An Introduction to Quantum Field Theory - Part I

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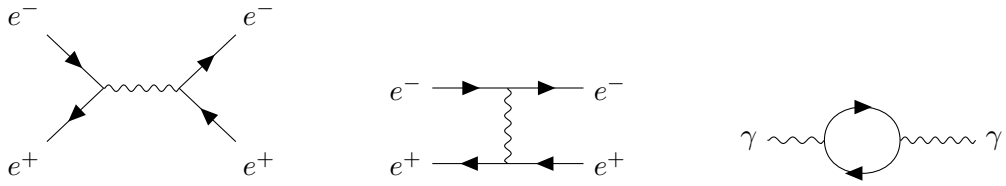

"The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing abstraction." – Sidney Coleman

Quantum field theory is the most complete and consistent theoretical framework that provides a unified description of quantum mechanics and special relativity. In the limit $\hbar \to 0$ we obtain relativistic classical fields which can be used to describe for example electrodynamics. In the limit of small velocities $v/c \ll 1$, we obtain non-relativistic quantum mechanics. Finally in the limit $\hbar \to 0$ and $v/c \ll 1$ we obtain the description of non-relativistic classical fields such as sound waves.

Fields are described by a function which assigns a value to every point on the base manifold which we consider. In this course we will be mostly looking at relativistic quantum field theory, which is defined in 3+1 -dimensional Minkowski space-time. Most results can be straightforwardly translated to other systems, like 2-dimensional systems in condensed matter physics.

The notes are very brief and obviously only give an introduction to quantum field theory. The following books provide more detailed explanations and also more in-depth knowledge about quantum field theory.

1. M. Srednicki, "Quantum Field Theory"
2. M. Schwartz, "Quantum Field Theory and the Standard Model"
3. L. Ryder, "Quantum Field Theory"
4. M. Peskin, D. Schroeder, "An Introduction to Quantum Field Theory"
5. S. Weinberg, "The Quantum Theory of Fields", Vol. 1
6. C. Itzykson, J-B. Zuber, "Quantum Field Theory"
7. D. Bailin, "Introduction to Gauge Field Theory"
8. A. Zee, "Quantum Field Theory in a Nutshell"

The notes follow the same notation as Srednicki's book. In particular we will use natural units

$$\hbar = c = k_B = 1$$

and the signature $(-+++)$ for the metric. Thus length scales and time are measured with the same units

$$[\ell] = [t] = \frac{1}{[m]} = \frac{1}{[E]} = \frac{1}{[T]} = \mathrm{eV}^{-1}$$

which is the inverse of energy. Temperature is measure in the same units as energy.

Often we will use formal manipulations which require a more mathematical treatment. In quantum field theory we will encounter many divergences which have to be regulated. This is generally possible by considering a finite volume V and to quantize the theory in this finite volume. In the infinite volume limit we obtain $V \to (2\pi)^3 \delta^{(3)}(0)$ which is divergent. See arXiv:1201.2714 [math-ph] for some lecture notes with a more rigorous discussion of different mathematical issues in quantum field theory.

Contents

1 Spin 0 – scalar field theory

1.1 Relativistic classical field theory

An intuitive derivation of the Klein-Gordon equation. Any theory of fundamental physics has to be consistent with relativity as well as quantum theory. In particular a particle with 4-momentum p^μ have to satisfy the relativistic dispersion relation

$$p^\mu p_\mu = -E^2 + \vec{p}^2 = -m^2 \ . \tag{1.1}$$

Following the standard practice in quantum mechanics we replace the energy and momentum by operators

$$E \to i\hbar\partial_t \qquad\qquad \vec{p} \to -i\hbar\vec{\nabla} \tag{1.2}$$

and postulate the wave equation for a relativistic spin-0 particle

$$-m^2\phi = -(i\partial_t)^2 + (-i\vec{\nabla})^2\phi = (\partial_t^2 - \vec{\nabla}^2)\phi \qquad\qquad (\Box - m^2)\phi = 0 \tag{1.3}$$

which is the so-called Klein-Gordon equation. The Klein-Gordon equation is solved in terms of plane waves $\exp(i\vec{k}\cdot\vec{x} \pm i\omega_k t)$ with $\omega_k = (\vec{k}^2 + m^2)^{1/2}$ and thus the general solution is given by their superposition

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a(k)e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + b(k)e^{i\vec{k}\cdot\vec{x} + i\omega_k t} \right) \tag{1.4}$$

The factor $1/\omega_k$ ensures that the integral measure $\int d^3k/(2\pi)^3 2\omega_k$ is Lorentz invariant, i.e. invariant under orthochronous Lorentz transformations ($\Lambda^0_{\ 0} \geq 1$). This can be directly seen from noticing that[1]

$$\int_{-\infty}^{\infty} dk^0 \delta(k^2 + m^2)\theta(k^0) = \frac{1}{2\omega_k} \ , \tag{1.6}$$

since there is only one zero at $k^0 = \omega_k$ for $k^0 > 0$. Thus we can rewrite the integration in terms of $d^4k\delta(k^2 + m^2)\theta(k^0)$ with the Dirac δ-function and the Heaviside step function θ which is manifestly Lorentz invariant. For a real scalar field [$\phi(x,t) = \phi^*(x,t)$], the coefficients a and b are related by $b(k) = a^*(-k)$. Thus for a real scalar field we find

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a(k)e^{ik_\mu x^\mu} + a^*(k)e^{-ik_\mu x^\mu} \right) \tag{1.7}$$

where we changed the integration variable $\vec{k} \to -\vec{k}$ for the second term. If we were to interpret the solution as a quantum wave function, the second term would correspond to "negative energy" contributions.

[1] Note that

$$\int dx \delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|} \tag{1.5}$$

where x_i are zeros of the function f.

Using the representation of the δ function

$$(2\pi)^3\delta^{(3)}(\vec{k}) = \int d^3x e^{i\vec{k}\cdot\vec{x}} \tag{1.8}$$

we can invert Eq. (1.7) to obtain

$$\int d^3x e^{-ik_\mu x^\mu}\phi(x) = \frac{1}{2\omega_k}a(k) + \frac{1}{2\omega_k}e^{2i\omega_k t}a^*(-k) \tag{1.9}$$

$$\int d^3x e^{-ik_\mu x^\mu}\partial_0\phi(x) = -\frac{i}{2}a(k) + \frac{i}{2}e^{2i\omega_k t}a^*(-k) \tag{1.10}$$

and thus we obtain for the coefficients $a(k)$

$$a(k) = \int d^3x e^{-ik_\mu x^\mu}\left[\omega_k + i\partial_0\right]\phi(x) = i\int d^3x\, e^{-ik_\mu x^\mu} \overset{\leftrightarrow}{\partial_0}\, \phi(x)\,. \tag{1.11}$$

with $f(x) \overset{\leftrightarrow}{\partial_x} g(x) \equiv f(x)\partial_x g(x) - (\partial_x f(x))g(x)$.

1.2 Stationary action principle

This equation can also be derived from the least action principle using the following Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{1.12}$$

and thus the action $S = \int dt \int d^3x\mathcal{L}$. Consider a variation of the action with respect to the field ϕ and the coordinates x^μ

$$x'^\mu = x^\mu + \delta x^\mu \qquad\qquad \phi'(x) = \phi(x) + \delta\phi(x) \tag{1.13}$$

and thus the total variation of ϕ is $\Delta\phi = \delta\phi + (\partial_\mu\phi)\delta x^\mu$. Variation of the action yields

$$\delta S = \int_R d^4x'\mathcal{L}(\phi', \partial_\mu\phi', x') - \int_R d^4x\mathcal{L}(\phi, \partial_\mu\phi, x) \tag{1.14}$$

$$= \int_R d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \mathcal{L}\partial_\mu\delta x^\mu\right) \tag{1.15}$$

$$= \int_R d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \int_R d^4x\partial_\mu\left(\mathcal{L}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) \tag{1.16}$$

$$= \int_R d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \int_{\partial R} d\sigma_\mu\left(\mathcal{L}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) \tag{1.17}$$

where we used $\det(\frac{\partial x'^\mu}{\partial x^\lambda}) = \delta^\mu_\lambda + \partial_\lambda\delta x^\mu$ in the second line. If we demand on the boundary ∂R that there is no variation in the fields $\delta x^\mu = 0$ and $\delta\phi = 0$, we obtain the Euler-Lagrange equations

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \tag{1.18}$$

For a real scalar field the Euler-Lagrange equation is

$$0 = \frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} = \partial_\mu\partial^\mu\phi - m^2\phi\,. \tag{1.19}$$

