rangle\right) = 0 \tag{4.34}$$

This is commonly denoted as *Heisenberg picture.* In this picture, all operators

$$\hat{A}_H(t) = e^{i\hat{H}t}\hat{A}e^{-i\hat{H}t} \,, \tag{4.35}$$

e.g. momentum operator, will be time-dependent and satisfy

$$\frac{d}{dt}\hat{A}_H(t) = i[\hat{H}_H(t), \hat{A}_H(t)] + \left(\frac{\partial \hat{A}}{\partial t}\right)_H \,. \tag{4.36}$$

Thus the eigenvalues of any operator, which commutes with the Hamiltonian, are contants and provide good quantum numbers to describe the system. In quantum field theory we typically work in the Heisenberg picture, where all of the time-dependence is in the operators.

4.3 Free real scalar field

As we previously discussed the Klein-Gordon equation is solved by plane waves. Thus the general solution is given by their superposition. In order to avoid difficulties with infinities, we will consider the Klein-Gordon equation in a box, a finite volume V, and impose periodic boundary conditions $\phi(0, y, z, t) = \phi(L, y, z, t)$, etc., where L is the size of the box in x-direction. The wave vectors are of the form $\mathbf{k} = \frac{2\pi}{L}\mathbf{n}$ with $n_i = 0, \pm 1, \ldots$. In this case on

$$\phi(x^\mu) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_k}}\left(a(\mathbf{k})e^{-ik_\mu x^\mu} + a^*(\mathbf{k})e^{+ik_\mu x^\mu}\right) \tag{4.37}$$

The coefficients of the positive energy solution $a(\mathbf{k})$ and the negative energy solution $a^*(\mathbf{k})$ are related by complex conjugation for a real scalar field. For the quantization we proceed in the analogous way

[8]Note that $e^{\hat{A}} = \sum_{n=0}^{\infty} \frac{1}{n!}\hat{A}^n$.

to the quantum harmonic oscillator. The Lagrangian density for a free real scalar field, generalized momentum, Hamiltonian density and Poisson brackets are given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{4.38}$$

$$\pi = \dot{\phi} \tag{4.39}$$

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{m^2}{2}\phi^2 \tag{4.40}$$

$$\{\phi(\mathbf{x},t),\pi(\mathbf{y},t)\} = \delta^3(\mathbf{x}-\mathbf{y}) \qquad \{\pi(\mathbf{x},t),\pi(\mathbf{y},t)\} = \{\phi(\mathbf{x},t),\phi(\mathbf{y},t)\} = 0\ . \tag{4.41}$$

The Hamiltonian is positive definite and thus there is no problem with negative energies. We now quantize the real scalar field and replace the field and its conjugate by hermitian operators which satisfy the canonical (equal-time) commutation relations

$$[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\delta^3(\mathbf{x}-\mathbf{y}) \qquad [\pi(\mathbf{x},t),\pi(\mathbf{y},t)] = [\phi(\mathbf{x},t),\phi(\mathbf{y},t)] = 0\ . \tag{4.42}$$

Thus the coefficients $a(\mathbf{k})$ in Eq. (4.37) become operators

$$\phi(x^\mu) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_k}}\left(a(\mathbf{k})e^{-ik_\mu x^\mu} + a^\dagger(\mathbf{k})e^{+ik_\mu x^\mu}\right) \tag{4.43}$$

with creation operators $a^\dagger(\mathbf{k})$ and annihilation operators $a(\mathbf{k})$ for every momentum mode. The operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ satisfy the commutation relations

$$[a(\mathbf{k}),a^\dagger(\mathbf{k}')] = \delta_{\mathbf{k}\mathbf{k}'} \qquad\qquad [a(\mathbf{k}),a(\mathbf{k}')] = [a^\dagger(\mathbf{k}),a^\dagger(\mathbf{k}')] = 0\ . \tag{4.44}$$

Exercise: Show this by directly evaluating the commutators. For this you first have to use the field and the conjugate momentum $\pi = \partial^0\phi$ to find an expression for $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ in terms of the field and the conjugate momentum π. Similarly we can derive the other commutation relations.

The operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ play a similar role to the ladder operators for the quantum harmonic oscillator. There is a ground state which is annihilated by all annihilation operators $a(\mathbf{k})$

$$a(\mathbf{k})\,|0\rangle = 0 \quad \forall\mathbf{k} \tag{4.45}$$

and we can define number operators

$$N(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k}) \tag{4.46}$$

which have as their eigenvalues occupation numbers, i.e. the number of particles with a given 3-momentum $\mathbf{k}$

$$n(\mathbf{k}) = 0,1,2,\ldots \tag{4.47}$$

The set of all quantum states forms a *Fock space*, which consists of a direct sum of Hilbert spaces with $0,1,2,\ldots$ particles.

The Hamiltonian can be rewritten in terms of the number operators as

$$H = \sum_{\mathbf{k}} \omega_{\mathbf{k}}\left(N(\mathbf{k}) + \frac{1}{2}\right) \tag{4.48}$$

Note that the $\frac{1}{2}\omega_{\mathbf{k}}$ leads to an infinite vacuum energy in quantum field theory, when summing over all 3-momenta $\mathbf{k}$. In the absence of gravity, only energy differences are measured, thus we can define the energy relative to the ground state. The non-vanishing vacuum energy is due to the non-vanishing commutator $[a(\mathbf{k}), a^\dagger(\mathbf{k})] = 1$. This is formally equivalent to writing all annihilation to the right of the creation operators. It is denoted *normal ordering* and we define the normal ordered product of operators AB by $: AB :$, e.g.

$$: a(\mathbf{k}_1)a(\mathbf{k}_2)a^\dagger(\mathbf{k}_3) := a^\dagger(\mathbf{k}_3)a(\mathbf{k}_1)a(\mathbf{k}_2) \tag{4.49}$$

Note that the energy is always positive given that the particle number does not become negative. This does not occur given that the norm of the states in the Hilbert space have to be non-negative. Let $|n(\mathbf{k})\rangle$ be a state with occupation number $n(\mathbf{k})$ for momentum $\mathbf{k}$, then

$$[a(\mathbf{k})\,|n(\mathbf{k})\rangle]^\dagger\,[a(\mathbf{k})\,|n(\mathbf{k})\rangle] = \left\langle n(\mathbf{k})|a^\dagger(\mathbf{k})a(\mathbf{k})|n(\mathbf{k})\right\rangle = \langle n(\mathbf{k})|N(\mathbf{k})|n(\mathbf{k})\rangle \simeq n(\mathbf{k})\,\langle n(\mathbf{k})|n(\mathbf{k})\rangle \geq 0\,. \tag{4.50}$$

Let us finally discuss the normalization of states. We normalize the vacuum state to $1 = \langle 0|0\rangle$. In order to obtain a Lorentz-invariant normalization of the 1-particle state, we define it as

$$|k\rangle = \sqrt{2k^0V}a^\dagger(\mathbf{k})\,|0\rangle \tag{4.51}$$

and thus find for

$$\langle k|p\rangle = 2V\sqrt{k^0p^0}\left\langle 0|a(\mathbf{k})a^\dagger(\mathbf{p})|0\right\rangle = 2V\sqrt{k^0p^0}\left\langle 0|[a(\mathbf{k}), a^\dagger(\mathbf{p})]|0\right\rangle = 2k^0V\delta_{\mathbf{kp}}\,. \tag{4.52}$$

Aside, many textbooks don't quantize the scalar field in a box but in an infinite volume. Then the normalization of states becomes

$$\langle k|p\rangle = 2k^0(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{p})\,. \tag{4.53}$$

4.4 Free complex scalar field

We briefly sketch the main differences for a complex scalar field. The Lagrangian of a complex scalar field is given by

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \tag{4.54}$$

and thus there are two independent fields ϕ and ϕ^*. As there is no reality condition for the complex scalar field, there are two sets of creation and annihilation operators, one for particles (positive energy solutions) $a(\mathbf{k})$ and one for antiparticles (negative energy solutions) $b(\mathbf{k})$ and the field operator is given by

$$\phi(x^\mu) = \sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_k}}\left(a(\mathbf{k})e^{-ik_\mu x^\mu} + b^\dagger(\mathbf{k})e^{+ik_\mu x^\mu}\right)\,. \tag{4.55}$$

