Notation 1 *(Preliminaries)***.** Let $\mathcal{L}$ be a countable language. We will also let $\mathcal{A}$ and $\mathcal{B}$ denote $\mathcal{L}$-structures with universes A and B respectively.

- Let S be the set of all $\mathcal{L}$-formulas. Throughout this write-up, we will let Δ be a set of $\mathcal{L}$-formulas, so $\Delta \subseteq S$.
- Let S_n be the set of all $\mathcal{L}$-formulas with at most one free variable x.
- For any set X, let $\mathcal{L}_X = \mathcal{L} \cup \{c_x : x \in X\}$, where we augment $\mathcal{L}$ with constant symbols for every element of X.
- Let $S(X)$ be the set of all $\mathcal{L}_X$ formulas.
- Let $S_1(X)$ be defined similarly.
- If $X \subseteq A$, we define $\mathrm{Th}(\mathcal{A}, X)$ to be the theory of $\mathcal{A}$ interpreted as an $\mathcal{L}_X$-structure. In particular, $\mathrm{Th}(\mathcal{A}, A)$ is the elementary diagram of $\mathcal{A}$.
- An $\mathcal{A}$*-instance* of a formula φ is any formula obtained from φ by substituting $c_a \in \mathcal{L}_A$ for each variable.

Definition 1 *(Boolean closure)***.** Let Δ be a set of $\mathcal{L}$-formulas and let X be a set. We define $\Delta^b(X)$ to be the set of formulas in $S_1(X)$ obtained from formulas in Δ using conjunction, disjunction, negation, and substituting instances of c_x for a variable. Trivially, $S^b(A) = S(A)$.

Definition 2 *(Partitioning)***.** Let $\varphi \in S_1(B)$ and $\Gamma = \{\psi_1, \dots, \psi_n\} \subset S_1(B)$. We say that Γ *partitions* φ if we have the following.

$$\begin{aligned}
&\mathcal{B} \models \forall x(\psi_1(x) \vee \cdots \vee \psi_n(x) \vee \neg\varphi(x)) \\
&\mathcal{B} \models \forall x((\psi_i(x) \wedge \psi_j(x)) \to \neg\varphi(x)) \qquad \text{for all } i \neq j \\
&\mathcal{B} \models \exists x(\varphi(x) \wedge \psi_i(x)) \qquad\qquad\qquad\quad \text{for all } i
\end{aligned}$$

In other words, whenever $\varphi(x)$ is true in $\mathcal{B}$, exactly one of the $\psi_i(x)$'s is true.

Definition 3 *(Generalized rank and degree)***.** Let $\Delta \subseteq A$. In order to define a generalized version of Morley rank and degree, we will need to define two sets $S^\alpha(\mathcal{A}, \Delta)$ and $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ for any ordinal α inductively.

- Let $S^0(\mathcal{A}, \Delta) = \{\varphi \in S_1(A) : \mathcal{A} \models \exists x \varphi(x)\}$, i.e. the set of all $\mathcal{L}$-formulas that have witnesses in $\mathcal{A}$.
- If $S^\alpha(\mathcal{A}, \Delta)$ is already defined, we may define $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ as follows. Given a formula φ, we say that $\varphi \in \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ if and only if there exists a finite k such that for all $\mathcal{B} \succeq \mathcal{A}$ and any finite $\Gamma \subset \Delta^b(B)$ partitioning $\mathcal{A}$, there are no more than k formulas ψ

in Γ such that $(\varphi \wedge \psi) \in S^\alpha(\mathcal{A}, \Delta)$.

- Let $S^{\alpha+1}(\mathcal{A}, \Delta) = S^\alpha(\mathcal{A}, \Delta) \setminus \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$.
- For α limit, let $S^\alpha(\mathcal{A}, \Delta) = \bigcap_{\beta<\alpha} S^\beta(\mathcal{A}, \Delta)$.
- The Δ-*rank* of φ in $\mathcal{A}$ is the least ordinal α such that $\varphi \in \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$.
- The Δ-*degree* of φ in $\mathcal{A}$ is the least number k witnessing that $\varphi \in \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$.

Proposition 1 *(Δ-rank generalizes Morley rank)***.** Let p be a 1-type in $\mathrm{Th}(\mathcal{A}, A)$. The Morley rank of p is the least p such that $p \cap \mathrm{Tr}^\alpha(\mathcal{A}, S) \neq \varnothing$. The Morley degree of p is the minimum of the S-degrees of the formulas in $p \cap \mathrm{Tr}^\alpha(\mathcal{A}, S)$.

Proof. □

Definition 4 *(Minimality)***.** A formula φ is *minimal* in $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ if $\varphi \in \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ and there is no $\psi \in \Delta$ with some $\mathcal{A}$-instance ψ' such that $(\varphi \wedge \psi')$ and $(\varphi \wedge \neg\psi')$ are both in $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$.

