Notation 1 *(Preliminaries)***.** Let $\mathcal{L}$ be a countable language. We will also let $\mathcal{A}$ and $\mathcal{B}$ denote $\mathcal{L}$-structures with universes A and B respectively.

- Let S be the set of all $\mathcal{L}$-formulas. Throughout this write-up, we will let Δ be a set of $\mathcal{L}$-formulas, so $\Delta \subseteq S$.
- Let S_n be the set of all $\mathcal{L}$-formulas with at most one free variable x.
- For any set X, let $\mathcal{L}_X = \mathcal{L} \cup \{c_x : x \in X\}$, where we augment $\mathcal{L}$ with constant symbols for every element of X.
- Let $S(X)$ be the set of all $\mathcal{L}_X$ formulas.
- Let $S_1(X)$ be defined similarly.
- If $X \subseteq A$, we define $\mathrm{Th}(\mathcal{A}, X)$ to be the theory of $\mathcal{A}$ interpreted as an $\mathcal{L}_X$-structure. In particular, $\mathrm{Th}(\mathcal{A}, A)$ is the elementary diagram of $\mathcal{A}$.
- An $\mathcal{A}$*-instance* of a formula φ is any formula obtained from φ by substituting $c_a \in \mathcal{L}_A$ for each variable.

Definition 1 *(Boolean closure)***.** Let Δ be a set of $\mathcal{L}$-formulas and let X be a set. We define $\Delta^b(X)$ to be the set of formulas in $S_1(X)$ obtained from formulas in Δ using conjunction, disjunction, negation, and substituting instances of c_x for a variable. Trivially, $S^b(A) = S(A)$.

Definition 2 *(Partitioning)***.** Let $\varphi \in S_1(B)$ and $\Gamma = \{\psi_1, \ldots, \psi_n\} \subset S_1(B)$. We say that Γ *partitions* φ if we have the following.

$$\begin{aligned}
&\mathcal{B} \models \forall x(\psi_1(x) \vee \cdots \vee \psi_n(x) \vee \neg\varphi(x)) \\
&\mathcal{B} \models \forall x((\psi_i(x) \wedge \psi_j(x)) \to \neg\varphi(x)) \qquad \text{for all } i \neq j \\
&\mathcal{B} \models \exists x(\varphi(x) \wedge \psi_i(x)) \qquad \text{for all } i
\end{aligned}$$

In other words, whenever $\varphi(x)$ is true in $\mathcal{B}$, exactly one of the $\psi_i(x)$'s is true.

Definition 3 *(Generalized rank and degree)***.** Let $\Delta \subseteq A$. In order to define a generalized version of Morley rank and degree, we will need to define two sets $S^\alpha(\mathcal{A}, \Delta)$ and $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ for any ordinal α inductively.

- Let $S^0(\mathcal{A}, \Delta) = \{\varphi \in S_1(A) : \mathcal{A} \models \exists x\varphi(x)\}$, i.e. the set of all $\mathcal{L}$-formulas that have witnesses in $\mathcal{A}$.
- If $S^\alpha(\mathcal{A}, \Delta)$ is already defined, we may define $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ as follows. Given a formula φ, we say that $\varphi \in \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ if and only if there exists a finite k such that for all $\mathcal{B} \succeq \mathcal{A}$ and any finite $\Gamma \subset \Delta^b(B)$ partitioning $\mathcal{A}$, there are no more than k formulas ψ

in Γ such that $(\varphi \wedge \psi) \in S^\alpha(\mathcal{A}, \Delta)$.

- Let $S^{\alpha+1}(\mathcal{A}, \Delta) = S^\alpha(\mathcal{A}, \Delta) \setminus \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$.
- For α limit, let $S^\alpha(\mathcal{A}, \Delta) = \bigcap_{\beta<\alpha} S^\beta(\mathcal{A}, \Delta)$.
- The Δ-*rank* of φ in $\mathcal{A}$ is the least ordinal α such that $\varphi \in \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$.
- The Δ-*degree* of φ in $\mathcal{A}$ is the least number k witnessing that $\varphi \in \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$.

Proposition 1 *(Δ-rank generalizes Morley rank)***.** Let p be a 1-type in $\mathrm{Th}(\mathcal{A}, A)$. The Morley rank of p is the least p such that $p \cap \mathrm{Tr}^\alpha(\mathcal{A}, S) \neq \varnothing$. The Morley degree of p is the minimum of the S-degrees of the formulas in $p \cap \mathrm{Tr}^\alpha(\mathcal{A}, S)$.

