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Homework A

Part 1

1. (a) $\frac{d}{dx}[(3x^2-7x+1)^5(4x-3)^3]$

$= (3x^2-7x+1)^5 \frac{d}{dx}(4x-3)^3 + (4x-3)^3 \frac{d}{dx}(3x^2-7x+1)^5$

$= (3x^2-7x+1)^5[3(4x-3)^2](4) + (4x-3)^3[5(3x^2-7x+1)^4(6x-7)]$

$= (4x-3)^2(3x^2-7x+1)^4[12(3x^2-7x+1)+5(4x-3)(6x-7)]$

$= (4x-3)^2(3x^2-7x+1)^4(36x^2-84x+12+120x^2-230x+105)$

$= (4x-3)^2(3x^2-7x+1)^4(156x^2-314x+117)$

(b) $\frac{d}{dx}\left[\frac{4x^2-1}{(2x+7)^3}\right]$

$= \frac{(2x+7)^3\frac{d}{dx}(4x^2-1)-(4x^2-1)\frac{d}{dx}(2x+7)^3}{(2x+7)^6}$

$= \frac{(2x+7)^3(8x)-(4x^2-1)[3(2x+7)^2(2)]}{(2x+7)^6}$

$= \frac{(2x+7)(8x)-(4x^2-1)(6)}{(2x+7)^6}$

$= \frac{16x^2+56x-24x^2+6}{(2x+7)^6}$

$= \frac{-8x^2+56x+6}{(2x+7)^6}$

(c) $\frac{d}{dx}\sin^2(1-3x)$

$= [2\sin(1-3x)][\cos(1-3x)]\frac{d}{dx}(1-3x)$

$= -6\sin(1-3x)\cos(1-3x)$

$= -6(\frac{1}{2})[\sin(2-6x)]$

$= -3\sin(2-6x)$

(d) $\frac{d}{dx}\sqrt{\cos 2x}$

$= \frac{1}{2\sqrt{\cos 2x}}\left(\frac{d}{dx}\cos 2x\right)$

$= \frac{-2\sin 2x}{2\sqrt{\cos 2x}}$

$= -\frac{\sin 2x}{\sqrt{\cos 2x}}$

(e) $\frac{d}{dx}\tan(\cos^{-1}x)$

$= [\sec^2(\cos^{-1}x)]\left(-\frac{1}{\sqrt{1-x^2}}\right)$

$= \frac{1}{\cos^2(\cos^{-1}x)}\left(-\frac{1}{\sqrt{1-x^2}}\right)$

$= -\frac{1}{x^2\sqrt{1-x^2}}$

Part 1 (cont.)

1. (cont.) (f) $\frac{d}{dx}\sin^{-1}(x^2)$

$= \frac{1}{\sqrt{1-x^4}}\left(\frac{d}{dx}x^2\right)$

$= \frac{2x}{\sqrt{1-x^4}}$

(g) $\frac{d}{dx}\tan(e^{3x})$

$= \frac{1}{1+e^{6x}}\left(\frac{d}{dx}e^{3x}\right)$

$= \frac{3e^{3x}}{1+e^{6x}}$

(h) $\frac{d}{dx}\ln\left(\frac{e^{x^2+7x}\sin 3x}{\sqrt{x}}\right)$

$= \frac{d}{dx}(\ln e^{x^2+7x} + \ln\sin 3x - \ln\sqrt{x})$

$= \frac{d}{dx}(x^2+7x + \ln\sin 3x - \ln\sqrt{x})$

$= 2x+7 + \frac{3\cos 3x}{\sin 3x} - \left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{2\sqrt{x}}\right)$

$= 2x+7+3\cot 3x - \frac{1}{2x}$

(i) $\frac{d}{dx}\log_a x$

$= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$

$= \frac{1}{\ln a}\left(\frac{d}{dx}\ln x\right)$

$= \frac{1}{x\ln a}$

(j) $\frac{d}{dx}\tan^{-1}(\cos x)$

$= \frac{1}{1+\cos^2 x}(-\sin x)$

$= \frac{-\sin x}{1+\cos^2 x}$

2. (a) $e^{\sin x} + xy = y^3 + \sec x$

Differentiating both sides:

$e^{\sin x}\cos x + y + x\frac{dy}{dx} = 3y^2\frac{dy}{dx} + \sec x\tan x$

$e^{\sin x}\cos x + y - \sec x\tan x = (3y^2-x)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{e^{\sin x}\cos x + y - \sec x\tan x}{3y^2-x}$

(b) $y = 3^x$

$\ln y = \ln 3^x = x\ln 3$

Differentiating both sides:

$\frac{1}{y}\left(\frac{dy}{dx}\right) = \ln 3$

$\frac{dy}{dx} = y\ln 3 = 3^x\ln 3$

Part 1 (cont.)