The conjugate momentum to the field ϕ is

$$\pi = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} \tag{1.20}$$

and the Hamiltonian density is obtained by a Legendre transformation

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} \tag{1.21}$$

The Poisson brackets for two functionals $L_{1,2}$ are defined as[2]

$$\{L_1, L_2\}_{\phi,\pi} = \int d^3x \left[\frac{\delta L_1}{\delta \phi(x,t)} \frac{\delta L_2}{\delta \pi(x,t)} - \frac{\delta L_1}{\delta \pi(x,t)} \frac{\delta L_2}{\delta \phi(x,t)} \right] . \tag{1.24}$$

and thus the Poisson brackets for the field ϕ and conjugate momentum π are

$$\{\phi(x,t), \pi(y,t)\} = \delta^3(x-y) \tag{1.25}$$

$$\{\pi(x,t), \pi(y,t)\} = \{\phi(x,t), \phi(y,t)\} = 0 \tag{1.26}$$

and Hamilton's equation for $f = f(\phi, \pi)$ reads

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \tag{1.27}$$

1.3 Symmetries and Noether's theorem

For an arbitrary surface ∂R we can rewrite the second integral as

$$\int_{\partial R} d\sigma_\mu \left(\mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \int_{\partial R} d\sigma_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - \Theta^\mu{}_\nu \delta x^\nu \right] \tag{1.28}$$

where we defined the energy-momentum tensor (see below)

$$\Theta^\mu{}_\nu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu \, . \tag{1.29}$$

For example if the action is invariant (i.e. $\delta S = 0$) under the following symmetry transformation

$$\delta x^\mu = X^\mu{}_\nu \delta\omega^\nu \qquad\qquad \Delta\phi = \Phi_\nu \delta\omega^\nu \tag{1.30}$$

The surface term vanishes because $\delta S = 0$ and since $\delta\omega^\nu$ is arbitrary

$$0 = \int_{\partial R} J^\mu{}_\nu d\sigma_\mu \delta\omega^\nu = \int_R d^4x \partial_\mu J^\mu{}_\nu \delta\omega^\nu \tag{1.31}$$

[2]The functional derivative of a functional is the straightforward generalisation of the ordinary derivative

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\epsilon \to 0} \frac{F[f(x) + \epsilon\delta(x-y)] - F[f(x)]}{\epsilon} \tag{1.22}$$

and in particular

$$\frac{\delta}{\delta f(t)} f(t') = \delta(t - t') \, . \tag{1.23}$$

with the Noether current (The infinitesimal parameter is typically factored out of the Noether current.)

$$J^\mu_{\ \nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\nu - \Theta^\mu_{\ \kappa} X^\kappa_{\ \nu} \ . \tag{1.32}$$

As $\delta\omega^\mu$ is arbitrary function and for sufficiently smooth J^μ_ν we obtain

$$\partial_\mu J^\mu_\nu = 0 \ . \tag{1.33}$$

Thus we can define a conserved charge

$$Q_\nu = \int_V d^3x J^0_{\ \nu} \tag{1.34}$$

of the system which is conserved when integrating over the whole volume

$$\frac{dQ_\nu}{dt} = \int d^3x \, \partial_0 J^0_{\ \nu} = \int d^3x \, \partial_i J^i_{\ \nu} = \int d\sigma^i \, J^i_{\ \nu} = 0 \ , \tag{1.35}$$

where we used the conservation of the Noether current, $\partial_\mu J^\mu_\nu = 0$, in the second step. The last equation holds because the current has to vanish at infinity.

1.3.1 Examples

Consider $\Delta\phi = 0$ and $\delta x^\mu = a^\mu$, where a^μ is a translation in space-time. Noether's theorem tells us that the Noether current is

$$J^\mu_\nu = -\Theta^\mu_\kappa a^\kappa \tag{1.36}$$

Time-translations, $a^0 \neq 0$, $a^i = 0$ imply the conservation of energy

$$H = \int d^3x \Theta^0_{\ 0} = \int d^3x \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_0 \phi - \mathcal{L} \right] \tag{1.37}$$

and spatial translations similarly imply the conservation of momentum

$$P_i = \int d^3x \Theta^0_{\ i} = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_i \phi \tag{1.38}$$

and thus we can interpret Θ^μ_ν as energy-momentum tensor.

The energy-momentum tensor as it is defined in Eq. (1.29) is not symmetric and also not unique. We may add a term $\partial_\lambda f^{\lambda\mu\nu}$ with $f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$, because $\partial_\mu \partial_\lambda f^{\lambda\mu\nu}$ vanishes due to f being antisymmetric in μ, λ while the derivatives are symmetric. By choosing $f^{\lambda\mu\nu}$ appropriately we obtain the canonical energy-momentum tensor

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu} \tag{1.39}$$

which is symmetric in μ, ν. The total 4-momentum of the system is the same, since

$$\int d^3x \partial_\lambda f^{\lambda 0\nu} = \int d^3x \partial_i f^{i0\nu} = \int d\sigma_i f^{i0\nu} = 0 \tag{1.40}$$

irrespective of the choice of f. The last equality holds because towards infinity the fields have to sufficiently quickly go to zero, in order for the action to be normalizable and thus also any combination of the fields $\partial_i f^{i0\nu}$ has to go to zero.

1.4 Canonical quantization – free real scalar field (3)

Before moving to quantizing a scalar field let us review the quantization of the harmonic oscillator. The quantum harmonic oscillator is defined by the Lagrangian

$$L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2 \tag{1.41}$$

We can directly determine the conjugate momentum, Hamiltonian and Poisson brackets using standard techniques

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \tag{1.42}$$

$$H = p\dot{q} - L = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2 \tag{1.43}$$

$$\{q, p\} = 1 \qquad \{p, p\} = \{q, q\} = 0 \tag{1.44}$$

We obtain the quantum harmonic oscillator by replacing q and p by operators $\hat{q}$ and $\hat{p} = -i\frac{d}{dq}$ and the Poisson bracket by the commutator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{q}^2 \tag{1.45}$$

$$[\hat{q}, \hat{p}] = i \qquad [\hat{p}, \hat{p}] = [\hat{q}, \hat{q}] = 0\ . \tag{1.46}$$

We proceed in the analogous way for the real scalar field. The Lagrangian density for a free real scalar field, generalized momentum, Hamiltonian density and Poisson brackets are given by

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \Omega_0 \tag{1.47}$$

$$\pi = \dot{\phi} \tag{1.48}$$

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{m^2}{2}\phi^2 - \Omega_0 \tag{1.49}$$

$$\{\phi(x,t), \pi(y,t)\} = \delta^3(x-y) \qquad \{\pi(x,t), \pi(y,t)\} = \{\phi(x,t), \phi(y,t)\} = 0\ . \tag{1.50}$$

The Hamiltonian is positive definite and thus there is no problem with negative energies. We now quantize the real scalar field and replace the field and its conjugate by hermitian operators which satisfy the canonical (equal-time) commutation relations

$$[\phi(x,t), \pi(y,t)] = i\delta^3(x-y) \qquad [\pi(x,t), \pi(y,t)] = [\phi(x,t), \phi(y,t)] = 0\ . \tag{1.51}$$

Thus the coefficients $a(k)$ in Eq. (1.7) become operators

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a(\vec{k})e^{ik_\mu x^\mu} + a^\dagger(\vec{k})e^{-ik_\mu x^\mu}\right] \tag{1.52}$$

$$= \int \frac{d^3k}{((2\pi)^3 2\omega_k)^{1/2}}\left[a(\vec{k})f_k(x) + a^\dagger(\vec{k})f_k^*(x)\right] \tag{1.53}$$

with the positive frequency (energy) solutions

$$f_k(x) = \frac{e^{ikx}}{((2\pi)^3 2\omega_k)^{1/2}} \tag{1.54}$$

which form an orthonormal set

$$\int d^3x f_k^*(x) i \overset{\leftrightarrow}{\partial_0} f_{k'}(x) = \delta^{(3)}(\vec{k} - \vec{k}') \; . \tag{1.55}$$

The operators $a(k)$ and $a^\dagger(k)$ satisfy the commutation relations

$$[a(k), a^\dagger(k')] = \int d^3x d^3y (2\pi)^3 (4\omega_k \omega_{k'})^{1/2} [f_k^*(x) i \overset{\leftrightarrow}{\partial_0} \phi(x), \phi(y) i \overset{\leftrightarrow}{\partial_0} f_{k'}(y)] \tag{1.56}$$

$$= i(2\pi)^3 \int d^3x d^3y (4\omega_k \omega_{k'})^{1/2} \tag{1.57}$$

$$\times \Big(f_k^*(x,t) i\partial_0 f_{k'}(y,t) [\pi(x,t), \phi(y,t)] + i\partial_0 f_k^*(x,t) f_{k'}(y,t) [\phi(x,t), \pi(y,t)] \Big) \tag{1.58}$$

$$= (2\pi)^3 \int d^3x (4\omega_k \omega_{k'})^{1/2} f_k^*(x,t) i \overset{\leftrightarrow}{\partial_0} f_{k'}(y,t) \tag{1.59}$$

$$= (2\pi)^3 2\omega_k \delta^{(3)}(k - k') \tag{1.60}$$

Note that for $\vec{k} = \vec{k}'$, there is a divergence, $\delta^{(3)}(0)$. We can formulate the quantum field theory in a mathematically more rigorous way by quantizing the theory in a finite volume V. In this case, the integrals are replaced by sums and $(2\pi)^3 \delta^{(3)}(0) \to V$. See the textbook "Quantum Field Theory" by Mandl and Shaw for a discussion in terms of fields quantized in a finite volume. However this breaks Lorentz invariance and for simplicity in this course we will always work with the infinite volume limit directly using formal manipulations, but keep in mind that we can always go to a finite volume to regularize the theory. Similarly we can derive the other commutation relations

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0 \; . \tag{1.61}$$

The operators $a(k)$ and $a^\dagger(k)$ play a similar role to the ladder operators for the quantum harmonic oscillator.