Proof. Let $\mathcal{B} \succeq \mathcal{A}$ be an $\aleph_0$-saturated model, so p is realized in $\mathcal{B}$. To prove the first part, we show by induction on α that for all $\varphi \in p$,

$$\mathrm{RM}(\varphi) = \mathrm{RM}^{\mathcal{B}}(\varphi) \geq \alpha \Leftrightarrow \varphi \in S^\alpha(\mathcal{A}, S).$$

If $\alpha = 0$, we clearly have by definition that $\mathcal{A} \vDash \exists x \varphi(x)$ if and only if $\varphi(\mathcal{A})$ is nonempty. The limit case follows easily by induction.

As for the successor case, we need to show that $\mathrm{RM}(\varphi) \geq \alpha + 1$ if and only if ϕ is in $S^\alpha(\mathcal{A}, S) \setminus \mathrm{Tr}^\alpha(\mathcal{A}, S)$. First suppose that $\mathrm{RM}(\varphi) \geq \alpha + 1$, so there exist infinitely many $\mathcal{L}_B$-formulas $\psi_1, \psi_2, \ldots$ defining pairwise disjoint subsets of $\varphi(\mathcal{B})$ such that $\mathrm{RM}(\psi_i) \geq \alpha$ for all $i < \omega$. Let k be any finite number. Then, define the following finite set of formulas $\Gamma \subset S^b(B)$.

$$\Gamma = \left\{ \psi_1, \ldots, \psi_{k+1}, \left(\varphi \wedge \bigwedge_{i=1}^{k+1} \neg \psi_i \right) \right\}$$

This set clearly partitions φ. We can see that for at least $k+1$ formulas $\psi_i \in \Gamma$, $\psi_i \wedge \varphi = \psi_i$ has rank α, which by induction means $\psi_i \in S^\alpha(\mathcal{B}, S)$. Thus, we conclude that φ cannot be in $\mathrm{Tr}^\alpha(\mathcal{A}, S)$.

Now suppose that $\mathrm{RM}(\varphi) = \alpha$. Then, we can see that the Morley degree of φ as an $\mathcal{L}_B$-formula is a k witnessing that $\varphi \in \mathrm{Tr}^\alpha(\mathcal{A}, S)$. In fact, by the definition of the Morley degree of a type, the second part of the proposition is trivial. $\square$

Definition 4 *(Minimality)***.** A formula φ is *minimal* in $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ if $\varphi \in \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ and there is no $\psi \in \Delta$ with some $\mathcal{A}$-instance ψ' such that $(\varphi \wedge \psi')$ and $(\varphi \wedge \neg \psi')$ are both in $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$.

Lemma 1 *(Properties of rank)***.** Let φ, φ_0, φ_1, and ψ be formulas in $S_1(A)$.

(i) If $\varphi(\mathcal{A}) \subseteq \psi(\mathcal{A})$ and $\varphi \in S^\alpha(\mathcal{A}, \Delta)$, then $\psi \in S^\alpha(\mathcal{A}, \Delta)$.
(ii) Suppose $\mathcal{B} \succeq \mathcal{A}$. Then $\varphi \in S^\alpha(\mathcal{A}, \Delta)$ if and only if $\varphi \in S^\alpha(\mathcal{B}, \Delta)$.

(iii) If $(\varphi_0 \vee \varphi_1)(\mathcal{A}) \supseteq \varphi(\mathcal{A})$ and $\varphi \in S^\alpha(\mathcal{A}, \Delta)$, then one of φ_0 and φ_1 is in $S^\alpha(\mathcal{A}, \Delta)$.

Proof. (i) Clearly $\varphi \in S^0(\mathcal{A}, \Delta)$ implies that $\psi(\mathcal{A}) \supseteq \varphi(\mathcal{A}) \neq \varnothing$, so $\psi \in S^0(\mathcal{A}, \Delta)$. For the successor case, given a set which partitions φ with k formulas in $S^\alpha(\mathcal{A}, \Delta)$, we easily also have a set which partitions ψ with k formulas in $S^\alpha(\mathcal{A}, \Delta)$. Thus, if $\varphi \notin \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$, then $\psi \notin \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$. The limit case follows easily from induction.

(ii) Since $\mathcal{A} \vDash \exists x \varphi(x)$ if and only if $\mathcal{B} \vDash \exists x \varphi(x)$, we immediately have the case of $\alpha = 0$. In fact, since the property of a set partitioning a formula depends only on the satisfaction of finitely many sentences, a set partitioning φ in $\mathcal{A}$ with k formulas in $S^\alpha(\mathcal{A}, \Delta)$ will also be a set partitioning φ in $\mathcal{B}$ with k formulas in $S^\alpha(\mathcal{B}, \Delta)$ and vice versa by induction. The limit case also follows easily.

