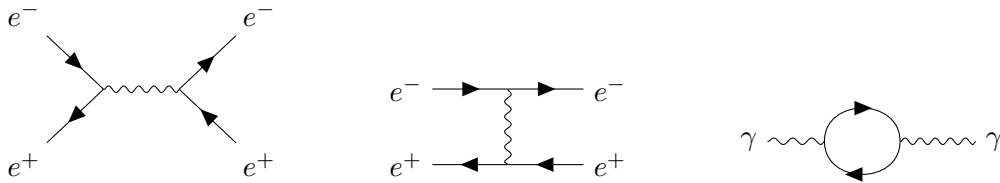


# An Introduction to Quantum Field Theory

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"The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing abstraction." – Sidney Coleman

Quantum field theory is the most complete and consistent theoretical framework that provides a unified description of quantum mechanics and special relativity. In the limit  $\hbar \rightarrow 0$  we obtain relativistic classical fields which can be used to describe for example electrodynamics. In the limit of small velocities  $v/c \ll 1$ , we obtain non-relativistic quantum mechanics. Finally in the limit  $\hbar \rightarrow 0$  and  $v/c \ll 1$  we obtain the description of non-relativistic classical fields such as sound waves.

Fields are described by a function which assigns a value to every point on the base manifold which we consider. In this course we will be mostly looking at relativistic quantum field theory, which is defined in 3+1 -dimensional Minkowski space-time. Most results can be straightforwardly translated to other systems, like 2-dimensional systems in condensed matter physics.

The notes are very brief and obviously only give an introduction to quantum field theory. The following books provide more detailed explanations and also more in-depth knowledge about quantum field theory.

1. L. Ryder, "Quantum Field Theory"
2. M. Srednicki, "Quantum Field Theory"
3. S. Weinberg, "The Quantum Theory of Fields", Vol. 1
4. M. Peskin, D. Schroeder, "An Introduction to Quantum Field Theory"
5. D. Bailin, "Introduction to Gauge Field Theory"
6. C. Itzykson, J-B. Zuber, "Quantum Field Theory"
7. A. Zee, "Quantum Field Theory in a Nutshell"
8. M. Schwartz, "Quantum Field Theory and the Standard Model"

Throughout the notes we will use natural units

$$\hbar = c = k_B = 1$$

and the signature  $(+ - - -)$  for the metric. Thus length scales and time are measured with the same units

$$[\ell] = [t] = \frac{1}{[m]} = \frac{1}{[E]} = \frac{1}{[T]} = \text{eV}^{-1}$$

which is the inverse of energy. Temperature is measure in the same units as energy.

Often we will use formal manipulations which require a more mathematical treatment. In quantum field theory we will encounter many divergences which have to be regulated. This is generally possible by considering a finite volume  $V$  and to quantize the theory in this finite volume. In the infinite volume limit we obtain  $V \rightarrow (2\pi)^3 \delta^{(3)}(0)$  which is divergent. See arXiv:1201.2714 [math-ph] for some lecture notes with a more rigorous discussion of different mathematical issues in quantum field theory.

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# 1 Spin 0 – scalar field theory

## 1.1 Relativistic classical field theory

An intuitive derivation of the Klein-Gordon equation. Any theory of fundamental physics has to be consistent with relativity as well as quantum theory. In particular a particle with 4-momentum  $p^\mu$  have to satisfy the relativistic dispersion relation

$$p^\mu p_\mu = E^2 - \vec{p}^2 = m^2. \quad (1.1)$$

Following the standard practice in quantum mechanics we replace the energy and momentum by operators

$$E \rightarrow i\hbar\partial_t \quad \vec{p} \rightarrow -i\hbar\vec{\nabla} \quad (1.2)$$

and postulate the wave equation for a relativistic spin-0 particle

$$m^2\phi = (i\partial_t)^2 - (-i\vec{\nabla})^2\phi = -(\partial_t^2 - \vec{\nabla}^2)\phi \quad (\square + m^2)\phi = 0 \quad (1.3)$$

which is the so-called Klein-Gordon equation. The Klein-Gordon equation is solved in terms of plane waves  $\exp(i\vec{k} \cdot \vec{x} \pm i\omega_k t)$  with  $\omega_k = (\vec{k}^2 + m^2)^{1/2}$  and thus the general solution is given by their superposition

$$\phi(x, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left( a(k) e^{i\vec{k} \cdot \vec{x} - i\omega_k t} + b(k) e^{i\vec{k} \cdot \vec{x} + i\omega_k t} \right) \quad (1.4)$$

The factor  $1/\omega_k$  ensures that the field  $\phi$  is a Lorentz scalar, i.e. invariant under orthochronous Lorentz transformations ( $\Lambda_0^0 \geq 1$ ). This can be directly seen from noticing that<sup>1</sup>

$$\int_{-\infty}^{\infty} dk^0 \delta(k^2 - m^2) \theta(k^0) = \frac{1}{2\omega_k}, \quad (1.6)$$

since there is only one zero at  $k^0 = \omega_k$  for  $k^0 > 0$ . Thus we can rewrite the integration in terms of  $d^4k \delta(k^2 - m^2) \theta(k^0)$  with the Dirac  $\delta$ -function and the Heaviside step function  $\theta$  which is manifestly Lorentz invariant. For a real scalar field [ $\phi(x, t) = \phi^*(x, t)$ ], the coefficients  $a$  and  $b$  are related by  $b(k) = a^*(-k)$ . Thus for a real scalar field we find

$$\phi(x, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a(k) e^{-ik_\mu x^\mu} + a^*(k) e^{ik_\mu x^\mu} \quad (1.7)$$

where we changed the integration variable  $\vec{k} \rightarrow -\vec{k}$  for the second term. If we were to interpret the solution as a quantum wave function, the second term would correspond to "negative energy" contributions.

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<sup>1</sup>Note that

$$\int dx \delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|} \quad (1.5)$$

where  $x_i$  are zeros of the function  $f$ .

Using the representation of the  $\delta$  function

$$(2\pi)^3 \delta^{(3)}(\vec{k}) = \int d^3x e^{i\vec{k} \cdot \vec{x}} \quad (1.8)$$

we can invert Eq. (1.7) to obtain

$$\int d^3x e^{ik_\mu x^\mu} \phi(x) = \frac{1}{2\omega_k} a(k) + \frac{1}{2\omega_k} e^{2i\omega_k t} a^*(k) \quad (1.9)$$

$$\int d^3x e^{ik_\mu x^\mu} \partial_0 \phi(x) = -\frac{i}{2} a(k) + \frac{i}{2} e^{2i\omega_k t} a^*(k) \quad (1.10)$$

and thus we obtain for the coefficients  $a(k)$

$$a(k) = \int d^3x e^{ik_\mu x^\mu} [\omega_k + i\partial_0] \phi(x) = i \int d^3x e^{ik_\mu x^\mu} \overleftrightarrow{\partial}_0 \phi(x) . \quad (1.11)$$

with  $f(x) \overleftrightarrow{\partial}_x g(x) \equiv f(x) \partial_x g(x) - (\partial_x f(x)) g(x)$ .

## 1.2 Least action principle

This equation can also be derived from the least action principle using the following Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \quad (1.12)$$

and thus the action  $S = \int dt \int d^3x \mathcal{L}$ . Consider a variation of the action with respect to the field  $\phi$  and the coordinates  $x^\mu$

$$x'^\mu = x^\mu + \delta x^\mu \quad \phi'(x) = \phi(x) + \delta \phi(x) \quad (1.13)$$

and thus the total variation of  $\phi$  is  $\Delta \phi = \delta \phi + (\partial_\mu \phi) \delta x^\mu$ . Variation of the action yields

$$\delta S = \int_R d^4x' \mathcal{L}(\phi', \partial_\mu \phi', x') - \int_R d^4x \mathcal{L}(\phi, \partial_\mu \phi, x) \quad (1.14)$$

$$= \int_R d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu + \mathcal{L} \partial_\mu \delta x^\mu \right) \quad (1.15)$$

$$= \int_R d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi + \int_R d^4x \partial_\mu \left( \mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi \right) \quad (1.16)$$

$$= \int_R d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi + \int_{\partial R} d\sigma_\mu \left( \mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi \right) \quad (1.17)$$

where we used  $\det(\frac{\partial x'^\mu}{\partial x^\lambda}) = \delta^\mu_\lambda + \partial_\lambda \delta x^\mu$  in the second line. If we demand on the boundary  $\partial R$  that there is no variation in the fields  $\delta x^\mu = 0$  and  $\delta \phi = 0$ , we obtain the Euler-Lagrange equations

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \quad (1.18)$$

For a real scalar field the Euler-Lagrange equation is

$$0 = \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} - \frac{\delta \mathcal{L}}{\delta \phi} = \partial_\mu \partial^\mu \phi + m^2 \phi . \quad (1.19)$$

The conjugate momentum to the field  $\phi$  is

$$\pi = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} \quad (1.20)$$

and the Hamiltonian density is obtained by a Legendre transformation

$$\mathcal{H} = \pi \partial_0 \phi - \mathcal{L} \quad (1.21)$$

The Poisson brackets for two functionals  $L_{1,2}$  are defined as<sup>2</sup>

$$\{L_1, L_2\} = \int d^3x \left[ \frac{\delta L_1}{\delta \pi(x, t)} \frac{\delta L_2}{\delta \phi(x, t)} - \frac{\delta L_1}{\delta \phi(x, t)} \frac{\delta L_2}{\delta \pi(x, t)} \right] . \quad (1.23)$$

and thus the Poisson brackets for the field  $\phi$  and conjugate momentum  $\pi$  are

$$\{\pi(x, t), \phi(y, t)\} = \delta^3(x - y) \quad (1.24)$$

$$\{\pi(x, t), \pi(y, t)\} = \{\phi(x, t), \phi(y, t)\} = 0 \quad (1.25)$$

The equations of motion are as usual determined by Hamilton's equations

$$\partial_0 \phi(x, t) = \{H, \phi(x, t)\} \quad \partial_0 \pi(x, t) = \{H, \pi(x, t)\} . \quad (1.26)$$

The generalisation to an arbitrary number of fields is straightforward.

### 1.3 Symmetries and Noether's theorem

For an arbitrary surface  $\partial R$  we can rewrite the second integral as

$$\int_{\partial R} d\sigma_\mu \left( \mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi \right) = \int_{\partial R} d\sigma_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Delta \phi - \Theta^\mu_\nu \delta x^\nu \right] \quad (1.27)$$

where we defined the energy-momentum tensor

$$\Theta^\mu_\nu \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu . \quad (1.28)$$

If the action is invariant (i.e.  $\delta S = 0$ ) under the following symmetry transformation

$$\delta x^\mu = X^\mu_\nu \delta \omega^\nu \quad \Delta \phi = \phi_\nu \delta \omega^\nu \quad (1.29)$$

The surface term vanishes because  $\delta S = 0$  and since  $\delta \omega^\nu$  is arbitrary

$$0 = \int_{\partial R} J^\mu_\nu d\sigma_\mu \delta \omega^\nu = \int_R d^4x \partial_\mu J^\mu_\nu \delta \omega^\nu \quad (1.30)$$

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<sup>2</sup>The functional derivative of a functional is the straightforward generalisation of the ordinary derivative

$$\frac{\delta}{\delta f(t)} f(t') = \delta(t - t') . \quad (1.22)$$

with the Noether current

$$J_\nu^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\nu - \Theta_\kappa^\mu X_\nu^\kappa. \quad (1.31)$$

As  $\delta\omega^\mu$  is arbitrary function and for sufficiently smooth  $J_\nu^\mu$  we obtain

$$\partial_\mu J_\nu^\mu = 0. \quad (1.32)$$

Thus we can define a conserved charge

$$Q_\nu = \int_V d^3x J_\nu^0 \quad (1.33)$$

of the system which is conserved when integrating over the whole volume

$$\frac{dQ_\nu}{dt} = \int d^3x \partial_0 J_\nu^0 = \int d^3x \partial_i J_\nu^i = \int d\sigma^i J_\nu^i = 0, \quad (1.34)$$

where we used the conservation of the Noether current,  $\partial_\mu J_\nu^\mu = 0$ , in the second step. The last equation holds because the current has to vanish at infinity.

### 1.3.1 Examples

Consider  $\Delta\phi = 0$  and  $\delta x^\mu = a^\mu$ , where  $a^\mu$  is a translation in space-time. Noether's theorem tells us that the Noether current is

$$J_\nu^\mu = -\Theta_\kappa^\mu a^\kappa \quad (1.35)$$

Time-translations,  $a^0 \neq 0$ ,  $a^i = 0$  imply the conservation of energy

$$H = \int d^3x \Theta_0^0 = \int d^3x \left[ \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_0 \phi - \mathcal{L} \right] \quad (1.36)$$

and spatial translations similarly imply the conservation of momentum

$$P_i = \int d^3x \Theta_i^0 = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_i \phi \quad (1.37)$$

and thus we can interpret  $\Theta_\nu^\mu$  as energy-momentum tensor.

The energy-momentum tensor as it is defined in Eq. (1.28) is not symmetric and also not unique. We may add a term  $\partial_\lambda f^{\lambda\mu\nu}$  with  $f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$ , because  $\partial_\mu \partial_\lambda f^{\lambda\mu\nu}$  vanishes due to  $f$  being antisymmetric in  $\mu, \lambda$  while the derivatives are symmetric. By choosing  $f^{\lambda\mu\nu}$  appropriately we obtain the canonical energy-momentum tensor

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu} \quad (1.38)$$

which is symmetric in  $\mu, \nu$ . The total 4-momentum of the system is the same, since

$$\int d^3x \partial_\lambda f^{\lambda 0\nu} = \int d^3x \partial_i f^{i 0\nu} = \int d\sigma_i f^{i 0\nu} = 0 \quad (1.39)$$

irrespective of the choice of  $f$ . The last equality holds because towards infinity the fields have to sufficiently quickly go to zero, in order for the action to be normalizable and thus also any combination of the fields  $\partial_i f^{i 0\nu}$  has to go to zero.

## 1.4 Canonical quantization – free real scalar field

Before moving to quantizing a scalar field let us review the quantization of the harmonic oscillator. The quantum harmonic oscillator is defined by the Lagrangian

$$L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2 \quad (1.40)$$

We can directly determine the conjugate momentum, Hamiltonian and Poisson brackets using standard techniques

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \quad (1.41)$$

$$H = p\dot{q} - L = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2 \quad (1.42)$$

$$\{p, q\} = 1 \quad \{p, p\} = \{q, q\} = 0 \quad (1.43)$$

We obtain the quantum harmonic oscillator by replacing  $q$  and  $p$  by operators  $\hat{q}$  and  $\hat{p} = -i\frac{d}{dq}$  and the Poisson bracket by the commutator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{q}^2 \quad (1.44)$$

$$[\hat{p}, \hat{q}] = -i \quad [\hat{p}, \hat{p}] = [\hat{q}, \hat{q}] = 0 . \quad (1.45)$$

We proceed in the analogous way for the real scalar field. The Lagrangian density for a free real scalar field, generalized momentum, Hamiltonian density and Poisson brackets are given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \quad (1.46)$$

$$\pi = \partial_0\phi \quad (1.47)$$

$$\mathcal{H} = \frac{1}{2}\partial_0\phi\partial_0\phi + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{m^2}{2}\phi^2 \quad (1.48)$$

$$\{\pi(x, t), \phi(y, t)\} = \delta^3(x - y) \quad \{\pi(x, t), \pi(y, t)\} = \{\phi(x, t), \phi(y, t)\} = 0 . \quad (1.49)$$

The Hamiltonian is positive definite and thus there is no problem with negative energies. We now quantize the real scalar field and replace the field and its conjugate by hermitian operators which satisfy the canonical (equal-time) commutation relations

$$[\pi(x, t), \phi(y, t)] = -i\delta^3(x - y) \quad [\pi(x, t), \pi(y, t)] = [\phi(x, t), \phi(y, t)] = 0 . \quad (1.50)$$

Thus the coefficients  $a(k)$  in Eq. (1.7) become operators

$$\phi(x, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[ a(\vec{k})e^{-ik_\mu x^\mu} + a^\dagger(\vec{k})e^{ik_\mu x^\mu} \right] \quad (1.51)$$

$$= \int \frac{d^3k}{((2\pi)^3 2\omega_k)^{1/2}} \left[ a(\vec{k})f_k(x) + a^\dagger(\vec{k})f_k^*(x) \right] \quad (1.52)$$



with the positive frequency (energy) solutions

$$f_k(x) = \frac{e^{-ikx}}{((2\pi)^3 2\omega_k)^{1/2}} \quad (1.53)$$

which form an orthonormal set

$$\int d^3x f_k^*(x) i \overset{\leftrightarrow}{\partial}_0 f_{k'}(x) = \delta^{(3)}(\vec{k} - \vec{k}') . \quad (1.54)$$

The operators  $a(k)$  and  $a^\dagger(k)$  satisfy the commutation relations

$$[a(k), a^\dagger(k')] = \int d^3x d^3y (2\pi)^3 (4\omega_k \omega_{k'})^{1/2} [f_k^*(x) i \overset{\leftrightarrow}{\partial}_0 \phi(x), \phi(y) i \overset{\leftrightarrow}{\partial}_0 f_{k'}(y)] \quad (1.55)$$

$$= i(2\pi)^3 \int d^3x d^3y (4\omega_k \omega_{k'})^{1/2} \quad (1.56)$$

$$\times \left( f_k^*(x, t) i \partial_0 f_{k'}(y, t) [\pi(x, t), \phi(y, t)] + i \partial_0 f_k^*(x, t) f_{k'}(y, t) [\phi(x, t), \pi(y, t)] \right) \quad (1.57)$$

$$= (2\pi)^3 \int d^3x (4\omega_k \omega_{k'})^{1/2} f_k^*(x, t) i \overset{\leftrightarrow}{\partial}_0 f_{k'}(y, t) \quad (1.58)$$

$$= (2\pi)^3 2\omega_k \delta^{(3)}(k - k') \quad (1.59)$$

Note that for  $\vec{k} = \vec{k}'$ , there is a divergence,  $\delta^{(3)}(0)$ . We can formulate the quantum field theory in a mathematically more rigorous way by quantizing the theory in a finite volume  $V$ . In this case, the integrals are replaced by sums and  $(2\pi)^3 \delta^{(3)}(0) \rightarrow V$ . See the textbook "Quantum Field Theory" by Mandl and Shaw for a discussion in terms of fields quantized in a finite volume. However this breaks Lorentz invariance and for simplicity in this course we will always work with the infinite volume limit directly using formal manipulations, but keep in mind that we can always go to a finite volume to regularize the theory. Similarly we can derive the other commutation relations

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0 . \quad (1.60)$$

The operators  $a(k)$  and  $a^\dagger(k)$  play a similar role to the ladder operators for the quantum harmonic oscillator. We define the number operator for the wave with momentum  $k$

$$(2\pi)^3 2\omega_k \delta^3(0) N(k) = a^\dagger(k) a(k) . \quad (1.61)$$

The number operator  $N(k)$  satisfies the following commutation relations

$$[N(k), a^\dagger(k)] = a^\dagger(k) \quad [N(k), a(k)] = -a(k) \quad (1.62)$$

which are straightforward to show, e.g. the first one follows from

$$(2\pi)^3 2\omega_k \delta^3(0) [N(k), a^\dagger(k)] = [a^\dagger(k) a(k), a^\dagger(k)] \quad (1.63)$$

$$= a^\dagger(k) [a(k), a^\dagger(k)] + [a^\dagger(k), a^\dagger(k)] a(k) \quad (1.64)$$

$$= (2\pi)^3 2\omega_k \delta^3(0) a^\dagger(k) \quad (1.65)$$

Thus if  $|n(k)\rangle$  is an eigenstate of  $N(k)$  with eigenvalue  $n(k)$  then the states  $a^\dagger |n(k)\rangle$  and  $a(k) |n(k)\rangle$  are also eigenstates of  $N(k)$  with eigenvalues  $n(k) + 1$  and  $n(k) - 1$  respectively. This justifies the interpretation of the operator  $N(k)$  as a particle number (density) operator for momentum  $k$ . The number operators for different momenta  $k$  commute

$$[N(k), N(k')] = 0 \quad (1.66)$$

and thus eigenstates of the operators  $N(k)$  form a basis. Similar to the quantum harmonic oscillator there is a ground state energy. It gives an infinite contribution to the energy when integrating over all momenta

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k a^\dagger(k) a(k) + \epsilon_0 V \quad (1.67)$$

where we interpreted  $(2\pi)^3 \delta^3(0)$  as volume  $V$  and  $\epsilon_0 V$  denotes the zero-point energy with

$$\epsilon_0 = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega_k . \quad (1.68)$$

As only energy differences matter in the absence of gravity, we can subtract the zero point energy and redefine the Hamiltonian as

$$: H := \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k a^\dagger(k) a(k) \quad (1.69)$$

which is formally equivalent to writing all annihilation to the right of the creation operators. This is denoted *normal ordering* and we define the normal ordered product of operators  $AB$  by  $: AB :$ . Similarly the normal-ordered momentum is given by

$$: P_i := \int \frac{d^3k}{(2\pi)^3 2\omega_k} k_i a^\dagger(k) a(k) . \quad (1.70)$$

Thus the energy is always positive given that the particle number does not become negative. This does not occur given that the norm of the states in the Hilbert space have to be non-negative

$$\begin{aligned} [a(k) |n(k)\rangle]^\dagger [a(k) |n(k)\rangle] &= \langle n(k) | a^\dagger(k) a(k) | n(k) \rangle = (2\pi)^3 2\omega_k \delta^3(0) \langle n(k) | N(k) | n(k) \rangle \\ &= (2\pi)^3 2\omega_k \delta^3(0) n(k) \langle n(k) | n(k) \rangle \geq 0 . \end{aligned} \quad (1.71)$$

Thus as  $a(k)$  reduces  $n(k)$  by 1, there is a ground state  $|0\rangle$  with

$$a(k) |0\rangle = 0 \quad (1.72)$$

which does not contain any particles  $N(k) |0\rangle = 0$  of momentum  $k$ . The operator  $a(k)$  and  $a^\dagger(k)$  are commonly called annihilation and creation operators, because they annihilate and create one particle with momentum  $k$ , respectively.