Similar to the quantum harmonic oscillator there is a ground state energy. It gives an infinite contribution to the energy when integrating over all momenta

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \, \omega_k a^\dagger(k) a(k) + (\epsilon_0 - \Omega_0) V \tag{1.62}$$

where we interpreted $(2\pi)^3 \delta^3(0)$ as volume V and $\epsilon_0 V$ denotes the zero-point energy of all harmonic oscillators with

$$\epsilon_0 = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega_k \; . \tag{1.63}$$

As only energy differences matter in the absence of gravity, we can subtract the zero point energy by choosing $\Omega_0 = \epsilon_0$ and thus the Hamiltonian becomes

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \, \omega_k a^\dagger(k) a(k) \tag{1.64}$$

This is formally equivalent to writing all annihilation to the right of the creation operators and setting $\Omega_0 = 0$. It is denoted *normal ordering* and we define the normal ordered product of operators AB by $:AB:$. Similarly the normal-ordered momentum is given by

$$:P_i: = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, k_i a^\dagger(k) a(k)\ . \tag{1.65}$$

Thus the energy is always positive given that the particle number does not become negative. This does not occur given that the norm of the states in the Hilbert space have to be non-negative. Let $|n(k)\rangle$ be a state with occupation number $n(k)$ for momentum k, then

$$\begin{aligned}\left[a(k)\,|n(k)\rangle\right]^\dagger \left[a(k)\,|n(k)\rangle\right] &= \Big\langle n(k)|a^\dagger(k)a(k)|n(k)\Big\rangle = (2\pi)^3 2\omega_k \delta^3(0)\,\langle n(k)|N(k)|n(k)\rangle \\ &\simeq 2\omega_k V n(k)\,\langle n(k)|n(k)\rangle \geq 0\ ,\end{aligned} \tag{1.66}$$

where we employed the finite volume limit in the second line. Thus as $a(k)$ reduces n(k) by 1, there is a ground state $|0\rangle$ with

$$a(k)\,|0\rangle = 0 \tag{1.67}$$

which does not contain any particles of momentum k. The operator $a(k)$ and $a^\dagger(k)$ are commonly called annihilation and creation operators, because they annihilate and create one particle with momentum k, respectively. The vacuum expectation value of one field operator vanishes

$$\langle 0|\phi|0\rangle = 0\ . \tag{1.68}$$

The one-particle state $|k\rangle = a^\dagger(k)\,|0\rangle$ is normalized as

$$\langle k|k'\rangle = \Big\langle 0|a(k)a^\dagger(k')|0\Big\rangle = \Big\langle 0|[a(k), a^\dagger(k')]|0\Big\rangle = (2\pi)^3 2\omega_k \delta^3(k-k') \tag{1.69}$$

where we used the commutation relation[3] of the creation and annihilation operators and the normalization of the vacuum state $\langle 0|0\rangle = 1$. The one-particle wave function $\psi(x)$ for a particle with momentum p is

$$\psi(x) \equiv \langle 0|\phi(x)|p\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\langle 0|a(k)|p\rangle\, e^{ikx} + \Big\langle 0|a^\dagger(k)|p\Big\rangle\, e^{-ikx}\right] = e^{ipx}\ . \tag{1.70}$$

1.5 Lehmann-Symanzik-Zimmermann (LSZ) reduction formula (5)

Consider a real scalar field theory. States in a free theory (i.e. which contain at most terms quadratic in the fields) are constructed by acting with the creation operators on the vacuum. A one-particle state is given by

$$|k\rangle = a^\dagger(k)\,|0\rangle \tag{1.71}$$

where the creation operator can be written in terms of the field operator as derived in Eq. (1.11)

$$a^\dagger(k) = \int d^3x\, e^{ikx}\,(\omega_k - i\partial_0)\,\phi(x) = -i\int d^3x\, e^{ikx} \overset{\leftrightarrow}{\partial_0}\, \phi(x) \tag{1.72}$$

[3]Note that some textbooks use a non-covariant normalization of the commutation relation $[a(k), a^\dagger(k')] = \delta(k-k')$.

and similarly

$$a(k) = \int d^3x \; e^{-ikx} \left(\omega_k + i\partial_0\right) \phi(x) \; . \tag{1.73}$$

We can define the creation operator of a particle localized in momentum space near $\vec{k}_1$ and in position space near the origin by

$$a_1^\dagger = \int d^3k f_1(k) a^\dagger(k) \tag{1.74}$$

where the function f_1 describes the wave packet of the particle, which we take to be a Gaussian with width σ

$$f_1(k) \propto \exp\left(-\frac{(\vec{k}-\vec{k}_1)^2}{2\sigma^2}\right) \tag{1.75}$$

In the Schrödinger picture the state $a_1^\dagger |0\rangle$ will propagate and spread out. The two particles in a two-particle state $a_1^\dagger a_2^\dagger |0\rangle$ with $k_1 \neq k_2$ are widely separated for $t \to \infty$. The creation and annihilation operators are time-dependent. A suitable initial (final) state of a scattering experiment is

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t) a_2^\dagger(t) |0\rangle \equiv a_1^\dagger(-\infty) a_2^\dagger(-\infty) |0\rangle \tag{1.76}$$

$$|f\rangle = \lim_{t\to\infty} a_{1'}^\dagger(t) a_{2'}^\dagger(t) |0\rangle \equiv a_{1'}^\dagger(\infty) a_{2'}^\dagger(\infty) |0\rangle \tag{1.77}$$

The scattering amplitude is then given by $\langle f|i\rangle$. Note however that the creation and annihilation operators in the scattering amplitude are evaluated at different times. We thus have to relate them to each other.

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{\infty} dt \partial_0 a_1^\dagger(t) \qquad \text{fundamental theorem of calculus} \tag{1.78}$$

$$= \int d^3k f_1(k) \int d^4x \partial_0 \left(e^{ikx}(\omega_k - i\partial_0)\phi(x)\right) \tag{1.79}$$

$$= -i \int d^3k f_1(k) \int d^4x e^{ikx} \left(\partial_0^2 + \omega^2\right) \phi(x) \tag{1.80}$$

$$= -i \int d^3k f_1(k) \int d^4x e^{ikx} \left(\partial_0^2 + \vec{k}^2 + m^2\right) \phi(x) \tag{1.81}$$

$$= -i \int d^3k f_1(k) \int d^4x e^{ikx} \left(\partial_0^2 - \overleftarrow{\nabla}^2 + m^2\right) \phi(x) \tag{1.82}$$

$$= -i \int d^3k f_1(k) \int d^4x e^{ikx} \left(\partial_0^2 - \overrightarrow{\nabla}^2 + m^2\right) \phi(x) \tag{1.83}$$

$$= -i \int d^3k f_1(k) \int d^4x e^{ikx} \left(-\partial_\mu \partial^\mu + m^2\right) \phi(x) \tag{1.84}$$

and similarly for the annihilation operator

$$a_1(\infty) - a_1(-\infty) = i \int d^3k f_1(k) \int d^4x e^{-ikx} \left(-\partial_\mu \partial^\mu + m^2\right) \phi(x) \tag{1.85}$$

Thus we are in a position to evaluate the scattering amplitude, the matrix elements of the scattering operator S,

$$\begin{aligned} S_{fi} \equiv \langle f|i\rangle &= \left\langle 0|T\left(a_{2'}(\infty)a_{1'}(\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)\right)|0\right\rangle \\ &= i^4 \int d^4x_1 e^{ik_1x_1}(-\partial_1^2+m^2)\ldots d^4x_{2'}d^4x_{2'}e^{-ik_{2'}x_{2'}}(-\partial_{2'}^2+m^2)\,\langle 0|T\phi(x_1)\ldots\phi(x_{2'})|0\rangle \end{aligned} \tag{1.86}$$

where we inserted the time-ordering operator in the first step, then used the derived relation of the creation and annihilation operators. The expression directly generalizes for more particles in the initial and/or final state. The general result is called the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula, which relates the scattering amplitude to the expectation value of the time-ordered product of field operators. The same holds in an interacting theory if the fields satisfies the following two conditions

$$\langle 0|\phi(x)|0\rangle = 0 \qquad\qquad \langle 0|\phi(x)|k\rangle = e^{ikx}\ . \tag{1.87}$$

These conditions can always be satisfied by shifting and rescaling the field operator appropriately. The two conditions ensure that the interacting one-particle states behave like free one-particle states. In general the creation operator will create a mixture of one-particle and multi-particle states in an interacting theory. However, it can be shown that multi-particle states separate from one-particle states at the infinite past and future and thus we can consider scattering between (quasi-)free particles. See the discussion in the book by Mark Srednicki for a more rigorous argument why the same also holds in an interacting field theory.

Next we develop tools to calculate correlation functions such as $\langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle$.