(iii) We have the following deduction to show that this holds for the case of $\alpha = 0$.

$$\mathcal{A} \vDash \exists x \varphi(x) \Rightarrow \mathcal{A} \vDash \exists x (\varphi_0(x) \vee \varphi_1(x)) \Rightarrow \mathcal{A} \vDash \exists x \varphi_0(x) \vee \exists x \varphi_1(x)$$

Suppose the lemma holds for α, suppose $\varphi \in S^{\alpha+1}(\mathcal{A}, \Delta)$, and let $k \in \omega$. There exists a set Γ which partitions φ with formulas $\psi_1, \ldots, \psi_{2k} \in \Gamma$ such that each ψ_i is in $S^\alpha(\mathcal{A}, \Delta)$. For each ψ_i, at least one of $(\psi_i \wedge \varphi_0)$ and $(\psi_i \wedge \varphi_1)$ is in $S^\alpha(\mathcal{A}, \Delta)$ by induction. Since we have $2k$ formulas, for at least one of the φ_j's there are at least k formulas of the form $(\psi_i \wedge \varphi_j)$ in $S^\alpha(\mathcal{A}, \Delta)$. Then we can easily construct a partitioning set showing that φ_j cannot have degree k. We are able to show that for any $k \in \omega$ one of φ_0 and φ_1 is unable to have degree k. Thus, for one of the φ_j's, φ_j fails to have degree k for infinitely many k, meaning $\varphi_j \in S^{\alpha+1}(\mathcal{A}, \Delta)$.

Finally, for the limit case, assuming the lemma holds for all $\beta < \alpha$, it is true for at least one of the φ_j's that there is a cofinal sequence of ordinals in α where for each entry β, $\varphi_j \in S^\beta(\mathcal{A}, \Delta)$. Thus, $\varphi_j \in S^\alpha(\mathcal{A}, \Delta)$. □

Definition 5 *(Weak satisfiability)***.** First, for any set of formulas Θ, we define

$$\Theta^- = \{\neg\phi : \phi \in \Theta\}$$

to be the set of negations of all formulas in Θ. The set $\Gamma \subset S(A)$ is *weakly satisfiable* in $\mathcal{A}$ if no finite disjunctions of formulas in Γ^- is valid in $\mathcal{A}$.

Remark 1. A set $\Gamma \subset S(A)$ is weakly satisfiable in $\mathcal{A}$ if and only if Γ is satisfiable in some $\mathcal{B} \succeq \mathcal{A}$.

Proof. For the rightward direction, $\Gamma \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{A})$ is finitely satisfiable and therefore satisfiable. For the leftward direction, if $\Gamma \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{A})$ were not satisfiable, then there would exist formulas $\psi_1, \ldots, \psi_n \in \Gamma^-$ such that $\mathrm{Diag}_{\mathrm{el}}(\mathcal{A}) \models \forall \overline{x}(\psi_1 \vee \cdots \vee \psi_n)$, so Γ would not be weakly satisfiable in $\mathcal{A}$. $\square$

Lemma 2 *(A characterization of rank)***.** Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{L}$-structures. Let $\varphi \in S^\alpha(\mathcal{A}, \Delta)$. Let Γ be the set of all formulas obtained by switching around variables in the formulas of $\{\phi\} \cup \Delta \cup \Delta^-$. There exists $\Gamma^* \subseteq \Gamma$ weakly satisfiable in $\mathcal{A}$ such that if $\psi \in S_1(B)$ and $\Gamma^\dagger$ is weakly satisfiable in $\mathcal{B}$, where $\Gamma^\dagger$ is obtained from Γ^* by replacing each instance of φ by the corresponding instance of ψ, then $\psi \in S^\alpha(\mathcal{B}, \Delta)$.

Proof. We proceed by induction on α. For $\alpha = 0$, the set $\Gamma' = \{\varphi\}$ works, as well as its obvious counterpart $\Gamma^\dagger = \{\psi\}$.

If α is a limit ordinal, suppose the lemma holds for all $\beta < \alpha$. Then, for each β, there exists some $\Gamma^*_\beta \subseteq \Gamma$ weakly satisfiable in $\mathcal{A}$ such that the corresponding set $\Gamma^\dagger_\beta$ has the desired property of the lemma. We may change the variables in each Γ_β such that if $\beta, \gamma < \alpha$ and $\beta \neq \gamma$, then Γ_β and Γ_γ have no variables in common. Then, let $\Gamma^* = \bigcup_{\beta<\alpha} \Gamma^*_\beta$, so $\Gamma^\dagger = \bigcup_{\beta<\alpha} \Gamma^\dagger_\beta$. Clearly, Γ^* is weakly satisfiable in $\mathcal{A}$. If $\Gamma^\dagger$ is weakly satisfiable in $\mathcal{B}$, then each $\Gamma^\dagger_\beta$ is weakly satisfiable. By induction, that means $\psi \in S^\beta(\mathcal{B}, \Delta)$ for all $\beta < \alpha$ and so $\psi \in S^\alpha(\mathcal{B}, \Delta)$.