2. (c) (cont.) $y = x^x$

$= e^{\ln(x^x)}$

$= e^{x\ln x}$

$\frac{dy}{dx} = e^{x\ln x}\left(\frac{d}{dx} x\ln x\right)$

$= e^{x\ln x}(1+\ln x)$

$= x^x(1+\ln x)$

(d) $y = (\sin x)(\sin^{-1}x)(e^x)(x^2)$

$= uv$ (where $u = (\sin x)(\sin^{-1}x)$; $v = (e^x)(x^2)$) → for simplicity.

$\frac{du}{dx} = \sin x\left(\frac{1}{\sqrt{1-x^2}}\right) + \sin^{-1}x(\cos x)$

$= \frac{\sin x}{\sqrt{1-x^2}} + \cos x \sin^{-1}x$

$\frac{dv}{dx} = e^x(2x) + e^x(x^2) = e^x(x^2+2x)$

$\frac{dy}{dx} = \frac{du}{dx}(v) + \frac{dv}{dx}(u)$

$= \left(\frac{\sin x}{\sqrt{1-x^2}} + \cos x \sin^{-1}x\right)(e^x)(x^2) + [e^x(x^2+2x)](\sin x)(\sin^{-1}x)$

$= (\sin x)(\sin^{-1}x)(e^x)(x^2)\left(\frac{1}{\sin^{-1}x\sqrt{1-x^2}} + \cot x + 1 + \frac{2}{x}\right)$

Part 2

3. (a) $\int (x^2+1)^3\,dx$

$= \int (x^4+2x^2+1)(x^2+1)\,dx$

$= \int (x^6+2x^4+x^2+x^4+2x^2+1)\,dx$

$= \int (x^6+3x^4+3x^2+1)\,dx$

$= \frac{x^7}{7} + \frac{3x^5}{5} + x^3 + x + C$, where C is an arbitrary integer

(b) $\int \frac{x^3-2x+1}{x^2}\,dx$

$= \int \left(x - \frac{2}{x} + x^{-2}\right)dx$

$= \frac{x^2}{2} - 2\ln|x| - \frac{1}{x} + C$, where C is an arbitrary integer

(c)
$$
\begin{array}{r}
x - 4 \\
x^2+4x+3 \overline{\smash{)}\, x^3+0x^2+0x+0} \\
-(x^3+4x^2+3x) \\
\hline
-4x^2-3x+0 \\
-(-4x^2-16x-12) \\
\hline
13x+12
\end{array}
$$

Hence, $x^3 = (x^2+4x+3)(x-4) + 13x+12$.

→ answer continues

Part 2 (cont.)

3. (c) (cont.) $\int \frac{x^3}{x^2+4x+3}\,dx$

$= \int (x-4+\frac{13x+12}{x^2+4x+3})\,dx$

$= \int (x-4)\,dx + \boxed{\int \frac{13x+12}{(x+1)(x+3)}\,dx}$

Let $\frac{13x+12}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$

$Ax+3A+Bx+B = 13x+12$

$A+B=13$

$3A+B= \quad 12$

By comparing coefficients,

$A = -\frac{1}{2}$; $B = \frac{27}{2}$

$= \int (x-4)\,dx + \int [-\frac{1}{2(x+1)} + \frac{27}{2(x+3)}]\,dx$

$= \frac{x^2}{2} - 4x - \frac{1}{2}|x+1| + \frac{27}{2}|x+3| + C$, where C is an arbitrary constant