The one-particle state  $|k\rangle = a^\dagger(k) |0\rangle$  is normalized as

$$\langle k | k' \rangle = \langle 0 | a(k) a^\dagger(k') | 0 \rangle = \langle 0 | [a(k), a^\dagger(k')] | 0 \rangle = (2\pi)^3 2\omega_k \delta^3(k - k') \quad (1.73)$$

where we used the commutation relation<sup>3</sup> of the creation and annihilation operators and the normalization of the vacuum state  $\langle 0|0\rangle = 1$ . The one-particle wave function  $\psi(x)$  for a particle with momentum  $p$  is

$$\psi(x) \equiv \langle 0|\phi(x)|p\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[ \langle 0|a(k)|p\rangle e^{-ikx} + \langle 0|a^\dagger(k)|p\rangle e^{ikx} \right] = e^{-ipx} . \quad (1.74)$$

## 1.5 Complex scalar field

So far we considered a real scalar field. For a complex scalar field the discussion is very similar, but there are some important differences. The Lagrangian of a free complex scalar field  $\phi$  is given by

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \quad (1.75)$$

The Lagrangian is invariant under an internal symmetry  $\phi \rightarrow e^{i\alpha} \phi$  and thus there is a Noether current. For convenience we replace  $\alpha \rightarrow -q$  and thus obtain

$$J_\mu = -iq \left[ (\partial_\mu \phi^\dagger) \phi - (\partial_\mu \phi) \phi^\dagger \right] = iq \overleftrightarrow{\partial}_\mu \phi \quad (1.76)$$

and a conserved charge

$$Q = iq \int d^3x \phi^\dagger \overleftrightarrow{\partial}_t \phi . \quad (1.77)$$

The Euler-Lagrange equations and conjugate momenta are

$$(\square + m^2)\phi = 0 \quad (\square + m^2)\phi^\dagger = 0 \quad (1.78)$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^\dagger \quad \pi^\dagger = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger} = \dot{\phi} \quad (1.79)$$

The general solution for the fields  $\phi$  and  $\phi^\dagger$  are given by

$$\phi(x, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left( a(k) e^{-ikx} + b^\dagger(k) e^{ikx} \right) \quad \phi^\dagger(x, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left( a^\dagger(k) e^{ikx} + b(k) e^{-ikx} \right) . \quad (1.80)$$

We quantize it in the usual way, but we have to keep in mind that  $\phi$  and  $\phi^\dagger$  are independent. The equal time commutation relations are

$$[\pi(x, t), \phi(y, t)] = [\pi^\dagger(x, t), \phi^\dagger(y, t)] = -i\delta^3(x - y) \quad (1.81)$$

while all other equal time commutation relations vanish. This leads to the following commutation relations for the operators  $a(k)$  and  $b(k)$

$$[a(k), a^\dagger(k')] = [b(k), b^\dagger(k')] = (2\pi)^3 2\omega_k \delta(\vec{k} - \vec{k}') \quad (1.82)$$

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<sup>3</sup>Note that some textbooks use a non-covariant normalization of the commutation relation  $[a(k), a^\dagger(k')] = \delta(k - k')$ .

where  $a(k)$  and  $b(k)$  are annihilation operators. Thus there are two types of particles,  $a$  and  $b$ , with number operators and Hamiltonian

$$(2\pi)^3 2\omega_k \delta^3(0) N_a(k) = a^\dagger(k) a(k) \quad (2\pi)^3 2\omega_k \delta^3(0) N_b(k) = b^\dagger(k) b(k) \quad (1.83)$$

$$H = \int d^3k \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k \left[ a^\dagger(k) a(k) + b^\dagger(k) b(k) \right] . \quad (1.84)$$

Thus the particles both have the same positive energy, but they have equal and opposite charges, because the charge operator is given by

$$Q = q \int \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[ a^\dagger(k) a(k) - b^\dagger(k) b(k) \right] . \quad (1.85)$$

Otherwise their properties including mass are equivalent. They form a pair of particle and antiparticle. The existence of antiparticles is a general feature of relativistic quantum field theory which we will encounter later as well for Dirac fermions.

## 1.6 Lehmann-Symanzik-Zimmermann (LSZ) reduction formula

Consider a real scalar field theory. States in a free theory (i.e. which contain at most terms quadratic in the fields) are constructed by acting with the creation operators on the vacuum. A one-particle state is given by

$$|k\rangle = a^\dagger(k) |0\rangle \quad (1.86)$$

where the creation operator can be written in terms of the field operator as derived in Eq. (1.11)

$$a^\dagger(k) = \int d^3x e^{-ikx} (\omega_k - i\partial_0) \phi(x) = -i \int d^3x e^{-ikx} \overleftrightarrow{\partial}_0 \phi(x) \quad (1.87)$$

and similarly

$$a(k) = \int d^3x e^{ikx} (\omega_k + i\partial_0) \phi(x) . \quad (1.88)$$

We can define the creation operator of a particle localized in momentum space near  $\vec{k}_1$  and in position space near the origin by

$$a_1^\dagger = \int d^3k f_1(k) a^\dagger(k) \quad (1.89)$$

where the function  $f_1$  describes the wave packet of the particle, which we take to be a Gaussian with width  $\sigma$

$$f_1(k) \propto \exp\left(-\frac{(\vec{k} - \vec{k}_1)^2}{2\sigma^2}\right) \quad (1.90)$$

In the Schrödinger picture the state  $a_1^\dagger |0\rangle$  will propagate and spread out. The two particles in a two-particle state  $a_1^\dagger a_2^\dagger |0\rangle$  with  $k_1 \neq k_2$  will be widely separated for  $t \rightarrow \infty$ . The creation and annihilation

operators will be time-dependent. A suitable initial (final) state of a scattering experiment will be

$$|i\rangle = \lim_{t \rightarrow -\infty} a_1^\dagger(t) a_2^\dagger(t) |0\rangle \equiv a_1^\dagger(-\infty) a_2^\dagger(-\infty) |0\rangle \quad (1.91)$$

$$|f\rangle = \lim_{t \rightarrow \infty} a_1^\dagger(t) a_2^\dagger(t) |0\rangle \equiv a_1^\dagger(\infty) a_2^\dagger(\infty) |0\rangle \quad (1.92)$$

The scattering amplitude is then given by  $\langle f|i\rangle$ . Note however that the creation and annihilation operators in the scattering amplitude are evaluated at different times. We thus have to relate them to each other.

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{\infty} dt \partial_0 a_1^\dagger(t) \quad \text{fundamental theorem of calculus} \quad (1.93)$$

$$= \int d^3k f_1(k) \int d^4x \partial_0 \left( e^{-ikx} (\omega_k - i\partial_0) \phi(x) \right) \quad (1.94)$$

$$= -i \int d^3k f_1(k) \int d^4x e^{-ikx} (\partial_0^2 + \omega^2) \phi(x) \quad (1.95)$$

$$= -i \int d^3k f_1(k) \int d^4x e^{-ikx} (\partial_0^2 + \vec{k}^2 + m^2) \phi(x) \quad (1.96)$$

$$= -i \int d^3k f_1(k) \int d^4x e^{-ikx} \left( \partial_0^2 - \overleftarrow{\nabla}^2 + m^2 \right) \phi(x) \quad (1.97)$$

$$= -i \int d^3k f_1(k) \int d^4x e^{-ikx} \left( \partial_0^2 - \overrightarrow{\nabla}^2 + m^2 \right) \phi(x) \quad (1.98)$$

$$= -i \int d^3k f_1(k) \int d^4x e^{-ikx} (\partial_\mu \partial^\mu + m^2) \phi(x) \quad (1.99)$$

and similarly for the annihilation operator

$$a_1(\infty) - a_1(-\infty) = i \int d^3k f_1(k) \int d^4x e^{ikx} (\partial_\mu \partial^\mu + m^2) \phi(x) \quad (1.100)$$

Thus we are in a position to evaluate the scattering amplitude, the matrix elements of the scattering operator  $S$ ,

$$\begin{aligned} S_{fi} &\equiv \langle f|i\rangle = \left\langle 0|T(a_2(\infty)a_1(\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)|0\rangle \right. \\ &= i^4 \int d^4x_1 e^{-ik_1x_1} (\partial_1^2 + m^2) \dots d^4x_2 e^{ik_2x_2} (\partial_2^2 + m^2) \langle 0|T\phi(x_1) \dots \phi(x_2)|0\rangle \end{aligned} \quad (1.101)$$

where we inserted the time-ordering operator in the first step, then used the derived relation of the creation and annihilation operators. The expression directly generalizes for more particles in the initial and/or final state. The general result is called the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula, which relates the scattering amplitude to the expectation value of the time-ordered product of field operators. The same holds in an interacting theory if the fields satisfies the following two conditions

$$\langle 0|\phi(x)|0\rangle = 0 \quad \langle 0|\phi(x)|k\rangle = e^{-ikx} . \quad (1.102)$$

These conditions can always be satisfied by shifting and rescaling the field operator appropriately. The two conditions ensure that the interacting one-particle states behave like free one-particle states. In general the creation operator will create a mixture of one-particle and multi-particle states in an interacting theory. However, it can be shown that multi-particle states separate from one-particle states at the infinite past and future and thus we can consider scattering between (quasi-)free particles. See the discussion in the book by Mark Srednicki for a rigorous argument why the same also holds in an interacting field theory.

The  $S$  matrix is unitary because for two in-states  $|i\rangle$  and  $|j\rangle$  and the final (out-state)  $|f\rangle$  we find

$$\sum_f S_{jf}^* S_{fi} = \sum_f \langle j|f\rangle \langle f|i\rangle = \langle j|i\rangle = \delta_{ij} \quad (1.103)$$

where we used the completeness of the out states  $|f\rangle$ . Thus the probability to scatter to any final state  $|f\rangle$  is unity. It is often convenient to write the  $S$  matrix as

$$S = 1 + iT \quad (1.104)$$

where the identity refers to no scattering and the  $T$  matrix describes any (non-trivial) scattering. Then unitarity translates into

$$1 = S^\dagger S = (1 - iT^\dagger)(1 + iT) = 1 + iT - iT^\dagger + T^\dagger T \quad \Rightarrow \quad T^\dagger T = -i(T - T^\dagger) = 2\text{Im}(T) \quad (1.105)$$

which is one form of the optical theorem. Next we develop tools to calculate correlation functions such as  $\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle$ .

## 1.7 Spin-statistics connection

Could we quantize the scalar field also with anticommutators instead of commutators? Consider spin-0 particles with the Hamiltonian

$$H_0 = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k a^\dagger(k) a(k). \quad (1.106)$$

The creation and annihilation operators either satisfy commutation or anticommutation relations

$$[a(k), a(k')]_{\mp} = [a^\dagger(k), a^\dagger(k')]_{\mp} = 0 \quad [a(k), a^\dagger(k')]_{\mp} = (2\pi)^3 2\omega_k \delta^3(k - k') \quad (1.107)$$

Now construct a theory with local Lorentz-invariant interactions out of the non-hermitian fields

$$\phi^+(x, 0) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-i\vec{k}\cdot\vec{x}} a(k) \quad \phi^-(x, 0) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i\vec{k}\cdot\vec{x}} a^\dagger(k) \quad (1.108)$$

$$\phi^\pm(x, t) = e^{iH_0 t} \phi^\pm(x, 0) e^{-iH_0 t} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{\mp i k x} a^{(\dagger)}(k) \quad (1.109)$$

which are related by hermitian conjugation  $\phi^+ = (\phi^-)^\dagger$ .  $\phi^\pm(x, t)$  is the time-evolved field  $\phi^\pm(x, 0)$ . The field  $\phi$  is invariant under proper orthochronous Lorentz transformations  $\Lambda$

$$U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x) \quad (1.110)$$

and thus also the creation ( $a^\dagger(k)$ ) and annihilation ( $a(k)$ ) operators as well as the fields  $\phi^\pm(x)$  are Lorentz scalars. Construct an interaction Lagrangian density  $\mathcal{L}_1$  which is local and Lorentz invariant. The corresponding Hamiltonian density is denoted by  $\mathcal{H}_1$  and  $H_1$  denotes the interaction Hamiltonian in the Schrödinger picture

$$H_1 = \int d^3x \mathcal{H}_1(x, 0) \quad (1.111)$$

where  $\mathcal{H}_1(x, 0)$  is a hermitian function of  $\phi^\pm(x, 0)$ . The corresponding Hamiltonian density in the interaction picture  $\mathcal{H}_I(x, t)$  is defined by the same function, but with  $\phi^\pm(x, 0)$  replaced by  $\phi^\pm(x, t)$ , since the interaction Hamiltonian is given by  $H_I(t) = \exp(-iH_0t)H_1 \exp(iH_0t)$ .

Consider now a transition amplitude from a state  $|i\rangle$  at  $t = -\infty$  to state  $|f\rangle$  at  $t = \infty$

$$\mathcal{T}_{f \leftarrow i} = \left\langle f \left| T \exp \left[ -i \int_{-\infty}^{\infty} dt H_I(t) \right] \right| i \right\rangle. \quad (1.112)$$

The transition amplitude is only Lorentz invariant, if the time-ordering is frame independent. This is trivially satisfied in the forward and backward lightcone for time-like separations of two events  $(x - x')^2 > 0$ . However events with a space-like separation can have different temporal ordering. Thus we have to require that the interaction Hamiltonian commutes with itself at different times for space-like separations

$$[\mathcal{H}_I(x), \mathcal{H}_I(x')] = 0 \quad (x - x')^2 < 0 \quad (1.113)$$

The interaction Hamiltonian densities are constructed of the fields  $\phi^\pm(x, t)$  and thus we have to consider the commutator (anticommutator) of the fields  $\phi^\pm$  for space-like separations  $r^2 \equiv (x - x')^2 < 0$

$$[\phi^+(x), \phi^-(x')]_{\mp} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} e^{-i(kx - k'x')} [a(k), a^\dagger(k')]_{\mp} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x-x')} \quad (1.114)$$

The integral is best evaluated for  $t = t'$  using spherical coordinates. For  $t = t'$ ,  $k = |\vec{k}|$  and  $r = |\vec{x} - \vec{x}'|$

$$\frac{1}{8\pi^2} \int_0^\infty dk \int_{-1}^1 d\cos\theta \frac{k^2}{E(k)} e^{ikr \cos\theta} = \frac{1}{8\pi^2} \int_0^\infty dk \frac{k^2}{E(k)} \frac{e^{ikr} - e^{-ikr}}{ikr} = -\frac{i}{8\pi^2 r} \int_{-\infty}^\infty dk \frac{k}{\sqrt{k^2 + m^2}} e^{ikr}. \quad (1.115)$$

The integrand has branch cuts along the imaginary axis starting at  $\pm im$ . We deform the integration contour such that it wraps around the upper branch cut by defining  $\rho = ik/m$  and thus

$$[\phi^+(x), \phi^-(x')]_{\mp} = \frac{m}{4\pi^2 r} \int_1^\infty d\rho \frac{\rho e^{-\rho mr}}{\sqrt{\rho^2 - 1}} = \frac{m}{4\pi^2 r} \int_1^\infty d\rho \left( \frac{\partial}{\partial \rho} \sqrt{\rho^2 - 1} \right) e^{-\rho mr} \quad (1.116)$$

$$= \frac{m}{4\pi^2 r} \left\{ \left[ \sqrt{\rho^2 - 1} e^{-\rho mr} \right]_1^\infty - \int_1^\infty d\rho \sqrt{\rho^2 - 1} \frac{\partial}{\partial \rho} e^{-\rho mr} \right\} \quad (1.117)$$

$$= \frac{m^2}{4\pi^2} \int_1^\infty d\rho \sqrt{\rho^2 - 1} e^{-\rho mr} = \frac{m}{4\pi^2 r} K_1(mr) \equiv C(r) \quad (1.118)$$

where  $K_1$  denotes the modified Bessel function. Thus the (anti)commutator is non-zero for any  $r > 0$ . Consider now the linear combination  $\phi_\lambda(x) = \phi^+(x) + \lambda\phi^-(x)$  with an arbitrary complex number  $\lambda$ . The (anti)commutators of  $\phi_\lambda$  for space-like separations are then

$$[\phi_\lambda(x), \phi_\lambda^\dagger(x')]_\mp = [\phi^+(x), \phi^-(x')]_\mp + |\lambda|^2[\phi^-(x), \phi^+(x')]_\mp = (1 - |\lambda|^2)C(r) \quad (1.119)$$

$$[\phi_\lambda(x), \phi_\lambda(x')]_\mp = \lambda[\phi^+(x), \phi^-(x')]_\mp + \lambda[\phi^-(x), \phi^+(x')]_\mp = \lambda(1 \mp 1)C(r) \quad (1.120)$$

Thus in order for both (anti)commutators to vanish and to have a suitable interaction Hamiltonian built out of the field  $\phi_\lambda$  we have to choose commutators with  $|\lambda| = 1$ . This brings us back to a real scalar field if we choose  $\lambda = e^{i\alpha}$ , then  $e^{-i\alpha/2}\phi_\lambda(x) = \phi(x)$ .

The same argument can be made for higher spin fields in any number of space-time dimensions<sup>4</sup>. The allowed choice is always commutators for integer spin and anticommutators for half-integer spin. Hence particles with integer spin are bosons and particles with half-integer spin fermions.

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<sup>4</sup>For  $d = 2$  there are more possibilities apart from fermions and bosons, so-called anyons.



## 2 Feynman path integral

We develop the formulation of a quantum theory in terms of the Feynman path integral. Consider the generalized coordinate operators  $Q_a$  and their conjugate momentum operators  $P_a$  with the canonical commutation relations

$$[Q_a, P_b] = i\delta_{ab} \quad [Q_a, Q_b] = [P_a, P_b] = 0 \quad (2.1)$$

In a field theory the operators depend on the position  $x$ . Thus the index  $a$  labels on the one hand the position  $x$  and on the other hand a discrete species index  $m$ . Thus the Kronecker delta function  $\delta_{ab}$  should be interpreted as  $\delta_{x,m;y,n} = \delta(x-y)\delta_{m,n}$ . As the  $Q_a$  ( $P_a$ ) commute among each other there are simultaneous eigenstates  $|q\rangle$  ( $|p\rangle$ ) with eigenvalues  $q_a$  ( $p_a$ )

$$Q_a |q\rangle = q_a |q\rangle \quad P_a |p\rangle = p_a |p\rangle \quad (2.2)$$

which are orthogonal and satisfy a completeness relation

$$\langle q'|q\rangle = \Pi_a \delta(q'_a - q_a) \equiv \delta(q' - q) \quad \langle p'|p\rangle = \Pi_a \delta(p'_a - p_a) \equiv \delta(p' - p) \quad (2.3)$$

$$1 = \int \Pi_a dq_a |q\rangle \langle q| \equiv \int d\underline{q} |q\rangle \langle q| \quad 1 = \int \Pi_a dp_a |p\rangle \langle p| \equiv \int d\underline{p} |p\rangle \langle p| \quad (2.4)$$

From the commutation relations (2.1) the scalar product

$$\langle q|p\rangle = \Pi_a \frac{1}{\sqrt{2\pi}} e^{iq_a p_a} \quad (2.5)$$

follows. In the Heisenberg picture<sup>5</sup>, where operators  $P$  and  $Q$  depend on time, the corresponding completeness and orthogonality relations apply

$$\langle q'; t|q; t\rangle = \Pi_a \delta(q'_a - q_a) \equiv \delta(q' - q) \quad \langle p'; t|p; t\rangle = \Pi_a \delta(p'_a - p_a) \equiv \delta(p' - p) \quad (2.7)$$

$$1 = \int d\underline{q} |q; t\rangle \langle q; t| = \int d\underline{p} |p; t\rangle \langle p; t| \quad \langle q; t|p; t\rangle = \Pi_a \frac{1}{\sqrt{2\pi}} e^{iq_a p_a} . \quad (2.8)$$

Now we want to calculate the probability amplitude for a system to go from state  $|q; t\rangle$  to state  $|q'; t'\rangle$ , i.e. the scalar product  $\langle q'; t'|q; t\rangle$ . For an infinitesimally small time difference  $dt = t' - t$ , it is given by<sup>6</sup>

$$\langle q'; t'|q; t\rangle = \langle q'; t|e^{-iH(Q(t), P(t))dt}|q; t\rangle = \int d\underline{p} \langle q'; t|e^{-iH(Q(t), P(t))dt}|p; t\rangle \langle p; t|q; t\rangle \quad (2.9)$$

$$= \int \Pi_a \frac{dp_a}{2\pi} e^{-iH(q', p)dt + i\sum_a (q'_a - q_a)p_a} \quad (2.10)$$

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<sup>5</sup>States  $|a\rangle$  and operators  $A$  in the Heisenberg picture are related to operators in the "Schödinger picture at fixed time  $t = 0$ " by

$$|a; t\rangle_H = e^{iHt} |a\rangle_S \quad A_H(t) = e^{iHt} A_S e^{-iHt} . \quad (2.6)$$

<sup>6</sup>The Hamiltonian is a function of the operators  $Q$  and  $P$ . In the Heisenberg picture, it is given by the same function of  $Q(t)$  and  $P(t)$ . We adopt a standard-form of the Hamiltonian where all operators  $Q$  appear to the left of the operators  $P$ . Other conventions are possible (e.g. Weyl ordering) but will lead to slight differences in the definition of the Feynman path integral.

The last step is only possible for an infinitesimal time step, where  $\exp(-iHdt) \simeq 1 - iHdt$ , because we can neglect higher orders in  $dt$ . For a finite time interval, we split the time interval in many intermediate steps,  $t \equiv \tau_0, \tau_1, \tau_2, \dots, \tau_N, \tau_{N+1} \equiv t'$  with  $\tau_{k+1} - \tau_k = \delta\tau = (t' - t)/(N + 1)$  and insert the identity using the completeness relation for the states  $q$ . Hence the scalar product is given by the constrained<sup>7</sup> path integral

$$\begin{aligned} \langle q'; t' | q; t \rangle &= \int \Pi_{k=1}^N d\bar{q}(\tau_k) \langle q(\tau_{N+1}); \tau_{N+1} | q(\tau_N); \tau_N \rangle \langle q(\tau_N); \tau_N | q(\tau_{N-1}); \tau_{N-1} \rangle \dots \langle q(\tau_1); \tau_1 | q(\tau_0); \tau_0 \rangle \\ &= \int [\Pi_{k=1}^N d\bar{q}(\tau_k)] \left[ \Pi_{k=0}^N \Pi_a \frac{dp_{k,a}}{2\pi} \right] \end{aligned} \quad (2.11)$$

$$\begin{aligned} &\times \exp \left( i \sum_{k=1}^{N+1} \left[ \sum_a (q_a(\tau_k) - q_a(\tau_{k-1})) p_a(\tau_{k-1}) - H(q(\tau_k), p(\tau_{k-1})) \right] d\tau \right) \\ &\xrightarrow{N \rightarrow \infty} \int_{q(t^{(v)})=q^{(v)}} \mathcal{D}q \mathcal{D}p \exp \left( i \int_t^{t'} \left[ \sum_a \dot{q}_a(\tau) p_a(\tau) - H(q(\tau), p(\tau)) \right] d\tau \right) \end{aligned} \quad (2.12)$$

where we implicitly defined the integration measure

$$\int \mathcal{D}q \mathcal{D}p = \lim_{N \rightarrow \infty} \int [\Pi_{k=1}^N \Pi_a dq_a(\tau_k)] \left[ \Pi_{k=0}^N \Pi_a \frac{dp_a(\tau_k)}{2\pi} \right]. \quad (2.13)$$

Note that  $q$  and  $p$  in the path integral are ordinary functions (instead of operators). They are variables of integration and do not generally satisfy the classical Hamiltonian dynamics. For convenience we define the function

$$S = \int_t^{t'} d\tau [\dot{q}(\tau)p(\tau) - H], \quad (2.14)$$

which reduces to the action if the Hamiltonian dynamics holds.