Now suppose that α is a successor with $\alpha = \beta + 1$. We will want to construct a nice sequence of formulas in order to construct Γ^*.

Claim. There exists $\mathcal{A}' \succeq \mathcal{A}$ and a sequence $\langle \varphi_n : n \in \omega \rangle$ of formulas in $\Delta^b(A')$ such that, for all $m, n \in \omega$,

- $(\varphi \wedge \varphi_n) \in S^\beta(\mathcal{A}', \Delta)$,
- $\neg(\varphi_m \wedge \varphi_n)$ is a tautology, i.e. valid in every structure, if $m \neq n$, and
- φ_n is a conjunction of $\mathcal{A}'$-instances of formulas in $\Delta \cup \Delta^-$.

First, since $\varphi \in S^\alpha(\mathcal{A}, \Delta)$, there exists $\mathcal{A}_0 \succeq \mathcal{A}$ and mutually exclusive $\theta_0, \theta_1 \in \Delta^b(A_0)$ such that $(\varphi \wedge \theta_0), (\varphi \wedge \theta_1) \in S^\beta(\mathcal{A}_0, \Delta)$. By Lemma 1(iii), we can let θ be one of θ_0, θ_1, and $(\neg\theta_0 \wedge \neg\theta_1)$ such that $(\varphi \wedge \theta) \in S^\alpha(\mathcal{A}_0, \Delta)$. If $\theta = \theta_0$, let $\varphi_0^* = \theta_1$. Otherwise, let $\varphi_0^* = \theta_0$. By repeated use of Lemma 1(iii) and laws of Boolean algebra, we can find φ_0, a conjunction of $\mathcal{A}_0$-instances of formulas in $\Delta \cup \Delta^-$ such that $\varphi_0(\mathcal{A}_0) \subseteq \varphi_0^*(\mathcal{A}_0)$ and $\varphi \in S^\beta(\mathcal{A}_0, \Delta)$.

Since $\theta \in S^\alpha(\mathcal{A}, \Delta)$, we can repeat this process on θ instead of φ and, indeed, continue it indefinitely. This gives us a sequence of formulas $\langle \varphi_n : n \in \omega \rangle$ and an increasing chain of models

$$\mathcal{A} \preceq \mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \cdots$$

where, for all n, φ_n is a conjunction of $\mathcal{A}_n$-instances of formulas in $\Delta \cup \Delta^-$. Finally, let $\mathcal{A}' = \bigcup_{n\in\omega} \mathcal{A}_n$. Then, for all n, $(\varphi \wedge \varphi_n) \in S^\beta(\mathcal{A}', \Delta)$ by Lemma 1(ii). The sequence also satisfies the other two desired properties, meaning we have proven the claim.

There is an issue with this proof: I have shown that these formulas are mutually exclusive in $\mathcal{A}'$, but this is a weaker notion than the second part of the claim. To salvage this: make it so φ_{n+1} has the negation of a formula $\delta_m \in \Delta \cup \Delta^-$ for all $m \leq n$, where δ_m appears in φ_m but not φ_{m-1}.

For each n, let Γ_n be the set of all formulas obtained by switching around variables in the formulas of $\{(\varphi \wedge \varphi_n)\} \cup \Delta \cup \Delta^-$. Since we have assumed the lemma for β, there exists $\Gamma'_n \subseteq \Gamma_n$ weakly satisfiable in $\mathcal{A}'$ with our desired property. We can also assume that Γ_m and Γ_n share no variables if $m \neq n$ and that there are infinitely many variables not occurring in any Γ_n.

For each $c_a \in \mathcal{L}_A$ which occurs in one of our formulas φ_n we can set aside a variable x_a; this variable is independent of n. Let Γ_n^* be obtained from Γ_n by replacing each occurrence of c_a with a corresponding variable x_a. Let $\Gamma^* = \bigcup_{n\in\omega} \Gamma_n^*$. We can see that Γ^* is weakly satisfiable in $\mathcal{A}$. Now, let $\Gamma^\dagger$ be the set obtained from Γ^* by replacing each instance of φ with ψ.

