Notation 1 *(Preliminaries)***.** Let $\mathcal{L}$ be a countable language and let T be a stable theory in $\mathcal{L}$ which permits infinite models. We will also let $\mathcal{A}$ and $\mathcal{B}$ denote $\mathcal{L}$-structures with universes A and B respectively. When relevant, let $\mathcal{A} \vDash T$. We will also be using the monster model $\mathbb{M} \succeq \mathcal{A}$ for many of the following results.

- Let S be the set of all $\mathcal{L}$-formulas. Throughout this write-up, we will let Δ be a set of $\mathcal{L}$-formulas, so $\Delta \subseteq S$.
- Let Σ_n be the set of all $\mathcal{L}$-formulas with at most one free variable x.
- For any set X, let $\mathcal{L}_X = \mathcal{L} \cup \{c_x : x \in X\}$, where we augment $\mathcal{L}$ with constant symbols for every element of X.
- Let $S(X)$ be the set of all $\mathcal{L}_X$ formulas.
- Let $\Sigma_1(X)$ be defined similarly.
- If $X \subseteq A$, we define $\mathrm{Th}(\mathcal{A}, X)$ to be the theory of $\mathcal{A}$ interpreted as an $\mathcal{L}_X$-structure. In particular, $\mathrm{Th}(\mathcal{A}, A)$ is the elementary diagram of $\mathcal{A}$.
- An *A-instance* of a formula φ is any formula obtained from φ by substituting $c_a \in \mathcal{L}_A$ for each variable.

Definition 1 *(Boolean closure)***.** Let Δ be a set of $\mathcal{L}$-formulas and let X be a set. We define $\Delta^b(X)$ to be the set of formulas in $\Sigma_1(X)$ obtained from formulas in Δ using conjunction, disjunction, negation, and substituting instances of c_x for a variable. Trivially, $S^b(A) = S(A)$.

Definition 2 *(Partitioning)***.** Let $\varphi \in \Sigma_1(B)$ and $\Gamma = \{\psi_1, \dots, \psi_n\} \subset \Sigma_1(B)$. We say that Γ *partitions* φ if we have the following.

$$\begin{aligned}
&\mathcal{B} \vDash \forall x(\psi_1(x) \vee \cdots \vee \psi_n(x) \vee \neg\varphi(x)) \\
&\mathcal{B} \vDash \forall x((\psi_i(x) \wedge \psi_j(x)) \to \neg\varphi(x)) \qquad \text{for all } i \neq j \\
&\mathcal{B} \vDash \exists x(\varphi(x) \wedge \psi_i(x)) \qquad\qquad\qquad\quad \text{for all } i
\end{aligned}$$

In other words, whenever $\varphi(x)$ is true in $\mathcal{B}$, exactly one of the $\psi_i(x)$'s is true.

Definition 3 *(Generalized rank and degree)***.** Let $\Delta \subseteq A$. In order to define a generalized version of Morley rank and degree, we will need to define two sets $S^\alpha(\mathcal{A}, \Delta)$ and $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ for any ordinal α inductively.

- Let $S^0(\mathcal{A}, \Delta) = \{\varphi \in \Sigma_1(A) : \mathcal{A} \vDash \exists x\varphi(x)\}$, i.e. the set of all $\mathcal{L}$-formulas that have witnesses in $\mathcal{A}$.
- If $S^\alpha(\mathcal{A}, \Delta)$ is already defined, we may define $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ as follows. Given a formula

φ, we say that $\varphi \in \mathrm{Tr}^{\alpha}(\mathcal{A}, \Delta)$ if and only if there exists a finite k such that for all $\mathcal{B} \succeq \mathcal{A}$ and any finite $\Gamma \subset \Delta^{b}(B)$ partitioning φ, there are no more than k formulas ψ in Γ such that $(\varphi \wedge \psi) \in S^{\alpha}(\mathcal{B}, \Delta)$.

