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Homework B

Part 1

1. (i) $A_R = \int_0^1 \ln(1+x^2)\,dx$

$= (\ln 2 + \frac{\pi}{2} - 2)$ units^2

(ii) $V_y = 2\pi \int_0^1 x \ln(1+x^2)\,dx$

$= (2\pi \ln 2 - \pi)$ units^3

2. (i) $A_R = \int_1^4 \frac{\ln x}{x^2}\,dx$ $\rightarrow$ let $u = \ln x$, $du = \frac{1}{x}dx$, $dv = \frac{1}{x^2}dx$, $v = -\frac{1}{x}$

$= \left[-\frac{\ln x}{x}\right]_1^4 - \int_1^4 -\frac{1}{x}\left(\frac{1}{x}\right)dx$

$= \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^4$

$= \left(-\frac{\ln 4}{4} - \frac{1}{4}\right) - \left(-\frac{\ln 1}{1} - 1\right)$

$= \left(\frac{3 - \ln 4}{4}\right)$ units^2

(ii) $V_x = \pi \int_1^4 \left(\frac{\ln x}{x^2}\right)^2 dx$

$= 0.183$ units^3 (3 d.p.)

3. (i) The point of intersection of $y = x$ and $y = \frac{2x}{1+x^2}$ occurs when

$x = \frac{2x}{1+x^2}$ $\rightarrow$ $1 = \frac{2}{1+x^2}$; $x = 1$. Hence,

$A_R = \int_0^1 \left(\frac{2x}{1+x^2} - x\right) dx$

$= \left[\ln(1+x^2) - \frac{x^2}{2}\right]_0^1$

$= (\ln 2 - \frac{1}{2})$ units^2

(ii) $V_x = \pi \int_0^1 \left[\left(\frac{2x}{1+x^2}\right)^2 - x^2\right] dx$ $\rightarrow$ let $x = \tan\theta$, then $dx = \sec^2\theta\, d\theta$ and $1 + x^2 = \sec^2\theta$

upper limit changes to $\tan^{-1} 1 = \frac{\pi}{4}$

$= \pi \int_0^{\frac{\pi}{4}} \left[\left(\frac{2\tan\theta}{\sec^2\theta}\right)^2 - (\tan^2\theta)\right] \sec^2\theta\, d\theta$

$= \pi \int_0^{\frac{\pi}{4}} \left[(2\tan\theta\cos^2\theta)^2 - \tan^2\theta\right] \sec^2\theta\, d\theta$

$= \pi \left(\int_0^{\frac{\pi}{4}} \sin^2 2\theta \sec^2\theta\, d\theta - \int_0^{\frac{\pi}{4}} \tan^2\theta \sec^2\theta\, d\theta\right)$

$= \pi \left[\int_0^{\frac{\pi}{4}} 4\sin^2\theta\cos^2\theta\left(\frac{1}{\cos^2\theta}\right) - \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)(\sec^2\theta)\, d\theta\right]$

$= \pi \left[4\int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta - \int_0^{\frac{\pi}{4}} (\sec^4\theta - \sec^2\theta)\, d\theta\right]$

$= \pi \left[2\int_0^{\frac{\pi}{4}} (1 - \cos 2\theta)\, d\theta - \int_0^1 (x^2)\, dx\right]$

$= \pi \left(2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} - \left[\frac{x^3}{3}\right]_0^1\right)$

$= \pi \left[2\left(\frac{\pi}{4} - \frac{1}{2}\right) - \frac{1}{3}\right]$

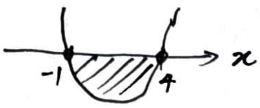
$= \left(\frac{\pi^2}{2} - \frac{4\pi}{3}\right)$ units^3

Part 1 (cont.)

4. $x^2-3x-4<0$

$(x-4)(x+1)<0$

$-1<x<4$



Hence,

$$\int_{-1}^{3\sqrt{2}} |x^2-3x-4|\,dx$$

$$= -\int_{-1}^{4}(x^2-3x-4)\,dx + \int_{4}^{3\sqrt{2}}(x^2-3x-4)\,dx$$

$$= -\left[\frac{x^3}{3}-\frac{3x^2}{2}-4x\right]_{-1}^{4} + \left[\frac{x^3}{3}-\frac{3x^2}{2}-4x\right]_{4}^{3\sqrt{2}}$$

$$= F(3\sqrt{2}) + F(-1) - 2F(4)$$

$$= \frac{(3\sqrt{2})^3}{3} - \frac{3(3\sqrt{2})^2}{2} - 4(3\sqrt{2}) - \frac{1}{3} - \frac{3}{2} + 4 - \frac{2(4)^3}{3} + \frac{6(4)^2}{2} + 8(4)$$

$$= 18\sqrt{2} - 27 - 12\sqrt{2} + \frac{13}{6} - 2\left(\frac{64}{3} - 24 - 16\right)$$

$$= 6\sqrt{2} - 27 + \frac{13}{6} - 2\left(-\frac{56}{3}\right)$$

$$= 6\sqrt{2} + \frac{25}{2}$$

5. (a) Set $\ln 2(x-1)^2 = \ln x$. Then,

$$e^{\ln 2(x-1)^2} = e^{\ln x}$$

$$2^{(x-1)^2} = x$$

$x=1$ works since $2^0=1$

$x=2$ also works since $2^1=2$.

$\}$ From the premises of the question, we can infer that these are the only sol^{n}s.

At the midpoint $x=1.5$,

$2^{(1.5-1)^2} = 2^{0.25} = \sqrt{\sqrt{2}} \approx \sqrt{1.414} < 1.5$ (since $\sqrt{x}<x$ when $x>1$).