## 2.1 Expectation values of operators

The Feynman path integral formalism allows to calculate expectation values of operators for intermediate times. Consider an operator<sup>8</sup>  $O(t_1)$  at time  $t_1$  with  $t < t_1 < t'$ . For an infinitesimal time interval  $[t, t + dt]$  with an insertion of an operator at time  $t$  we find

$$\langle q'; t + dt | O(P(t), Q(t)) | q; t \rangle = \int \Pi_a \frac{dp_a}{2\pi} \langle q'; t | e^{-iH(Q(\tau), P(\tau))dt} | p; t \rangle \langle p; t | O(P(t), Q(t)) | q; t \rangle \quad (2.15)$$

$$= \int \Pi_a \frac{dp_a}{2\pi} O(p, q) e^{-iH(q', p)dt + i \sum_a (q'_a - q_a)p_a}, \quad (2.16)$$

i.e. for every operator we insert the operator with  $Q$  and  $P$  replaced by the integration variables  $q$  and  $p$ , respectively. Thus for a finite time interval  $t < t_1 < t'$  we simply obtain

$$\langle q'; t' | O(P(t_1), Q(t_1)) | q; t \rangle = \int \mathcal{D}q \mathcal{D}p O(p(t_1), q(t_1)) e^{iS}. \quad (2.17)$$

<sup>7</sup>The initial and final state are fixed and not varied.

<sup>8</sup>The operators  $Q_a$  have to be ordered to the right of the conjugate operators  $P_a$ .

Then its expectation value can be expressed in terms of a path integral. Similarly for multiple operators  $O_1(t_1)$  and  $O_2(t_2)$

$$\int \mathcal{D}q \mathcal{D}p O_1(p(t_1), q(t_1)) O_2(p(t_2), q(t_2)) e^{iS} = \begin{cases} \langle q'; t' | O_2(P(t_2), Q(t_2)) O_1(P(t_1), Q(t_1)) | q; t \rangle & t_1 < t_2 \\ \langle q'; t' | O_1(P(t_1), Q(t_1)) O_2(P(t_2), Q(t_2)) | q; t \rangle & t_2 < t_1 \end{cases} \quad (2.18)$$

The right-hand side can be expressed in terms of the time-ordered product of the operators

$$TO(t_1)O(t_2) = \begin{cases} O(t_1)O(t_2) & t_2 < t_1 \\ O(t_2)O(t_1) & t_1 < t_2 \end{cases}. \quad (2.19)$$

It straightforwardly generalizes to the time-ordered product of an arbitrary number of operators.

$$\langle q'; t' | TO_1(t_1) \dots O_n(t_n) | q; t \rangle = \int \mathcal{D}q \mathcal{D}p O_1(t_1) \dots O_n(t_n) e^{iS}. \quad (2.20)$$

This motivates the introduction of external sources  $J$  and  $K$  for a the particle in quantum mechanics (field in quantum field theory)

$$\langle q'; t' | q; t \rangle_{J,K} = \int \mathcal{D}q \mathcal{D}p e^{i(S+Jq+Kp)} \quad (2.21)$$

where the products  $Jq$  and  $Kp$  should be understood  $Jq = \int d\tau J(\tau)q(\tau)$ . Functional derivatives (See footnote 2) with respect to the external sources evaluated for vanishing external sources then generates the time-ordered product of the corresponding operators. For example there is

$$\frac{1}{i} \frac{\delta}{\delta J(t_1)} \langle q'; t' | q; t \rangle_{J,K} \Big|_{J=K=0} = \int \mathcal{D}q \mathcal{D}p q(t_1) e^{iS} = \langle q'; t' | Q(t_1) | q; t \rangle \quad (2.22)$$

$$\frac{1}{i} \frac{\delta}{\delta K(t_1)} \langle q'; t' | q; t \rangle_{J,K} \Big|_{J=K=0} = \int \mathcal{D}q \mathcal{D}p p(t_1) e^{iS} = \langle q'; t' | P(t_1) | q; t \rangle \quad (2.23)$$

$$\frac{1}{i} \frac{\delta}{\delta J(t_2)} \frac{1}{i} \frac{\delta}{\delta J(t_1)} \langle q'; t' | q; t \rangle_{J,K} \Big|_{J=K=0} = \int \mathcal{D}q \mathcal{D}p q(t_1) q(t_2) e^{iS} = \langle q'; t' | TQ(t_1)Q(t_2) | q; t \rangle \quad (2.24)$$

and in general

$$\frac{1}{i} \frac{\delta}{\delta J(t_1)} \dots \frac{1}{i} \frac{\delta}{\delta K(t_n)} \langle q'; t' | q; t \rangle_{J,K} \Big|_{J=K=0} = \langle q'; t' | TQ(t_1) \dots P(t_n) \dots | q; t \rangle. \quad (2.25)$$

## 2.2 Vacuum-vacuum transition

So far we only considered position eigenstates as initial and final states. For other initial and final states we have to multiply with the wave functions  $\langle q | \phi \rangle = \phi(q)$  of these states and integrate over the generalized positions. In particular if we consider transition amplitudes from the infinite past to the infinite future

$$\langle \phi; \infty | \psi; -\infty \rangle_{J,K} = \int dq' dq \int_{q(\infty)=q', q(-\infty)=q} \mathcal{D}q \mathcal{D}p \phi^*(q(\infty)) \psi(q(-\infty)) e^{i(S+Jq+Kp)} \quad (2.26)$$

$$= \int \mathcal{D}q \mathcal{D}p \phi^*(q(\infty)) \psi(q(-\infty)) e^{i(S+Jq+Kp)} \quad (2.27)$$

where we used that the unconstrained Feynman path integral is equivalent to the constrained path integral when we integrate over the boundary conditions. An analogous expression holds for the expectation value of operators.

As discussed in the derivation of the LSZ reduction formula we are specifically interested in vacuum to vacuum transition amplitudes

$$\langle 0; \infty | 0; -\infty \rangle_{J,K} = \int \mathcal{D}\underline{q} \mathcal{D}\underline{p} e^{i(S+Jq+Kp)} \langle 0; \text{out} | q(\infty); \infty \rangle \langle q(-\infty); -\infty | 0; \text{in} \rangle \quad (2.28)$$

Thus we have to evaluate the ground state wave functions in the infinite past and future. We consider two concrete simple examples of free field theories, a quantum theory which reduces to the quantum harmonic oscillator in the absence of interactions and a real scalar field, but before discussing them we first introduce the Lagrangian version of the path integral.

### 2.3 Lagrangian version

In case the Hamiltonian is quadratic in the conjugate momenta it is a Gaussian integral and the integral over the conjugate momenta can be performed exactly.<sup>9</sup> If additionally the quadratic term in the conjugate momenta does not depend on the generalized variables  $q$  the result will only involve constants which can be absorbed in the definition of the integration measure

$$\langle q'; t' | q; t \rangle = \int_{q(t^{(\prime)})=q^{(\prime)}} \mathcal{D}q \exp \left( i \int_t^{t'} d\tau L(q(\tau), \dot{q}(\tau)) \right) \quad (2.30)$$

The function  $L(\dot{q}, q)$  is computed by first determining the stationary point, i.e. by solving

$$0 = \frac{\delta}{\delta p_a(\tau)} \left( \int_t^{t'} \left[ \sum_a \dot{q}_a(\tau') p_a(\tau') - H(q(\tau'), p(\tau')) \right] d\tau' \right) = \dot{q}_a(\tau) - \frac{\partial H(q(\tau), p(\tau))}{\partial p_a(\tau)} \quad (2.31)$$

for  $p$  in terms of  $q$  and  $\dot{q}$ , and then inserting the solution in integrand. This exactly mirrors the procedure in classical mechanics, when we move from the Hamiltonian formulation to Lagrangian formulation. The function  $L$  is the Lagrangian of the system. In the Lagrangian formulation the path integral is explicitly Lorentz invariant. This situation is very common and in particular it applies for the theories we are interested in for the rest of the lecture. Thus we will mostly use the Lagrangian version in the following.

### 2.4 Quantum harmonic oscillator

The Hamiltonian and the Lagrangian of the harmonic oscillator take the form

$$H = \frac{p^2}{2} + \frac{1}{2} \omega^2 q^2 \quad L = \frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega^2 q^2 \quad (2.32)$$

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<sup>9</sup> Note that the Gaussian integral is simply

$$\int dx e^{-(\frac{1}{2} x A x + b x + c)} = \int dx' e^{-(\frac{1}{2} x' A x' + c - \frac{1}{2} b A^{-1} b)} = \sqrt{\frac{2\pi}{A}} e^{\frac{1}{2} b A^{-1} b - c} \quad (2.29)$$

with  $x' = x + A^{-1}b$ .

using convenient units where  $m = 1$ . The relevant commutation relations are

$$[p, q] = -i \quad [p, p] = [q, q] = 0 \quad (2.33)$$

and the annihilation operator is defined by

$$a = \sqrt{\frac{\omega}{2}} \left( q + i \frac{p}{\omega} \right). \quad (2.34)$$

In position space the momentum operator is represented by  $p = -i\partial/\partial q$ . Then the ground state wave function is determined by  $(a|0\rangle = 0$  in position space)

$$0 = \left[ \omega q + \frac{\partial}{\partial q} \right] \langle q|0\rangle \quad \Rightarrow \quad \langle q|0\rangle = N e^{-\frac{1}{2}\omega q^2} \quad (2.35)$$

with some normalization constant  $N$ . In particular this expression holds for the wave function in the infinite future and past and thus the product of the two wave functions is given by.

$$\langle 0; \text{out} | q(\infty); \infty \rangle \langle q(-\infty); -\infty | 0; \text{in} \rangle = |N|^2 e^{-\frac{1}{2}\omega [q(-\infty)^2 + q(\infty)^2]} \quad (2.36)$$

$$= \lim_{\epsilon \rightarrow 0^+} |N|^2 e^{-\frac{\epsilon}{2}\omega \int_{-\infty}^{\infty} d\tau q(\tau)^2 e^{-\epsilon|\tau|}} \quad (2.37)$$

where we used the final value theorem of the Laplace transform. Note that the wave function is independent of the sources and thus only contributes a constant factor to the Feynman path integral. As we are only interested in the limit  $\epsilon \rightarrow 0$  (and  $q(\tau)^2$  is integrable for the harmonic oscillator), this motivates the definition of the partition function

$$Z[J, K] = \langle 0; \infty | 0; -\infty \rangle_{J, K} = \frac{\lim_{\epsilon \rightarrow 0} \int \mathcal{D}\underline{q} \mathcal{D}\underline{p} e^{i(S + Jq + Kp + i\epsilon \int d\tau \frac{1}{2} q^2)}}{\lim_{\epsilon \rightarrow 0} \int \mathcal{D}\underline{q} \mathcal{D}\underline{p} e^{i(S + i\epsilon \int d\tau \frac{1}{2} q^2)}} \quad (2.38)$$

as a generating functional of all correlation functions for vacuum to vacuum transitions. The correlation functions are obtained by taking derivatives with respect to the sources of  $Z[J, K]$ . Equivalently we could discard the normalization factor and consider  $\ln Z[J, K]$ , because derivatives of  $\ln Z[J, K]$  are automatically correctly normalized such that vacuum-vacuum transition amplitude in the absence of sources is normalized to unity, i.e.  $\langle 0|0 \rangle_{J=K=0} = 1$ . Similarly the partition function in the Lagrangian formulation is given by

$$Z[J] = \frac{\lim_{\epsilon \rightarrow 0} \int \mathcal{D}\underline{q} e^{i(S + Jq + i\epsilon \int d\tau \frac{1}{2} q^2)}}{\lim_{\epsilon \rightarrow 0} \int \mathcal{D}\underline{q} e^{i(S + i\epsilon \int d\tau \frac{1}{2} q^2)}}. \quad (2.39)$$

It is convenient to consider the Fourier-transform of  $q$

$$\tilde{q}(E) = \int dt e^{iEt} q(t) \quad q(t) = \int \frac{dE}{2\pi} e^{-iEt} \tilde{q}(E) \quad (2.40)$$

and equivalently for the source  $J(t)$ . Thus the term in the exponent becomes

$$\begin{aligned} S + Jq + \int \frac{i\epsilon}{2} q(\tau)^2 &= \frac{1}{2} \int d\tau \frac{dE}{2\pi} \frac{dE'}{2\pi} e^{-i(E+E')\tau} \left[ (-EE' - \omega^2 + i\epsilon) \tilde{q}(E) \tilde{q}(E') + \tilde{J}(E) \tilde{q}(E') + \tilde{J}(E') \tilde{q}(E) \right] \\ &= \frac{1}{2} \int \frac{dE}{2\pi} \left[ \tilde{q}(E) (E^2 - \omega^2 + i\epsilon) \tilde{q}(-E) + \tilde{J}(E) \tilde{q}(-E) + \tilde{J}(-E) \tilde{q}(E) \right] \end{aligned} \quad (2.41)$$

where we used that the  $\tau$ -integration yields a factor  $2\pi\delta(E + E')$  which has been used to perform the  $E'$ -integration. The path integral measure is invariant under a shift of the integration variables,  $\tilde{q}(E) \rightarrow \tilde{q}(E) - \frac{J(E)}{E^2 - \omega^2 + i\epsilon}$  and thus the amplitude is given by

$$Z[J] = \frac{\exp\left(-\frac{i}{2} \int \frac{dE}{2\pi} \frac{\tilde{J}(E)\tilde{J}(-E)}{E^2 - \omega^2 + i\epsilon}\right) \times \int \mathcal{D}\tilde{q} \exp\left(\frac{i}{2} \int \frac{dE}{2\pi} \tilde{q}(E) [E^2 - \omega^2 + i\epsilon] \tilde{q}(-E)\right)}{Z[0]} \quad (2.42)$$

$$= \exp\left(-\frac{i}{2} \int \frac{dE}{2\pi} \frac{\tilde{J}(E)\tilde{J}(-E)}{E^2 - \omega^2 + i\epsilon}\right). \quad (2.43)$$

Taking an inverse Fourier transform we obtain

$$Z[J] = \exp\left(-\frac{i}{2} \int dt dt' J(t) G(t - t') J(t')\right) \quad (2.44)$$

The Fourier transform of the kernel

$$G(t - t') \equiv \int \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{E^2 - \omega^2 + i\epsilon} \quad (2.45)$$

is a Green's function of the harmonic oscillator

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) G(t - t') = -\delta(t - t') \quad (2.46)$$

which can be shown by a straightforward calculation of the left-hand side. The integral over the energy  $E$  in the definition of the Green's function can also be evaluated via a contour integral in the complex  $E$  plane via the residue theorem.<sup>10</sup> The two simple poles are at  $E = \pm(\omega - i\epsilon)$ , i.e. the pole for positive (negative) energy is shifted slightly to the lower (upper) half of the complex plane. For  $t - t' > 0$  ( $t - t' < 0$ ) we close the contour via the lower (upper) half of the complex plane and thus the residue theorem results in

$$G(t - t') = \frac{1}{2\omega i} e^{-i\omega|t-t'|}. \quad (2.49)$$

The Green's function can similarly be expressed in terms of the 2-point correlation function. The second derivative with respect to the sources of the partition function is

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle = \frac{1}{i} \frac{\delta}{\delta J(t_1)} \frac{1}{i} \frac{\delta}{\delta J(t_2)} Z[J] \Big|_{J=0} = \frac{1}{i} \frac{\delta}{\delta J(t_1)} \left[ - \int dt' G(t_2 - t') J(t') \right] Z[J] \Big|_{J=0} \quad (2.50)$$

$$= iG(t_2 - t_1). \quad (2.51)$$

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<sup>10</sup>The residue theorem relates the contour integral along a positively oriented simple closed curve to the sum of the residues of the enclosed poles

$$\oint_{\gamma} f(z) dz = 2\pi i \sum \text{Res}(f, a_k). \quad (2.47)$$

In particular for a holomorphic function  $f$  the Cauchy integral theorem states

$$2\pi i f(a) = \oint_{\gamma} dz \frac{f(z)}{z - a}. \quad (2.48)$$

We can similarly evaluate higher-order correlation functions. For the four-point function we find

$$\langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = i^2 [G(t_2 - t_1)G(t_4 - t_3) + G(t_3 - t_1)G(t_4 - t_2) + G(t_4 - t_1)G(t_3 - t_2)] . \quad (2.52)$$

Thus the 4-point correlation function is given by the sum of the Green's functions of all possible pairings. More generally  $2n$ -point function is given by

$$\langle 0|TQ(t_1) \dots Q(t_{2n})|0\rangle = i^n \sum_{\text{pairings}} G(t_{i_1} - t_{i_2}) \dots G(t_{i_{2n-1}} - t_{i_{2n}}) . \quad (2.53)$$

All other  $n$ -point functions, i.e. with odd  $n$  vanish, because there is always one remaining source term. Interactions from terms in the Lagrangian with higher powers of the generalized coordinate  $q^n$ ,  $n > 2$  will change this result.

## 2.5 Free scalar field

The Hamiltonian and Lagrangian densities of a free real scalar field are given by

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 \quad \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \quad (2.54)$$

with the conjugate momentum  $\pi = \partial_0\phi$ . The relevant equal-time commutation relations are

$$[\pi(x, t), \phi(y, t)] = -i\delta(x - y) \quad [\pi(x, t), \pi(y, t)] = [\phi(x, t), \phi(y, t)] = 0 \quad (2.55)$$

and the annihilation operator is defined as

$$a(k) = \int d^3x e^{ikx} [i\pi(x) + \omega_k\phi(x)] . \quad (2.56)$$

We follow the discussion of the quantum harmonic oscillator to obtain the ground state for a given time  $t$ . In the field  $\phi$  basis, the conjugate momentum is given by  $\pi(x, t) = -i\frac{\delta}{\delta\phi(x, t)}$  and thus the defining equation of the ground state is

$$0 = \int d^3x e^{-i\vec{k}\cdot\vec{x}} \left[ \frac{\delta}{\delta\phi(x, t)} + \omega_k\phi(x, t) \right] \langle \phi; t|0\rangle \quad (2.57)$$

In analogy to the quantum harmonic oscillator we use a Gaussian ansatz for the ground state wave function

$$\langle \phi; t|0\rangle = N \exp \left( -\frac{1}{2} \int d^3x d^3y \mathcal{E}(x, y) \phi(x, t) \phi(y, t) \right) . \quad (2.58)$$

It provides a solution provided that

$$0 = \int d^3x e^{i\omega_k t} e^{-i\vec{k}\cdot\vec{x}} \left[ - \int d^3y \mathcal{E}(x, y) \phi(y, t) + \omega_k \phi(x, t) \right] \quad (2.59)$$

is satisfied for all  $\phi$ . We can rewrite it as follows

$$0 = \int d^3x e^{i\omega_k t} e^{-i\vec{k}\cdot\vec{x}} \left[ - \int d^3y \mathcal{E}(x, y) \phi(y, t) + \int d^3y \delta(x - y) \omega_k \phi(y, t) \right] \quad (2.60)$$

$$= \int d^3y \left[ - \int d^3x e^{-i\vec{k}\cdot\vec{x}} \mathcal{E}(x, y) + e^{-i\vec{k}\cdot\vec{y}} \omega_k \right] \phi(y, t) e^{i\omega_k t} . \quad (2.61)$$

As the equation holds for all  $\phi$  and the expression in the square brackets is a smooth function, the expression in the square brackets has to vanish identically. Inverting the Fourier transform we find for the kernel

$$\mathcal{E}(x, y) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \sqrt{\vec{k}^2 + m^2} . \quad (2.62)$$

This also holds in the infinite past and future (even for an interacting real scalar field theory) and thus

$$\langle 0; \text{out} | \phi(x, \infty); \infty \rangle \langle \phi(x, -\infty); -\infty | 0; \text{in} \rangle \quad (2.63)$$

$$= |N|^2 \exp \left( -\frac{1}{2} \int d^3x d^3y \mathcal{E}(x, y) [\phi(x, \infty) \phi(y, \infty) + \phi(x, -\infty) \phi(y, -\infty)] \right) \quad (2.64)$$

$$= \lim_{\epsilon \rightarrow 0^+} |N|^2 \exp \left( -\frac{\epsilon}{2} \int d^3x d^3y \int_{-\infty}^{\infty} d\tau \mathcal{E}(x, y) \phi(x, \tau) \phi(y, \tau) e^{-\epsilon|\tau|} \right) \quad (2.65)$$

This term will slightly shift the poles in the complex  $E$  plane and thus will determine the form of the Green's function like in the case of the quantum harmonic oscillator. The generating functional in this case is given by

$$Z[J] = \frac{\int \mathcal{D}\phi \exp \left( i \int d^4x (\mathcal{L} + J(x)\phi(x)) - \frac{\epsilon}{2} \int d^3x d^3y d\tau \mathcal{E}(x, y) \phi(x, \tau) \phi(y, \tau) \right)}{\int \mathcal{D}\phi \exp \left( i \int d^4x \mathcal{L} - \frac{\epsilon}{2} \int d^3x d^3y d\tau \mathcal{E}(x, y) \phi(x, \tau) \phi(y, \tau) \right)} \quad (2.66)$$

$$= \frac{\int \mathcal{D}\phi \exp \left( \frac{i}{2} \int d^4x d^4x' \phi(x) \phi(x') D(x, x') + i \int d^4x J(x) \phi(x) \right)}{\int \mathcal{D}\phi \exp \left( \frac{i}{2} \int d^4x d^4x' \phi(x) \phi(x') D(x, x') \right)} \quad (2.67)$$

where we dropped the factor  $e^{-\epsilon|\tau|}$  in the first line and it is understood that in the end the limit  $\epsilon \rightarrow 0$  is taken.  $D(x, y)$  in the second line is defined as

$$D(x, x') = \frac{\partial}{\partial x'_\mu} \frac{\partial}{\partial x^\mu} \delta(x - x') - m^2 \delta(x - x') + i\epsilon \mathcal{E}(\vec{x}, \vec{x}') \delta(t - t') . \quad (2.68)$$

All three terms only depend on the difference  $x - x'$  and thus the result is translation invariant. In momentum space with

$$\tilde{\phi}(p) = \int d^4x e^{-ipx} \phi(x) \quad \phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} \tilde{\phi}(p) \quad (2.69)$$

we thus obtain

$$\tilde{D}(p) = p^2 - m^2 + i\epsilon E(p) \quad (2.70)$$

$$S + J\phi + i\epsilon \text{ terms} = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \left( \tilde{\phi}(p) \tilde{D}(p) \tilde{\phi}(-p) + \tilde{J}(p) \tilde{\phi}(-p) + \tilde{J}(-p) \tilde{\phi}(p) \right) \quad (2.71)$$



