Defintion

ring_def

Ring - A *ring* is a set R with two binary operations called $+$ *addition* and $\times$ *multiplication* such that

1. $(R, +)$ is an abelian group.
2. $\times$ is associative, that is, $(a \times b) \times c = a \times (b \times c), \forall a, b, c \in R$.
3. Multiplcation distributes over addition, that is, $(a + b) \times c = a \times c + b \times c$ and $a \times (b + c) = a \times b + a \times c, \forall a, b, c \in R$.

Special Properties

- If multiplication is commutative, we say that R is a *commutative ring*.
- I there exists a $1 \in R$ such that $1 \times a = a \times 1 = a, \forall a \in R$, then R is a *ring with identity*.

Special Rings

- ## division_ring_def

  Division Ring - Let R be a nontrivial ring with identity. If every $a \in R \setminus \{0\}$ has a multiplicative inverse, then R is a *division ring*.

- A commutative division ring is a *field.*

Propositions

Proposition: Let R be a ring.

1. For all $a \in R, 0a = a0 = 0$.
2. For all $a, b \in R, (-a)b = a(-b) = -(ab)$.
3. For all $a, b \in R, (-a)(-b) = ab$.
4. Let $1 \in R$, then 1 is unique.
5. Let $1 \in R, \$then\$ - a = (-1)a$.

Element Properties

- ## zero_divisor_def

  Zero Divisor - Let R be a ring. An element $a \in R$ is a zero divisor if $a \neq 0$ and there exists a $b \in R$ such that $ab = 0$ or $ba = 0$.

- ## unit_def

  Unit - Let R be a nontrivial ring with identity. An element $u \in R$ is a *unit* in R if there exists some $v \in R$ such that $uv = vu = 1$. The set of units in R is $R^\times$.

  $R^\times$ forms a group under multiplication.

More Speicial Rings

- ## integral_domain_def

  Integral Domains - A nontrivial commutative ring with no zero divisors is an *integral domain.*

Examples

- $(\mathbb{Z}, +, \times), (\mathbb{Q}, +, \times), (\mathbb{R}, +, \times)$, and $(\mathbb{C}, +, \times)$ under standard addition and multiplication are all commutative rings with identity.
- The set of matrices under standard matrix addition and multiplication is a noncommutative ring with identity.
- $(2\mathbb{Z}, +, \times)$ under standard addition and multiplication is a commutative ring without identity.
- $(\mathbb{Q}, +, \times), (\mathbb{R}, +, \times)$, and $(\mathbb{C}, +, \times)$ are all fields.
- The ring of real quaternions is a division ring that is not a field.