So $y=\ln 2(x-1)^2$ is the bottom curve and $y=\ln x$ is the top curve.

$$A_R = \int_1^2 \left[\ln x - \ln 2(x-1)^2\right]dx$$

$$= \int_1^2 \ln x\,dx - \ln 2\int_1^2 (x-1)^2\,dx \rightarrow \text{let } u=x-1,\ du=dx.$$

$$= [x\ln x - x]_1^2 - \ln 2\int_0^1 u^2\,du$$

$$= (2\ln 2 - 2) - (1\ln 1 - 1) - \ln 2\left[\frac{u^3}{3}\right]_0^1$$

$$= 2\ln 2 - 2 + 1 - \frac{\ln 2}{3}$$

$$= \left(\frac{5\ln 2}{3} - 1\right) \text{ units}^2$$

5. (cont.) (b) When rotating S about $y=1$, the disk method with $r=(x^2+1)-1=x^2$ yields

$$V_x = \pi\int_0^b (x^2)^2\,dx = \pi\left[\frac{x^5}{5}\right]_0^b = \pi\frac{b^5}{5} \text{ units}^3$$

When rotating S about the y-axis, the shell method with $r=x$, $h=x^2$ yields

$$V_y = 2\pi\int_0^b x(x^2)\,dx = 2\pi\left[\frac{x^4}{4}\right]_0^b = \pi\frac{b^4}{2} \text{ units}^3$$

We want $V_x = V_y \rightarrow \pi\frac{b^5}{5} = \pi\frac{b^4}{2}$

$2b^5 = 5b^4$

$2b = 5$

$b = \frac{5}{2}$

6. (a) $\int_0^6 f(x)\,dx$

$\approx L(n)$

$= \frac{6-0}{3}(-3-3+5)$

$= -2$

(b) $\int_0^6 f(x)\,dx$

$\approx R(n)$

$= \frac{6-0}{3}(-3+5+21)$

$= 46$

(c) $\int_0^6 f(x)\,dx$

$\approx M(n)$

$= \frac{6-0}{3}(-4+0+12)$

$= 16$

(d) $\int_0^6 f(x)\,dx$

$\approx T(n)$

$= \frac{1}{2}(L(n)+R(n))$

$= \frac{1}{2}(-2+46)$

$= 22$

7. (a) $\int_0^6 g(x)\,dx$

$\approx L(n)$

$= (2-0)g(0) + (3-2)g(2) + (6-3)g(3)$

$= 6 + 4 + 24$

$= 34$

(b) $\int_0^6 g(x)\,dx$

$\approx R(n)$

$= (2-0)g(2) + (3-2)g(3) + (6-3)g(6)$

$= 8 + 8 - 21$

$= -5$

Part 1 (cont.)

7. (cont.) (c) $\int_0^6 g(x)\,dx$

$\approx T(n)$

$= \left(\frac{g(0)+g(2)}{2}\right)(2-0) + \left(\frac{g(2)+g(3)}{2}\right)(3-2) + \left(\frac{g(3)+g(6)}{2}\right)(6-3)$

$= \frac{7}{2}(2) + 6(1) + \frac{1}{2}(3)$

$= 14\frac{1}{2}$

Part 2

8. (i) Being at the origin implies $x = 0$.

We set $x(t) = t^3 - 4t^2 + 5t = 0$

$t(t^2 - 4t + 5) = 0$

$t^2 - 4t + 5 = 0$, $t = 0$ is a sol^n

↳ no real sol^{n}s of t here.

So P is only at the origin at $t = 0$ and never after that.

(ii) $v(t) = \frac{d}{dt}x(t) = \frac{d}{dt}(t^3 - 4t^2 + 5t) = 3t^2 - 8t + 5$.

Given that P travels along the x-axis, its motion is 1D and so

$s(t) = |v(t)| = |3t^2 - 8t + 5|$

$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}(3t^2 - 8t + 5) = 6t - 8$

(iii) P changes direction when the sign of $v(t)$ changes.

We set $v(t) = 3t^2 - 8t + 5 = 0$

$(3t - 5)(t - 1) = 0$

$t = 1$ and $t = \frac{5}{3}$

At $t = 1$, $x(t) = 1 - 4 + 5 = 2$ units, so from $t = 0$ to $t = 1$, $d_{0\to1} = 2$ units

At $t = \frac{5}{3}$, $x(t) = \left(\frac{5}{3}\right)^3 - 4\left(\frac{5}{3}\right)^2 + 5\left(\frac{5}{3}\right) = \frac{50}{27}$ units,

so from $t = 1$ to $t = \frac{5}{3}$, $d_{1\to\frac{5}{3}} = 2 - \frac{50}{27} = \frac{4}{27}$ units.

At $t = 2$, $x(t) = 2^3 - 4(2^2) + 5(2) = 2$ units,

so from $t = \frac{5}{3}$ to $t = 2$, $d_{\frac{5}{3}\to2} = 2 - \frac{50}{27} = \frac{4}{27}$ units

$d_{0\to2} = d_{0\to1} + d_{1\to\frac{5}{3}} + d_{\frac{5}{3}\to2} = 2 + \frac{4}{27} + \frac{4}{27} = \frac{62}{27}$ units

9. (i) $v(t) = \int a(t)\,dt = \int e^{-t}dt - \int t\,dt = -e^{-t} - \frac{t^2}{2} + C$, where C is an arbitrary constant

$v(0) = -e^{-0} - \frac{0^2}{2} + C = 2\text{m/s} \to C - 1 = 2$ so $C = 3$.

Hence $v(t) = (-e^{-t} - \frac{t^2}{2} + 3)$ m/s.

(ii) $d_{1\to3} = \int_1^3 |v(t)|\,dt = \int_1^3 |-e^{-t} - \frac{t^2}{2} + 3|\,dt = 2.223\text{m}$ (3d.p.) (using GC)