This implies that the field operator ϕ destroys particles and creates antiparticles and vice versa for $\phi^\dagger$. The creation and annihilation operators satisfy the following non-trivial (non-vanishing) commutation relations

$$[a(\mathbf{k}), a^\dagger(\mathbf{k'})] = [b(\mathbf{k}), b^\dagger(\mathbf{k'})] = \delta_{\mathbf{k}\mathbf{k'}} \ . \tag{4.56}$$

The normal ordered Hamiltonian is

$$: H := \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) + b^\dagger(\mathbf{k}) b(\mathbf{k}) \right) \tag{4.57}$$

and thus both particles and antiparticles have positive energy. As we discussed in the section on classical field theory, the complex Klein-Gordon field is invariant under $\phi \to e^{-i\alpha}\phi$ with conserved current

$$: J^\mu := -i : \left[(\partial^\mu \phi^\dagger)\phi - (\partial^\mu \phi)\phi^\dagger \right] : \ . \tag{4.58}$$

The operator corresponding to the conserved charge is obtained by inserting the field expansion (4.55) into

$$: Q := \int_V d^3x : J^0 := - \sum_{\mathbf{k}} \left[a^\dagger(\mathbf{k}) a(\mathbf{k}) - b^\dagger(\mathbf{k}) b(\mathbf{k}) \right] \tag{4.59}$$

and thus antiparticles carry the opposite charge compared to particles, while the mass and other properties are the same. The existence of antiparticles is a general feature of relativistic quantum field theory.

5 Scattering and decay

5.1 Interaction picture

So far we considered free theories without interactions. Let us assume that there is an additional term in the Hamiltonian, which describes the interaction between fields

$$H = H_0 + H_{int} \ , \tag{5.1}$$

where H_0 describes the free theory without interactions, e.g. the Hamiltonian of the Klein-Gordon field, and the interaction Hamiltonian H_{int}, e.g. $H_{int} = \frac{\lambda}{4!}\int d^3x\phi^4$. The interacting theory described by H is generally not possible to solve analytically. In the following we assume that the interaction (coupling λ) is small that that we can expand in the small interaction and express the result in a perturbation around the free theory described by H_0.

Like in the free theory, the Heisenberg field is given

$$\phi(x^\mu) = e^{iHt}\phi(\mathbf{x})e^{-iHt} \tag{5.2}$$

At any given moment of time t_0 we can express the field in terms of creation and annihilation operators like for the free field theory

$$\phi(t_0, \mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2Vk^0}} \left(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \ . \tag{5.3}$$

At a different time $t \neq t_0$, the Heisenberg field is given as

$$\phi(t, \mathbf{x}) = e^{iH(t-t_0)}\phi(t_0, \mathbf{x})e^{-iH(t-t_0)} \tag{5.4}$$

Similarly to the Heiselberg picture field, we can introduce the interaction picture field

$$\phi_I(t, \mathbf{x}) = e^{iH_0(t-t_0)}\phi(t_0, \mathbf{x})e^{-iH_0(t-t_0)} \ , \tag{5.5}$$

which corresponds to the Heisenberg field in the limit of no interactions. As it is possible to diagonalize the Hamiltonian H_0, it is easy to construct ϕ_I explicitly

$$\phi_I(t, \mathbf{x}) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2Vk^0}} \left(a(\mathbf{k})e^{-ik_\mu x^\mu} + a^\dagger(\mathbf{k})e^{ik_\mu x^\mu} \right) \Big|_{x^0 = t - t^0} \ . \tag{5.6}$$

Now we have to express the full Heisenberg field ϕ in terms of ϕ_I. Formally, it is

$$\phi(t, \mathbf{x}) = U^\dagger(t, t_0)\phi_I(t, \mathbf{x})U(t, t_0) \qquad\qquad U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \ , \tag{5.7}$$

where U is the unitary time-evolution operator. We would like to find $U(t, t_0)$ in terms of ϕ_I. It can be obtained from the corresponding Schödinger equation

$$i\frac{\partial}{\partial t}U(t, t_0) = e^{iH_0(t-t_0)}(H - H_0)e^{-iH(t-t_0)} = e^{iH_0(t-t_0)}H_{int}e^{-iH(t-t_0)} \tag{5.8}$$

$$= H_I(t)e^{iH_0(t-t_0)}e^{-iH(t-t_0)} = H_I(t)U(t - t_0) \tag{5.9}$$

where we wrote the interaction Hamiltonian in the interaction picture

$$H_I(t) \equiv e^{iH_0(t-t_0)} H_{int} e^{-iH_0(t-t_0)} = \frac{\lambda}{4!} \int d^3x \phi_I^4 \tag{5.10}$$