2 Feynman path integral

3 Additional Material

3.1 Cross sections and Decay rates (11)

The probability for a transition from state $|i\rangle$ to state $|f\rangle$ is then given by

$$P = \frac{|\langle f|i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} \tag{3.1}$$

We will first consider a finite volume V and time interval T in order to avoid infinities and then take the continuum limit. Thus

$$(2\pi)^3\delta^3(0) = \int d^3x e^{i0x} = V \qquad\qquad (2\pi)^4\delta^4(0) = \int d^4x e^{i0x} = VT \tag{3.2}$$

For definiteness we will consider the scattering of 2 particles to an arbitrary final state with n particles. The norm of a one-particle state is given by

$$\langle k|k\rangle = 2k^0(2\pi)^3\delta^3(0) = 2k^0V \tag{3.3}$$

and as the two-particle state in the infinite past and future can be effectively described by the product of two one-particle states, we obtain

$$\langle i|i\rangle = 4E_1E_2V^2 \qquad\qquad \langle f|f\rangle = \Pi_i(2k_i^0V)\ . \tag{3.4}$$

Similarly for the squared transition amplitude we obtain

$$|\langle f|i\rangle|^2 = |(2\pi)^4\delta(k_{in}-k_{out})|^2|M|^2 = (2\pi)^4\delta(k_{in}-k_{out})VT|M|^2\ . \tag{3.5}$$

Finally we have to sum over all possible final states. In the box with length L the 3-momenta are quantized $\vec{k}_i = \frac{2\pi}{L}\vec{n}_i$ and thus summing over the different modes corresponds to an integration

$$\sum_{n_i} \to \frac{L^3}{(2\pi)^3}\int d^3k_i\ . \tag{3.6}$$

The volume $V = L^3$ cancels against the volume factor from the normalization. Thus the transition rate is

$$\dot{P} = \frac{(2\pi)^4\delta(k_{in}-k_{out})}{4E_1E_2V}\int \Pi_{i=1}^n \frac{d^3k_i}{(2\pi)^3 2\omega_i}|M|^2\ . \tag{3.7}$$

The Lorentz-invariant differential cross section can be obtained by dividing by the incoming particle flux. In the rest frame of the second particle, it is simply given by the velocity of the first particle per volume. In the centre-of-mass frame (where the 3-momenta of the incoming particles add to zero), it is the relative velocity per volume $\sigma = \dot{P}V/v_{rel}$ where the relative velocity can be expressed by

$$v_{rel} = |\vec{v}_1 - \vec{v}_2| = \left|\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2}\right| = \frac{|\vec{p}_1|}{E_1E_2}(E_1+E_2) = \frac{|\vec{p}_1|}{E_1E_2}\sqrt{s} \tag{3.8}$$

In the last equation we introduced the so-called Mandelstam variable s. Scattering of 2 particles into 2 particles can be described in terms of 3 Lorentz-invariant quantities, the so-called Mandelstam variables

$$s = -(k_1 + k_2)^2 \qquad t = -(k_1 - k_3)^2 \qquad u = -(k_1 - k_4)^2 \tag{3.9}$$

The differential cross section can also be written in terms of the flux factor[4]

$$F = E_1 E_2 v_{rel} = \sqrt{s}\sqrt{E_1^2 - m_1^2} = \frac{1}{2}\sqrt{(s + m_1^2 - m_2^2)^2 - 4sm_1^2} = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \tag{3.11}$$

and thus expressed as

$$4F d\sigma = |M|^2 d\mathrm{LIPS}_n(k_1 + k_2) \; . \tag{3.12}$$

The Lorentz-invariant n-body phase space is defined by

$$d\mathrm{LIPS}_n(k) = (2\pi)^4 \delta\left(k - \sum_{i=1}^{n} k_i\right) \Pi_{i=1}^{n} \frac{d^3 k_i}{(2\pi)^3 2\omega_i} \; . \tag{3.13}$$

The 2-body phase space is particularly simple to evaluate in the centre-of-mass frame

$$\int d\mathrm{LIPS}_2(k) = \int (2\pi)^4 \delta(k - k_1 - k_2) \frac{d^3 k_1}{(2\pi)^3 2\omega_1} \frac{d^3 k_2}{(2\pi)^3 2\omega_2} \tag{3.14}$$

$$= \int (2\pi) \delta(\sqrt{s} - E_1 - E_2) \frac{d^3 k_1}{(2\pi)^3 2E_1 2E_2(k_1)} \tag{3.15}$$

$$= \int \delta(\sqrt{s} - E_1 - E_2) \frac{|\vec{k}_1| dE_1 d\Omega}{16\pi^2 E_2(E_1)} = \int \frac{d\Omega}{16\pi^2} \frac{|\vec{k}_1|}{\sqrt{s}} \tag{3.16}$$

$$= \int_{-1}^{1} \frac{d\cos\theta}{8\pi} \frac{|\vec{k}_1|}{\sqrt{s}} \tag{3.17}$$

where the last line holds if the integrand does not depend on the azimuthal angle ϕ.

In case that there are n identical particles in the final state, we have to divide by the symmetry factor $S = n!$ in the cross section in order to avoid double-counting the different final state configurations. Thus the total cross section is given by

$$\sigma = \frac{1}{S} \int d\sigma \; . \tag{3.18}$$

If there is one particle in the initial state and we are studying decays we have to slightly modify our assumptions, because we assumed all particles to be stable in the previous discussion. However it

[4]The energy of one particle in the centre-of-mass frame is

$$E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \tag{3.10}$$

which can be derived by using the definition of s and $\vec{p}_1 + \vec{p}_2 = 0$. For the last step in Eq. 3.11 we used $s = m_1^2 + m_2^2 - 2p_1 \cdot p_2$.

turns out that the LSZ reduction formula also holds in this case. We only have to modify the initial state normalization $\langle i|i\rangle = 2E_1 V$ and find for the differential decay rate

$$d\Gamma = \frac{|M|^2}{2E_1} d\mathrm{LIPS}_n(k_1) \tag{3.19}$$

and the total decay rate is obtained by summing over all outgoing momenta and dividing by the symmetry factor

$$\Gamma = \frac{1}{S}\int d\Gamma \ . \tag{3.20}$$

Note that the decay rate is not a Lorenz scalar. In the centre-of-mass frame of the particle, there is $E_1 = m_1$, while the decay rate is smaller in any other frame by a factor $\gamma = E_1/m_1$, the relativistic boost factor, which accounts for the relativistic time dilation. Faster particles have a longer lifetime, e.g. muons generated in the atmosphere reach the Earth's surface due to this time dilation factor.

3.2 Optical theorem

The S matrix is unitary because for two in-states $|i\rangle$ and $|j\rangle$ and the final (out-state) $|f\rangle$ we find

$$\sum_f S^*_{jf} S_{fi} = \sum_f \langle j|f\rangle \langle f|i\rangle = \langle j|i\rangle = \delta_{ij} \tag{3.21}$$

where we used the completeness of the out states $|f\rangle$. Thus the probability to scatter to any final state $|f\rangle$ is unity. It is often convenient to write the S matrix as

$$S = 1 + iT \tag{3.22}$$

where the identity refers to no scattering and the T matrix describes any (non-trivial) scattering. Then unitarity translates into

$$1 = S^\dagger S = (1 - iT^\dagger)(1 + iT) = 1 + iT - iT^\dagger + T^\dagger T \qquad \Rightarrow \qquad T^\dagger T = -i(T - T^\dagger) = 2\mathrm{Im}(T) \tag{3.23}$$

which is one form of the optical theorem.

3.3 Spin-statistics connection (4)

Could we quantize the scalar field also with anticommutators instead of commutators? Consider spin-0 particles with the Hamiltonian

$$H_0 = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k a^\dagger(k) a(k). \tag{3.24}$$

The creation and annihilation operators either satisfy commutation or anticommutation relations

$$[a(k), a(k')]_\mp = [a^\dagger(k), a^\dagger(k')]_\mp = 0 \qquad\qquad [a(k), a^\dagger(k')]_\mp = (2\pi)^3 2\omega_k \delta^3(k - k') \tag{3.25}$$

Now construct a theory with local Lorentz-invariant interactions out of the non-hermitian fields

$$\phi^+(x,0) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, e^{i\vec{k}\cdot\vec{x}} a(k) \qquad\qquad \phi^-(x,0) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, e^{-i\vec{k}\cdot\vec{x}} a^\dagger(k) \tag{3.26}$$

$$\phi^\pm(x,t) = e^{iH_0 t}\phi^\pm(x,0)e^{-iH_0 t} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{\pm ikx} a^{(\dagger)}(k) \tag{3.27}$$

which are related by hermitian conjugation $\phi^+ = (\phi^-)^\dagger$. $\phi^\pm(x,t)$ is the time-evolved field $\phi^\pm(x,0)$. The field ϕ is invariant under proper orthochronous Lorentz transformations Λ

$$U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x) \tag{3.28}$$

and thus also the creation ($a^\dagger(k)$) and annihilation ($a(k)$) operators as well as the fields $\phi^\pm(x)$ are Lorentz scalars. Construct an interaction Lagrangian density $\mathcal{L}_1$ which is local and Lorentz invariant. The corresponding Hamiltonian density is denoted by $\mathcal{H}_1$ and H_1 denotes the interaction Hamiltonian in the Schrödinger picture

$$H_1 = \int d^3x \mathcal{H}_1(x,0) \tag{3.29}$$

where $\mathcal{H}_1(x,0)$ is a hermitian function of $\phi^\pm(x,0)$. The corresponding Hamiltonian density in the interaction picture $\mathcal{H}_I(x,t)$ is defined by the same function, but with $\phi^\pm(x,0)$ replaced by $\phi^\pm(x,t)$, since the interaction Hamiltonian is given by $H_I(t) = \exp(-iH_0t)H_1\exp(iH_0t)$.