Lemma 1 *(Properties of rank)***.** Let φ, φ_0, φ_1, and ψ be formulas in $S_1(A)$.

(i) If $\varphi(\mathcal{A}) \subseteq \psi(\mathcal{A})$ and $\varphi \in S^\alpha(\mathcal{A}, \Delta)$, then $\psi \in S^\alpha(\mathcal{A}, \Delta)$.
(ii) Suppose $\mathcal{B} \succeq \mathcal{A}$. Then $\varphi \in S^\alpha(\mathcal{A}, \Delta)$ if and only if $\varphi \in S^\alpha(\mathcal{B}, \Delta)$.
(iii) If $(\varphi_0 \vee \varphi_1)(\mathcal{A}) \supseteq \varphi(\mathcal{A})$ and $\varphi \in S^\alpha(\mathcal{A}, \Delta)$, then one of φ_0 and φ_1 is in $S^\alpha(\mathcal{A}, \Delta)$.

Proof. □

Definition 5 *(Weak satisfiability)***.** First, for any set of formulas Θ, we define

$$\Theta^- = \{\neg\phi : \phi \in \Theta\}$$

to be the set of negations of all formulas in Θ. The set $\Gamma \subset S(A)$ is *weakly satisfiable* in $\mathcal{A}$ if no finite disjunctions of formulas in Γ^- is valid in $\mathcal{A}$.

Remark 1. A set $\Gamma \subset S(A)$ is weakly satisfiable in $\mathcal{A}$ if and only if Γ is satisfiable in some $\mathcal{B} \succeq \mathcal{A}$.

Proof. □

Lemma 2 *(A characterization of rank)***.** Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{L}$-structures. Let $\varphi \in S^\alpha(\mathcal{A}, \Delta)$. Let Γ be the set of all formulas obtained by switching around variables in the formulas of $\{\phi\} \cup \Delta \cup \Delta^-$. There exists $\Gamma^* \subseteq \Gamma$ weakly satisfiable in $\mathcal{A}$ such that if $\psi \in S_1(B)$ and $\Gamma^\dagger$ is weakly satisfiable in $\mathcal{B}$, where $\Gamma^\dagger$ is obtained from Γ^* by replacing each instance of φ by the corresponding instance of ψ, then $\psi \in S^\alpha(\mathcal{B}, \Delta)$.

Proof. □

Lemma 3. Let $n \in \omega$, let Δ be finite, and φ be an $\mathcal{L}_X$-formula containing at most one variable x free and possibly names for the elements of the universe of some model. There exists $\Gamma^* \subseteq S(X)$ depending only on n, φ, and Δ such that for any $\mathcal{A}$, if $\varphi \in S_1(A)$, then $\varphi \in S^n(\mathcal{A}, \Delta)$ if and only if Γ^* is weakly satisfied in $\mathcal{A}$.

Proof. □

Lemma 4. Let Δ be finite, $n \in \omega$, and let $\varphi \in \mathrm{Tr}^n(\mathcal{A}, \Delta)$ have Δ-degree 1. For each $\psi \in (\Delta \cup \Delta^-) \cap S_{k+1}$ there exists $\theta \in S_k(A)$ such that if $\mathcal{B} \succeq \mathcal{A}$, then

$$(\varphi \wedge \psi)(x, b_1, \ldots, b_k) \in \mathrm{Tr}^n(\mathcal{B}, \Delta) \Leftrightarrow \mathcal{B} \vDash \theta(b_1, \ldots, b_k),$$

where $b_1, \ldots, b_k$ are arbitrary in B.

Proof. □

Lemma 5. If Δ is finite then $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta) = \varnothing$ if $\alpha \geq \omega$.

Proof. □

Definition 6 *(Δ-stable)*.

Lemma 6. Let $\mathcal{A} \preceq \mathcal{B}$, let φ be a formula in $S_1(A)$ which is Δ-stable, and let $\psi \in \Delta^b(B)$. Then there exists $\theta \in S_1(A)$ such that $\varphi(\mathcal{A}) \cap \psi(\mathcal{B}) = \theta(\mathcal{A})$.

Proof. No proof but some references listed, including Shelah. □

Lemma 7. If φ is minimal in $\mathrm{Tr}^\alpha(\mathcal{A}, S)$, then the S-degree of φ is 1.

Proof. □

Lemma 8. If Δ is finite and φ is minimal in $\mathrm{Tr}^n(\mathcal{A}, \Delta)$, then the Δ-degree of φ in $\mathcal{A}$ is 1.

Proof. □

Theorem 1 *(Model extension)*. Let $\mathcal{A}$ and $\mathcal{B}$ be models of a countable stable theory and suppose that $\mathcal{A} \prec \mathcal{B}$ and $P(\mathcal{A}) = P(\mathcal{B})$ where P is a unary predicate symbol. There exists $\mathcal{C} \succ \mathcal{B}$ such that $P(\mathcal{C}) = P(\mathcal{B})$.

Proof. □