- Let $S^{\alpha+1}(\mathcal{A}, \Delta) = S^{\alpha}(\mathcal{A}, \Delta) \setminus \mathrm{Tr}^{\alpha}(\mathcal{A}, \Delta)$.
- For α limit, let $S^{\alpha}(\mathcal{A}, \Delta) = \bigcap_{\beta<\alpha} S^{\beta}(\mathcal{A}, \Delta)$.
- The Δ-*rank* of φ in $\mathcal{A}$ is the least (and unique) ordinal α such that $\varphi \in \mathrm{Tr}^{\alpha}(\mathcal{A}, \Delta)$. In general we denote this as $\mathrm{rank}_{\mathcal{A},\Delta}(\varphi)$.
- The Δ-*degree* of φ in $\mathcal{A}$ is the least number k witnessing that $\varphi \in \mathrm{Tr}^{\alpha}(\mathcal{A}, \Delta)$. In general we denote this as $\deg_{\mathcal{A},\Delta}(\varphi)$.
- When $\mathcal{A}$ or Δ are unambiguous, we may omit them when writing rank and deg.
- For a set of formulas Γ (often when Γ is a type), we define the Δ-rank of Γ to be $\inf\{\mathrm{rank}(\varphi) : \Gamma \vDash \varphi\}$.

Proposition 1 *(Δ-rank generalizes Morley rank)***.** Let p be a 1-type in $\mathrm{Th}(\mathcal{A}, A)$. The Morley rank of p is the least p such that $p \cap \mathrm{Tr}^{\alpha}(\mathcal{A}, S) \neq \varnothing$. The Morley degree of p is the minimum of the S-degrees of the formulas in $p \cap \mathrm{Tr}^{\alpha}(\mathcal{A}, S)$.

Proof. Let $\mathcal{B} \succeq \mathcal{A}$ be an $\aleph_0$-saturated model, so p is realized in $\mathcal{B}$. To prove the first part, we show by induction on α that for all $\varphi \in p$,

$$\mathrm{RM}(\varphi) = \mathrm{RM}^{\mathcal{B}}(\varphi) \geq \alpha \Leftrightarrow \varphi \in S^{\alpha}(\mathcal{A}, S).$$

If $\alpha = 0$, we clearly have by definition that $\mathcal{A} \vDash \exists x \varphi(x)$ if and only if $\varphi(\mathcal{A})$ is nonempty. The limit case follows easily by induction.

As for the successor case, we need to show that $\mathrm{RM}(\varphi) \geq \alpha + 1$ if and only if ϕ is in $S^{\alpha}(\mathcal{A}, S) \setminus \mathrm{Tr}^{\alpha}(\mathcal{A}, S)$. First suppose that $\mathrm{RM}(\varphi) \geq \alpha + 1$, so there exist infinitely many $\mathcal{L}_B$-formulas $\psi_1, \psi_2, \ldots$ defining pairwise disjoint subsets of $\varphi(\mathcal{B})$ such that $\mathrm{RM}(\psi_i) \geq \alpha$ for all $i < \omega$. Let k be any finite number. Then, define the following finite set of formulas $\Gamma \subset S^{b}(B)$.

$$\Gamma = \left\{ \psi_1, \ldots, \psi_{k+1}, \left(\varphi \wedge \bigwedge_{i=1}^{k+1} \neg\psi_i \right) \right\}$$

This set clearly partitions φ. We can see that for at least $k+1$ formulas $\psi_i \in \Gamma$, $\psi_i \wedge \varphi = \psi_i$ has rank α, which by induction means $\psi_i \in S^{\alpha}(\mathcal{B}, S)$. Thus, we conclude that φ cannot be in $\mathrm{Tr}^{\alpha}(\mathcal{A}, S)$.

Now suppose that $\mathrm{RM}(\varphi) = \alpha$. Then, we can see that the Morley degree of φ as an

$\mathcal{L}_B$-formula is a k witnessing that $\varphi \in \mathrm{Tr}^\alpha(\mathcal{A}, S)$. In fact, by the definition of the Morley degree of a type, the second part of the proposition is trivial. $\square$

Definition 4 *(Minimality)***.** A formula φ is *minimal* in $\mathcal{A}$ if $\varphi \in \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ and there is no $\psi \in \Delta$ with some A-instance ψ' such that $(\varphi \wedge \psi')$ and $(\varphi \wedge \neg\psi')$ are both in $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$.

Lemma 1 *(Properties of rank)***.** Let φ, φ_0, φ_1, and ψ be formulas in $\Sigma_1(A)$.