Analogously to the derivation in case of the quantum harmonic oscillator the  $d^4x$  integration yields a factor  $(2\pi)^4 \delta^{(4)}(p + p')$  which introduces the different signs for the momenta. Note that  $\tilde{D}(p)$  is symmetric,  $\tilde{D}(p) = \tilde{D}(-p)$  and thus using linearity of the path integral measure by shifting the field  $\tilde{\phi}(p) \rightarrow \tilde{\phi}(p) - \tilde{D}(p)^{-1} \tilde{J}(p)$  we obtain

$$Z[J] = \frac{1}{Z[0]} \exp \left( -\frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \tilde{J}(p) \tilde{D}(p)^{-1} \tilde{J}(-p) \right) \times \int \mathcal{D}\phi \exp \left( \frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \tilde{\phi}(p) \tilde{D}(p) \tilde{\phi}(-p) \right) \quad (2.72)$$

$$= \exp \left( -\frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \tilde{J}(p) \tilde{D}(p)^{-1} \tilde{J}(-p) \right) \quad (2.73)$$

In position space we obtain

$$Z[J] = \exp \left( -\frac{i}{2} \int d^4x d^4x' J(x) \Delta_F(x - x') J(x') \right) \quad (2.74)$$

where we implicitly defined the Feynman propagator

$$\Delta_F(x - x') \equiv \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-x')}}{p^2 - m^2 + i\epsilon E(p)} . \quad (2.75)$$

As we are only interested in the limit  $\epsilon \rightarrow 0^+$ , it is common to replace  $i\epsilon E(p)$  by  $i\epsilon$  which we will do in the following for simplicity. The Feynman propagator is a Green's function of the Klein-Gordon equation and thus satisfies (in the limit  $\epsilon \rightarrow 0$ )

$$(\square_x + m^2) \Delta_F(x - x') = -\delta^{(4)}(x - x') \quad (2.76)$$

which follows directly from its definition. In the complex plane of the energy there are two simple poles at  $\pm (\sqrt{p^2 + m^2} - i\epsilon)$ . Using the residue theorem we can evaluate the integral over the energy and obtain

$$\Delta_F(x - x') = -i \int \frac{d^3p}{(2\pi)^3 2E(p)} e^{-i\omega_p |t-t'| + i\vec{p}(\vec{x}-\vec{x}')} \quad (2.77)$$

$$= -i \theta(t - t') \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{-ip(x-x')} - i\theta(t' - t) \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{-ip(x'-x)} \quad (2.78)$$

using the result from the same calculation for the harmonic oscillator. The two remaining integrals can be evaluated in terms of modified Bessel functions which we will see when looking at the spin-statistics connection theorem. Finally analogously to the discussion for the harmonic oscillator we obtain the expectation values of the time-ordered products by taking functional derivatives with respect to the sources

$$\langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle = \frac{1}{i} \frac{\delta}{\delta J(x_1)} \dots \frac{1}{i} \frac{\delta}{\delta J(x_n)} Z[J] \Big|_{J=0} \quad (2.79)$$

In particular the 2-point correlation function is given in terms of the Feynman propagator

$$\langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle = i \Delta_F(x_2 - x_1) . \quad (2.80)$$

or explicitly writing out the time-ordered product we find

$$i\Delta_F(x_2 - x_1) = \theta(t_1 - t_2) \langle 0 | \phi(x_1) \phi(x_2) | 0 \rangle + \theta(t_2 - t_1) \langle 0 | \phi(x_2) \phi(x_1) | 0 \rangle \quad (2.81)$$

$$= \theta(t_1 - t_2) \langle 0 | \phi^+(x_1) \phi^-(x_2) | 0 \rangle + \theta(t_2 - t_1) \langle 0 | \phi^+(x_2) \phi^-(x_1) | 0 \rangle . \quad (2.82)$$

with the positive and negative frequency solutions

$$\phi^+(x, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a(k) e^{-ikx} \quad (2.83)$$

$$\phi^-(x, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a^\dagger(k) e^{ikx} . \quad (2.84)$$

Thus the Feynman propagator for  $t_2 < t_1$  can be interpreted as the creation of a particle at  $x_2$  and the annihilation of a particle at  $x_1$  and vice versa for  $t_1 < t_2$ . The time-ordered product of  $2n$  field operators is given by the sum over all pairings

$$\langle 0 | T \phi(x_1) \dots \phi(x_{2n}) | 0 \rangle = i^n \sum_{\text{pairings}} \Delta_F(x_{i_1} - x_{i_2}) \dots \Delta_F(x_{i_{2n-1}} - x_{i_{2n}}) . \quad (2.85)$$

This result is known as *Wick's theorem*.

## 2.6 Interacting scalar field theory

Finally we will be considering interactions. Consider a Lagrangian  $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$  where  $\mathcal{L}_0$  is the Lagrangian of a free field (or exactly solvable) and  $\mathcal{L}_1$  denotes the interaction Lagrangian.<sup>11</sup> Then the generating functional can be written as

$$Z[J] = \frac{\int \mathcal{D}\phi e^{i \int d^4x (\mathcal{L}_0 + \mathcal{L}_1 + J\phi + i\frac{\epsilon}{2}\phi^2)}}{\int \mathcal{D}\phi e^{i \int d^4x (\mathcal{L}_0 + \mathcal{L}_1 + i\frac{\epsilon}{2}\phi^2)}} \quad (2.86)$$

We are going to use our solution for the free field theory to obtain a more convenient expression for the interacting theory<sup>12</sup>

$$Z[J] \propto \int \mathcal{D}\phi e^{i \int d^4x \mathcal{L}_1} e^{i \int d^4x (\mathcal{L}_0 + J\phi + i\frac{\epsilon}{2}\phi^2)} = \exp \left[ i \int d^4x \mathcal{L}_1 \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right] Z_0[J] \quad (2.87)$$

with the generating functional of the free theory

$$Z_0[J] = \exp \left( -\frac{i}{2} \int d^4x d^4x' J(x) \Delta_F(x - x') J(x') \right) \quad (2.88)$$

Thus vacuum expectation values of time-ordered products of operators are given by

$$\langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle = \frac{1}{Z[0]} \frac{1}{i} \frac{\delta}{\delta J(x_1)} \dots \frac{1}{i} \frac{\delta}{\delta J(x_n)} \exp \left[ i \int d^4x \mathcal{L}_1 \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right] Z_0[J] \Big|_{J=0} . \quad (2.89)$$

<sup>11</sup>Recall that we made several assumptions when going to the Lagrangian formulation:  $H$  is no more than quadratic in the conjugate momenta and this term does not depend on the generalized coordinates. If this is not satisfied or if we want to calculate expectation values of conjugate momenta, we have to resort to the Hamiltonian formulation.

<sup>12</sup>The derivation is intuitive. A more rigorous derivation can be found in Ryder chapter 6.4.



Figure 1:  $E = 0$  and  $V = 0$



Figure 2:  $E = 0$  and  $V = 2$

In order to better understand the expression we will do a double expansion of the generating functional in terms of the number of interactions and the number of propagators  $\Delta_F$ . For concreteness we will consider the simplest interaction Lagrangian

$$\mathcal{L}_1 = \frac{1}{3!} g \phi^3. \quad (2.90)$$

This theory is not bounded from below and thus there is no ground state (similar to the harmonic oscillator with a  $q^3$  correction). This however does not become obvious in perturbation theory, when we expand in the small coupling  $g$ . We will ignore it in the following and illustrate the perturbative expansion of the generating functional. The expansion of Eq. (2.87) then results in

$$Z[J] \propto \sum_{V=0}^{\infty} \frac{1}{V!} \left[ \frac{ig}{6} \int d^4x \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right)^3 \right]^V \sum_{P=0}^{\infty} \frac{1}{P!} \left[ -\frac{i}{2} \int d^4x d^4x' J(x) \Delta_F(x-x') J(x') \right]^P. \quad (2.91)$$

For a given term in the expansion with  $P$  propagators and  $V$  vertices the number of surviving sources is  $E = 2P - 3V$ , where  $E$  stands for external legs. In case of the expectation value of a time-ordered product  $E$  has to match the number of operators in the time-ordered product. The overall phase factor of each term is  $i^V \frac{1}{i^{3V}} \frac{1}{i^P} = i^{-P-2V}$ . There are several identical terms for a given set of  $(V, P, E)$ , because the functional derivatives can act on the propagators in different combinations. We can represent each term in the expansion diagrammatically. The diagrams are called *Feynman diagrams* which have been first introduced by R. Feynman. We represent each propagator by a line and each interaction by a vertex where three lines meet. For example there is one connected<sup>13</sup> diagram with  $E = 0$  and  $V = 0$  as shown in Fig. 1 and there are 2 connected diagrams with  $E = 0$  and  $V = 2$  as shown in Fig. 2. The multiplicity of each diagram can be obtained by the following considerations: (i) we can rearrange the three functional derivatives of each vertex without changing the diagram. This results in a factor  $3!$  for each vertex. (ii) We can similarly rearrange the vertices which results in a factor  $V!$ . (iii) For each propagator we can rearrange the ends which results in a factor  $2!$  for each propagator. (iv) Finally we can rearrange the propagators themselves and thus obtain another  $P!$  diagrams. These factors exactly cancel the factors in the double expansion. Thus we represent each vertex by a factor  $ig \int d^4x$ , a propagator by  $i\Delta_F(x-x')$  and an external source by  $i \int d^4x J(x)$ .

<sup>13</sup>A connected diagram is a diagram, where I can trace a path along the diagram between any two vertices.

This outlined counting of the diagrams generally leads to an overcounting for diagrams which possess a symmetry which corresponds to cases when a rearrangement of the functional derivatives can be exactly reverted by a change of the sources. For example for the first diagram in Fig. 2 we can swap the propagators which are connected to the same vertex and compensate it by swapping 2 derivatives yielding a factor  $2^2$ . Furthermore the propagator connecting the two vertices can be swapped and compensated for by swapping the two vertices. For the second diagram we can arrange the propagators in  $3!$  ways and compensate this by exchanging the derivatives at each vertex. In addition, it is possible to simultaneously swap all propagators and compensate it by exchanging the vertices which yields another factor of 2. Thus the term corresponding to each diagram has to be divided by the symmetry factor  $S$  of the diagram.

The shown diagrams are all connected. In general there are also disconnected diagrams. A general diagram can be thought of as the product of the different connected subdiagrams and thus a general diagram

$$D = \frac{1}{S_D} \Pi_I C_I^{n_I} \quad (2.92)$$

where  $C_I$  denotes a connected diagram including its symmetry factor,  $n_I$  is the multiplicity for each connected diagram and  $S_D$  the symmetry factor from exchanging the different connected diagrams. As we already considered the symmetry factors of each individual connected diagram, the symmetry factor  $S_D$  can only account for rearrangements between diagrams. However we only end up with the same diagram, if we exchange all propagators and vertices of one diagram with an identical one and hence  $S_D = \Pi_I n_I!$ . The generating functional up to normalization is given by the sum over all diagrams

$$Z[J] \propto \sum D = \sum_{\{n_I\}} \Pi_I \frac{1}{n_I!} C_I^{n_I} = \Pi_I \sum_{n_I=0}^{\infty} \frac{1}{n_I!} C_I^{n_I} = \Pi_I e^{C_I} = e^{\sum_I C_I} . \quad (2.93)$$

We find that the generating functional is given by the exponential of the sum of all connected diagrams. If we omit diagrams without any external sources in the sum, the so-called vacuum diagrams, we automatically obtain the correct normalization  $Z[0] = 1$  and thus define the generating functional  $W$  for fully connected diagrams

$$iW[J] = \ln Z[J] = \sum_{I \neq \{0\}} C_I \quad (2.94)$$

where the notation  $I \neq \{0\}$  implies that vacuum diagrams are omitted.

We now calculate the vacuum expectation value of  $\phi$ .

$$\langle 0 | \phi(x) | 0 \rangle = \frac{1}{i} \frac{\delta}{\delta J(x)} Z[J] \Big|_{J=0} = \frac{\delta}{\delta J(x)} W[J] \Big|_{J=0} . \quad (2.95)$$

The leading order contribution at order  $\mathcal{O}(g)$  can be obtained from the diagram on the right-hand side of Fig. 3, where the filled circle denotes a source. The source is removed by the derivative and thus we obtain

$$\langle 0 | \phi(x) | 0 \rangle = \frac{1}{i} \frac{\delta}{\delta J(x)} \frac{1}{2} \left[ ig \int d^4 y \right] \left[ i \int d^4 y' J(y') \right] i \Delta_F(y-y') i \Delta_F(y-y) = -i \frac{g}{2} \Delta_F(0) \int d^4 y \Delta_F(y-x) \quad (2.96)$$

with symmetry factor  $\frac{1}{2}$ . This is in contradiction with the requirement for the LSZ reduction formula which requires that  $\langle 0|\phi(x)|0\rangle = 0$ . Thus we have to modify our theory by introducing an additional term in the interaction Lagrangian, a so-called tadpole term,  $Y\phi$

$$\mathcal{L}_1 = \frac{g}{6}\phi^3 + Y\phi. \quad (2.97)$$

It will lead to a new vertex which contributes a factor  $iY \int d^4y$ . In this modified theory there is a second contribution from the diagram on the left-hand side in Fig. 3 and thus

$$\langle 0|\phi(x)|0\rangle = \left(Y - i\frac{g}{2}\Delta_F(0)\right) \int d^4y \Delta_F(y-x). \quad (2.98)$$

By choosing  $Y$  we can ensure that the vacuum expectation value of the field vanishes as required by the LSZ reduction formula. This procedure is called renormalization. Let us evaluate  $\Delta_F(0)$  to obtain the required value of  $Y$

$$\Delta_F(0) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon}. \quad (2.99)$$

This integral diverges quadratically! Thus we have to regularize the integral first before evaluating it. We are using cutoff regularization, where we impose a cutoff on the momentum in Euclidean space  $|k_E| < \Lambda$ . We first perform a so-called Wick rotation,  $k^0 \rightarrow ik_E^4$ ,  $k^i \rightarrow k_E^i$  to go from Minkowski space to Euclidean space, then use spherical coordinates to evaluate the integral.

$$i \int_{|k_E| < \Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} = \int_{|k_E| < \Lambda} \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} \quad (2.100)$$

$$= \int d\Omega_4 \int_0^{\Lambda^2} \frac{k_E^2 dk_E^2}{2(2\pi)^4} \frac{1}{k_E^2 + m^2} \quad (2.101)$$

$$= \frac{2\pi^2}{2(2\pi)^4} \int_{m^2}^{\Lambda^2+m^2} dx \left(1 - \frac{m^2}{x}\right) \quad (2.102)$$

$$= \frac{1}{16\pi^2} [x - m^2 \ln x]_{m^2}^{\Lambda^2+m^2} \quad (2.103)$$

$$= \frac{1}{16\pi^2} \left[ \Lambda^2 + m^2 \ln \left(1 + \frac{\Lambda^2}{m^2}\right) \right] \quad (2.104)$$

where we used that the area of a d-dimensional unit sphere is given by

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad (2.105)$$

and the substitution  $x = k_E^2 + m^2$ . Thus the leading divergence is quadratic and the second term diverges logarithmically. The tadpole coupling  $Y = \frac{g}{2}i\Delta_F(0)$  is real as it is required for a hermitian Lagrangian.

Divergences in the evaluation of loop integrals are a common feature in quantum field theory. They are dealt with by first regularizing the integrals and then renormalizing couplings in the Lagrangian

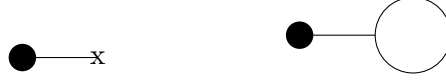


Figure 3: Leading contribution to tadpole. Filled circle correspond to external source.  $x$  denotes vertex  $Y$ .

to absorb the divergences. Time-permitting we will discuss it in more detail at the end of the lecture course.

Summarising, the generating functional  $Z[J]$  can be expressed in terms of the generating functional  $W[J]$ , which is given by the sum of all connected diagrams with no tadpoles and at least two sources  $J$ . The Lagrangian has to include all relevant couplings such that all divergences can be absorbed.

## 2.7 Scattering matrix

We are now in the position to calculate the probability for a transition from an initial state  $|i\rangle$  to a final state  $|f\rangle$ . We consider the example of  $2 \rightarrow 2$  scattering  $1 + 2 \rightarrow 3 + 4$  and thus need the vacuum expectation value of the time-ordered product of four field operators

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = \delta_1\delta_2\delta_3\delta_4 Z[J]|_{J=0} \quad (2.106)$$

$$= \left[ \delta_1\delta_2\delta_3\delta_4 iW[J] \right] \quad (2.107)$$

$$+ \delta_1\delta_2 iW[J]\delta_3\delta_4 iW[J] + \delta_1\delta_3 iW[J]\delta_2\delta_4 iW[J] + \delta_1\delta_4 iW[J]\delta_2\delta_3 iW[J] \Big]_{J=0} \\ = \langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle_C - \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) \quad (2.108) \\ - \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) - \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3)$$

where we used  $Z[J] = \exp(iW[J])$ ,  $\delta_i W[0] = 0$  and defined a short-hand for the functional derivative with respect to the source

$$\delta_i \equiv \frac{1}{i} \frac{\delta}{\delta J(x_i)} . \quad (2.109)$$

The first term corresponds to a fully connected diagram, where all four particles are connected to each other, while the others are products of 2-point functions and thus represent disconnected diagrams where always 2 particles are connected with a propagator. The LSZ reduction formula then determines the transition matrix element. Let us first consider how the disconnected diagrams contribute to the scattering amplitude. For example the contribution from  $-\Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4)$  can be separated

$$i^4 \int e^{ik_1x_1 + ik_2x_2 - ik_3x_3 - ik_4x_4} \Pi_i d^4x_i (\partial_i^2 + m^2) i\Delta_F(x_1 - x_3) i\Delta_F(x_2 - x_4) = F_{13}F_{24} \quad (2.110)$$

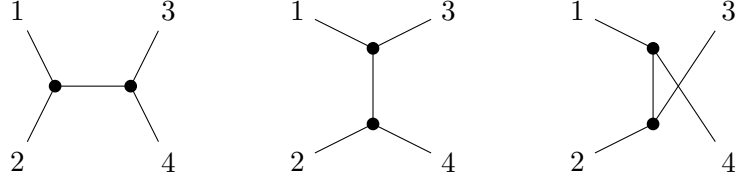


Figure 4: Tree-level diagrams contributing to  $2 \rightarrow 2$  scattering

with

$$F_{13} = i \int d^4x_1 d^4x_3 \frac{d^4p}{(2\pi)^4} (p^2 - m^2) e^{ik_1x_1 - ik_3x_3 - ip(x_1 - x_3)} \quad (2.111)$$

$$= i \int \frac{d^4p}{(2\pi)^4} (p^2 - m^2) (2\pi)^4 \delta(k_1 - p) (2\pi)^4 \delta(k_3 - p) \quad (2.112)$$

$$= i(2\pi)^4 \delta(k_3 - k_1) (k_1^2 - m^2) \quad (2.113)$$

$$F_{24} = i(2\pi)^4 \delta(k_4 - k_2) (k_2^2 - m^2) . \quad (2.114)$$

Thus the 4-momenta of the outgoing particles are equal to the ones of the incoming particles and it only contributes to forward scattering when the final state is the same as the initial state and thus does not contribute to the  $T$  matrix. The other two disconnected contributions are proportional to  $\delta$  functions like  $\delta(k_1 + k_2)$  with  $k_1^0 + k_2^0 \geq 2m > 0$  and thus do not contribute. Generally only fully connected diagrams which are generated by taking derivatives of  $W$  are of interest.

Here the leading order is generated by the tree-level diagrams in Fig. 4. They translate to (We drop the subscript  $F$  from the Feynman propagator in the following.)

$$\begin{aligned} \delta_1 \delta_2 \delta_3 \delta_4 iW = (ig)^2 i^5 \int d^4x d^4y \Delta(x - y) & \left[ \Delta(x_1 - x) \Delta(x_2 - x) \Delta(y - x_3) \Delta(y - x_4) \right. \\ & + \Delta(x_1 - x) \Delta(x_3 - x) \Delta(y - x_2) \Delta(y - x_4) \\ & \left. + \Delta(x_1 - x) \Delta(x_4 - x) \Delta(y - x_2) \Delta(y - x_3) \right] \end{aligned} \quad (2.115)$$

and thus it contributes to the  $S$  matrix as follows

$$\begin{aligned} -ig^2 \int d^4x d^4y e^{ik_1x_1 + ik_2x_2 - ik_3x_3 - ik_4x_4} \Pi_i d^4x_i \Delta(x - y) & \left[ \delta(x_1 - x) \delta(x_2 - x) \delta(y - x_3) \delta(y - x_4) \right. \\ & + \delta(x_1 - x) \delta(x_3 - x) \delta(y - x_2) \delta(y - x_4) \\ & \left. + \delta(x_1 - x) \delta(x_4 - x) \delta(y - x_2) \delta(y - x_3) \right] \end{aligned} \quad (2.116)$$

$$= -ig^2 \int d^4x d^4y \Delta(x-y) \left[ e^{i(k_1+k_2)x-i(k_3+k_4)y} + e^{i(k_1-k_3)x+i(k_2-k_4)y} + e^{i(k_1-k_4)x+i(k_2-k_3)y} \right] \quad (2.117)$$

$$= -ig^2 \int d^4x d^4y \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon} \left[ e^{i(k_1+k_2)x-i(k_3+k_4)y} + e^{i(k_1-k_3)x+i(k_2-k_4)y} + e^{i(k_1-k_4)x+i(k_2-k_3)y} \right] \quad (2.118)$$

$$= -ig^2 \int d^4x d^4y \frac{d^4p}{(2\pi)^4} \frac{e^{i(k_1+k_2-p)x-i(k_3+k_4-p)y} + e^{i(k_1-k_3-p)x+i(k_2-k_4+p)y} + e^{i(k_1-k_4-p)x+i(k_2-k_3+p)y}}{p^2 - m^2 + i\epsilon} \quad (2.119)$$

$$= -i(2\pi)^4 g^2 \int d^4p \frac{\delta(k_1+k_2-p)\delta(k_3+k_4-p) + \delta(k_1-k_3-p)\delta(k_2-k_4+p) + \delta(k_1-k_4-p)\delta(k_2-k_3+p)}{p^2 - m^2 + i\epsilon} \quad (2.120)$$

$$= -ig^2(2\pi)^4 \delta(k_1+k_2-k_3-k_4) \left[ \frac{1}{s-m^2+i\epsilon} + \frac{1}{t-m^2+i\epsilon} + \frac{1}{u-m^2+i\epsilon} \right] \quad (2.121)$$

where we introduced the so-called Mandelstam variables

$$s = (k_1 + k_2)^2 \quad t = (k_1 - k_3)^2 \quad u = (k_1 - k_4)^2 \quad (2.122)$$

In the centre-of-mass frame  $s$  is simply given by the square of the total energy of the system  $s = (E_1 + E_2)^2$ . The Mandelstam variables satisfy

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2. \quad (2.123)$$

For general scattering processes it is convenient to define the matrix element  $M$  via

$$\langle f|i \rangle = iT_{fi} = i(2\pi)^4 \delta(k_{in} - k_{out}) M \quad (2.124)$$

where  $k_{in}$  ( $k_{out}$ ) is the sum of incoming (outgoing) momenta. Looking at our calculation we can derive a simple set of Feynman rules to calculate the matrix element:

1. Each external line corresponds to a factor 1.
2. For each internal line with momentum  $p$  write  $\frac{i}{p^2 - m^2 + i\epsilon}$ .
3. For each vertex write  $ig$ .
4. At each vertex the four-momentum is conserved.
5. Integrate over internal free momenta  $k_i$  with measure  $d^4k_i/(2\pi)^4$ .
6. Divide by symmetry factor of the diagram.
7. Sum over the expressions of the different diagrams.