Consider now a transition amplitude from a state $|i\rangle$ at $t=-\infty$ to state $|f\rangle$ at $t=\infty$

$$\mathcal{T}_{f\leftarrow i} = \left\langle f|T\exp\left[-i\int_{-\infty}^{\infty} dt H_I(t)\right]|i\right\rangle \ . \tag{3.30}$$

The transition amplitude is only Lorentz invariant, if the time-ordering is frame independent. This is trivially satisfied in the forward and backward lightcone for time-like separations of two events $(x-x')^2 > 0$. However events with a space-like separation can have different temporal ordering. Thus we have to require that the interaction Hamiltonian commutes with itself at different times for space-like separations

$$[\mathcal{H}_I(x), \mathcal{H}_I(x')] = 0 \qquad\qquad (x-x')^2 > 0 \tag{3.31}$$

The interaction Hamiltonian densities are constructed of the fields $\phi^\pm(x,t)$ and thus we have to consider the commutator (anticommutator) of the fields $\phi^\pm$ for space-like separations $r^2 \equiv (x-x')^2 > 0$

$$[\phi^+(x), \phi^-(x')]_\mp = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\frac{d^3k'}{(2\pi)^3 2\omega'_k} e^{i(kx-k'x')}[a(k), a^\dagger(k')]_\mp = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{ik(x-x')} \tag{3.32}$$

The integral is best evaluated for $t=t'$ using spherical coordinates. For $t=t'$, $k=|\vec{k}|$ and $r=|\vec{x}-\vec{x}'|$

$$\frac{1}{8\pi^2}\int_0^\infty dk \int_{-1}^1 d\cos\theta \frac{k^2}{E(k)} e^{ikr\cos\theta} = \frac{1}{8\pi^2}\int_0^\infty dk \frac{k^2}{E(k)}\frac{e^{ikr}-e^{-ikr}}{ikr} = -\frac{i}{8\pi^2 r}\int_{-\infty}^\infty dk \frac{k}{\sqrt{k^2+m^2}} e^{ikr} \ . \tag{3.33}$$

The integrand has branch cuts along the imaginary axis starting at $\pm im$. We deform the integration contour such that it wraps around the upper branch cut by defining $\rho = ik/m$ and thus

$$[\phi^+(x), \phi^-(x')]_\mp = \frac{m}{4\pi^2 r}\int_1^\infty d\rho \frac{\rho e^{-\rho m r}}{\sqrt{\rho^2-1}} = \frac{m}{4\pi^2 r}\int_1^\infty d\rho \left(\frac{\partial}{\partial\rho}\sqrt{\rho^2-1}\right) e^{-\rho m r} \tag{3.34}$$

$$= \frac{m}{4\pi^2 r}\left\{\left[\sqrt{\rho^2-1}e^{-\rho m r}\right]_1^\infty - \int_1^\infty d\rho\sqrt{\rho^2-1}\frac{\partial}{\partial\rho}e^{-\rho m r}\right\} \tag{3.35}$$

$$= \frac{m^2}{4\pi^2}\int_1^\infty d\rho\sqrt{\rho^2-1}e^{-\rho m r} = \frac{m}{4\pi^2 r}K_1(mr) \equiv C(r) \tag{3.36}$$

where K_1 denotes the modified Bessel function. Thus the (anti)commutator is non-zero for any $r > 0$. Consider now the linear combination $\phi_\lambda(x) = \phi^+(x) + \lambda\phi^-(x)$ with an arbitrary complex number λ. The (anti)commutators of ϕ_λ for space-like separations are then

$$[\phi_\lambda(x), \phi_\lambda^\dagger(x')]_\mp = [\phi^+(x), \phi^-(x')]_\mp + |\lambda|^2[\phi^-(x), \phi^+(x')]_\mp = (1-|\lambda|^2)C(r) \tag{3.37}$$

$$[\phi_\lambda(x), \phi_\lambda(x')]_\mp = \lambda[\phi^+(x), \phi^-(x')]_\mp + \lambda[\phi^-(x), \phi^+(x')]_\mp = \lambda(1\mp 1)C(r) \tag{3.38}$$

Thus in order for both (anti)commutators to vanish and to have a suitable interaction Hamiltonian built out of the field ϕ_λ we have to choose commutators with $|\lambda| = 1$. This brings us back to a real scalar field if we choose $\lambda = e^{i\alpha}$, then $e^{-i\alpha/2}\phi_\lambda(x) = \phi(x)$.

The same argument can be made for higher spin fields in any number of space-time dimensions[5]. The allowed choice is always commutators for integer spin and anticommutators for half-integer spin. Hence particles with integer spin are bosons and particles with half-integer spin fermions.

3.4 Complex scalar field (22)

So far we considered a real scalar field. For a complex scalar field the discussion is very similar, but there are a few important differences. The Lagrangian of a free complex scalar field ϕ is given by

$$\mathcal{L} = -\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi \tag{3.39}$$

The Lagrangian is invariant under an internal symmetry $\phi \to e^{-iq\alpha}\phi$, where q is the charge of the field, and thus there is a Noether current and we obtain

$$J_\mu = -iq\left[(\partial_\mu\phi^\dagger)\phi - (\partial_\mu\phi)\phi^\dagger\right] = iq\phi^\dagger\overset{\leftrightarrow}{\partial_\mu}\phi \tag{3.40}$$

and a conserved charge

$$Q = iq\int d^3x\phi^\dagger\overset{\leftrightarrow}{\partial_t}\phi\,. \tag{3.41}$$

The Euler-Lagrange equations and conjugate momenta are

$$(\Box - m^2)\phi = 0 \qquad\qquad (\Box - m^2)\phi^\dagger = 0 \tag{3.42}$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi^\dagger \qquad\qquad \pi^\dagger = \frac{\partial\mathcal{L}}{\partial\dot\phi^\dagger} = \dot\phi \tag{3.43}$$

[5]For $d = 2$ there are more possibilities apart from fermions and bosons, so-called anyons.

The general solution for the fields ϕ and $\phi^\dagger$ are given by

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a(k)e^{ikx} + b^\dagger(k)e^{-ikx} \right) \tag{3.44}$$

$$\phi^\dagger(x,t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a^\dagger(k)e^{-ikx} + b(k)e^{ikx} \right) . \tag{3.45}$$

We quantize it in the usual way, but we have to keep in mind that ϕ and $\phi^\dagger$ are independent. The equal time commutation relations are

$$[\pi(x,t), \phi(y,t)] = [\pi^\dagger(x,t), \phi^\dagger(y,t)] = -i\delta^3(x-y) \tag{3.46}$$

while all other equal time commutation relations vanish. This leads to the following commutation relations for the operators $a(k)$ and $b(k)$

$$[a(k), a^\dagger(k')] = [b(k), b^\dagger(k')] = (2\pi)^3 2\omega_k \delta(\vec{k} - \vec{k}') \tag{3.47}$$

where $a(k)$ and $b(k)$ are annihilation operators. Thus there are two types of particles, a and b, with Hamiltonian

$$H = \int d\frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k \left[a^\dagger(k)a(k) + b^\dagger(k)b(k) \right] . \tag{3.48}$$

Thus the particles both have the same positive energy, but they have equal and opposite charges, because the charge operator is given by

$$Q = q \int \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a^\dagger(k)a(k) - b^\dagger(k)b(k) \right] . \tag{3.49}$$

Otherwise their properties including mass are equivalent. They form a pair of particle and antiparticle. The existence of antiparticles is a general feature of relativistic quantum field theory.

A Review

A.1 Quantum harmonic oscillator

The last 1D example which we are studying is the harmonic oscillator. It is very important approximation to many physical phenomena such as the vibrational modes of a diatomic molecule as shown in Fig. 1. A particle with mass m is subject to a restoring force $-kx$, where x is the displacement from the equilibrium position. The potential energy of the particle is

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2 \tag{A.1}$$

where $\omega = \sqrt{k/m}$ is the angular oscillation frequency. Thus its Hamiltonian is

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 \tag{A.2}$$

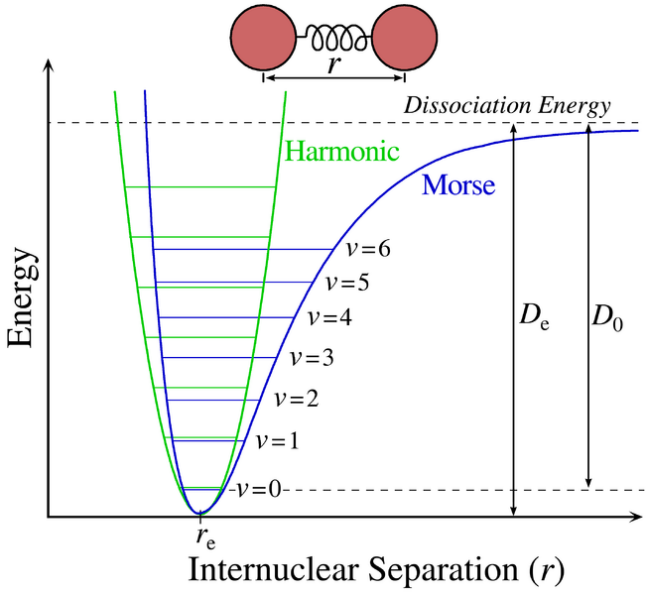

Figure 1: First few vibrational modes of a diatomic molecule are well described by a harmonic oscillator. Figure taken from `https://en.wikipedia.org/wiki/File:Morse-potential.png`.

and the time-independent Schrödinger equation is given by

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2\right)\varphi_E(x)=E\varphi_E(x)\ . \tag{A.3}$$

Now we will be going one step and solve the quantum harmonic oscillator using the ladder operator method.