This differential equation can be solved iteratively starting from the initial condition $U(t_0, t_0) = 1$ and results in

$$U(t, t_0) = 1 - i \int_{t_0}^{t} dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) + \dots \, . \tag{5.11}$$

Note that the interaction Hamiltonians are ordered according to time. We can write the expression more compactly. For example the integrals of the second order term can be rewritten as follows

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) = \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} \mathcal{T} \left(H_I(t_1) H_I(t_2) \right) \tag{5.12}$$

where we introduced time ordering $\mathcal{T}$ of the operators. The factor of $1/2$ originates from the fact that the expression on the left only integrates over the lower-right triangle of the square. A similar argument holds for all higher order terms and we find

$$U(t, t_0) = \mathcal{T} \left(e^{-i \int_{t_0}^{t} dt' H_I(t')} \right) \, . \tag{5.13}$$

Note that $t \geq t_0$. The result is the *Dyson series*. The expression for the time evolution operator can be generalized and the second argument replaced by an arbitrary time $t' \leq t$. The time-evolution operator satisfies the following identities

$$U(t_1, t_2) U(t_2, t_3) = U(t_1, t_3) \qquad\qquad U(t_1, t_3) [U(t_2, t_3)]^\dagger = U(t_1, t_2) \, . \tag{5.14}$$

Using the time evolution operator we can obtain the probability amplitude for scattering of particles in an initial state $|i\rangle$ into a final state $|f\rangle$. The transition probability is generally given by

$$S_{fi} \equiv \langle f | U(\infty, -\infty) | i \rangle = \left\langle f | \mathcal{T} \left(e^{-i \int d^4x \mathcal{H}_I} \right) | i \right\rangle \, , \tag{5.15}$$

where $\mathcal{H}_I$ denotes the interaction Hamiltonian density ($H_I = \int d^3x \mathcal{H}_I$) and we assume that particles come from the infinite past and go to the infinite future. The matrix formed by S_{fi} is called S-matrix or scattering matrix.

In the previous chapter we have seen how to construct the intitial and final state by acting with the creation operators on the vacuum state. Although the vacuum state of an interacting theory $|\Omega\rangle$ generally differs from the vacuum state of the free theory $|0\rangle$, in the infinite past and future the vacuum state of the interacting theory can be related to the one of the free theory under quite general assumptions. Intuitively, if the theory is weakly interacting and there are no long-range interactions [9]

[9]There are subtleties in strongly interacting theories and theories with long-range interactions. In quantum chromodynamics for example we typically do not have free quarks, but observe jets of particles and thus the initial and final states are in principle constructed in terms of jets, but the amplitude generally separates in a function describing the formation of jets and an amplitude for the elementary quarks (QCD factorization theorem).

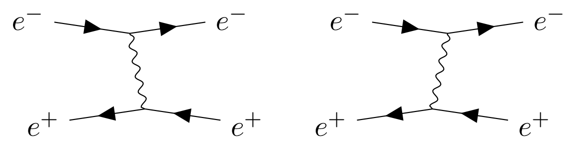


Figure 6: Electron-positron scattering

the multi-particle state separate in a product of free one-particle states (since they are sufficiently separated from each other). We will not further dwell on these complications in quantum field theory, but try to obtain an intuitive understanding of the S-matrix.

The interaction Hamiltonian consists of terms with multiple (more than 2) field operators. Each field operator contains both creation and annihilation operators. Thus when it acts on state in Fock space, it will create and annihilate particles and the output will be a different state in Fock space with generally different number of particles.

Let us illustrate the physical interpretation of the S-matrix and the Dyson series using the example of e^-e+ scattering which is pictorially shown in Fig. 6. As there are two interactions, we have to go to second order in the interaction Hamiltonian for the expression of Eq. (5.15). The diagrams in Fig. 6 have to be read from left to right. The electron and positron in the initial state first propagate (evolve in time) from the distant past to the first interaction point following the description of the free Hamiltonian H_0, when either the electron (left diagram) or the positron (right diagram) emit a photon (wavy line). There are two possible ways for the scattering to occur due to the time-ordering of the interaction Hamiltonians. The electron, positron and photon of the intermediate state propagate again following H_0 until the second interaction occurs, where the photon is absorbed either by the positron (left) or the electron (right). The resulting electron and positron in the final state then propagate according to H_0 to the distant future.