(d) $\int \frac{x^2-1}{x^3-3x-1}\,dx$

$= \frac{1}{3}\int \frac{3x^2-3}{x^3-3x-1}\,dx$

$= \frac{1}{3}\ln|x^3-3x-1| + C$, where C is an arbitrary constant

(e) $\int \frac{1}{x^2+4x+4}\,dx$

$= \int \frac{1}{(x+2)^2}\,dx$

$= -\frac{1}{x+2} + C$, where C is an arbitrary constant

(f) $\int \frac{1}{x^2+4x+13}\,dx$

$= \int \frac{1}{(x+2)^2+3^2}\,dx$

$= \frac{1}{3}\tan^{-1}(\frac{x+2}{3}) + C$, where C is an arbitrary constant

(g) $\int \frac{x+4}{x^2+4x+13}\,dx$

$= \int \frac{x+2}{x^2+4x+13}\,dx - \int \frac{2}{x^2+4x+13}\,dx$ from 3(f)

$= \frac{1}{2}\ln|x^2+4x+13| - \frac{2}{3}\tan^{-1}(\frac{x+2}{3}) + C$, where C is an arbitrary constant

(h) $\int \frac{1}{\sqrt{15+2x-x^2}}\,dx$

$= \int \frac{1}{\sqrt{4^2-(x-1)^2}}\,dx$

$= \sin^{-1}(\frac{x-1}{4}) + C$, where C is an arbitrary constant

(i) $\int \frac{x}{\sqrt{1-x^2}}\,dx$ is the integral. Let $u = \sqrt{1-x^2}$, then $\frac{du}{dx} = \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$

$= \int -\frac{du}{dx}\,dx$

$= \int -1\,du$

$= -u + C$

$= -\sqrt{1+x^2} + C$, where C is an arbitrary constant

Part 2 (cont.)

3. (cont.) (j) $\int \frac{x}{\sqrt{15+2x-x^2}}\,dx$

$= \int \frac{x}{\sqrt{16-(x-1)^2}}\,dx$ $\rightarrow$ let $u = x-1$, then $\frac{du}{dx} = 1$ and $x = u+1$

$= \int \frac{u+1}{\sqrt{16-u^2}}\,du$

$= \int \frac{u}{\sqrt{16-u^2}}\,du + \int \frac{1}{\sqrt{16-u^2}}\,du$ $\rightarrow$ let $v = 16-u^2$, then $\frac{dv}{du} = -2u$, $u = -\frac{1}{2}\left(\frac{dv}{du}\right)$

$= \int \frac{-\frac{1}{2}\left(\frac{dv}{du}\right)}{\sqrt{v}}\,du + \int \frac{1}{\sqrt{16-u^2}}\,du$

$= \int -\frac{1}{2\sqrt{v}}\,dv + \int \frac{1}{\sqrt{16-u^2}}\,du$

$= -\sqrt{v} + \sin^{-1}\left(\frac{u}{4}\right) + c$

$= -\sqrt{16-u^2} + \sin^{-1}\left(\frac{x-1}{4}\right) + C$

$= -\sqrt{16-(x-1)^2} + \sin^{-1}\left(\frac{x-1}{4}\right) + C$

$= -\sqrt{15+2x-x^2} + \sin^{-1}\left(\frac{x-1}{4}\right) + C$, where C is an arbitrary constant

4. (a) $\int \sec^2(4x)\,dx$

$= \frac{\tan(4x)}{4} + C$, where C is an arbitrary constant

(b) $\int \tan^2 x\,dx$

$= \int (\sec^2 x - 1)\,dx$

$= \tan x - x + C$, where C is an arbitrary constant

(c) $\int \tan 2x\,dx$

$= \int \frac{\sin 2x}{\cos 2x}\,dx$

$= -\frac{1}{2}\int \frac{-2\sin 2x}{\cos 2x}\,dx$

$= -\frac{1}{2}\ln|\cos 2x| + C$, where C is an arbitrary constant

(d) $\int \frac{4\sin x}{1-\cos x}\,dx$

$= 4\int \frac{\sin x}{1-\cos x}\,dx$

$= 4\ln|1-\cos x| + C$, where C is an arbitrary constant

(e) $\int \sin 3x \sin 2x\,dx$

$= -\frac{1}{2}\int (\cos 5x - \cos x)\,dx$

$= -\frac{1}{2}\left(\frac{\sin 5x}{5} - \sin x\right) + C$

$= \frac{5\sin x - \sin 5x}{10} + C$, where C is an arbitrary constant

Part 2 (cont.)