Note momenta of incoming (outgoing) particles are going in (out).



The probability for a transition from state  $|i\rangle$  to state  $|f\rangle$  is then given by

$$P = \frac{|\langle f|i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} \quad (2.125)$$

We will first consider a finite volume  $V$  and time interval  $T$  in order to avoid infinities and then take the continuum limit. Thus

$$(2\pi)^3 \delta^3(0) = \int d^3x e^{i0x} = V \quad (2\pi)^4 \delta^4(0) = \int d^4x e^{i0x} = VT \quad (2.126)$$

For definiteness we will consider the scattering of 2 particles to an arbitrary final state with  $n$  particles. The norm of a one-particle state is given by

$$\langle k|k\rangle = 2k^0(2\pi)^3 \delta^3(0) = 2k^0V \quad (2.127)$$

and as the two-particle state in the infinite past and future can be effectively described by the product of two one-particle states, we obtain

$$\langle i|i\rangle = 4E_1E_2V^2 \quad \langle f|f\rangle = \Pi_i(2k_i^0V) . \quad (2.128)$$

Similarly for the squared transition amplitude we obtain

$$|\langle f|i\rangle|^2 = |(2\pi)^4 \delta(k_{in} - k_{out})|^2 |M|^2 = (2\pi)^4 \delta(k_{in} - k_{out}) VT |M|^2 . \quad (2.129)$$

Finally we have to sum over all possible final states. In the box with length  $L$  the 3-momenta are quantized  $\vec{k}_i = \frac{2\pi}{L} \vec{n}_i$  and thus summing over the different modes corresponds to an integration

$$\sum_{n_i} \rightarrow \frac{L^3}{(2\pi)^3} \int d^3k_i . \quad (2.130)$$

The volume  $V = L^3$  cancels against the volume factor from the normalization. Thus the transition rate is

$$\dot{P} = \frac{(2\pi)^4 \delta(k_{in} - k_{out})}{4E_1E_2V} \int \Pi_{i=1}^n \frac{d^3k_i}{(2\pi)^3 2\omega_i} |M|^2 . \quad (2.131)$$

The Lorentz-invariant differential cross section can be obtained by dividing by the incoming particle flux. In the rest frame of the second particle, it is simply given by the velocity of the first particle per volume. In the centre-of-mass frame (where the 3-momenta of the incoming particles add to zero), it is the relative velocity per volume  $\sigma = \dot{P}V/v_{rel}$  where the relative velocity can be expressed by

$$v_{rel} = |\vec{v}_1 - \vec{v}_2| = \left| \frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \right| = \frac{|\vec{p}_1|}{E_1E_2} (E_1 + E_2) = \frac{|\vec{p}_1|}{E_1E_2} \sqrt{s} \quad (2.132)$$

or in terms of the flux factor<sup>14</sup>

$$F = E_1E_2v_{rel} = \sqrt{s} \sqrt{E_1^2 - m_1^2} = \frac{1}{2} \sqrt{(s + m_1^2 - m_2^2)^2 - 4sm_1^2} = \sqrt{(p_1 \cdot p_2)^2 - m_1^2m_2^2} \quad (2.134)$$

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<sup>14</sup> The energy of one particle in the centre-of-mass frame is

$$E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \quad (2.133)$$

which can be derived by using the definition of  $s$  and  $\vec{p}_1 + \vec{p}_2 = 0$ .

where we used  $s = m_1^2 + m_2^2 + 2p_1 \cdot p_2$ . and thus the differential cross section is determined by

$$4F d\sigma = |M|^2 d\text{LIPS}_n(k_1 + k_2) \quad (2.135)$$

where the Lorentz-invariant  $n$ -body phase space is defined by

$$d\text{LIPS}_n(k) = (2\pi)^4 \delta\left(k - \sum_{i=1}^n k_i\right) \prod_{i=1}^n \frac{d^3 k_i}{(2\pi)^3 2\omega_i} . \quad (2.136)$$

The 2-body phase space is particularly simple to evaluate in the centre-of-mass frame

$$\int d\text{LIPS}_2(k) = \int (2\pi)^4 \delta(k - k_1 - k_2) \frac{d^3 k_1}{(2\pi)^3 2\omega_1} \frac{d^3 k_2}{(2\pi)^3 2\omega_2} \quad (2.137)$$

$$= \int (2\pi) \delta(\sqrt{s} - E_1 - E_2) \frac{d^3 k_1}{(2\pi)^3 2E_1 2E_2(k_1)} \quad (2.138)$$

$$= \int \delta(\sqrt{s} - E_1 - E_2) \frac{|\vec{k}_1| dE_1 d\Omega}{16\pi^2 E_2(E_1)} = \int \frac{d\Omega}{16\pi^2} \frac{|\vec{k}_1|}{\sqrt{s}} \quad (2.139)$$

$$= \int_{-1}^1 \frac{d \cos \theta}{8\pi} \frac{|\vec{k}_1|}{\sqrt{s}} \quad (2.140)$$

where the last line holds if the integrand does not depend on the azimuthal angle  $\phi$ .

In case that there are  $n$  identical particles in the final state, we have to divide by the symmetry factor  $S = n!$  in the cross section in order to avoid double-counting the different final state configurations. Thus the total cross section is given by

$$\sigma = \frac{1}{S} \int d\sigma . \quad (2.141)$$

If there is one particle in the initial state and we are studying decays we have to slightly modify our assumptions, because we assumed all particles to be stable in the previous discussion. However it turns out that the LSZ reduction formula also holds in this case. We only have to modify the initial state normalization  $\langle i|i \rangle = 2E_1 V$  and find for the differential decay rate

$$d\Gamma = \frac{|M|^2}{2E_1} d\text{LIPS}_n(k_1) \quad (2.142)$$

and the total decay rate is obtained by summing over all outgoing momenta and dividing by the symmetry factor

$$\Gamma = \frac{1}{S} \int d\Gamma . \quad (2.143)$$

Note that the decay rate is not a Lorentz scalar. In the centre-of-mass frame of the particle, there is  $E_1 = m_1$ , while the decay rate is smaller in any other frame by a factor  $\gamma = E_1/m_1$ , the relativistic boost factor, which accounts for the relativistic time dilation. Faster particles have a longer lifetime, e.g. muons generated in the atmosphere reach the Earth's surface due to this time dilation factor.

### 3 Spin 1 – gauge theories

Consider complex scalar field  $\phi$ . The Lagrangian

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi^\dagger \phi) \quad (3.1)$$

with potential  $V$  is invariant under global phase transformations  $\phi \rightarrow e^{i\alpha} \phi$ :  $\phi^\dagger \phi \rightarrow \phi^\dagger \phi$  and also  $|\partial_\mu \phi|^2 \rightarrow |\partial_\mu \phi|^2$ . The potential is also invariant under local phase transformations, a so-called gauge transformation, i.e. transformations which depend on the position

$$\phi(x) \rightarrow e^{i\alpha(x)} \phi(x) , \quad (3.2)$$

but the kinetic term is not invariant under gauge transformations, because

$$\partial_\mu \phi(x) \rightarrow \partial_\mu e^{i\alpha(x)} \phi(x) = e^{i\alpha(x)} [\partial_\mu \phi(x) + i(\partial_\mu \alpha(x)) \phi(x)] . \quad (3.3)$$

In order to obtain a field theory which is invariant under gauge transformations, we introduce the covariant derivative

$$D_\mu = \partial_\mu + igA_\mu \quad (3.4)$$

with a gauge field (or vector field)  $A_\mu$ , i.e. the field transforms as a vector under the Lorentz group. The Lagrangian

$$\mathcal{L} = (D_\mu \phi)^\dagger D^\mu \phi - V(\phi^\dagger \phi) \quad (3.5)$$

is invariant under gauge transformations if the covariant derivative transforms as

$$D_\mu \phi(x) \rightarrow e^{i\alpha(x)} D_\mu \phi(x) \quad (3.6)$$

or the gauge field  $A_\mu$  transforms as follows with respect to a gauge transformation

$$gA_\mu(x) \rightarrow gA_\mu(x) - \partial_\mu \alpha(x) . \quad (3.7)$$

From the gauge field  $A_\mu$  we can form the field strength tensor

$$F_{\mu\nu} = \frac{1}{ig} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu , \quad (3.8)$$

and its dual field strength tensor  $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ . The field strength tensor is invariant under gauge transformations, i.e.  $F_{\mu\nu} \rightarrow F_{\mu\nu}$ . The components of the field strength tensor  $F^{\mu\nu}$  can be identified with electric  $\vec{E}$  and magnetic  $\vec{B}$  fields

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \quad (3.9)$$

and thus we can write Maxwell's equations as

$$\partial_\mu F^{\mu\nu} = J^\nu \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 \quad (3.10)$$

with external source  $(J^\nu) = (\rho, \vec{j})$  [scalar potential  $\rho$  and current  $\vec{j}$ ]. The first set of equations can be derived from the Lagrangian of quantum electrodynamics (QED) together with a source term

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu. \quad (3.11)$$

and the second set of Maxwell equations is a direct consequence of the Bianchi identity.

In addition to the kinetic term there is another term which can be formed out of the field strength tensor and its dual

$$\mathcal{L} = c \frac{i}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} = c \frac{i}{8\pi^2} \partial_\mu (A_\nu \tilde{F}^{\mu\nu}). \quad (3.12)$$

As it can be written as a total derivative using the Bianchi identity, it is a topological term.

The Lagrangian is invariant under an infinite number of symmetries defined by the functions  $\alpha(x)$ . However there is not an infinite number of conserved charges! We should interpret a gauge symmetry as a redundancy in our description of the fields. In electrodynamics there are only two physical degrees of freedom, the two polarisation states of the photon. This has to be compared to the 4 components of the gauge field  $A_\mu(x)$ . This requires us to fix the gauge, i.e. to impose conditions on the gauge field. Common choices are:

- Lorentz gauge:  $\partial_\mu A^\mu = 0$ , which is Lorentz invariant as the name suggests. It however does not fix the gauge completely, because there are non-trivial solutions to  $\partial_\mu \partial^\mu \alpha = 0$ .
- Coulomb gauge:  $\nabla \cdot \vec{A} = 0$ , is not Lorentz invariant, but fixes the gauge completely. In the absence of charged matter, the zeroth component of the gauge field vanishes and thus the only remaining degrees of freedom are the two transverse components of  $\vec{A}$ . Thus it is straightforward to quantize a gauge field in Coulomb gauge using canonical quantization for the two physical degrees of freedom.

### 3.1 Gupta-Bleuler quantization

We would like to obtain a Lorentz covariant quantum theory of free electrodynamics

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) \quad (3.13)$$

$$= \frac{1}{2}A_\nu (\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu) + \text{total derivative} \quad (3.14)$$

$$= \frac{1}{2}A_\mu [\Box g^{\mu\nu} - \partial^\mu \partial^\nu] A_\nu + \text{total derivative}. \quad (3.15)$$

As we require Lorentz invariance we have to consider all four components of the gauge field  $A_\mu$  and its conjugate momentum  $\pi_\nu$ . They should satisfy the canonical equal-time commutation relations

$$[A_\mu(x, t), \pi_\nu(y, t)] = -ig_{\mu\nu} \delta^{(3)}(x - y) \quad [A_\mu(x, t), A_\nu(y, t)] = [\pi_\mu(x, t), \pi_\nu(y, t)] = 0. \quad (3.16)$$

However there is a problem, since the conjugate momenta

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} \quad (3.17)$$

do not have a time-component,  $\pi^0 = 0$ . Thus it is impossible to define an operator which satisfies the equal-time commutation relations. We thus have to modify the Lagrangian. One way forward due to Fermi is to start from the Lagrangian (with  $\xi = 1$ <sup>15</sup>)

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \quad (3.18)$$

It is equivalent to the original Lagrangian after imposing the Lorentz gauge condition and leads to the following equations of motion

$$\left[ \square g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] A_\nu = 0 \quad (3.19)$$

and in particular for  $\xi = 1$  the equations of motion is simply the massless Klein-Gordon operator applied to the gauge field

$$\square A_\nu = 0 \quad (3.20)$$

and thus the propagator is  $\tilde{\Delta}_{\mu\nu}(p) = -g_{\mu\nu}/(p^2 + i\epsilon)$ . The conjugate momentum now becomes

$$\pi^\mu = F^{\mu 0} - g^{\mu 0} \partial_\nu A^\nu \quad (3.21)$$

and the equations of motion (3.20) are solved by a plane wave expansion

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2k^0} \sum_{\lambda=0}^3 \left[ \epsilon_\mu^{(\lambda)}(k) a_\lambda e^{-ikx} + \epsilon_\mu^{(\lambda)*}(k) a_\lambda^\dagger e^{ikx} \right] \quad (3.22)$$

with the four polarisation vectors  $\epsilon_\mu^{(\lambda)}$  with a Lorentz invariant normalization

$$\epsilon_\mu^{(\lambda)*} \epsilon_\nu^{(\lambda')} g^{\mu\nu} = g^{\lambda\lambda'}, \quad (3.23)$$

i.e.  $\epsilon^{(0)}$  is time-like, while the others are space-like. For a photon moving along the third axis, i.e. with momentum  $k^\mu = (k, 0, 0, k)$ , the polarisation vectors can be chosen as  $\epsilon_\mu^{(\lambda)} = \delta_\mu^\lambda$ , and thus  $\vec{k} \cdot \epsilon^{(1,2)} = 0$ . Thus the polarisations  $\epsilon^{(0)}$  is 'time-like' or 'scalar',  $\epsilon^{(3)}$  longitudinal and thus clearly unphysical, while the polarisations  $\epsilon^{(1,2)}$  are transverse. Let us go ahead and try to quantize the theory. We calculate the commutation relation

$$[a_\lambda, a_{\lambda'}^\dagger] = -g_{\lambda\lambda'} 2k_0 (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}'). \quad (3.24)$$

This however poses a problem. The time-like polarisation has an opposite sign. This implies that 1-particle time-like states

$$|1\rangle = \int \frac{d^3k}{(2\pi)^3 2k^0} a_0^\dagger |0\rangle \quad (3.25)$$

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<sup>15</sup>We keep  $\xi$  general because we will recover it from the Faddeev-Popov trick which is discussed in the next section.

have a negative norm  $\langle 1|1 \rangle < 1$ , also the Hamiltonian

$$H = \int \frac{d^3k}{(2\pi)^3 2k^0} k^0 \left[ -g^{\lambda\lambda'} a_\lambda^\dagger a_{\lambda'} \right] \quad (3.26)$$

has a negative expectation value for a 1-particle time-like state,  $\langle 1|H|1 \rangle < 0$ . This indicates a problem with time-like states. They are clearly unphysical and have to be removed, also we did not impose the Lorentz gauge condition yet. The Lorentz gauge condition can not be applied at the operator level, because otherwise the commutation relations can not be satisfied. Gupta and Bleuler suggested to impose the following weaker condition

$$\partial_\mu A^{+\mu} |\psi\rangle = 0 \quad (3.27)$$

on any physical state  $|\psi\rangle$  where  $A^+$  is the positive frequency part of the gauge field expansion, i.e. it contains the annihilation operator. This implies that the expectation value of  $\partial_\mu A^\mu$  vanishes for all physical states

$$\langle \psi | \partial_\mu A^\mu | \psi \rangle = \langle \psi | \partial_\mu A^{+\mu} + \partial_\mu A^{-\mu} | \psi \rangle = 0. \quad (3.28)$$

If we substitute the field expansion (3.22) into Eq. (3.27) we find that physical states contain an admixture of time-like and longitudinal degrees freedom

$$\sum_{\lambda=0}^3 k^\mu \epsilon_\mu^{(\lambda)} a_\lambda(k) |\psi\rangle = 0 \quad k^\mu = (k, 0, 0, k) \quad [a_3(k) - a_0(k)] |\psi\rangle = 0 \quad (3.29)$$

such that their contribution exactly cancels, e.g. in the Hamiltonian.

### 3.2 LSZ reduction formula for quantum electrodynamics

From the expansion of the gauge field in Eq. (3.22) together with the normalization of the polarisation vectors we obtain expressions for the annihilation/creation operators (no summation over  $\lambda$ )

$$\epsilon_\mu^{(\lambda)*}(k) \int d^3x e^{ikx} [k^0 + i\partial_0] A^\mu(x) = g^{\lambda\lambda} a_\lambda(k) \quad (3.30)$$

$$\epsilon_\mu^{(\lambda)}(k) \int d^3x e^{-ikx} [k^0 - i\partial_0] A^\mu(x) = g^{\lambda\lambda} a_\lambda^\dagger(k) \quad (3.31)$$

From here we can follow the derivation in Sec. 1.6 to deduce the LSZ reduction formula for quantum electrodynamics. Creation (annihilation) operators for incoming (outgoing) photons will be replaced by (no sum over  $\lambda$ )

$$a_\lambda^\dagger(k)_{\text{in}} \rightarrow -ig^{\lambda\lambda} \epsilon_\mu^{(\lambda)} \int d^4x e^{-ikx} \partial^2 A^\mu(x) \quad a_\lambda(k)_{\text{out}} \rightarrow -ig^{\lambda\lambda} \epsilon_\mu^{(\lambda)*} \int d^4x e^{ikx} \partial^2 A^\mu(x). \quad (3.32)$$

Thus the transition amplitude is given by replacing the creation and annihilation operators as outlined above and taking the vacuum expectation value of the time-ordered product of the field operators. Similar to the case of the real scalar field, the LSZ formula is valid if the field is normalized like a free field

$$\langle 0 | A_\mu(x) | 0 \rangle = 0 \quad \langle 0 | A_\mu(x) | k, \lambda \rangle = \epsilon_\mu^{(\lambda)}(k) e^{-ikx} \quad (3.33)$$

where  $|k, \lambda\rangle$  is a single photon state which is normalized to  $\langle k', \lambda' | k, \lambda \rangle = -g_{\lambda\lambda'} (2\pi)^3 2k^0 \delta^{(3)}(k - k')$ .

### 3.3 Path integral quantization of quantum electrodynamics

We now move on to look at the path integral for quantum electrodynamics. The gauge freedom poses a problem for the path integral: The integration over the redundant degrees of freedom in the path integral leads to a divergent result, which can be seen by considering a gauge field configuration which is pure gauge. In this case the kinetic term vanishes and thus the integration over this direction in field space diverges.

Moreover, even ignoring this infinity there is no Feynman propagator: The differential operator  $\square g_{\mu\nu} - \partial_\mu \partial_\nu$  is not invertible. It annihilates any pure gauge term  $\partial_\mu \alpha$  and thus it is impossible to derive a Green's function/Feynman propagator. The path integral diverges, because we are integrating over a continuous infinity of physically equivalent field configurations. Thus we would like to restrict the path integral to the physical configuration space, i.e. pick one (inequivalent) configuration from each gauge orbit. We follow the approach by Faddeev and Popov on how to isolate the gauge degrees of freedom. We choose a gauge fixing condition  $0 = G(A)$ . For example we obtain the Lorentz gauge for  $G(A) = \partial_\mu A^\mu$ . Let  $A_\mu^\alpha$  denote the gauge-transformed gauge field,  $A_\mu^\alpha = A_\mu + \frac{1}{g} \partial_\mu \alpha$ . Then the continuum generalisation of the identity

$$1 = \int \Pi_i da_i \det \left( \frac{\partial g_i}{\partial a_j} \right) \delta^{(n)}(g(a)) \quad \rightarrow \quad 1 = \int \mathcal{D}\alpha(x) \det \left( \frac{\delta G(A^\alpha)}{\delta \alpha} \right) \delta(G(A^\alpha)) \quad (3.34)$$

can be used to insert the identity into the path integral. The determinant is generally independent of  $A$ , e.g. in Lorentz gauge we find  $\det(\delta G(A^\alpha)/\delta \alpha) = \det(\partial^2/g)$ . We insert the identity into the path integral and thus obtain

$$\int \mathcal{D}A e^{iS} = \det \left( \frac{\delta G(A^\alpha)}{\delta \alpha} \right) \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS} \delta(G(A^\alpha)) \quad (3.35)$$

$$= \det \left( \frac{\delta G(A^\alpha)}{\delta \alpha} \right) \int \mathcal{D}\alpha \int \mathcal{D}A^\alpha e^{iS} \delta(G(A^\alpha)) \quad (3.36)$$

$$= \det \left( \frac{\delta G(A^\alpha)}{\delta \alpha} \right) \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS} \delta(G(A)) \quad (3.37)$$

where we shift the integration variable  $A \rightarrow A^\alpha$  in the second line and then rename it in the third to  $A$ . In this way we isolated the integral over the gauge parameter. The remaining path integral satisfies the gauge fixing condition  $G(A) = 0$ . We choose the gauge condition

$$G(A) = \partial_\mu A^\mu(x) - \omega(x) \quad (3.38)$$

with a general scalar function  $\omega(x)$ . The usual Lorentz condition is obtained by setting  $\omega$  to zero

$$\int \mathcal{D}A e^{iS} = \det \left( \frac{1}{g} \partial^2 \right) \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS} \delta(\partial_\mu A^\mu - \omega). \quad (3.39)$$

As this holds for any  $\omega$ , we can integrate over it weighing it with a Gaussian:  $\exp[-i \int d^4x \omega^2(x)/2\xi]$

$$\int \mathcal{D}A e^{iS} = N(\xi) \int \mathcal{D}\omega \exp \left[ -i \int d^4x \frac{\omega^2}{2\xi} \right] \det \left( \frac{1}{g} \partial^2 \right) \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS} \delta(\partial_\mu A^\mu - \omega) \quad (3.40)$$

$$= N(\xi) \det \left( \frac{1}{g} \partial^2 \right) \int \mathcal{D}\alpha \int \mathcal{D}A e^{iS} \exp \left[ -i \int d^4x \frac{(\partial_\mu A^\mu)^2}{2\xi} \right] \quad (3.41)$$

where  $N(\xi)$  is an normalization constant. Note that the gauge condition is not satisfied exactly. Gauge-equivalent contributions are however exponentially suppressed. In the last line we evaluated the path integral over  $\omega$ . In the final result there is a new term in the Lagrangian, the so-called Faddeev-Popov term which is quadratic in the gauge fixing condition. We can repeat the same derivation for expectation values of *gauge-invariant* operators. The restriction to gauge-invariant operators is important for the step when we shifted the integration variables from  $A \rightarrow A^\alpha$ , i.e. performed a gauge transformation. The pre-factors  $N(\xi) \det(\partial^2/g) \int \mathcal{D}\alpha$  drop out after properly normalising the partition function

$$Z[J] = \int \mathcal{D}A e^{i \int d^4x \left( \mathcal{L} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + i\epsilon \text{ terms} \right)}. \quad (3.42)$$

Popular (convenient) choices of the gauge parameter  $\xi$  are  $\xi = 1$ , the so-called "Feynman gauge", and  $\xi = 0$ , unitary gauge, while the use of a general  $\xi$  is typically called  $R_\xi$  gauge. All final gauge-invariant (physical) results are independent of the parameter  $\xi$ .