As we are typically only interested, if some scattering occurs, the S matrix is often written in terms of the T-matrix and we usually factor out the overall energy-momentum conservation and define $\mathcal{M}$

$$S = 1 + i\,T = 1 + i(2\pi)^4\delta^{(4)}(k_i^\mu - k_f^\mu)\mathcal{M}\ , \tag{5.16}$$

where k_i^μ (k_f^μ) denotes the sum of the initial (final) state 4-momenta. The matrix element $i\mathcal{M}$ is Lorentz-invariant. Quantum field theory provides a set of rules to calculate the matrix element $i\mathcal{M}$, the so-called Feynman rules.

5.2 Feynman rules

1. Scalars are denoted by dashed lines, fermions by solid lines and gauge bosons by wavy lines. Charged particles have an arrow on the line, which indicates the direction of the flow of charge.

2. External scalar lines should be replaced by 1, ingoing (outgoing) external gauge bosons by the polarization vector of the gauge boson ϵ_μ (ϵ_μ^*), ingoing (outgoing) fermions by u ($\bar{u} = u^\dagger\gamma^0$)

spinor and ingoing (outgoing) antifermions by the corresponding $\bar{v}$ (v) spinor.

 1 $\epsilon_\mu(\epsilon^*_\mu)$  $u(\bar{v})$  $\bar{u}(v)$ (5.17)

3. At every vertex 4-momentum is conserved.

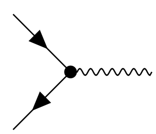
4. An internal line for a particle with 4-momentum p^μ (pointing from left to right) corresponds to a propagator[10]

 $\frac{i}{p^2 - m^2 + i\epsilon}$ μ ν $\frac{-i\eta^{\mu\nu}}{p^2 - m^2 + i\epsilon}$  $\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$ (5.18)

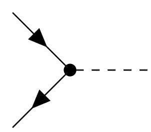
Note that virtual particles do not generally satisfy the dispersion relation $p_\mu p^\mu = m^2$.

5. Each term in the interaction Lagrangian corresponds to a vertex. Here are examples of a few common interaction terms together with their Feynman rules (ψ denotes spinors, ϕ a scalar and A_μ a gauge field)

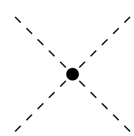
$$\mathcal{L} = eA_\mu \bar{\psi}\gamma^\mu\psi \tag{5.19}$$

 μ $\quad ie\gamma^\mu$

$$\mathcal{L} = y\phi\bar{\psi}P_L\psi \tag{5.20}$$

 $\quad iyP_L$

$$\mathcal{L} = \frac{\lambda}{4!}\phi^4 \tag{5.21}$$

 $\quad i\lambda$

6. Integrate over any free internal 4-momenta k_i

$$\int \frac{d^4k_i}{(2\pi)^4} \tag{5.22}$$

7. If the Feynman diagram exhibits any symmetry, divide by the number of possible symmetric configurations (symmetry factor).

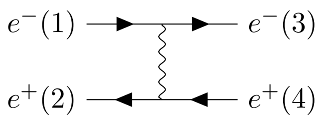
8. Sum over the expressions of all possible different diagrams. If external fermion lines are exchanged, there is a relative minus sign between diagrams for every exchange of fermions.

9. As fermions are descibed by spinors and to properly take into accout the matrix structure of the γ-matrices, fermion lines are read in the opposite direction of the arrows.

[10] The $+i\epsilon$ ensures the causality. ϵ is taken to zero at the end of the calculation.