4. (f) $\int_0^{\frac{\pi}{4}} \sec^4 x\,dx$
(cont.)

$= \int_0^{\frac{\pi}{4}} (\sec^2 x)(\sec^2 x)\,dx$

$= \int_0^{\frac{\pi}{4}} (1+\tan^2 x)(\sec^2 x)\,dx$ → let $u = \tan x$, $\frac{du}{dx} = \sec^2 x$; the lower bound remains 0 and upper bound becomes $\tan\frac{\pi}{4} = 1$.

$= \int_0^1 (1+u^2)\,du$

$= \left[u + \frac{u^3}{3}\right]_0^1$

$= \left[\tan x + \frac{\tan^3 x}{3}\right]_0^1$

$= 1 + \frac{1}{3}$

$= \frac{4}{3}$

5. (a) $\int (e^{2-x} + e^{3x-5} + 2^x + e)\,dx$

$= \int e^{2-x}\,dx + \int e^{3x-5}\,dx + \int 2^x\,dx + \int e\,dx$

$= -e^{2-x} + \frac{e^{3x-5}}{3} + \frac{2^x}{\ln 2} + ex + C$, where C is an arbitrary constant

(b) $\int \frac{1}{1+e^x}\,dx$

$= \int \frac{1+e^x - e^x}{1+e^x}\,dx$

$= \int 1\,dx - \int \frac{e^x}{1+e^x}\,dx$ → let $u = 1+e^x$, $\frac{du}{dx} = e^x$, $du = e^x dx$

$= \int 1\,dx - \int \frac{1}{u}\,du$

$= x - \ln|u| + C$

$= x - \ln(1+e^x) + C$, where C is an arbitrary constant

(c) $\int_0^2 x e^{x^2}\,dx$ → let $u = x^2$, $\frac{du}{dx} = 2x$, upper bound changes to 4

$= \int_0^4 \frac{1}{2}\left(\frac{du}{dx}\right) e^u\,dx$

$= \frac{1}{2}\int_0^4 e^u\,du$

$= \frac{1}{2}\left[e^u\right]_0^4$

$= \frac{1}{2}(e^4 - 1)$

Part 2 (cont.)

5. (cont.) (d) $\int xe^x \, dx \quad \rightarrow$ let $u = x$, $du = dx$, $\frac{dv}{dx} = e^x$, $v = e^x$

$= xe^x - \int e^x \, dx$

$= e^x(x-1) + C$, where C is an arbitrary constant.

6. (a) $\int \ln x \, dx \quad \rightarrow$ let $u = \ln x$, $du = \frac{1}{x} dx$, $\frac{dv}{dx} = 1$, $v = x$.

$= x \ln x - \int x\left(\frac{1}{x} dx\right)$

$= x \ln x - x + C$, where C is an arbitrary constant.

(b) $\int \tan^{-1} x \, dx \quad \rightarrow$ let $u = \tan^{-1} x$, $\frac{du}{dx} = \frac{1}{1+x^2}$, $\frac{dv}{dx} = 1$, $v = x$.

$= x \tan^{-1} x - \int x \left(\frac{1}{1+x^2}\right) dx$

$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$

$= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$, where C is an arbitrary constant.

Part 3

1. (i) Differentiating the equation of the curve on both sides:

$4x - 3\left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0$

$4x - 3y = (3x - 2y)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{4x - 3y}{3x - 2y}$

At the point $(4, 3)$, $x = 4$ and $y = 3$, hence

$\frac{dy}{dx} = \frac{4(4) - 3(3)}{3(4) - 2(3)} = \frac{7}{6}$

$y = \frac{7}{6}x + C \quad \rightarrow C = 3 - \frac{7}{6}(4) = -\frac{5}{3}$

Eqn. of tangent is $y = \frac{7}{6}x - \frac{5}{3} \rightarrow 6y = 7x - 10$

(ii) $m_{normal} = -\frac{1}{m_{tangent}} = -\frac{6}{7}$

$y = -\frac{6}{7}x + C \rightarrow \quad C = 3 + \frac{6}{7}(4) = \frac{45}{7}$

Eqn. of normal is $y = -\frac{6}{7}x + \frac{45}{7} \rightarrow 7y = -6x + 45$

2. (i) $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$\left.\frac{dy}{dx}\right|_{x=16} = \frac{1}{2\sqrt{16}} = \frac{1}{8}$