Similarly to the path integral of the scalar field we introduced some terms  $+i\epsilon$  which originate from the ground state wave function. Thus the quadratic term of the integrand in the exponent can be written as

$$\mathcal{L} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + i\epsilon \text{ terms} = \frac{1}{2} A_\mu \left[ \square g^{\mu\nu} - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right] A_\nu + i\epsilon \text{ terms} + \text{total derivative} \quad (3.43)$$

Analogously to the real scalar field, we introduce external source terms  $J_\mu$  and derive an expression for the partition function

$$Z[J_\mu] = \exp \left( -\frac{i}{2} \int d^4x d^4x' J^\mu(x) \Delta_{\mu\nu}(x-x') J^\nu(x') \right) \quad (3.44)$$

with the Feynman propagator of the photon field

$$\Delta_{\mu\nu}(x-x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-x')} \tilde{\Delta}_{\mu\nu}(p) \quad \tilde{\Delta}_{\mu\nu}(p) = \frac{-P_{\mu\nu}(p)}{p^2 + i\epsilon} \quad (3.45)$$

and the projection operator

$$P_{\mu\nu}(p) = g_{\mu\nu} - \left( 1 - \frac{1}{\xi} \right) \frac{p_\mu p_\nu}{p^2}. \quad (3.46)$$

In scattering processes we consider that the interactions vanish in the infinite past and future. In the absence of gauge interactions, single particle states of charged scalars and transversely polarized photons are gauge-invariant, while time-like and longitudinal photon polarizations transform under gauge transformations. We can thus define the scattering matrix by not considering any time-like or longitudinal photon polarizations in the initial or final state, i.e. project onto the physical subspace:

$$S = P_0 S_{FP} P_0 \quad (3.47)$$

where  $P_0$  is the projection onto the physical subspace,  $S_{FP}$  the  $S$  matrix defined by the Faddeev-Popov procedure and  $S$  the physical  $S$  matrix. It remains to show that the  $S$  matrix is unitary, i.e. that there is no scattering from a physical state to either a time-like or longitudinal polarized photon. We will come back to this later, time-permitting, when discussing the Ward-Takashi identity.



## 4 Lorentz group and spin $\frac{1}{2}$ fermions

I am assuming some basic knowledge of group theory as it is used in physics. See the appendices if you need to review some notions.

### 4.1 Rotations, spin and the $SU(2)$ group

Rotations in 3 dimensional space are described by the group  $SO(3)$  which leaves the scalar product  $\sum_i (x^i)^2$  of 3-vectors  $x^i$  invariant. There are three independent rotations around the three coordinate axes which are generated by

$$J^1 = -i \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix} \quad J^2 = -i \begin{pmatrix} 0 & -1 & \\ & 0 & \\ 1 & & 0 \end{pmatrix} \quad J^3 = -i \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{pmatrix} . \quad (4.1)$$

Thus a general rotation around the  $x^1$  axis is given by exponentiation of a multiple of the generator  $J^1$ ,  $\exp(i\theta_1 J^1)$ , and similarly for rotations around the other axes. A general rotation is given by exponentiating a linear combination of the  $J^i$ . We equivalently define differential operators through their action on functions of the three coordinates. The action of  $J^3$  for example is

$$J^3 F(x^1, x^2, x^3) = i \lim_{\theta_3 \rightarrow 0} \left[ \frac{F(x'^1, x'^2, x'^3) - F(x^1, x^2, x^3)}{\theta_3} \right] \quad (4.2)$$

$$= i \lim_{\theta_3 \rightarrow 0} \left[ \frac{F(x^1 + \theta_3 x^2, x^2 - \theta_3 x^1, x^3) - F(x^1, x^2, x^3)}{\theta_3} \right] \quad (4.3)$$

$$= i (x^2 \partial^1 - x^1 \partial^2) F(x^1, x^2, x^3) = -i [x^1, \partial^2] F(x^1, x^2, x^3) \quad (4.4)$$

such that we find

$$J^i = -i \epsilon^{ijk} x^j \partial^k . \quad (4.5)$$

with the totally antisymmetric tensor with  $\epsilon^{123} = 1$ . These are exactly the generators of the angular momentum operators which satisfy

$$[J^i, J^j] = i \epsilon^{ijk} J^k . \quad (4.6)$$

Moving on to  $SU(2)$ . It is the group of  $2 \times 2$  special unitary matrices

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad U^\dagger U = 1 \quad \det U = 1 \quad (4.7)$$

which act on a 2-dimensional complex space of (Weyl) spinors  $\chi$ . If  $\chi \rightarrow U\chi$ , the hermitian conjugate transforms as  $\chi^\dagger \rightarrow \chi^\dagger U^\dagger$ . Thus the inner product  $\chi^\dagger \chi$  remains invariant. It is also convenient to define the charge-conjugate spinor

$$\tilde{\chi} \equiv \begin{pmatrix} -\chi_2^* \\ \chi_1^* \end{pmatrix} = (\epsilon_{\alpha\beta} \chi^{*\beta}) \quad \epsilon_{\alpha\beta} = \begin{cases} 1 & \alpha = 1, \beta = 2 \\ -1 & \alpha = 2, \beta = 1 \\ 0 & \text{otherwise} \end{cases} \quad (4.8)$$

which transforms like  $\chi$ , while  $\tilde{\chi}^\dagger$  transforms like  $\chi^\dagger$ . The outer product  $\chi\tilde{\chi}^\dagger$  transforms as

$$\chi\tilde{\chi}^\dagger = \begin{pmatrix} -\chi_1\chi_2 & \chi_1^2 \\ \chi_2^2 & \chi_1\chi_2 \end{pmatrix} \rightarrow U\chi\tilde{\chi}^\dagger U^\dagger. \quad (4.9)$$

We can rewrite the outer product using the Pauli spin matrices as

$$\chi\tilde{\chi}^\dagger = x^i \sigma^i \quad (4.10)$$

with

$$x^1 = \frac{\chi_1^2 - \chi_2^2}{2} \quad x^2 = \frac{\chi_1^2 + \chi_2^2}{-2i} \quad x^3 = -\chi_1\chi_2 \quad (4.11)$$

The outer product transforms under a  $SU(2)$  transformation as follows

$$\sigma^i x'^i = U \sigma^i U^\dagger x^i. \quad (4.12)$$

If we take the determinant we find

$$-\det(\sigma^i x'^i) = \sum_i (x'^i)^2 = \sum_i (x^i)^2 \quad (4.13)$$

Thus  $SU(2)$  rotations leaves the inner product  $x^i$  invariant and thus induce  $SO(3)$  rotations. In fact the  $SU(2)$  rotations on  $\chi$  are equivalent to the  $SO(3)$  rotations on  $x^i$ .  $SU(2)$  transformations are also described by three real parameters. The matrices can be parameterised in terms of the underlying Lie algebra using the Pauli matrices as generators

$$U = e^{i \frac{\alpha_i}{2} \sigma^i}. \quad (4.14)$$

which satisfy the commutation relation

$$\left[ \frac{\sigma^i}{2}, \frac{\sigma^j}{2} \right] = i \epsilon^{ijk} \frac{\sigma^k}{2}. \quad (4.15)$$

In summary we have defined a 3-dimensional vector representation of  $SO(3)$  and a spinor representation of  $SU(2)$ . We can similarly define other representations of the group  $SU(2)$  with generators  $S^i$  which satisfy the commutation relation  $[S^i, S^j] = i \epsilon^{ijk} S^k$ . There is an infinite number of representations of  $SU(2)$  of dimensions  $n = 2, 3, 4, \dots$ .

## 4.2 Lorentz group

The Lorentz group is formed by the transformations which leave the spacetime interval  $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$  invariant. In addition to the three generators for rotations, there are three boosts

$$K^1 = -i \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad K^2 = -i \begin{pmatrix} 0 & & 1 & \\ & 0 & & \\ 1 & & 0 & \\ & & & 0 \end{pmatrix} \quad K^3 = -i \begin{pmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} \quad (4.16)$$

in  $x^1$ ,  $x^2$  and  $x^3$  direction. A finite Lorentz transformation is obtained by exponentiating the a linear combination of the generators. For example for boosts in  $x$ -direction we obtain

$$B_1 = \exp(i\phi_1 K^1) = \begin{pmatrix} \cosh \phi_1 & \sinh \phi_1 & 0 & 0 \\ \sinh \phi_1 & \cosh \phi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.17)$$

and thus  $K^1 = -i \frac{d}{d\phi_1} B_x|_{\phi_1=0}$ . Note that  $\cosh \phi_1 = \gamma_1 = \frac{1}{\sqrt{1-v_1^2}}$  and  $\sinh \phi_1 = \beta_1 \gamma_1$ . The generators satisfy the following commutation relations

$$[K^i, K^j] = -i\epsilon^{ijk} J^k \quad [J^i, J^j] = i\epsilon^{ijk} J^k \quad [K^i, J^j] = i\epsilon^{ijk} K^k \quad (4.18)$$

The 6 generators generate all proper orthochronous Lorentz transformations in  $L_+^\uparrow$  or  $SO(1,3)^+$ . In order to better understand the group structure, it is convenient to consider a complex linear combination of the generators. We define

$$N_\pm^j = \frac{1}{2} (J^j \pm iK^j) \quad (4.19)$$

The generators  $N_\pm^i$  satisfy the commutation relations

$$[N_\pm^i, N_\pm^j] = i\epsilon^{ijk} N_\pm^k \quad [N_+^i, N_-^j] = 0 \quad (4.20)$$

and thus we can consider the Lorentz group as a direct product  $SU(2) \times SU(2)$  of two  $SU(2)$  ("spin") groups. The irreducible representations are classified as  $(s_+, s_-)$ , where  $s_\pm$  denotes the representation of the  $SU(2)$  group formed by  $\{N_\pm^i\}$ . For example we can define two spinors

$$\chi \sim \left(\frac{1}{2}, 0\right) \quad \text{with } J^i = \frac{1}{2}\sigma^i \text{ and } K^i = -\frac{i}{2}\sigma^i \quad (4.21)$$

$$\eta \sim \left(0, \frac{1}{2}\right) \quad \text{with } J^i = \frac{1}{2}\sigma^i \text{ and } K^i = \frac{i}{2}\sigma^i \quad (4.22)$$

A general Lorentz transformation can be written as

$$U = \exp(i\phi^i K^i + i\theta^i J^i) \quad (4.23)$$

Using the generators for the spinor representation in Eqs. (4.21) we see that the transformation is not unitary,  $UU^\dagger \neq 1$  which seems problematic in the context of quantum mechanics. Note however that rotations and Lorentz boosts only make up part of the Poincaré group which is the fundamental group for relativistic systems. In particular there are two discrete transformations, parity  $P$  and time-reversal  $T$ , which transform between the different disconnected parts of the Lorentz group:

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \quad (4.24)$$

Time-reversal belongs to the improper non-orthochronous Lorentz transformations  $L_-^\downarrow$ ,  $\det \Lambda = -1$  and  $\Lambda_0^0 = -1$  and parity (space inversion) belongs to the improper orthochronous Lorentz transformations  $L_-^\uparrow$ ,  $\det \Lambda = -1$  and  $\Lambda_0^0 = 1$ . Parity transformation forms a discrete  $Z_2$  group and thus we can classify states according to the eigenvalues of the parity operator  $\pm 1$ , e.g. scalars have parity  $+1$  and pseudo-scalars, have parity  $-1$ . Parity changes the sign of boosts, i.e.  $K^i \rightarrow -K^i$ , because the parameters of the boosts (velocities) change signs. Thus parity exchanges the generators  $N_+^i \leftrightarrow N_-^i$  and the spinor representations  $\chi \leftrightarrow \eta$ .

### 4.3 Poincaré group

In addition to the Lorentz transformations, the Poincaré group  $ISO(1,3)$  contains translations by a constant vector  $a^\mu$

$$x^\mu \rightarrow x'^\mu = x^\mu + a^\mu. \quad (4.25)$$

Thus a general group transformation in the Poincaré group is given by a combined Lorentz transformation with a translation

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \quad (4.26)$$

Translations do not commute with Lorentz transformations

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a_1^\mu \quad x''^\mu = \Lambda^\mu_\nu x'^\nu + a_2^\mu \quad (4.27)$$

$$= \Lambda^\mu_\nu (\Lambda^\nu_\rho x^\rho + a_1^\nu) + a_2^\mu \quad (4.28)$$

$$= (\Lambda_2 \Lambda_1)^\mu_\nu x^\nu + \Lambda^\mu_\nu a_1^\nu + a_2^\mu \quad (4.29)$$

The Poincaré group is a so-called *semi-direct product group*,  $ISO(1,3) = P_4 \rtimes SO(1,3)$ . Let us now find the generators for the Poincaré group. The generators of  $P_4$  are translations

$$P^\mu = -i\partial^\mu \quad (4.30)$$

and the generators of Lorentz transformations are

$$L^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \quad (4.31)$$

which satisfy the following commutation relations

$$[P^\mu, P^\nu] = 0 \quad (4.32)$$

$$[L^{\mu\nu}, L^{\rho\sigma}] = -i(g^{\mu\rho} L^{\nu\sigma} - g^{\mu\sigma} L^{\nu\rho} - g^{\nu\rho} L^{\mu\sigma} + g^{\nu\sigma} L^{\mu\rho}) \quad (4.33)$$

$$[L^{\mu\nu}, P^\rho] = -i(g^{\mu\rho} P^\nu - g^{\nu\rho} P^\mu). \quad (4.34)$$

More generally, we can define the spin-orbital momentum operator

$$M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \quad (4.35)$$

out of the generators of the Lorentz group  $L^{\mu\nu}$  and the hermitian spin generators  $S^{\mu\nu}$ . The spin operators which can be considered as generators of 'internal rotations' satisfy the same commutation relations as  $L^{\mu\nu}$  and commute with  $L^{\mu\nu}$

$$[L^{\mu\nu}, S^{\rho\sigma}] = 0 \quad (4.36)$$

and are thus generators of  $ISO(1, 3)$ .

We next construct the representations of the Poincaré group. Lorentz transformations change the 4-momentum  $p^\mu$  of a state  $|p\rangle$ , but leave  $p_\mu p^\mu$  invariant. Consider a state  $|p\rangle$  with  $P^\mu |p\rangle = p^\mu |p\rangle$  then the transformed state  $U(\Lambda, a) |p\rangle = |\Lambda p\rangle$  has momentum

$$P^\mu |\Lambda p\rangle = (\Lambda p)^\mu |\Lambda p\rangle, \quad (4.37)$$

while the interval of the 4-momentum remains the same

$$(\Lambda p)^\mu (\Lambda p)_\mu = \Lambda^\mu_\nu p^\nu \Lambda^\rho_\mu p_\rho = \delta^\rho_\nu p^\nu p_\rho = p^2 \quad (4.38)$$

where we used  $\Lambda^\mu_\rho \Lambda^\rho_\nu = \delta^\mu_\nu$ . Thus Lorentz transformations leave the operator  $P_\mu P^\mu$  invariant. It is a quadratic Casimir operator, since it commutes with all generators of the Poincaré group, and is associated with the mass of the particle.

There is another Casimir operator  $W_\mu W^\mu$  formed from the generators: the Pauli-Lubanski vector

$$W_\mu = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} S^{\nu\rho} P^\sigma. \quad (4.39)$$

As it satisfies  $W_\mu P^\mu = 0$ , we find a simple expression for the Pauli-Lubanski vector in a frame where  $P_\mu = (m, 0, 0, 0)$  and thus  $W_\mu = (0, W_1, W_2, W_3)$

$$W_i = -\frac{m}{2} \epsilon_{ijk0} S^{jk} \equiv \frac{m}{2} S_i \quad (4.40)$$

where we defined the spin operator  $S_i$ . The spin operator satisfies the  $SU(2)$  commutation relations and can thus be represented by  $\sigma^i/2$ . Thus particle states are classified by their mass and spin. In case of proper orthochronous Poincaré transformations  $P^\dagger_+$  there are in total six cases

1. time-like 4-momentum and positive energy:  $p^2 = m^2 > 0, p^0 > 0$
2. light-like 4-momentum and positive energy:  $p^2 = 0, p^0 > 0$
3. zero 4-momentum:  $p^\mu = 0$
4. time-like 4-momentum and negative energy:  $p^2 = m^2 > 0, p^0 < 0$
5. light-like 4-momentum and negative energy:  $p^2 = 0, p^0 < 0$
6. space-like 4-momentum:  $p^2 < 0$

The first two cases correspond to physical states (particles), the third one corresponds to the vacuum and the last can represent virtual particles which may have a space-like 4-momentum, while the remaining ones are unphysical. In the first case the eigenvalue of the mass is real and the eigenvalue of the second Casimir operator  $W^2$  is given by  $-m^2s(s+1)$ , where  $s$  denotes the spin. There are  $2s+1$  degrees of freedom for a given spin  $s$  which are distinguished by the value of the third component  $S^3$ . The second case corresponds to a massless particle: Both  $P^2 = W^2 = 0$  and they are proportional to each other due to  $P_\mu W^\mu = 0$ . The proportionality constant is called helicity and equal to  $\pm s$ , where  $s$  is the spin of the representation.

The variation of a field  $F$  under a general Poincaré transformation is given by

$$\delta F \equiv F'(x') - F(x) = F'(x + \delta x) - F(x) = F'(x) - F(x) + \partial_\mu F(x) \delta x^\mu \equiv \bar{\delta} F + \partial_\mu F \delta x^\mu \quad (4.41)$$

Scalar fields remain the same in different inertial frames,  $\phi'(x') = \phi(x)$  and thus

$$\delta\phi = 0 = \bar{\delta}\phi + \partial_\mu \phi \delta x^\mu. \quad (4.42)$$

For an infinitesimal Lorentz transformation  $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$  we find  $\delta x^\mu = \omega^{\mu\nu} x_\nu$  and thus  $\bar{\delta}\phi = -\frac{i}{2}\omega^{\mu\nu} L_{\mu\nu}$  or  $M_{\mu\nu} = L_{\mu\nu}$  and  $S_{\mu\nu} = 0$ . The vector field transforms like a Lorentz vector  $\delta A_\mu = \omega_\mu^\nu A_\nu$ .

#### 4.4 Dirac equation

As we already pointed out there are two irreducible spinor representations of the Lorentz group. Thus we consider two component spinor fields (Weyl spinors)

$$\psi'_R(x') \equiv \exp\left(\frac{i\sigma^i}{2}(\theta^i - i\phi^i)\right) \psi_R(x) \quad \psi'_L(x') \equiv \exp\left(\frac{i\sigma^i}{2}(\theta^i + i\phi^i)\right) \psi_L(x) \quad (4.43)$$

which transform as  $(\frac{1}{2}, 0)$  and  $(0, \frac{1}{2})$  states, respectively. A Dirac (4-component) spinor is defined as

$$\psi(x) = \begin{pmatrix} \psi_R(x) \\ \psi_L(x) \end{pmatrix}, \quad (4.44)$$

which transforms under Lorentz transformations  $[\omega_{0i} = \phi_i$  and  $\omega_{ij} = \epsilon_{ijk}\theta_k]$

$$\psi'(x') = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right) \psi(x) \quad (4.45)$$

with the generators

$$\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] \quad (4.46)$$

constructed out of the  $\gamma$  matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \quad (4.47)$$

with  $\sigma^0 = \bar{\sigma}^0 = 1$  and  $\bar{\sigma}^i = -\sigma^i$ . The fifth  $\gamma$  matrix,  $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ , is diagonal

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.48)$$

It is the chirality operator and is used to define the two projection operators  $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$  which project onto the left-handed and right-handed Weyl spinors.

The  $\gamma$  matrices form a Clifford algebra and satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \quad (4.49)$$

The Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \quad (4.50)$$

is the equation of motion for a spin  $\frac{1}{2}$  fermion which has both left-handed and right-handed components. It is convenient to define the Dirac conjugate spinor

$$\bar{\psi} \equiv \psi^\dagger\gamma^0 \quad (4.51)$$

which transforms as  $\bar{\psi} \rightarrow \bar{\psi}U(\Lambda)^{-1}$  under a Lorentz transformation  $\psi \rightarrow U(\Lambda)\psi$ .

The Dirac equation is solved by the particle  $u$  (positive energy) and antiparticle  $v$  (negative energy) spinors

$$(\not{p} - m)u_s(p) = 0 \quad (\not{p} + m)v_s(p) = 0 \quad (4.52)$$

In the rest frame of the particle the solutions are straightforward to solve. There are two solutions for each of the equations. We distinguish the two solutions by their eigenvalue of the spin matrix

$$S_3 = \begin{pmatrix} \frac{\sigma_3}{2} & 0 \\ 0 & \frac{\sigma_3}{2} \end{pmatrix} \quad (4.53)$$

and denote

$$S_3 u_\pm(0) = \pm \frac{1}{2} u_\pm(0) \quad S_3 v_\pm(0) = \mp \frac{1}{2} v_\pm(0) \quad (4.54)$$

Note the different sign for the antiparticle spinor. This is such that both  $b^\dagger$  and  $d^\dagger$  create a particle with spin up in the z-direction, i.e.  $[J_3, d_\pm^\dagger(0)] = \pm \frac{1}{2} d_\pm^\dagger(0)$ . Thus in the rest frame we find

$$u_+(0) = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u_-(0) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad v_+(0) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad v_-(0) = \sqrt{m} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (4.55)$$

The corresponding spinors at finite momentum are found by boosting the spinors which are defined at rest. The spinors are normalised as

$$\bar{u}_s(p)u_{s'}(p) = -\bar{v}_s(p)v_{s'}(p) = 2m\delta_{ss'} \quad \bar{u}_s(p)v_{s'}(p) = 0 \quad (4.56)$$

We can construct a projection operator onto the particle states

$$2mP_+ = \sum_s u_s(p) \bar{u}_s(p) \quad (4.57)$$

It satisfies

$$P_+^2 = P_+ \quad (\not{p} - m)P_+ = 0 \quad (4.58)$$

If we take  $P_+ = a + b\not{p}$  as ansatz for  $P_+$ , we obtain

$$0 = (\not{p} - m)(a + b\not{p}) = (a - mb)\not{p} + b\not{p}\not{p} - am = (a - mb)\not{p} + bp^2 - am \quad (4.59)$$

$$\Rightarrow a = mb \quad (4.60)$$

from the second relation. The first relation  $P_+^2 = P_+$  results in  $b = 1/2m$  and thus

$$P_+ = \frac{\not{p} + m}{2m} \quad (4.61)$$

Similarly the projection operator onto the antiparticle states is given by

$$2mP_- = - \sum_s v_s(p) \bar{v}_s(p) = -(\not{p} - m) \quad (4.62)$$

and their combination satisfies  $P_+ + P_- = 1$ . The most general solution of the Dirac equation is then a linear combination

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2k^0} \sum_s \left[ b_s(k) u_s(k) e^{-ikx} + d_s^\dagger(k) v_s(k) e^{ikx} \right] \quad (4.63)$$

$$\bar{\psi}(x) = \int \frac{d^3k}{(2\pi)^3 2k^0} \sum_s \left[ b_s^\dagger(k) \bar{u}_s(k) e^{ikx} + d_s \bar{v}_s(k) e^{-ikx} \right] \quad (4.64)$$

with coefficients  $b_s$  and  $d_s$ .