In order to simplify our following discussion, we define the variable

$$\xi=\sqrt{\frac{m\omega}{\hbar}}x \qquad\qquad \frac{d}{d\xi}=\sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx} \tag{A.4}$$

and can rewrite our Hamiltonian in Eq. (A.2) as follows

$$H=\frac{1}{2}\hbar\omega\left(-\frac{d^2}{d\xi^2}+\xi^2\right) \tag{A.5}$$

In the next step we want to factorise the Hamiltonian in analogy to the identity

$$u^2-v^2=(u-v)(u+v)\ . \tag{A.6}$$

Hence we form two new operators as linear combination of the old ones

$$a=\frac{1}{\sqrt{2}}\left(\xi+\frac{d}{d\xi}\right)=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+i\frac{\hat{p}}{m\omega}\right) \tag{A.7}$$

$$a^\dagger=\frac{1}{\sqrt{2}}\left(\xi-\frac{d}{d\xi}\right)=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}-i\frac{\hat{p}}{m\omega}\right)\ . \tag{A.8}$$

The operators satisfy the following commutation relation

$$[a, a^\dagger] = 1 \ . \tag{A.9}$$

Looking at the product of the two operators

$$a^\dagger a = \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)\frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right) \tag{A.10}$$

$$= \frac{1}{2}\left(\xi^2 - \frac{d^2}{d\xi^2} + \xi\frac{d}{d\xi} - \frac{d}{d\xi}\xi\right) \tag{A.11}$$

$$= \frac{1}{2}\left(\xi^2 - \frac{d^2}{d\xi^2} + [\xi, \frac{d}{d\xi}]\right) \tag{A.12}$$

$$= \frac{1}{2}\left(\xi^2 - \frac{d^2}{d\xi^2} - 1\right) \tag{A.13}$$

we observe that we almost obtain the result which we wanted to obtain. Hence, the Hamiltonian can be rewritten as

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \ . \tag{A.14}$$

Before interpreting the operators $\hat{a}$ and $\hat{a}^\dagger$ physically, we have to show two more commutation relations:

$$[H, a] = Ha - aH \tag{A.15}$$

$$= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)a - a\hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \tag{A.16}$$

$$= \hbar\omega\left(a^\dagger a a - a a^\dagger a\right) \tag{A.17}$$

$$= \hbar\omega\left(a^\dagger a a - \left(a^\dagger a + [a, a^\dagger]\right)a\right) \tag{A.18}$$

$$= -\hbar\omega a \tag{A.19}$$

Now given an energy eigenstate $|E\rangle$ with a given energy E, we can calculate the energy eigenvalue of the states $a\,|E\rangle$ as follows

$$H(a\,|E\rangle) = Ha\,|E\rangle \tag{A.20}$$

$$= (aH + [H, a])\,|E\rangle \tag{A.21}$$

$$= (aE - \hbar\omega a)\,|E\rangle \tag{A.22}$$

$$= (E - \hbar\omega)(a\,|E\rangle) \tag{A.23}$$

Similarly for the operator $a^\dagger$

$$[H, a^\dagger] = +\hbar\omega a^\dagger \ . \tag{A.24}$$

and

$$H(a^\dagger\,|E\rangle) = (E + \hbar\omega)(a^\dagger\,|E\rangle) \ . \tag{A.25}$$

Hence the states $a\,|E\rangle$, $a^\dagger\,|E\rangle$ are also energy eigenstates with energies $E \pm \hbar\omega$, respectively. The operators a and $a^\dagger$ transform a state with energy E into a state with energy $E \pm \hbar\omega$. They are denoted *ladder operators*, more specifically $a^\dagger$ is denoted *raising operator* and *a lowering operator*.

Next we have to find the lowest energy eigenstate or ground state. Classically we observe that there is a minimum energy of the harmonic oscillator. Hence there has to be a lowest energy eigenstate

$$a\,|E_{lowest}\rangle = 0\ . \tag{A.26}$$

This is called the *ladder termination condition*. The energy of this lowest energy eigenstate is given by

$$H\,|E_{lowest}\rangle = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|E_{lowest}\rangle = \frac{1}{2}\hbar\omega\,|E_{lowest}\rangle\ . \tag{A.27}$$

Note that lowest energy level is not zero as it would be for a classical harmonic oscillator, but $\frac{1}{2}\hbar\omega$. It is known as *zero point energy* and ultimately due to the non-vanishing commutator of the ladder operators $[a, a^\dagger] = 1$. The lowest energy eigenstate is commonly denoted $|0\rangle$. The energy of the n^{th} state $|n\rangle$ is given by

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)\ , \tag{A.28}$$

because applying the raising operator $a^\dagger$ n times increases the energy with respect to the lowest energy eigenstate by $n \times \hbar\omega$. In addition to the ladder operators it is convenient to introduce the *number operator*,

$$\hat{N} = \hat{a}^\dagger\hat{a}\ , \tag{A.29}$$

which counts the energy quanta. It fulfils the following eigenvalue equation

$$\hat{N}\,|n\rangle = n\,|n\rangle\ , \tag{A.30}$$

where the n in $|n\rangle$ denotes the number of energy quanta. We can rewrite the Hamiltonian as

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)\ . \tag{A.31}$$

See Fig. 2 for an illustration of the action of the ladder operator on the energy eigenstates. All other energy eigenstates can be constructed from the lowest energy eigenstate using the raising operator. By demanding that all states $|n\rangle$ are properly normalised,

$$\langle n|n\rangle = 1\ , \tag{A.32}$$

it is possible to show[6] that the raising and lowering operators act on a state $|n\rangle$

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle \tag{A.33}$$

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\ . \tag{A.34}$$

[6]See the discussion in McIntyre Chap.9.

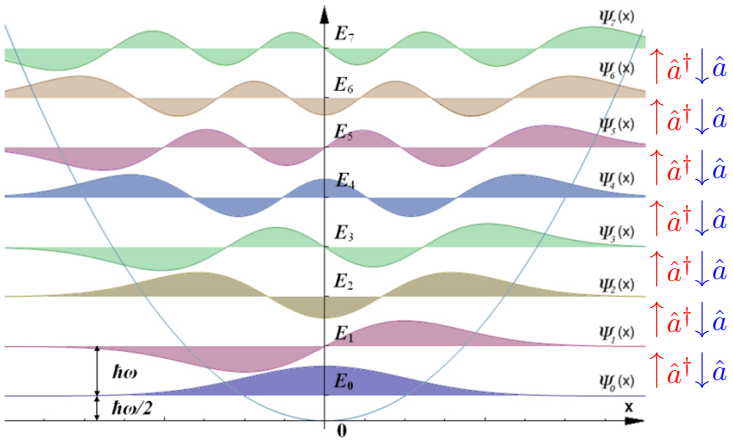

Figure 2: Energy levels of harmonic oscillator. Raising operator $a^\dagger$ increases energy by $\hbar\omega$ and lowering operator a lowers it. Figure taken from `https://commons.wikimedia.org/w/index.php?curid=11623546`.

Thus we can write the state $|n\rangle$ as follows

$$|n\rangle \equiv \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |0\rangle \ , \tag{A.35}$$

where we denoted the lowest energy eigenstate by $|0\rangle$. The factor $1/\sqrt{n!}$ ensures that the states are correctly normalised.

The wave function of the lowest energy eigenstate $\phi_0(\xi)$ can be determined from the ladder termination condition in Eq. (A.26)

$$0 = a\phi_0(\xi) = \frac{1}{\sqrt{2}} \left(\xi + \frac{d}{d\xi}\right) \phi_0(\xi) \ . \tag{A.36}$$

It is an ODE, which can be solved using standard techniques

$$\phi_0(\xi) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2} \ . \tag{A.37}$$

The explicit form of the wave function involves a well studied special function, the Hermite polynomial, such that the normalised wave functions can be written as

$$\varphi_E(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2} \quad \text{with } \xi = \sqrt{\frac{m\omega}{\hbar}} x \text{ and } n = 0, 1, 2, 3, \ldots \tag{A.38}$$

where H_n is a Hermite polynomial of order n

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2 \qquad H_3(x) = 8x^3 - 12x \ . \tag{A.39}$$

Figure 3 shows wave functions and probability densities for the first four energy levels of a harmonic oscillator.

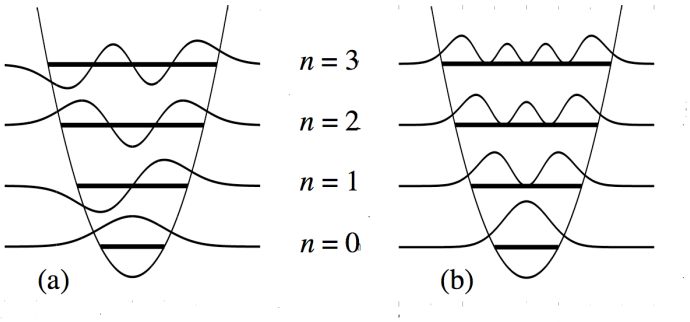

Figure 3: (a) Wave functions and (b) the probability density for the first four energy levels of a harmonic oscillator.