10. Closed loops of fermions give an additional factor -1.

As already mentioned the two diagrams in Fig. 6 correspond to two different time orderings. In terms of the Feynman rules listed above they are equivalent and thus we only have to consider one of the two, which can be evaluated to



$$i\mathcal{M}_1 = [\bar{u}_3 ie\gamma^\mu u_1][\bar{v}_2 ie\gamma^\nu v_4]\frac{-ig_{\mu\nu}}{(p_1-p_3)^2+i\epsilon} \tag{5.23}$$

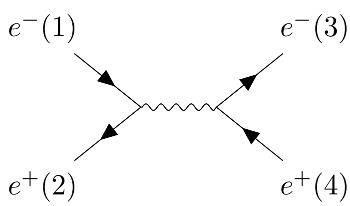
where the subscripts on the spinors u, v (momenta p_i) denote which of the external fermions it belongs to. It is convenient to introduce the so-called Mandelstam variables

$$s = (p_1+p_2)^2 \qquad t = (p_1-p_3)^2 \qquad u = (p_1-p_4)^2 \tag{5.24}$$

and rewrite the matrix element as

$$i\mathcal{M}_1 = ie^2\frac{[\bar{u}_3\gamma^\mu u_1][\bar{v}_2\gamma_\mu v_4]}{t+i\epsilon}\ , \tag{5.25}$$

where $\gamma_\mu \equiv g_{\mu\nu}\gamma^\nu$. There is another diagram, which can be interpreted as the annihilation of the initial state electron positron pair into a virtual photon which then splits into the final state electron-positron pair.



$$i\mathcal{M}_2 = -[\bar{u}_3 ie\gamma^\mu v_4][\bar{v}_2 ie\gamma^\nu u_1]\frac{-ig_{\mu\nu}}{s+i\epsilon} \tag{5.26}$$

The total matrix element is given by

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = e^2\left(\frac{[\bar{u}_3\gamma^\mu u_1][\bar{v}_2\gamma_\mu v_4]}{t+i\epsilon} - \frac{[\bar{u}_3\gamma^\mu v_4][\bar{v}_2\gamma_\mu u_1]}{s+i\epsilon}\right)\ . \tag{5.27}$$

Obviously, these are only the lowest-order diagrams which contribute to the scattering $e^-e^+ \to e^-e^+$. We could for example exchange multiple photons of the photon could split into a virtual e^+e^-pair which then recombines into a photon. These higher-order diagrams contain loops and are thus called loop diagrams.

5.3 Cross section

The probability for a transition from state $|i\rangle$ to state $|f\rangle$ is given by

$$P = \frac{|\langle f|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle} \tag{5.28}$$

For definiteness we will consider the scattering of 2 particles to an arbitrary final state with n particles. The norm of a one-particle state is

$$\langle k|k\rangle = 2k^0 V \tag{5.29}$$

and as the two-particle state in the infinite past and future can be effectively described by the product of two one-particle states, we obtain

$$\langle i|i\rangle = 4E_1E_2V^2 \qquad\qquad \langle f|f\rangle = \prod_i (2k_i^0 V)\ , \tag{5.30}$$

where E_1 and E_2 are the energies of the incoming particles and k_i^0 the energies of the outgoing particles. Similarly for the squared transition amplitude we obtain

$$|\langle f|i\rangle|^2 = |(2\pi)^4\delta^{(4)}(k^\mu_{in} - k^\mu_{out})|^2 |M|^2 = (2\pi)^4\delta^{(4)}(k^\mu_{in} - k^\mu_{out}) VT |M|^2\ , \tag{5.31}$$

where we used $(2\pi)^4\delta^{(4)}(0) = \int d^4x e^{i0} = VT$. Finally we have to sum over all possible final states. In the box with length L the 3-momenta are quantized $\mathbf{k_i} = \frac{2\pi}{L}\mathbf{n_i}$ and thus summing over the different modes corresponds to an integration

$$\sum_{n_i} \to \frac{L^3}{(2\pi)^3}\int d^3k_i\ . \tag{5.32}$$

The volume $V = L^3$ cancels against the volume factor from the normalization. Thus the transition rate is

$$\dot{P} = \frac{(2\pi)^4\delta^{(4)}(k_{in} - k_{out})}{4E_1E_2V}\int\prod_{i=1}^n \frac{d^3k_i}{(2\pi)^3 2k_i^0}|M|^2\ . \tag{5.33}$$