## 4.5 Canonical quantisation

The Dirac equation can be obtained from the Lagrangian<sup>16</sup>

$$\mathcal{L} = \bar{\psi} (i\not{\partial} - m) \psi \quad (4.66)$$

where we introduced the Dirac slash notation  $\not{\partial} \equiv \gamma^\mu \partial_\mu$ . The fields  $\psi$  and  $\bar{\psi}$  are treated as dynamically independent fields. The conjugate momentum is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger \quad (4.67)$$

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<sup>16</sup>The Lagrangian is not hermitian, but it is equivalent to a hermitian one up to a total derivative

$$\mathcal{L} + \text{total derivative} = \frac{i}{2} \bar{\psi} \overleftrightarrow{\not{\partial}} \psi - m\bar{\psi}\psi = \frac{i}{2} (\bar{\psi} \not{\partial} \psi - (\partial_\mu \bar{\psi}) \gamma^\mu \psi) - m\bar{\psi}\psi. \quad (4.65)$$



and thus the Hamiltonian density is

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \psi^\dagger \gamma^0 (-i\gamma^i \partial_i + m)\psi \quad (4.68)$$

$$= \psi^\dagger i\partial_t \psi \quad (4.69)$$

where the Dirac equation has been used in the last line. From this we can calculate the Hamiltonian

$$H = \int d^3x \mathcal{H} = \int \frac{d^3k}{(2\pi)^3 2k^0} k^0 \sum_s \left[ b_s^\dagger(k) b_s(k) - d_s(k) d_s^\dagger(k) \right]. \quad (4.70)$$

If we quantise with commutators, the  $d$ -quanta have negative energy. The way out is to quantise the field with anti-commutators  $\{A, B\} = AB + BA$

$$\{\psi_i(x, t), \pi(y, t)\} = i\delta_{ij}\delta^{(3)}(x - y) \quad \{\psi_i(x, t), \psi_j(y, t)\} = \{\pi_i(x, t), \pi(y, t)\} = 0 \quad (4.71)$$

The equal-time anticommutation relation of the field operators translate in the anticommutation relations of the creation and annihilation operators

$$\{b_s(k), b_{s'}^\dagger(k')\} = \{d_s(k), d_{s'}^\dagger(k')\} = (2\pi)^3 2k^0 \delta^{(3)}(\vec{k} - \vec{k}') \delta_{ss'} \quad (4.72)$$

$$\{b_s(k), b_{s'}(k')\} = \{b_s^\dagger(k), b_{s'}^\dagger(k')\} = 0 \quad (4.73)$$

$$\{d_s(k), d_{s'}(k')\} = \{d_s^\dagger(k), d_{s'}^\dagger(k')\} = 0 \quad (4.74)$$

$$\{d_s(k), b_{s'}^\dagger(k')\} = \{b_s(k), d_{s'}^\dagger(k')\} = 0 \quad (4.75)$$

and hence all particles have positive energy.

Similarly to the complex scalar field, the Lagrangian of a Dirac fermion is invariant under a global phase rotation  $\psi \rightarrow e^{i\alpha}\psi$ . The associated conserved charge is

$$Q = \int \frac{d^3k}{(2\pi)^3 2k^0} \sum_s \left[ b_s^\dagger(k) b_s(k) - d_s^\dagger(k) d_s(k) \right] \quad (4.76)$$

and thus  $b^\dagger$  creates particles with positive charge and  $d^\dagger$  creates the corresponding antiparticles with negative charge. Similarly to scalar quantum electrodynamics we can promote the global symmetry to a local symmetry by replacing the ordinary derivative by a covariant derivative  $D_\mu = \partial_\mu + iqeA_\mu$ , where  $q$  denotes the charge of the particle in units of  $e$ . This defines quantum electrodynamics

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi. \quad (4.77)$$

It is straightforward to derive an expression for the Feynman propagator. We have seen in the case of the scalar and gauge fields that the Feynman propagator is a Green's function for the differential operator, where we appropriately shift the poles in the complex plane summarised in the  $+i\epsilon$  prescription. Here the Green's function is

$$(i\not{\partial}_x - m)_{\alpha\gamma} S_{\gamma\beta}(x - y) = \delta^4(x - y) \delta_{\alpha\beta} \quad (4.78)$$

where  $\alpha, \beta$  are spinor indices, and thus we obtain for the Feynman propagator

$$iS_{\alpha\beta}(x - y) = \langle 0 | T \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle \quad (4.79)$$

$$S(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{\not{p} - m + i\epsilon}. \quad (4.80)$$

## 4.6 LSZ reduction formula

Similarly to the gauge field and the scalar fields we can derive the LSZ reduction formula. One-particle states are created by the creation operators where we distinguish between the particle and antiparticle states

$$|p, s, +\rangle = b_s^\dagger(p) |0\rangle \quad |p, s, -\rangle = d_s^\dagger(p) |0\rangle \quad (4.81)$$

with the normalisation  $\langle p, s, q | p', s', q' \rangle = (2\pi)^3 2k^0 \delta^{(3)}(p-p') \delta_{ss'} \delta_{qq'}$ . We can invert the field expansion to obtain the creation operators<sup>17</sup>

$$b_s^\dagger(p) = \int d^3x e^{-ipx} \bar{\psi}(x) \gamma^0 u_s(p) \quad d_s^\dagger(p) = \int d^3x e^{-ipx} \bar{v}_s(p) \gamma^0 \psi(x) \quad (4.83)$$

and the corresponding annihilation operators. This leads to the following replacements in scattering amplitudes

$$b_s^\dagger(p)_{\text{in}} \rightarrow i \int d^4x \bar{\psi}(x) (i \overleftarrow{\not{\partial}} - m) u_s(p) e^{-ipx} \quad b_s(p)_{\text{out}} \rightarrow i \int d^4x e^{ipx} \bar{u}_s(p) (i \not{\partial} - m) \psi(x) \quad (4.84)$$

$$d_s^\dagger(p)_{\text{in}} \rightarrow i \int d^4x e^{-ipx} \bar{v}_s(p) (i \not{\partial} + m) \psi(x) \quad d_s(p)_{\text{out}} \rightarrow i \int d^4x \bar{\psi}(x) (i \overleftarrow{\not{\partial}} + m) v_s(p) e^{ipx} . \quad (4.85)$$

This holds also for an interacting field theory, if the Dirac field is properly normalised

$$\langle 0 | \psi(x) | 0 \rangle = 0 \quad (4.86)$$

$$\langle 0 | \psi(x) | p, s, + \rangle = u_s(p) e^{-ipx} \quad \langle 0 | \psi(x) | p, s, - \rangle = 0 \quad (4.87)$$

$$\langle 0 | \bar{\psi}(x) | p, s, + \rangle = 0 \quad \langle 0 | \bar{\psi}(x) | p, s, - \rangle = \bar{v}_s(p) e^{-ipx} . \quad (4.88)$$

## 4.7 Fermionic path integral

In order to define the path integral for fermions, we first have to introduce Grassmann numbers, anticommuting c-numbers. See Sec. 6.7 in Ryder for a more detailed discussion. It is in fact a graded algebra. The generators  $C_i$  of an n-dimensional Grassmann algebra satisfy

$$\{C_i, C_j\} = 0 \quad \Rightarrow \quad C_i^2 = 0 , \quad (4.89)$$

i.e. the generators are nilpotent. This implies that the expansion of any function in terms of the generators only contains a finite number of terms, e.g.

$$f(C_1, C_2) = a_0 + a_1 C_1 + a_2 C_2 + a_3 C_1 C_2 = a_0 + a_1 C_1 + a_2 C_2 - a_3 C_2 C_1 \quad (4.90)$$

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<sup>17</sup>In order to show this we need the normalisation in terms of  $u^\dagger$  instead of  $\bar{u}$ . It can be shown that

$$u_s^\dagger(p) u_{s'}(p) = 2p^0 \delta_{ss'} \quad v_s^\dagger(p) v_{s'}(p) = -2p^0 \delta_{ss'} . \quad (4.82)$$

We define differentiation as a left-differentiation (acting from the left)

$$\frac{\partial f}{\partial C_1} = a_1 + a_3 C_2 \qquad \frac{\partial f}{\partial C_2} = a_2 - a_3 C_1 \quad (4.91)$$

The derivatives satisfy the following anticommutation relations

$$\{C_i, \frac{\partial}{\partial C_j}\} = \delta_{ij} \qquad \{\frac{\partial}{\partial C_i}, \frac{\partial}{\partial C_j}\} = 0 \quad (4.92)$$

which are straightforward to show. In order to define integration the infinitesimals also have to be Grassmann quantities

$$\{C_i, dC_j\} = \{dC_i, dC_j\} = 0 \quad (4.93)$$

Multi-dimensional integrals are defined by iteration

$$\int dC_1 dC_2 f(C_1, C_2) = \int dC_1 \left( \int dC_2 f(C_1, C_2) \right) . \quad (4.94)$$

Using this we can show

$$\left( \int dC_1 \right)^2 = \int dC_1 dC_2 = - \int dC_2 dC_1 = - \left( \int dC_1 \right)^2 \quad \Rightarrow \quad \int dC_1 = 0 \quad (4.95)$$

As there the fermionic integrals are scaleless we define the integral

$$\int dC_i C_i = 1 \quad \Rightarrow \quad \int dC_i C_j = \delta_{ij} . \quad (4.96)$$

Using these relations we can now evaluate integrals over general functions, for example the function  $f$  defined above

$$\int dC_1 f = \int dC_1 [a_0 + a_1 C_1 + a_2 C_2 + a_3 C_1 C_2] \quad (4.97)$$

$$= \int dC_1 a_0 + a_1 \int dC_1 C_1 - a_2 C_2 \int dC_1 + a_3 C_2 \int dC_1 C_1 = a_1 + a_3 C_2 \quad (4.98)$$

which exactly agrees with the result of the derivative. Changing variables from  $C_i$  to  $\alpha_i$ ,  $C = M\alpha$ , we find

$$C_1 C_2 = (M_{11}\alpha_1 + M_{12}\alpha_2)(M_{21}\alpha_1 + M_{22}\alpha_2) \quad (4.99)$$

$$= (M_{11}M_{22} - M_{12}M_{21})\alpha_1\alpha_2 = \det M \alpha_1\alpha_2 \quad (4.100)$$

and thus in order for the integral to preserve our integration rules, we have to demand that

$$dC_1 dC_2 = (\det M)^{-1} d\alpha_1 d\alpha_2 \quad (4.101)$$

Now considering 2 complex Grassmann quantities  $\eta$  and  $\bar{\eta}$ , we find for the exponential

$$\exp(-\bar{\eta}\eta) = 1 - \bar{\eta}\eta \quad \Rightarrow \quad \int d\bar{\eta} d\eta \exp(-\bar{\eta}\eta) = 1 . \quad (4.102)$$

Let us generalise the result to higher dimensions. In 2 dimensions the inner product reads  $\bar{\eta}\eta = \bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2$  and thus

$$(\bar{\eta}\eta)^2 = (\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2)^2 = 2\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2 \quad (4.103)$$

and thus

$$\int d\bar{\eta}d\eta e^{-\bar{\eta}\eta} = \int d\bar{\eta}_1d\eta_1d\bar{\eta}_2d\eta_2 e^{-\bar{\eta}\eta} = \int d\bar{\eta}_1d\eta_1d\bar{\eta}_2d\eta_2 \bar{\eta}_1\eta_1\bar{\eta}_2\eta_2 = 1. \quad (4.104)$$

The same result applies in n-dimensional space. In particular we find for the Gaussian integral

$$\int d\bar{\eta}d\eta e^{-\bar{\eta}_i A_{ij} \eta_j} = \det A. \quad (4.105)$$

Equipped with this we can define the path integral for the free Dirac field

$$Z_0[\eta, \bar{\eta}] = \frac{1}{N} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left( i \int d^4x \bar{\psi} (i\not{\partial} - m) \psi + \bar{\eta}\psi + \bar{\psi}\eta \right) \quad (4.106)$$

with some normalisation constant  $N$ . Completing the squares and performing the fermionic Gaussian integral leads to the partition function

$$Z_0[\eta, \bar{\eta}] = \exp \left( -i \int d^4x d^4y \bar{\eta}(x) S(x-y) \eta(y) \right). \quad (4.107)$$

Taking two derivatives we obtain the propagator

$$\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(x)} i \frac{\delta}{\delta \eta(y)} Z_0[\eta, \bar{\eta}] \Big|_{\eta=\bar{\eta}=0} = i S(x-y). \quad (4.108)$$

The different sign for the functional derivative with respect to  $\eta$  is chosen to cancel the sign coming from the ordering of the sources in the exponential. For an interacting fermionic field theory we can follow the same steps as for a bosonic interacting field theory, while carefully keeping track of signs from commuting the fermionic fields.

## 4.8 Quantum electrodynamics (QED)

In particular for quantum electrodynamics which is described by the Lagrangian

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i\not{D} - m) \psi \quad (4.109)$$

with  $D_\mu = \partial_\mu + iqeA_\mu$  we obtain the following set of Feynman rules

$$\begin{array}{ccc} \alpha \xleftarrow{p} \beta & \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\epsilon} & \mu \xleftarrow{p} \nu & \frac{-ig_{\mu\nu}}{p^2+i\epsilon} & \begin{array}{c} \alpha \\ \swarrow \\ \text{---} \mu \\ \nwarrow \\ \beta \end{array} & -iqe\gamma_{\alpha\beta}^\mu \end{array} \quad (4.110)$$

External incoming (outgoing) photons yield a factor  $\epsilon_\mu^\lambda$  ( $\epsilon_\mu^{\lambda*}$ ), incoming (outgoing) fermions yield a factor  $u_s$ , ( $\bar{u}_s$ ), incoming (outgoing) antifermions yield a factor  $\bar{v}_s$  ( $v_s$ ). Fermion lines always have to be evaluated against the arrow of the fermion line. Exchanging two external fermion lines yields a factor  $-1$ . Closed fermion loops yield a factor  $-1$ .

## A Review

### A.1 Quantum harmonic oscillator

The last 1D example which we are studying is the harmonic oscillator. It is very important approximation to many physical phenomena such as the vibrational modes of a diatomic molecule as shown in Fig. 5. A particle with mass  $m$  is subject to a restoring force  $-kx$ , where  $x$  is the displacement

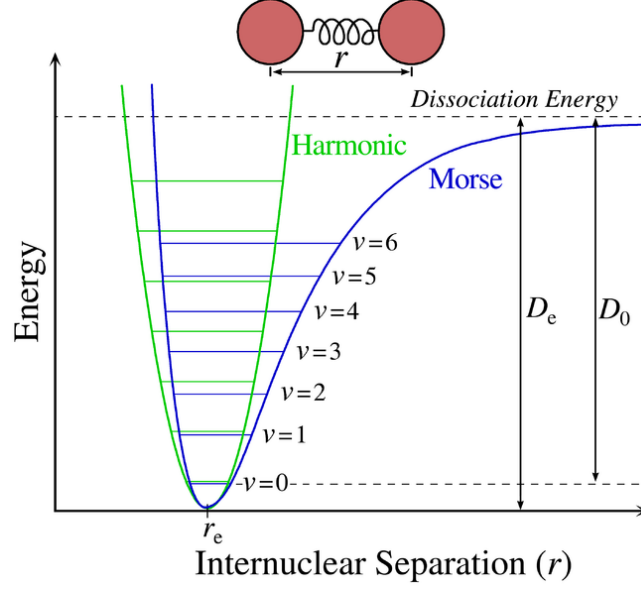


Figure 5: First few vibrational modes of a diatomic molecule are well described by a harmonic oscillator. Figure taken from <https://en.wikipedia.org/wiki/File:Morse-potential.png>.

from the equilibrium position. The potential energy of the particle is

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2 \quad (\text{A.1})$$

where  $\omega = \sqrt{k/m}$  is the angular oscillation frequency. Thus its Hamiltonian is

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2 \quad (\text{A.2})$$

and the time-independent Schrödinger equation is given by

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right)\varphi_E(x) = E\varphi_E(x). \quad (\text{A.3})$$

Now we will be going one step and solve the quantum harmonic oscillator using the ladder operator method.

In order to simplify our following discussion, we define the variable

$$\xi = \sqrt{\frac{m\omega}{\hbar}}x \quad \frac{d}{d\xi} = \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx} \quad (\text{A.4})$$

and can rewrite our Hamiltonian in Eq. (A.2) as follows

$$H = \frac{1}{2}\hbar\omega \left( -\frac{d^2}{d\xi^2} + \xi^2 \right) \quad (\text{A.5})$$

In the next step we want to factorise the Hamiltonian in analogy to the identity

$$u^2 - v^2 = (u - v)(u + v) . \quad (\text{A.6})$$

Hence we form two new operators as linear combination of the old ones

$$a = \frac{1}{\sqrt{2}} \left( \xi + \frac{d}{d\xi} \right) = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} + i\frac{\hat{p}}{m\omega} \right) \quad (\text{A.7})$$

$$a^\dagger = \frac{1}{\sqrt{2}} \left( \xi - \frac{d}{d\xi} \right) = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} - i\frac{\hat{p}}{m\omega} \right) . \quad (\text{A.8})$$

The operators satisfy the following commutation relation

$$[a, a^\dagger] = 1 . \quad (\text{A.9})$$

Looking at the product of the two operators

$$a^\dagger a = \frac{1}{\sqrt{2}} \left( \xi - \frac{d}{d\xi} \right) \frac{1}{\sqrt{2}} \left( \xi + \frac{d}{d\xi} \right) \quad (\text{A.10})$$

$$= \frac{1}{2} \left( \xi^2 - \frac{d^2}{d\xi^2} + \xi \frac{d}{d\xi} - \frac{d}{d\xi} \xi \right) \quad (\text{A.11})$$

$$= \frac{1}{2} \left( \xi^2 - \frac{d^2}{d\xi^2} + [\xi, \frac{d}{d\xi}] \right) \quad (\text{A.12})$$

$$= \frac{1}{2} \left( \xi^2 - \frac{d^2}{d\xi^2} - 1 \right) \quad (\text{A.13})$$

we observe that we almost obtain the result which we wanted to obtain. Hence, the Hamiltonian can be rewritten as

$$H = \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) . \quad (\text{A.14})$$

Before interpreting the operators  $\hat{a}$  and  $\hat{a}^\dagger$  physically, we have to show two more commutation relations:

$$[H, a] = Ha - aH \quad (\text{A.15})$$

$$= \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) a - a\hbar\omega \left( a^\dagger a + \frac{1}{2} \right) \quad (\text{A.16})$$

$$= \hbar\omega \left( a^\dagger aa - aa^\dagger a \right) \quad (\text{A.17})$$

$$= \hbar\omega \left( a^\dagger aa - \left( a^\dagger a + [a, a^\dagger] \right) a \right) \quad (\text{A.18})$$

$$= -\hbar\omega a \quad (\text{A.19})$$

Now given an energy eigenstate  $|E\rangle$  with a given energy  $E$ , we can calculate the energy eigenvalue of the states  $a|E\rangle$  as follows

$$H(a|E\rangle) = Ha|E\rangle \quad (\text{A.20})$$

$$= (aH + [H, a])|E\rangle \quad (\text{A.21})$$

$$= (aE - \hbar\omega a)|E\rangle \quad (\text{A.22})$$

$$= (E - \hbar\omega)(a|E\rangle) \quad (\text{A.23})$$

Similarly for the operator  $a^\dagger$

$$[H, a^\dagger] = +\hbar\omega a^\dagger. \quad (\text{A.24})$$

and

$$H(a^\dagger|E\rangle) = (E + \hbar\omega)(a^\dagger|E\rangle). \quad (\text{A.25})$$

Hence the states  $a|E\rangle$ ,  $a^\dagger|E\rangle$  are also energy eigenstates with energies  $E \pm \hbar\omega$ , respectively. The operators  $a$  and  $a^\dagger$  transform a state with energy  $E$  into a state with energy  $E \pm \hbar\omega$ . They are denoted *ladder operators*, more specifically  $a^\dagger$  is denoted *raising operator* and  $a$  *lowering operator*.