A.2 Time-dependent perturbation theory in quantum mechanics

As we are dealing with a time-dependent Hamiltonian, we have to return to the Schrödinger equation in its original form. So far we always considered the Hamiltonian to be time-independent, while the quantum states evolved with time. This is commonly denoted as *Schrödinger picture*. If we however consider the states[7]

$$|\psi\rangle_H \equiv e^{i\hat{H}t/\hbar}\,|\psi(t)\rangle = \left(1 + \frac{i}{\hbar}\hat{H}t + \frac{1}{2}\left[\frac{i}{\hbar}\hat{H}t\right]^2\right)|\psi(t)\rangle \tag{A.40}$$

we find that

$$i\hbar\frac{\partial}{\partial t}\,|\psi\rangle_H = i\hbar\frac{\partial}{\partial t}e^{i\hat{H}t/\hbar}\,|\psi(t)\rangle = e^{i\hat{H}t/\hbar}\left(-\hat{H}\,|\psi(t)\rangle + i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle\right) = 0 \tag{A.41}$$

This is commonly denoted as *Heisenberg picture*. In this picture, all operators

$$\hat{A}_H(t) = e^{i\hat{H}t/\hbar}\hat{A}e^{-i\hat{H}t/\hbar}\,, \tag{A.42}$$

e.g. momentum operator, will be time-dependent and satisfy

$$\frac{d}{dt}\hat{A}_H(t) = \frac{i}{\hbar}[\hat{H}_H(t), \hat{A}_H(t)] + \left(\frac{\partial\hat{A}}{\partial t}\right)_H\,. \tag{A.43}$$

Thus the eigenvalues of any operator, which commutes with the Hamiltonian, are contants and provide good quantum numbers to describe the system. We will now consider the *interaction picture*, in which both the Hamiltonian and the quantum state will depend on time. It is useful for cases, where we can split the Hamiltonian $\hat{H}(t) = \hat{H}_0 + \hat{H}'(t)$ in a time-independent part $\hat{H}_0$ and a time-dependent one $\hat{H}'$. It is defined by

$$|\psi(t)\rangle_I = e^{i\hat{H}_0t/\hbar}\,|\psi(t)\rangle \qquad\qquad \hat{A}_I(t) = e^{i\hat{H}_0t/\hbar}\hat{A}e^{-i\hat{H}_0t/\hbar}\,. \tag{A.44}$$

[7]Note that $e^{\hat{A}} = \sum_{n=0}^{\infty}\frac{1}{n!}\hat{A}^n$.

We rewrite the Schrödinger equation for the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}'(t)$ in the interaction picture

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle_I &= i\hbar\frac{\partial}{\partial t}e^{i\hat{H}_0 t/\hbar}\left|\psi(t)\right\rangle && \text{(A.45)}\\
&= -\hat{H}_0\left|\psi(t)\right\rangle_I + e^{i\hat{H}_0 t/\hbar} i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle && \text{(A.46)}\\
&= -\hat{H}_0\left|\psi(t)\right\rangle_I + e^{i\hat{H}_0 t/\hbar}\left(\hat{H}_0 + \hat{H}'(t)\right)\left|\psi(t)\right\rangle && \text{(A.47)}\\
&= -\hat{H}_0\left|\psi(t)\right\rangle_I + e^{i\hat{H}_0 t/\hbar}\left(\hat{H}_0 + \hat{H}'(t)\right)e^{-i\hat{H}_0 t/\hbar}\left|\psi(t)\right\rangle_I && \text{(A.48)}\\
&= e^{i\hat{H}_0 t/\hbar}\hat{H}'(t)e^{-i\hat{H}_0 t/\hbar}\left|\psi(t)\right\rangle_I && \text{(A.49)}\\
&= \hat{H}'_I(t)\left|\psi(t)\right\rangle_I \ . && \text{(A.50)}
\end{aligned}
$$

We used that the exponential $\exp(iH_0 t/\hbar)$ commutes with H_0. Finally, we can now formally integrate the equation and obtain the equivalent integral equation

$$
\left|\psi(t)\right\rangle_I = \left|\psi(t_0)\right\rangle_I + \frac{1}{i\hbar}\int_{t_0}^{t}\hat{H}'_I(t')\left|\psi(t')\right\rangle_I dt' \ . \tag{A.51}
$$

This integral equation can be iteratively solved by plugging the solution back in on the right-hand side of the equation, i.e.

$$
\left|\psi(t)\right\rangle_I = \left|\psi(t_0)\right\rangle_I + \frac{1}{i\hbar}\int_{t_0}^{t}\hat{H}'_I(t')\left|\psi(t_0)\right\rangle_I dt' + \left(\frac{1}{i\hbar}\right)^2\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\hat{H}'_I(t')\hat{H}'_I(t'')\left|\psi(t_0)\right\rangle_I + \dots \tag{A.52}
$$

We introduce the time-ordered product of operators to simplify the expression further. The time-ordered product of two operators A and B is defined as

$$
\mathcal{T}\left(A(t_1)B(t_2)\right) \equiv \begin{cases} A(t_1)B(t_2) & \text{for } t_1 \geq t_2 \\ B(t_2)A(t_1) & \text{otherwise} \end{cases} \ . \tag{A.53}
$$

This allows us to rewrite the quadratic term as

$$
\begin{aligned}
\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\hat{H}'_I(t')\hat{H}'_I(t'') &= \int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\mathcal{T}\left(\hat{H}'_I(t')\hat{H}'_I(t'')\right) = \mathcal{T}\left(\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\hat{H}'_I(t')\hat{H}'_I(t'')\right) && \text{(A.54)}\\
&= \mathcal{T}\left(\int_{t_0}^{t}dt''\int_{t_0}^{t''}dt'\hat{H}'_I(t'')\hat{H}'_I(t')\right) = \mathcal{T}\left(\int_{t_0}^{t}dt''\int_{t_0}^{t''}dt'\hat{H}'_I(t')\hat{H}'_I(t'')\right) && \text{(A.55)}\\
&= \frac{1}{2}\mathcal{T}\left(\int_{t_0}^{t}dt'\int_{t_0}^{t}dt''\hat{H}'_I(t')\hat{H}'_I(t'')\right) = \frac{1}{2}\mathcal{T}\left(\int_{t_0}^{t}dt'\hat{H}'_I(t')\right)^2 , && \text{(A.56)}
\end{aligned}
$$

where we introduced the time-ordered product in the first line, relabelled the integration variables $t' \leftrightarrow t''$ in the second line, and noticed that summing the last terms of the first and second line and

dividing by 2 can be rewritten as the first term of the third line. Hence we can write the quadratic term as the square of the integral and

$$|\psi(t)\rangle_I = |\psi(t_0)\rangle_I + \frac{1}{i\hbar}\int_{t_0}^{t} \hat{H}'_I(t')dt' \, |\psi(t_0)\rangle_I + \frac{1}{2}\mathcal{T}\left(\frac{1}{i\hbar}\int_{t_0}^{t} dt' H'_I(t')\right)^2 |\psi(t_0)\rangle_I + \dots \tag{A.57}$$

$$= \mathcal{T}\left(e^{-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}'_I(t')dt'}\right)|\psi(t_o)\rangle_I \ . \tag{A.58}$$

We will only consider the leading order transitions and neglect terms with more than one $\hat{H}'$. If we have the set of eigenstates $|n\rangle$ to the Hamiltonian $\hat{H}_0$ with eigenvalue E_n, we can calculate the transition amplitude to find our particle in state $|n\rangle$ at time t if we start in state $|m\rangle$ at time t_0. The system is initially in the state

$$|m(t)\rangle = e^{-iH_0 t/\hbar}|m\rangle = e^{-iE_m t/\hbar}|m\rangle \ , \tag{A.59}$$

where $|m(t)\rangle$ denotes the state $|m\rangle$ which has been evolved to time t with the Hamiltonian H_0. The probability amplitude that there is a transition to the state

$$|n(t)\rangle = e^{-iH_0 t/\hbar}|n\rangle = e^{-iE_n t/\hbar}|n\rangle \tag{A.60}$$

is given by

$$\langle n(t)|\psi(t)\rangle = \Big\langle n|e^{iH_0 t/\hbar}|\psi(t)\Big\rangle = \langle n|\psi(t)\rangle_I \tag{A.61}$$

where the subscript I indicates the interaction picture. The initial state $|\psi(t_0)\rangle_I = e^{iH_0 t/\hbar}|m(t)\rangle = |m\rangle$. Inserting the states in Eq. (A.52), we find

$$\langle n|\psi(t)\rangle_I = \langle n|m\rangle + \frac{1}{i\hbar}\int_{t_0}^{t}\Big\langle n|\hat{H}'_I(t')|m\Big\rangle dt' \tag{A.62}$$

$$= \delta_{nm} + \frac{1}{i\hbar}\int_{t_0}^{t} e^{i(E_n-E_m)t'/\hbar}\Big\langle n|\hat{H}'(t')|m\Big\rangle dt' \ ,$$

where we used the definition of an operator in the interaction picture in Eq. (A.44). Note that all expressions in the last line are given in the usual Schrödinger picture, which we worked with in the previous sections. The probability for a transition from the state $|m\rangle$ to the state $|n\rangle \neq |m\rangle$ is given by

$$P_{mn}(t) = \left|\frac{1}{i\hbar}\int_{t_0}^{t} dt' e^{i(E_n-E_m)t'/\hbar}\Big\langle n|\hat{H}'(t')|m\Big\rangle\right|^2 \ . \tag{A.63}$$

The first term in Eq. (A.62) corresponds to an unchanged quantum state $|m\rangle$, which has not been affected by any interaction. The second term describes one transition between two states induced by the Hamiltonian $\hat{H}'$:

1. We start with a quantum system in the eigenstate $|m\rangle$ of the Hamiltonian H_0.
2. Then we evolve the quantum system in time from t_0 to t' using the Hamiltonian H_0. Thus eigenstates $|m\rangle$ of H_0 do not change and only pick up a phase factor $\exp(iE_m(t'-t_0)/\hbar)$.