Let us denote the incoming momenta by p_1 and p_2 and the outgoing momenta by k_i. The differential cross section can be obtained by dividing by the incoming particle flux. In the rest frame of the second particle, it is simply given by the velocity of the first particle per volume. In the centre-of-mass frame (where the 3-momenta of the incoming particles add to zero), it is the relative velocity per volume $\sigma = \dot{P}V/v_{rel}$ where the relative velocity can be expressed by

$$v_{rel} = |\mathbf{v_1} - \mathbf{v_2}| = \left|\frac{\mathbf{p_1}}{E_1} - \frac{\mathbf{p_2}}{E_2}\right| = \frac{|\mathbf{p_1}|}{E_1E_2}(E_1 + E_2) = \frac{|\mathbf{p_1}|}{E_1E_2}\sqrt{s} \tag{5.34}$$

In the last equation we introduced the so-called Mandelstam variable $s = (p_1 + p_2)^2$. The differential cross section can also be written in terms of the flux factor $F = E_1E_2v_{rel}$. In order to simplify the expression, note that in the centre-of-mass frame of a 2-particle system the 3-momenta add to zero $\mathbf{p_1} + \mathbf{p_2} = 0$ and we can express the energy and momentum of the particles in terms of its masses and the Mandelstam variable $s = (p_1^\mu + p_2^\mu)^2 = (E_1 + E_2)^2$, where p_i are the 4-momenta. We can rewrite the Mandelstam variable s as follows

$$s = (p_1^\mu + p_2^\mu)^2 = m_1^2 + m_2^2 + 2p_1^\mu p_{2\mu} \tag{5.35}$$

$$= m_1^2 + m_2^2 + 2E_1E_2 - 2\mathbf{p_1}\cdot\mathbf{p_2} \tag{5.36}$$

$$= m_1^2 + m_2^2 + 2E_1E_2 + 2|\mathbf{p_1}|^2 \tag{5.37}$$

$$= m_1^2 + m_2^2 + 2E_1E_2 + 2E_1^2 - 2m_1^2 \tag{5.38}$$

$$= m_2^2 - m_1^2 + 2E_1(E_2 + E_1) = m_2^2 - m_1^2 + 2E_1\sqrt{s} \tag{5.39}$$

and thus find for the energy of one particle in the centre-of-mass frame

$$E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \ . \tag{5.40}$$

Using the expression for E_1 we find for the 3-momentum $|\mathbf{p_1}|$

$$|\mathbf{p_1}| = \sqrt{E_1^2 - m_1^2} = \frac{1}{2\sqrt{s}}\sqrt{(s + m_1^2 - m_2^2)^2 - 4sm_1^2} \tag{5.41}$$

$$= \frac{1}{2\sqrt{s}}\sqrt{s^2 + m_1^4 + m_2^4 - 2sm_1^2 - 2sm_2^2 - 2m_1^2 m_2^2} \tag{5.42}$$

$$= \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}} \tag{5.43}$$

which can be expressed in terms of the Källén function $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$. Thus we obtain for the flux factor

$$F = E_1 E_2 v_{rel} = \sqrt{s}|\mathbf{p_1}| = \frac{1}{2}\lambda^{1/2}(s, m_1^2, m_2^2) \tag{5.44}$$

and can express the differential cross section as

$$4F d\sigma = |M|^2 d\mathrm{LIPS}_n(p_1 + p_2) \ , \tag{5.45}$$

where the Lorentz-invariant n-body phase space is defined by

$$d\mathrm{LIPS}_n(k) \equiv (2\pi)^4 \delta^{(4)}\left(k - \sum_{i=1}^{n} k_i\right) \prod_{i=1}^{n} \frac{d^3 k_i}{(2\pi)^3 2k_i^0} \ . \tag{5.46}$$

The 2-body phase space is particularly simple to evaluate in the centre-of-mass frame

$$\int d\mathrm{LIPS}_2(k) = \int (2\pi)^4 \delta^{(4)}(k - k_1 - k_2) \frac{d^3 k_1}{(2\pi)^3 2k_1^0} \frac{d^3 k_2}{(2\pi)^3 2k_2^0} \tag{5.47}$$

$$= \int (2\pi)\delta(\sqrt{s} - k_1^0(|\mathbf{k_1}|) - k_2^0(|\mathbf{k_1}|)) \frac{d^3 k_1}{(2\pi)^3 2k_1^0 2k_2^0} \tag{5.48}$$

$$= \int \delta(\sqrt{s} - k_1^0 - k_2^0(k_1^0)) \frac{|\mathbf{k_1}| dk_1^0 d\Omega}{16\pi^2 k_2^0(k_1^0)} \tag{5.49}$$