(i) Suppose $\varphi(\mathcal{A}) \subseteq \psi(\mathcal{A})$. Then $\mathrm{rank}_{\mathcal{A}}(\varphi) \leq \mathrm{rank}_{\mathcal{A}}(\psi)$ and, if $\mathrm{rank}_{\mathcal{A}}(\varphi) = \mathrm{rank}_{\mathcal{A}}(\psi)$, then $\deg_{\mathcal{A}}(\varphi) \leq \deg_{\mathcal{A}}(\psi)$.
(ii) If $\mathcal{B} \succeq \mathcal{A}$, then $\mathrm{rank}_{\mathcal{A}}(\varphi) = \mathrm{rank}_{\mathcal{B}}(\varphi)$.
(iii) If $(\varphi_0 \vee \varphi_1)(\mathcal{A}) \supseteq \varphi(\mathcal{A})$ and $\mathrm{rank}_{\mathcal{A}}(\varphi) \geq \alpha$, then at least one of φ_0 and φ_1 has a Δ-rank of at least α. In particular, if $\mathrm{rank}_{\mathcal{A}}(\varphi) = \alpha$ for any ordinal α, then $\deg_{\mathcal{A}}(\varphi) \geq 1$.

Proof. (i) Clearly $\varphi \in S^0(\mathcal{A}, \Delta)$ implies that $\psi(\mathcal{A}) \supseteq \varphi(\mathcal{A}) \neq \varnothing$, so $\psi \in S^0(\mathcal{A}, \Delta)$. For the successor case, given a set which partitions φ with k formulas in $S^\alpha(\mathcal{A}, \Delta)$, we easily also have a set which partitions ψ with k formulas in $S^\alpha(\mathcal{A}, \Delta)$. Thus, if $\varphi \notin \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$, then $\psi \notin \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$. The limit case follows easily from induction. Also, if k formulas of rank α partition φ, then these same formulas partition ψ, giving the second part of the statement easily.

(ii) Since $\mathcal{A} \vDash \exists x \varphi(x)$ if and only if $\mathcal{B} \vDash \exists x \varphi(x)$, we immediately have the case of $\alpha = 0$. In fact, since the property of a set partitioning a formula depends only on the satisfaction of finitely many sentences, a set partitioning φ in $\mathcal{A}$ with k formulas in $S^\alpha(\mathcal{A}, \Delta)$ will also be a set partitioning φ in $\mathcal{B}$ with k formulas in $S^\alpha(\mathcal{B}, \Delta)$ and vice versa by induction. The limit case also follows easily.

(iii) We have the following deduction to show that this holds for the case of $\alpha = 0$.

$$\mathcal{A} \vDash \exists x \varphi(x) \Rightarrow \mathcal{A} \vDash \exists x(\varphi_0(x) \vee \varphi_1(x)) \Rightarrow \mathcal{A} \vDash \exists x \varphi_0(x) \vee \exists x \varphi_1(x)$$

Suppose the lemma holds for α, suppose $\varphi \in S^{\alpha+1}(\mathcal{A}, \Delta)$, and let $k \in \omega$. There exists a set Γ which partitions φ with formulas $\psi_1, \ldots, \psi_{2k} \in \Gamma$ such that each ψ_i is in $S^\alpha(\mathcal{A}, \Delta)$. For each ψ_i, at least one of $(\psi_i \wedge \varphi_0)$ and $(\psi_i \wedge \varphi_1)$ is in $S^\alpha(\mathcal{A}, \Delta)$ by induction. Since we have $2k$ formulas, for at least one of the φ_j's there are at least k formulas of the form $(\psi_i \wedge \varphi_j)$ in $S^\alpha(\mathcal{A}, \Delta)$. Then we can easily construct a partitioning set showing that φ_j cannot have degree k. We are able to show that for any $k \in \omega$ one of φ_0 and φ_1 is unable to have degree k. Thus, for one of the φ_j's, φ_j fails to have degree k for infinitely many k, meaning $\varphi_j \in S^{\alpha+1}(\mathcal{A}, \Delta)$.

Finally, for the limit case, assuming the lemma holds for all $\beta < \alpha$, it is true for at least one of the φ_j's that there is a cofinal sequence of ordinals in α where for each entry β, $\varphi_j \in S^\beta(\mathcal{A}, \Delta)$. Thus, $\varphi_j \in S^\alpha(\mathcal{A}, \Delta)$. $\square$

In light of Lemma 1(ii), we can always assume we are working in the monster $\mathbb{M}$ and omit $\mathbb{M}$ from our notation of rank and deg.

Lemma 2 *(Definability of Δ-rank)***.** Let Δ be finite, $n \in \omega$, and let ψ have Δ-rank n and Δ-degree 1. For each $\varphi \in (\Delta \cup \Delta^-) \cap \Sigma_{k+1}$ and for any $\mathcal{B} \succeq \mathcal{A}$, the set

$$D = \{\bar{b} \in \mathcal{B}^k : \mathrm{rank}_{\mathcal{B}}(\psi(x) \wedge \varphi(x, \bar{b})) = n\}$$

is definable over A.