3. At time t', the Hamiltonian $\hat{H}'$ affects the quantum system and leads to transitions between states, depending on the matrix elements $\left\langle n|\hat{H}'|m\right\rangle$.

4. After the interaction, the system is in a new state $|n\rangle$ and we evolve it with the Hamiltonian H_0 in time from t' to t and the quantum state picks up a phase factor $\exp(iE_n(t-t')/\hbar)$.

5. The transition probability is given by projecting this state at time t onto the eigenstates of H_0. We explicitly include the phase factors from time evolution.

Higher order terms in Eq. (A.52) can be interpreted in the same way. The only difference is that there will be multiple transitions at different times t' , t'', ..., and the quantum states between the transitions are evolved in time using the Hamiltonian H_0 leading to additional phase factors.

A.3 Green's function

Consider a differential equation of the form

$$L\psi(x) = f(x) \tag{A.64}$$

with a linear differential operator $L = L(x)$. The Green's function $G(x, y)$ of the differential operator is defined by

$$LG(x, y) = -\delta(x - y) \ . \tag{A.65}$$

It can be used to obtained a general solution for the differential equation Eq. (A.64)

$$\psi(x) = -\int dy G(x, y) f(y) \ . \tag{A.66}$$

This is straightforward to see

$$L\psi(x) = L\left(-\int dy G(x, y) f(y)\right) = -\int dy LG(x, y) f(y) = \int dy \delta(x - y) f(y) = f(x) \ . \tag{A.67}$$

A.4 Group theory

Symmetries are fundamental ingredients in describing physics. A symmetry transformation is a reversible operation which does not change the physical system. The set of symmetry transformations form a group.

Definition: A group G is a set of elements with an operation of multiplication that satisfies the following four properties:

1. Closure: $A, B \in G \Rightarrow AB \in G$

2. Associativity: $A, B, C \in G \Rightarrow (AB)C = A(BC)$

3. Identity: there is $E \in G$ such that $AE = EA = A$ for all $A \in G$

4. Inverse: For all $A \in G$ there is $A^{-1} : AA^{-1} = E$

Examples include

- Trivial group: $\{E\}$
- Discrete Z_2 group $\{A, E\} \in Z_2$, $A = A^{-1}$ and thus $E = AA^{-1} = A^2$
- $O(N)$ the orthogonal group: group of $n \times n$ orthogonal real matrices
- $SO(N)$ the special orthogonal group: group of $n \times n$ orthogonal real matrices with unit determinant
- $U(1)$ the unit complex numbers $\{e^{i\alpha}\}$, $\alpha \in \mathbb{R}$
- $U(N)$ the unitary group: group of $n \times n$ complex unitary matrices
- $SU(N)$ the special unitary group: group of $n \times n$ complex unitary matrices with unit determinant
- Lorentz group $SO(3,1)$ with its spin group $\mathrm{SL}(2, C)$.

The first two examples are so-called discrete finite groups. The remaining examples are Lie groups, which are continuous groups. In fact all of them (apart from the Lorentz group) are compact. $U(1)$, $SU(N)$ and $SO(N)$ are so-called simply Lie groups. The unitary group can be decomposed in terms of the special unitary group and the unit complex numbers: $U(N) \cong SU(N) \times U(1)$.

A.4.1 Lie groups

The connected component of Lie group which contains the identity can be described by the underlying Lie algebra. If A is an element in the Lie algebra, $\exp(\alpha A)$, $\alpha \in \mathbb{R}$ is a one-parameter family of group elements in the Lie group. The vector space of the Lie algebra can be described by the so-called "generators" which form a basis. Thus any element of the Lie algebra (and the connected component of the identity in the Lie group) can be described in terms of those generators. In physics we generally use $\exp(i\alpha_i T^i)$ with real parameters α_i and hermitean matrices T^i. For example the generators of the spin group $SU(2)$ are the Pauli matrices. For $SU(3)$ they are the so-called Gell-Mann matrices. As $SU(N)$ is a unitary group, the generators are hermitean matrices. In physics we typically normalize the generators as $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

A.4.2 Representations

Symmetries leave the system invariant, but transform states $|\psi\rangle \in \mathcal{H}$ in the Hilbert space. Representations are defined as mappings from the abstract group to linear transformations acting on a vector

space which preserve the group structure. For example the group $SU(2)$ can be represented by 2×2 special unitary matrices. In particular the generators are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.68}$$

We can however similarly construct a 3-dimensional representation by mapping the three generators of $SU(2)$ to the matrices

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{A.69}$$

A.4.3 Special relativitiy and the Lorentz group

The equivalence of inertial reference frames implies that space is isotropic and homogeneous and time is also homogeneous. It is not possible to distinguish inertial reference frames. In particular, light moves at the same speed in every reference frame F (F'): the distance travelled, $r^{(\prime)} = ct^{(\prime)}$. Hence $0 = -(ct)^2 + r^2 = -(ct')^2 + r'^2$ and more generally the interval s

$$s^2 = -c^2t^2 + x^i x^i = -c^2 t'^2 + x'^i x'^i \tag{A.70}$$

with $r^2 = x^i x^i = \sum_i (x^i)^2$ is the same in all inertial reference frames. Linear transformations which leave the interval s unchanged are known as Lorentz transformations.

We measure time in the same units as distance, i.e. $c = 1$, and combine both of them in a 4-vector x^μ, $\mu = 0, 1, 2, 3$ with

$$x^0 = t \qquad \text{and} \qquad (x^i) = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{A.71}$$

and thus the interval s is

$$s^2 = -x^0 x^0 + x^i x^i = \eta_{\mu\nu} x^\mu x^\nu \tag{A.72}$$

with the metric tensor

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{A.73}$$

From the condition that the interval s is invariant under Lorentz transformations $x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$

$$\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} x^\rho x^\sigma \equiv \eta_{\rho\sigma} x^\rho x^\sigma \tag{A.74}$$

it follows that Lorentz transformations satisfy

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} \qquad \Leftrightarrow \eta = \Lambda^T \eta \Lambda \tag{A.75}$$

Taking the determinant we find that

$$\det\Lambda = \begin{cases} 1 & \text{proper Lorentz transformations} \\ -1 & \text{improper Lorentz transformations} \end{cases} \tag{A.76}$$

and the 00 component yields the condition $|\Lambda^0_0| \geq 1$. If $\Lambda^0_0 \geq 1 (\leq -1)$ it is an orthochronous (non-orthochronous) Lorentz transformation. Thus the Lorentz group splits in 4 disconnected components

$$\text{proper orthochronous } L^{\uparrow}_{+}: \ \det\Lambda = 1, \Lambda^0_0 \geq 1 \tag{A.77}$$

$$\text{proper non-orthochronous } L^{\downarrow}_{+}: \ \det\Lambda = 1, \Lambda^0_0 \leq -1 \tag{A.78}$$

$$\text{improper orthochronous } L^{\uparrow}_{-}: \ \det\Lambda = -1, \Lambda^0_0 \geq 1 \tag{A.79}$$

$$\text{improper non-orthochronous } L^{\downarrow}_{-}: \ \det\Lambda = -1, \Lambda^0_0 \leq -1 \tag{A.80}$$

The set of events parameterized by x^μ coordinates together with the metric tensor $\eta_{\mu\nu}$ defines a space called the Minkowski space-time. Distances are invariant under Lorentz transformations

$$\Delta s^2 = \eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu) \qquad\qquad ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \tag{A.81}$$

Indices are lowered and raised with $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ (the components of the inverse of the metric, η^{-1}. 4-vectors with an upper index x^μ are called contravariant vectors and with a lower index x_μ covariant (co-vary with the basis) vectors. Covariant vectors form the dual vector space to the contravariant vectors. A general mixed tensor transforms as

$$T'^{\mu_1\mu_2\ldots mu_k}_{\nu_1\nu_2\ldots\nu_l} = \Lambda^{\mu_1}_{\rho_1}\Lambda^{\mu_2}_{\rho_2}\ldots\Lambda^{\mu_k}_{\rho_k}\Lambda^{\ \sigma_1}_{\nu_1}\Lambda^{\ \sigma_2}_{\nu_2}\ldots\Lambda^{\ \sigma_l}_{\nu_l}T^{\rho_1\rho_2\ldots\rho_k}_{\sigma_1\sigma_2\ldots\sigma_l} \ . \tag{A.82}$$