Next we have to find the lowest energy eigenstate or ground state. Classically we observe that there is a minimum energy of the harmonic oscillator. Hence there has to be a lowest energy eigenstate

$$a|E_{\text{lowest}}\rangle = 0. \quad (\text{A.26})$$

This is called the *ladder termination condition*. The energy of this lowest energy eigenstate is given by

$$H|E_{\text{lowest}}\rangle = \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) |E_{\text{lowest}}\rangle = \frac{1}{2}\hbar\omega |E_{\text{lowest}}\rangle. \quad (\text{A.27})$$

Note that lowest energy level is not zero as it would be for a classical harmonic oscillator, but  $\frac{1}{2}\hbar\omega$ . It is known as *zero point energy* and ultimately due to the non-vanishing commutator of the ladder operators  $[a, a^\dagger] = 1$ . The lowest energy eigenstate is commonly denoted  $|0\rangle$ . The energy of the  $n^{\text{th}}$  state  $|n\rangle$  is given by

$$E_n = \hbar\omega \left( n + \frac{1}{2} \right), \quad (\text{A.28})$$

because applying the raising operator  $a^\dagger$   $n$  times increases the energy with respect to the lowest energy eigenstate by  $n \times \hbar\omega$ . In addition to the ladder operators it is convenient to introduce the *number operator*,

$$\hat{N} = \hat{a}^\dagger \hat{a}, \quad (\text{A.29})$$

which counts the energy quanta. It fulfils the following eigenvalue equation

$$\hat{N}|n\rangle = n|n\rangle, \quad (\text{A.30})$$

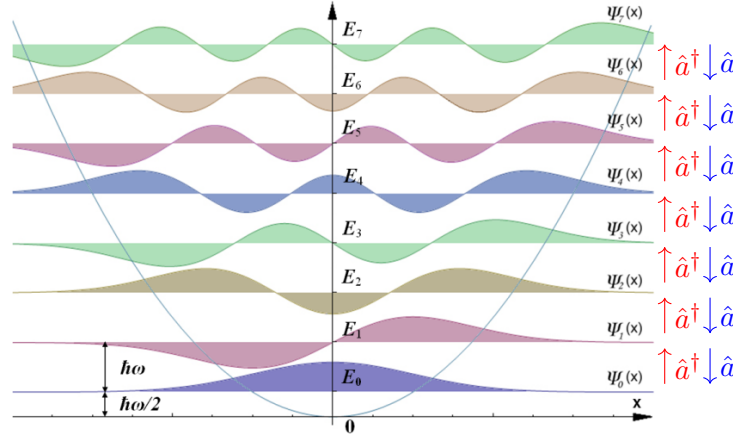


Figure 6: Energy levels of harmonic oscillator. Raising operator  $a^\dagger$  increases energy by  $\hbar\omega$  and lowering operator  $a$  lowers it. Figure taken from <https://commons.wikimedia.org/w/index.php?curid=11623546>.

where the  $n$  in  $|n\rangle$  denotes the number of energy quanta. We can rewrite the Hamiltonian as

$$\hat{H} = \hbar\omega \left( \hat{N} + \frac{1}{2} \right). \quad (\text{A.31})$$

See Fig. 6 for an illustration of the action of the ladder operator on the energy eigenstates. All other energy eigenstates can be constructed from the lowest energy eigenstate using the raising operator. By demanding that all states  $|n\rangle$  are properly normalised,

$$\langle n|n\rangle = 1, \quad (\text{A.32})$$

it is possible to show<sup>18</sup> that the raising and lowering operators act on a state  $|n\rangle$

$$a |n\rangle = \sqrt{n} |n-1\rangle \quad (\text{A.33})$$

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle. \quad (\text{A.34})$$

Thus we can write the state  $|n\rangle$  as follows

$$|n\rangle \equiv \frac{1}{\sqrt{n!}} \left( a^\dagger \right)^n |0\rangle, \quad (\text{A.35})$$

where we denoted the lowest energy eigenstate by  $|0\rangle$ . The factor  $1/\sqrt{n!}$  ensures that the states are correctly normalised.

The wave function of the lowest energy eigenstate  $\phi_0(\xi)$  can be determined from the ladder termination condition in Eq. (A.26)

$$0 = a\phi_0(\xi) = \frac{1}{\sqrt{2}} \left( \xi + \frac{d}{d\xi} \right) \phi_0(\xi). \quad (\text{A.36})$$

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<sup>18</sup>See the discussion in McIntyre Chap.9.



It is an ODE, which can be solved using standard techniques

$$\phi_0(\xi) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}. \quad (\text{A.37})$$

The explicit form of the wave function involves a well studied special function, the Hermite polynomial, such that the normalised wave functions can be written as

$$\varphi_E(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2} \quad \text{with } \xi = \sqrt{\frac{m\omega}{\hbar}} x \text{ and } n = 0, 1, 2, 3, \dots \quad (\text{A.38})$$

where  $H_n$  is a Hermite polynomial of order  $n$

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2 \quad H_3(x) = 8x^3 - 12x. \quad (\text{A.39})$$

Figure 7 shows wave functions and probability densities for the first four energy levels of a harmonic oscillator.

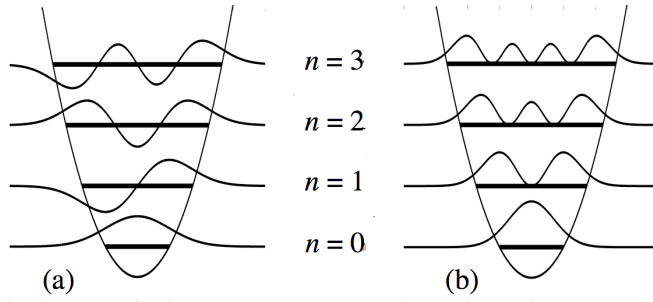


Figure 7: (a) Wave functions and (b) the probability density for the first four energy levels of a harmonic oscillator.

## A.2 Time-dependent perturbation theory in quantum mechanics

As we are dealing with a time-dependent Hamiltonian, we have to return to the Schrödinger equation in its original form. So far we always considered the Hamiltonian to be time-independent, while the quantum states evolved with time. This is commonly denoted as *Schrödinger picture*. If we however consider the states<sup>19</sup>

$$|\psi\rangle_H \equiv e^{i\hat{H}t/\hbar} |\psi(t)\rangle = \left(1 + \frac{i}{\hbar} \hat{H}t + \frac{1}{2} \left[\frac{i}{\hbar} \hat{H}t\right]^2\right) |\psi(t)\rangle \quad (\text{A.40})$$

we find that

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle_H = i\hbar \frac{\partial}{\partial t} e^{i\hat{H}t/\hbar} |\psi(t)\rangle = e^{i\hat{H}t/\hbar} \left(-\hat{H} |\psi(t)\rangle + i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle\right) = 0 \quad (\text{A.41})$$

<sup>19</sup>Note that  $e^{\hat{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{A}^n$ .

This is commonly denoted as *Heisenberg picture*. In this picture, all operators

$$\hat{A}_H(t) = e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar}, \quad (\text{A.42})$$

e.g. momentum operator, will be time-dependent and satisfy

$$\frac{d}{dt} \hat{A}_H(t) = \frac{i}{\hbar} [\hat{H}_H(t), \hat{A}_H(t)] + \left( \frac{\partial \hat{A}}{\partial t} \right)_H. \quad (\text{A.43})$$

Thus the eigenvalues of any operator, which commutes with the Hamiltonian, are constants and provide good quantum numbers to describe the system. We will now consider the *interaction picture*, in which both the Hamiltonian and the quantum state will depend on time. It is useful for cases, where we can split the Hamiltonian  $\hat{H}(t) = \hat{H}_0 + \hat{H}'(t)$  in a time-independent part  $\hat{H}_0$  and a time-dependent one  $\hat{H}'$ . It is defined by

$$|\psi(t)\rangle_I = e^{i\hat{H}_0 t/\hbar} |\psi(t)\rangle \quad \hat{A}_I(t) = e^{i\hat{H}_0 t/\hbar} \hat{A} e^{-i\hat{H}_0 t/\hbar}. \quad (\text{A.44})$$

We rewrite the Schrödinger equation for the Hamiltonian  $\hat{H} = \hat{H}_0 + \hat{H}'(t)$  in the interaction picture

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle_I = i\hbar \frac{\partial}{\partial t} e^{i\hat{H}_0 t/\hbar} |\psi(t)\rangle \quad (\text{A.45})$$

$$= -\hat{H}_0 |\psi(t)\rangle_I + e^{i\hat{H}_0 t/\hbar} i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle \quad (\text{A.46})$$

$$= -\hat{H}_0 |\psi(t)\rangle_I + e^{i\hat{H}_0 t/\hbar} (\hat{H}_0 + \hat{H}'(t)) |\psi(t)\rangle \quad (\text{A.47})$$

$$= -\hat{H}_0 |\psi(t)\rangle_I + e^{i\hat{H}_0 t/\hbar} (\hat{H}_0 + \hat{H}'(t)) e^{-i\hat{H}_0 t/\hbar} |\psi(t)\rangle_I \quad (\text{A.48})$$

$$= e^{i\hat{H}_0 t/\hbar} \hat{H}'(t) e^{-i\hat{H}_0 t/\hbar} |\psi(t)\rangle_I \quad (\text{A.49})$$

$$= \hat{H}'_I(t) |\psi(t)\rangle_I. \quad (\text{A.50})$$

We used that the exponential  $\exp(iH_0 t/\hbar)$  commutes with  $H_0$ . Finally, we can now formally integrate the equation and obtain the equivalent integral equation

$$|\psi(t)\rangle_I = |\psi(t_0)\rangle_I + \frac{1}{i\hbar} \int_{t_0}^t \hat{H}'_I(t') |\psi(t')\rangle_I dt'. \quad (\text{A.51})$$

This integral equation can be iteratively solved by plugging the solution back in on the right-hand side of the equation, i.e.

$$|\psi(t)\rangle_I = |\psi(t_0)\rangle_I + \frac{1}{i\hbar} \int_{t_0}^t \hat{H}'_I(t') |\psi(t_0)\rangle_I dt' + \left( \frac{1}{i\hbar} \right)^2 \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \hat{H}'_I(t') \hat{H}'_I(t'') |\psi(t_0)\rangle_I + \dots \quad (\text{A.52})$$

We introduce the time-ordered product of operators to simplify the expression further. The time-ordered product of two operators  $A$  and  $B$  is defined as

$$\mathcal{T}(A(t_1)B(t_2)) \equiv \begin{cases} A(t_1)B(t_2) & \text{for } t_1 \geq t_2 \\ B(t_2)A(t_1) & \text{otherwise} \end{cases}. \quad (\text{A.53})$$

This allows us to rewrite the quadratic term as

$$\int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \hat{H}'_I(t') \hat{H}'_I(t'') = \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \mathcal{T} \left( \hat{H}'_I(t') \hat{H}'_I(t'') \right) = \mathcal{T} \left( \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \hat{H}'_I(t') \hat{H}'_I(t'') \right) \quad (\text{A.54})$$

$$= \mathcal{T} \left( \int_{t_0}^t dt'' \int_{t_0}^{t''} dt' \hat{H}'_I(t'') \hat{H}'_I(t') \right) = \mathcal{T} \left( \int_{t_0}^t dt'' \int_{t_0}^{t''} dt' \hat{H}'_I(t') \hat{H}'_I(t'') \right) \quad (\text{A.55})$$

$$= \frac{1}{2} \mathcal{T} \left( \int_{t_0}^t dt' \int_{t_0}^t dt'' \hat{H}'_I(t') \hat{H}'_I(t'') \right) = \frac{1}{2} \mathcal{T} \left( \int_{t_0}^t dt' \hat{H}'_I(t') \right)^2, \quad (\text{A.56})$$

where we introduced the time-ordered product in the first line, relabelled the integration variables  $t' \leftrightarrow t''$  in the second line, and noticed that summing the last terms of the first and second line and dividing by 2 can be rewritten as the first term of the third line. Hence we can write the quadratic term as the square of the integral and

$$|\psi(t)\rangle_I = |\psi(t_0)\rangle_I + \frac{1}{i\hbar} \int_{t_0}^t \hat{H}'_I(t') dt' |\psi(t_0)\rangle_I + \frac{1}{2} \mathcal{T} \left( \frac{1}{i\hbar} \int_{t_0}^t dt' \hat{H}'_I(t') \right)^2 |\psi(t_0)\rangle_I + \dots \quad (\text{A.57})$$

$$= \mathcal{T} \left( e^{-\frac{i}{\hbar} \int_{t_0}^t \hat{H}'_I(t') dt'} \right) |\psi(t_0)\rangle_I. \quad (\text{A.58})$$

We will only consider the leading order transitions and neglect terms with more than one  $\hat{H}'$ . If we have the set of eigenstates  $|n\rangle$  to the Hamiltonian  $\hat{H}_0$  with eigenvalue  $E_n$ , we can calculate the transition amplitude to find our particle in state  $|n\rangle$  at time  $t$  if we start in state  $|m\rangle$  at time  $t_0$ . The system is initially in the state

$$|m(t)\rangle = e^{-iH_0 t/\hbar} |m\rangle = e^{-iE_m t/\hbar} |m\rangle, \quad (\text{A.59})$$

where  $|m(t)\rangle$  denotes the state  $|m\rangle$  which has been evolved to time  $t$  with the Hamiltonian  $H_0$ . The probability amplitude that there is a transition to the state

$$|n(t)\rangle = e^{-iH_0 t/\hbar} |n\rangle = e^{-iE_n t/\hbar} |n\rangle \quad (\text{A.60})$$

is given by

$$\langle n(t) | \psi(t) \rangle = \left\langle n | e^{iH_0 t/\hbar} | \psi(t) \right\rangle = \langle n | \psi(t) \rangle_I \quad (\text{A.61})$$

where the subscript  $I$  indicates the interaction picture. The initial state  $|\psi(t_0)\rangle_I = e^{iH_0 t/\hbar} |m(t)\rangle = |m\rangle$ . Inserting the states in Eq. (A.52), we find

$$\begin{aligned} \langle n | \psi(t) \rangle_I &= \langle n | m \rangle + \frac{1}{i\hbar} \int_{t_0}^t \left\langle n | \hat{H}'_I(t') | m \right\rangle dt' \\ &= \delta_{nm} + \frac{1}{i\hbar} \int_{t_0}^t e^{i(E_n - E_m)t'/\hbar} \left\langle n | \hat{H}'_I(t') | m \right\rangle dt', \end{aligned} \quad (\text{A.62})$$

where we used the definition of an operator in the interaction picture in Eq. (A.44). Note that all expressions in the last line are given in the usual Schrödinger picture, which we worked with in the previous sections. The probability for a transition from the state  $|m\rangle$  to the state  $|n\rangle \neq |m\rangle$  is given by

$$P_{mn}(t) = \left| \frac{1}{i\hbar} \int_{t_0}^t dt' e^{i(E_n - E_m)t'/\hbar} \langle n | \hat{H}'(t') | m \rangle \right|^2. \quad (\text{A.63})$$

The first term in Eq. (A.62) corresponds to an unchanged quantum state  $|m\rangle$ , which has not been affected by any interaction. The second term describes one transition between two states induced by the Hamiltonian  $\hat{H}'$ :

1. We start with a quantum system in the eigenstate  $|m\rangle$  of the Hamiltonian  $H_0$ .
2. Then we evolve the quantum system in time from  $t_0$  to  $t'$  using the Hamiltonian  $H_0$ . Thus eigenstates  $|m\rangle$  of  $H_0$  do not change and only pick up a phase factor  $\exp(iE_m(t' - t_0)/\hbar)$ .
3. At time  $t'$ , the Hamiltonian  $\hat{H}'$  affects the quantum system and leads to transitions between states, depending on the matrix elements  $\langle n | \hat{H}' | m \rangle$ .
4. After the interaction, the system is in a new state  $|n\rangle$  and we evolve it with the Hamiltonian  $H_0$  in time from  $t'$  to  $t$  and the quantum state picks up a phase factor  $\exp(iE_n(t - t')/\hbar)$ .
5. The transition probability is given by projecting this state at time  $t$  onto the eigenstates of  $H_0$ . We explicitly include the phase factors from time evolution.

Higher order terms in Eq. (A.52) can be interpreted in the same way. The only difference is that there will be multiple transitions at different times  $t'$ ,  $t''$ , ..., and the quantum states between the transitions are evolved in time using the Hamiltonian  $H_0$  leading to additional phase factors.

### A.3 Green's function

Consider a differential equation of the form

$$L\psi(x) = f(x) \quad (\text{A.64})$$

with a linear differential operator  $L = L(x)$ . The Green's function  $G(x, y)$  of the differential operator is defined by

$$LG(x, y) = -\delta(x - y). \quad (\text{A.65})$$

It can be used to obtain a general solution for the differential equation Eq. (A.64)

$$\psi(x) = - \int dy G(x, y) f(y). \quad (\text{A.66})$$

This is straightforward to see

$$L\psi(x) = L \left( - \int dy G(x, y) f(y) \right) = - \int dy LG(x, y) f(y) = \int dy \delta(x - y) f(y) = f(x). \quad (\text{A.67})$$

## A.4 Group theory

Symmetries are fundamental ingredients in describing physics. A symmetry transformation is a reversible operation which does not change the physical system. The set of symmetry transformations form a group.

**Definition:** A group  $G$  is a set of elements with an operation of multiplication that satisfies the following four properties:

1. Closure:  $A, B \in G \Rightarrow AB \in G$
2. Associativity:  $A, B, C \in G \Rightarrow (AB)C = A(BC)$
3. Identity: there is  $E \in G$  such that  $AE = EA = A$  for all  $A \in G$
4. Inverse: For all  $A \in G$  there is  $A^{-1} : AA^{-1} = E$

**Examples** include

- Trivial group:  $\{E\}$
- Discrete  $Z_2$  group  $\{A, E\} \in Z_2$ ,  $A = A^{-1}$  and thus  $E = AA^{-1} = A^2$
- $O(N)$  the orthogonal group: group of  $n \times n$  orthogonal real matrices
- $SO(N)$  the special orthogonal group: group of  $n \times n$  orthogonal real matrices with unit determinant
- $U(1)$  the unit complex numbers  $\{e^{i\alpha}\}$ ,  $\alpha \in \mathbb{R}$
- $U(N)$  the unitary group: group of  $n \times n$  complex unitary matrices
- $SU(N)$  the special unitary group: group of  $n \times n$  complex unitary matrices with unit determinant
- Lorentz group  $SO(3, 1)$  with its spin group  $SL(2, C)$ .

The first two examples are so-called discrete finite groups. The remaining examples are Lie groups, which are continuous groups. In fact all of them (apart from the Lorentz group) are compact.  $U(1)$ ,  $SU(N)$  and  $SO(N)$  are so-called simply Lie groups. The unitary group can be decomposed in terms of the special unitary group and the unit complex numbers:  $U(N) \cong SU(N) \times U(1)$ .

### A.4.1 Lie groups

The connected component of Lie group which contains the identity can be described by the underlying Lie algebra. If  $A$  is an element in the Lie algebra,  $\exp(\alpha A)$ ,  $\alpha \in \mathbb{R}$  is a one-parameter family of group elements in the Lie group. The vector space of the Lie algebra can be described by the so-called "generators" which form a basis. Thus any element of the Lie algebra (and the connected component of the identity in the Lie group) can be described in terms of those generators. In physics we generally

use  $\exp(i\alpha_i T^i)$  with real parameters  $\alpha_i$  and hermitean matrices  $T^i$ . For example the generators of the spin group  $SU(2)$  are the Pauli matrices. For  $SU(3)$  they are the so-called Gell-Mann matrices. As  $SU(N)$  is a unitary group, the generators are hermitean matrices. In physics we typically normalize the generators as  $\text{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$ .

#### A.4.2 Representations

Symmetries leave the system invariant, but transform states  $|\psi\rangle \in \mathcal{H}$  in the Hilbert space. Representations are defined as mappings from the abstract group to linear transformations acting on a vector space which preserve the group structure. For example the group  $SU(2)$  can be represented by  $2 \times 2$  special unitary matrices. In particular the generators are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A.68})$$

We can however similarly construct a 3-dimensional representation by mapping the three generators of  $SU(2)$  to the matrices

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (\text{A.69})$$

#### A.4.3 Special relativity and the Lorentz group

The equivalence of inertial reference frames implies that space is isotropic and homogeneous and time is also homogeneous. It is not possible to distinguish inertial reference frames. In particular, light moves at the same speed in every reference frame  $F$  ( $F'$ ): the distance travelled,  $r^{(t)} = ct^{(t)}$ . Hence  $0 = (ct)^2 - r^2 = (ct')^2 - r'^2$  and more generally the interval  $s$

$$s^2 = c^2 t^2 - x^i x^i = c^2 t'^2 - x'^i x'^i \quad (\text{A.70})$$

with  $r^2 = x^i x^i = \sum_i (x^i)^2$  is the same in all inertial reference frames. Linear transformations which leave the interval  $s$  unchanged are known as Lorentz transformations.

We measure time in the same units as distance, i.e.  $c = 1$ , and combine both of them in a 4-vector  $x^\mu$ ,  $\mu = 0, 1, 2, 3$  with

$$x^0 = t \quad \text{and} \quad (x^i) = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad (\text{A.71})$$

and thus the interval  $s$  is

$$s^2 = x^0 x^0 - x^i x^i = \eta_{\mu\nu} x^\mu x^\nu \quad (\text{A.72})$$

with the metric tensor

$$(\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (\text{A.73})$$

From the condition that the interval  $s$  is invariant under Lorentz transformations  $x'^\mu = \Lambda^\mu_\nu x^\nu$

$$\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho x^\sigma \equiv \eta_{\rho\sigma} x^\rho x^\sigma \quad (\text{A.74})$$

it follows that Lorentz transformations satisfy

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma \quad \Leftrightarrow \quad \eta = \Lambda^T \eta \Lambda \quad (\text{A.75})$$

Taking the determinant we find that

$$\det \Lambda = \begin{cases} 1 & \text{proper Lorentz transformations} \\ -1 & \text{improper Lorentz transformations} \end{cases} \quad (\text{A.76})$$

and the 00 component yields the condition  $|\Lambda^0_0| \geq 1$ . If  $\Lambda^0_0 \geq 1$  ( $\leq -1$ ) it is an orthochronous (non-orthochronous) Lorentz transformation. Thus the Lorentz group splits in 4 disconnected components

$$\text{proper orthochronous } L_+^\uparrow : \det \Lambda = 1, \Lambda^0_0 \geq 1 \quad (\text{A.77})$$

$$\text{proper non-orthochronous } L_+^\downarrow : \det \Lambda = 1, \Lambda^0_0 \leq -1 \quad (\text{A.78})$$

$$\text{improper orthochronous } L_-^\uparrow : \det \Lambda = -1, \Lambda^0_0 \geq 1 \quad (\text{A.79})$$

$$\text{improper non-orthochronous } L_-^\downarrow : \det \Lambda = -1, \Lambda^0_0 \leq -1 \quad (\text{A.80})$$

The set of events parameterized by  $x^\mu$  coordinates together with the metric tensor  $\eta_{\mu\nu}$  defines a space called the Minkowski space-time. Distances are invariant under Lorentz transformations

$$\Delta s^2 = \eta_{\mu\nu} (x^\mu - y^\mu)(x^\nu - y^\nu) \quad ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad (\text{A.81})$$

Indices are lowered and raised with  $\eta_{\mu\nu}$  and  $\eta^{\mu\nu}$  (the components of the inverse of the metric,  $\eta^{-1}$ ). 4-vectors with an upper index  $x^\mu$  are called contravariant vectors and with a lower index  $x_\mu$  covariant (co-vary with the basis) vectors. Covariant vectors form the dual vector space to the contravariant vectors. A general mixed tensor transforms as

$$T^{\mu_1 \mu_2 \dots \mu_k}_{\nu_1 \nu_2 \dots \nu_l} = \Lambda^{\mu_1}_{\rho_1} \Lambda^{\mu_2}_{\rho_2} \dots \Lambda^{\mu_k}_{\rho_k} \Lambda^{\sigma_1}_{\nu_1} \Lambda^{\sigma_2}_{\nu_2} \dots \Lambda^{\sigma_l}_{\nu_l} T^{\rho_1 \rho_2 \dots \rho_k}_{\sigma_1 \sigma_2 \dots \sigma_l}. \quad (\text{A.82})$$