Proof. We may first suppose that $\mathcal{B}$ is the monster model $\mathbb{M}$. We assume that each $\varphi \in \Delta$ is stable since the theory T is stable. Letting φ be any formula in Δ or Δ^- with $k+1$ free variables, we define the following global φ-type.

$$p = \{\varphi(x, \bar{b}) : \bar{b} \in D\} \cup \{\neg\varphi(x, \bar{c}) : \bar{c} \in \mathbb{M}^k \setminus D\}$$

First, we show that this type is consistent, i.e. that $\mathrm{Diag}_{\mathrm{el}}(\mathcal{A}) \cup p$ is satisfiable. Let $p_0 \subset p$ be finite; we'll write this partial φ-type as

$$\left\{\varphi(x, \bar{b}_1), \ldots, \varphi(x, \bar{b}_r), \neg\varphi(x, \bar{c}_1), \ldots, \neg\varphi(x, \bar{c}_s)\right\}$$

where $\bar{b}_1, \ldots, \bar{b}_r \in D$ and $\bar{c}_1, \ldots, \bar{c}_s \in \mathbb{M}^k \setminus D$. We will show that

$$\theta_i(x) = \psi(x) \wedge \varphi(x, \bar{b}_1) \wedge \cdots \wedge \varphi(x, \bar{b}_i)$$

has Δ-rank n for every $i \in \{1, \ldots, r\}$. It will follow, since $n \geq 0$, that this formula has a witness. First, we have that $\mathrm{rank}(\psi(x) \wedge \varphi(x, \bar{b}_1)) = n$ by definition. Now, let $i \geq 2$ and suppose $\mathrm{rank}(\theta_{i-1}(x)) = n$. We know that $\deg(\theta_{i-1}(x)) = 1$ by Lemma 1(i) and (iii). Thus, if $\mathrm{rank}(\theta_{i-1}(x) \wedge \varphi(x, \bar{b}_i)) < n$, then it must be the case that $\mathrm{rank}(\theta_{i-1}(x) \wedge \neg\varphi(x, \bar{b}_i)) = n$. However, by Lemma 1(i), this means that $\mathrm{rank}(\psi(x) \wedge \neg\varphi(x, \bar{b}_i)) = n$, contradicting that ψ has Δ-degree 1. So, it can only be the case that $\mathrm{rank}(\theta_i(x)) = n$. This along with analogous logic for each $\neg\varphi(x, \bar{c}_i)$ shows that

$$\mathrm{rank}_{\mathbb{M}}\left(\psi(x) \wedge \bigwedge \varphi(x, \bar{b}_i) \wedge \bigwedge \neg\varphi(x, \bar{c}_i)\right) = n$$

In particular, since $n \geq 0$, the formula has a witness in $\mathbb{M}$, meaning that p_0 is satisfiable and so p is realized in $\mathbb{M}$.

Since φ is stable, the φ-type p is definable by some formula $\theta(\overline{y})$ with parameters in the monster. This formula also defines D. To finish our proof, we only need to show that D is A-invariant. Let σ be any automorphism on $\mathbb{M}$ which fixes A pointwise. We must show that for any $\overline{b} \in D$, $\sigma(\overline{b}) \in D$. Since the only names in ψ are names of elements of A and the only names in $\varphi(x, \overline{b})$ are $\overline{b}$, we obtain the following.

$$\begin{aligned}\operatorname{rank}(\sigma(\psi(x) \wedge \varphi(x, \overline{b}))) &= \operatorname{rank}(\sigma(\psi(x)) \wedge \sigma(\varphi(x, \overline{b}))) \\ &= \operatorname{rank}(\psi(x) \wedge \varphi(x, \sigma(\overline{b}))) = n\end{aligned}$$

Thus, $\sigma(\overline{b}) \in D$, meaning D is A-invariant. Thus, D is A-definable. Since we are working in the monster model, whatever formula defines D in $\mathbb{M}$ will also define any $\mathcal{B}$ extending $\mathcal{A}$, completing our proof. □

Proposition 2 *(Encoding formulas)***.** Let A and B be sets with A containing at least two elements. Given finitely many formulas $\varphi_1(x, \overline{b}_1), \ldots, \varphi_n(x, \overline{b}_n)$ with fixed parameters in B, there exists a formula $\varphi(x, \overline{b}_1, \ldots, \overline{b}_n, \overline{z})$ such that each φ_i is equivalent to an A-instance of φ.