$$= \int \frac{d\Omega}{16\pi^2} \frac{|\mathbf{k_1}|}{\sqrt{s}} \tag{5.50}$$

$$= \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{32\pi^2 s} \int d\Omega \tag{5.51}$$

$$= \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{16\pi s} \int_{-1}^{1} d\cos\theta \tag{5.52}$$

with the Källén function $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$. In order to evaluate the energy integral over k_1^0, we used

$$\delta(f(x)) = \sum_{\mathrm{zeros} x'} \frac{\delta(x - x')}{|f'(x')|} \ . \tag{5.53}$$

For the second-last line we used that $|\mathbf{k_1}|$ can be expressed in terms of the Källén function. In the last line we used that the squared matrix element for a 2-body final state generally does not depend on the azimuthal angle ϕ in the centre-of-mass frame.

In case that there are n identical particles in the final state, we have to divide by the symmetry factor $S = n!$ in the cross section in order to avoid double-counting the different final state configurations. Thus the total cross section is given by

$$\sigma = \frac{1}{S}\int d\sigma \ . \tag{5.54}$$

Hence the $2 \to 2$ scattering cross section $1 + 2 \to 3 + 4$ in the centre of mass frame is given by

$$\sigma = \frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{64\pi\ sF}\frac{1}{S}\int_{-1}^{1}|M|^2 d\cos\theta \ . \tag{5.55}$$

The cross section for example can be used to determine the number of particles produced at a collider: For example the cross section to produce a Z boson for $\sqrt{s} = m_Z = 91$ GeV is given by $\sigma = 4 \times 10^4 pb$. The Large Electron Positron (LEP) collider at CERN had an instantaneous luminosity of $\mathcal{L} = 10^{31}\mathrm{cm}^{-2}s^{-1}$. One year has roughly $3 \times 10^7 s$. Thus LEP produced about

$$\sigma\mathcal{L}t = 4 \times 10^4 pb \times 10^{31}\mathrm{cm}^{-2}s^{-1} \times 3 \times 10^7 s \approx 10^7 \tag{5.56}$$

Z bosons per year, where we used that $1barn = 10^{-24}\mathrm{cm}^2$.

If there is one particle in the initial state and we are studying decays we have to slightly modify our assumptions, because we assumed all particles to be stable in the previous discussion. However it turns out that the LSZ reduction formula also holds in this case. We only have to modify the initial state normalization $\langle i|i\rangle = 2E_1 V$ and find for the differential decay rate of a particle with energy E_1 and 4-momentum p_1

$$d\Gamma = \frac{|M|^2}{2E_1} d\mathrm{LIPS}_n(p_1) \tag{5.57}$$

and the decay rate is obtained by summing over all outgoing momenta and dividing by the symmetry factor

$$\Gamma = \frac{1}{S}\int d\Gamma \ . \tag{5.58}$$

Note that the decay rate is not a Lorenz scalar. In the centre-of-mass frame of the particle, there is $E_1 = m_1$, while the decay rate is smaller in any other frame by a factor $\gamma = E_1/m_1$, the relativistic boost factor, which accounts for the relativistic time dilation. Faster particles have an apparent longer lifetime, e.g. muons generated in the atmosphere reach the Earth's surface due to this time dilation factor. For decays into two particles with masses m_1 and m_2 in the centre-of-mass frame we find

$$\Gamma = \frac{\lambda^{1/2}(M^2, m_1^2, m_2^2)}{32\pi M^3}\frac{1}{S}\int_{-1}^{1}|M|^2 d\cos\theta \ . \tag{5.59}$$

If there are multiple decay channels to different final state particle, the decay rates have to be summed

$$\Gamma_{tot} = \sum_{\text{all process}} \Gamma_i \tag{5.60}$$

For example charged pions usually decay to a muon and a neutrino

$$\pi^+ \to \mu^+ + \nu_\mu \ , \tag{5.61}$$

but there are also rarer decay channels such as

$$\pi^+ \to e^+ + \nu_e \qquad\qquad \pi^+ \to \mu^+ + \nu_\mu + \gamma \qquad\qquad \pi^+ \to e^+ + \nu_e + \gamma \ . \tag{5.62}$$