Suppose $\Gamma^\dagger$ is weakly satisfiable in $\mathcal{B}$. Then, by Remark 1, there exists $\mathcal{B}' \succeq \mathcal{B}$ where $\Gamma^\dagger$ is satisfied. By replacing each variable v_a with a constant symbol in $\mathcal{L}_{B'}$ appropriately, we have a sequence of formulas $\langle \psi_n : n \in \omega \rangle$ corresponding to each φ_n. This gives us infinitely many disjoint formulas $(\psi \wedge \psi_n) \in S^\beta(\mathcal{B}', \Delta)$, witnessing that $\psi \in S^\alpha(\mathcal{B}', \Delta)$. By Lemma 1(ii), $\psi \in S^\alpha(\mathcal{B}, \Delta)$. □

Lemma 3. Let Γ be defined as in Lemma 2. Let $n \in \omega$, let Δ be finite, and φ be an $\mathcal{L}_X$-formula containing at most one variable x free and possibly names for the elements of the universe of some model. There exists $\Gamma^* \subseteq S(X)$ depending only on n, φ, and Δ such that for any $\mathcal{A}$, if $\varphi \in S_1(A)$, then $\varphi \in S^n(\mathcal{A}, \Delta)$ if and only if Γ^* is weakly satisfied in $\mathcal{A}$.

Proof. We proceed by induction on n. For $n = 0$, $\Gamma^* = \{\varphi(x)\}$ clearly works. Now assume that the result is true for $n = m$ where $m \geq 0$.

For $\psi \in S^{m+1}(\mathcal{B}, \Delta)$, let $\Theta(\mathcal{B})$ be the set of all formulas $\theta \in \Delta \cup \Delta^-$ such that $(\psi \wedge \theta') \in S^{m+1}(\mathcal{B}', \Delta)$ and $(\psi \wedge \neg\theta') \in S^m(\mathcal{B}', \Delta)$ for some $\mathcal{B}' \succeq \mathcal{B}$ and some $\mathcal{B}'$-instance θ' of θ. Given $\varphi \in S^{m+1}(\mathcal{B}, \Delta)$, choose $\mathcal{B} \succeq \mathcal{A}$ and $\psi \in S^{m+1}(\mathcal{B}, \Delta)$ such that □

Lemma 4. Let Δ be finite, $n \in \omega$, and let $\varphi \in \mathrm{Tr}^n(\mathcal{A}, \Delta)$ have Δ-degree 1. For each $\psi \in$

$(\Delta \cup \Delta^-) \cap S_{k+1}$ there exists $\theta \in S_k(A)$ such that if $\mathcal{B} \succeq \mathcal{A}$, then

$$(\varphi \wedge \psi)(x, b_1, \ldots, b_k) \in \mathrm{Tr}^n(\mathcal{B}, \Delta) \Leftrightarrow \mathcal{B} \vDash \theta(b_1, \ldots, b_k),$$

where $b_1, \ldots, b_k$ are arbitrary in B.

Proof. Short proof, looks terse. □

Lemma 5. If Δ is finite then $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta) = \varnothing$ if $\alpha \geq \omega$.

Proof. No proof but explains some analogous results. □

Definition 6 *(Δ-stable)***.** To be added. Generalization of stability (duh).

Lemma 6. Let $\mathcal{A} \preceq \mathcal{B}$, let φ be a formula in $S_1(A)$ which is Δ-stable, and let $\psi \in \Delta^b(B)$. Then there exists $\theta \in S_1(A)$ such that $\varphi(\mathcal{A}) \cap \psi(\mathcal{B}) = \theta(\mathcal{A})$.

Proof. No proof but some references listed, including Shelah. □

Lemma 7. If φ is minimal in $\mathrm{Tr}^\alpha(\mathcal{A}, S)$, then the S-degree of φ is 1.

Proof. Long, seemingly full proof given. □

That last lemma says that all transcendental points in the Stone space of a model have degree 1.

Lemma 8. If Δ is finite and φ is minimal in $\mathrm{Tr}^n(\mathcal{A}, \Delta)$, then the Δ-degree of φ in $\mathcal{A}$ is 1.

Proof. Lachlan's experimental notation really shines here. □

Theorem 1 *(Model extension)***.** Let $\mathcal{A}$ and $\mathcal{B}$ be models of a countable stable theory and suppose that $\mathcal{A} \prec \mathcal{B}$ and $P(\mathcal{A}) = P(\mathcal{B})$ where P is a unary predicate symbol. There exists $\mathcal{C} \succ \mathcal{B}$ such that $P(\mathcal{C}) = P(\mathcal{B})$.

Proof. Long! Uses a theorem of Ehrenfeucht. Check Reference 8. □