Proof. We can define φ explicitly as the conjunction of the following formulas.

$$\begin{aligned}\left(\bigwedge_{i=2}^{n} z_1 \neq z_i\right) &\rightarrow \varphi_1(x, \overline{y}_1) \\ (z_1 = z_2) &\rightarrow \varphi_2(x, \overline{y}_2) \\ (z_1 = z_3 \wedge z_1 \neq z_2) &\rightarrow \varphi_3(x, \overline{y}_3) \\ &\vdots \\ \left(z_1 = z_n \wedge \left(\bigwedge_{i=2}^{n-1} z_1 \neq z_i\right)\right) &\rightarrow \varphi_n(x, \overline{y}_n)\end{aligned}$$

This formula works like a switch-case statement, checking for the leftmost z_i which is equal to z_1, where $2 \leq i \leq n$. If z_1 is not equal to any of the z_i's, then we default to evaluating φ_1. Taking two distinct elements $a_1, a_2 \in A$, we can easily express any of the φ_i's as an A-instance of φ. □

Corollary 1. If φ encodes each formula in Δ as in Proposition 2, then for any $\mathcal{L}_A$ formula ψ,

$$\operatorname{rank}_{\Delta}(\psi) = \operatorname{rank}_{\{\varphi\}}(\psi) \text{ and } \deg_{\Delta}(\psi) = \deg_{\{\varphi\}}(\psi).$$

Lemma 3. If Δ is finite then the Δ-rank of any formula is finite.

Proof. First, we define a version of Shelah 2-rank for a formula $\varphi(x)$ given a finite set of $\mathcal{L}$-formulas Δ.

- $\mathrm{R}_\Delta(\varphi) \geq 0$ if φ is consistent.
- $\mathrm{R}_\Delta(\varphi) \geq \alpha + 1$ if, for some $\psi(x, y) \in \Delta$ and $a \in \mathbb{M}^y$, we have both of the following.

$$\mathrm{R}_\Delta(\varphi(x) \wedge \psi(x, a)) \geq \alpha \, \mathrm{R}_\Delta(\varphi(x) \wedge \neg\psi(x, a)) \geq \alpha$$

- For a limit ordinal γ, $\mathrm{R}_\Delta(\varphi) \geq \gamma$ if $\mathrm{R}_\Delta(\varphi) \geq \gamma$ for all $\alpha < \gamma$.

By dint of Corollary 1, we may assume that $\Delta = \{\psi(x, y)\}$, where ψ is a stable $\mathcal{L}$-formula. It should be clear that for any φ, $\mathrm{rank}(\varphi) \leq \mathrm{R}(\varphi)$. Now, we show that $\mathrm{R}(\varphi)$ must be finite. Assume for the sake of contradiction that $\mathrm{R}(x = x) \geq \omega$. Then, using compactness, we can construct a binary tree with countably many parameters where each branch of the tree is a distinct and satisfiable ψ-type. But then ψ isn't stable, contradicting our assumption. $\square$

Lemma 4. If Δ is finite and φ is minimal in $\mathcal{A}$, then the Δ-degree of φ is 1.

Proof. Using Corollary 1, we can assume that $\Delta = \{\varphi(x, \overline{y})\}$ contains one stable $\mathcal{L}$-formula. Suppose ψ is minimal in $\mathcal{A}$ and has rank n. Then, using logic similar to the proof in Lemma 2, we can see that the following φ-type is consistent and complete in $\mathcal{A}$.

$$p = \{\varphi(x, \overline{a})^e : a \in A \text{ and } \mathrm{rank}(\varphi(x, \overline{a})^e) = n \text{ where } e = \pm 1\}$$

Note that $\mathrm{rank}(p) = n$. Suppose for the sake of contradiction that $\deg(\varphi) \geq 2$, so we can partition it into two formulas ψ_0 and ψ_1, each of which being in $\Delta^b(\mathbb{M})$ and having degree 1. Then, we may define the following global φ-type which extends p for $i = 0, 1$.

$$q_i = \{\varphi(x, \overline{a})^e : a \in A \text{ and } \mathrm{rank}(\psi_i(x) \wedge \varphi(x, \overline{a})^e) = n \text{ where } e = \pm 1\}$$

Note that each q_i also has rank n and that $q_i \restriction A = p$. This means that q_0 and q_1 are both nondividing extensions of p. However, a nondividing extension of p must be unique, meaning $q_0 = q_1$, a contradiction. Therefore, p cannot have degree larger than 1. $\square$