At $(16, 4)$, $y = \frac{1}{8}x + C \rightarrow C = 4 - \frac{16}{8} = 2$

Eqn. of tangent is $y = \frac{1}{8}x + 2 \rightarrow 8y = x + 16$

Part 3 (cont.)

2. (cont.) (ii) When $x = 16.01$,

$$y \approx y\big|_{x=16} + \left(\frac{dy}{dx}\Big|_{x=16}\right)(16.01 - 16)$$

$$= 4 + \frac{1}{8}(0.01)$$

$$= 4.00125$$

(iii) $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = -\frac{1}{4}x^{-\frac{3}{2}}$$

$$\frac{d^2y}{dx^2}\Big|_{x=16} = -\frac{1}{4}\left(\frac{16^{\frac{3}{2}}}{16}\right) = -\frac{1}{4}\left(\frac{64}{16}\right) = -\frac{1}{16} < 0$$

So at $(16, 4)$, the curve is concave down.

Hence the linear approximation is an overestimate.

3. (i) $$f'(x) = \frac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2}$$

$$= \frac{acx + ad - acx - bc}{(cx+d)^2}$$

$$= \frac{ad - bc}{(cx+d)^2}$$

If $ad - bc \neq 0$, then $f'(x) \neq 0$.

Hence the graph of $y = f(x)$ has no stationary points.

(ii) From (i), $f'(x) = \frac{3(1) - (-7)(2)}{(2x+1)^2} = \frac{17}{(2x+1)^2} > 0$, $x \neq \frac{1}{2}$.

Hence as $f'(x)$ is always positive on all points other than $x = -\frac{1}{2}$ (where it is undefined), the graph of y is increasing (for any interval on the graph that does not contain $x = -\frac{1}{2}$).

(iii) $$\lim_{x \to \infty} \frac{3x-7}{2x+1}$$

$$= \lim_{x \to \infty} \frac{3 - \frac{7}{x}}{2 + \frac{1}{x}}$$

$$= \frac{3}{2}$$ (the horizontal asymptote is thus $y = \frac{3}{2}$).

From (ii), the vertical asymptote is $x = -\frac{1}{2}$.

4. We have $\frac{dV}{dt} = -2\,cm^3/s$.

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{1}{27}\pi h^3$ ($r = \frac{h}{3}$ by similar triangles)

$$\frac{dV}{dh} = \frac{1}{9}\pi h^2$$

$$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$$

$$\frac{dh}{dt}\Big|_{h=6} = -2 \div \frac{1}{9}\pi(6^2) = -\frac{1}{2\pi}\ cm/s$$

Part 3 (cont.)

5. We have $x^2+y^2=r^2=25\,cm^2$ so $y=\sqrt{25-x^2}$

$A = 2xy = 2x\sqrt{25-x^2}$

$\frac{dA}{dx} = 2\left[x\frac{d}{dx}\sqrt{25-x^2} + \sqrt{25-x^2}\left(\frac{d}{dx}x\right)\right]$

$= 2\left[x(-2x)\left(\frac{1}{2\sqrt{25-x^2}}\right) + \sqrt{25-x^2}\right]$

$= 2\left(\frac{-x^2}{\sqrt{25-x^2}} + \frac{25-x^2}{\sqrt{25-x^2}}\right)$

$= \frac{50-4x^2}{\sqrt{25-x^2}}$

To find stationary points, we set $\frac{dA}{dx}=0$, i.e.

$50-4x^2=0$, hence $x^2=\frac{25}{2}$ and $x=\frac{5}{\sqrt{2}}$.

$A = 2x\sqrt{25-x^2} = 2\left(\frac{5}{\sqrt{2}}\right)\sqrt{25-\frac{25}{2}} = 5(\sqrt{2})\left(\sqrt{\frac{25}{2}}\right) = 25\,cm^2$