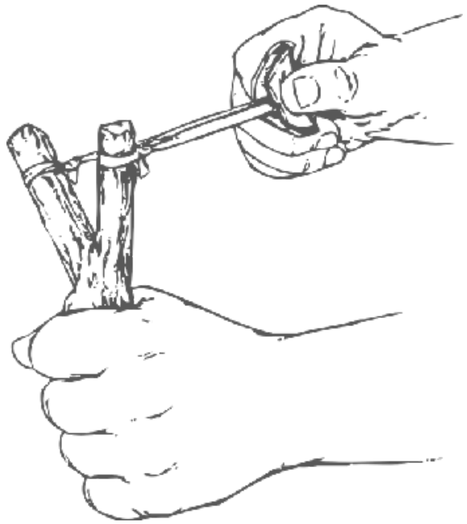
Bayesian data analysis: Theory & practice

Part 5a: Hypothesis testing

Michael Franke

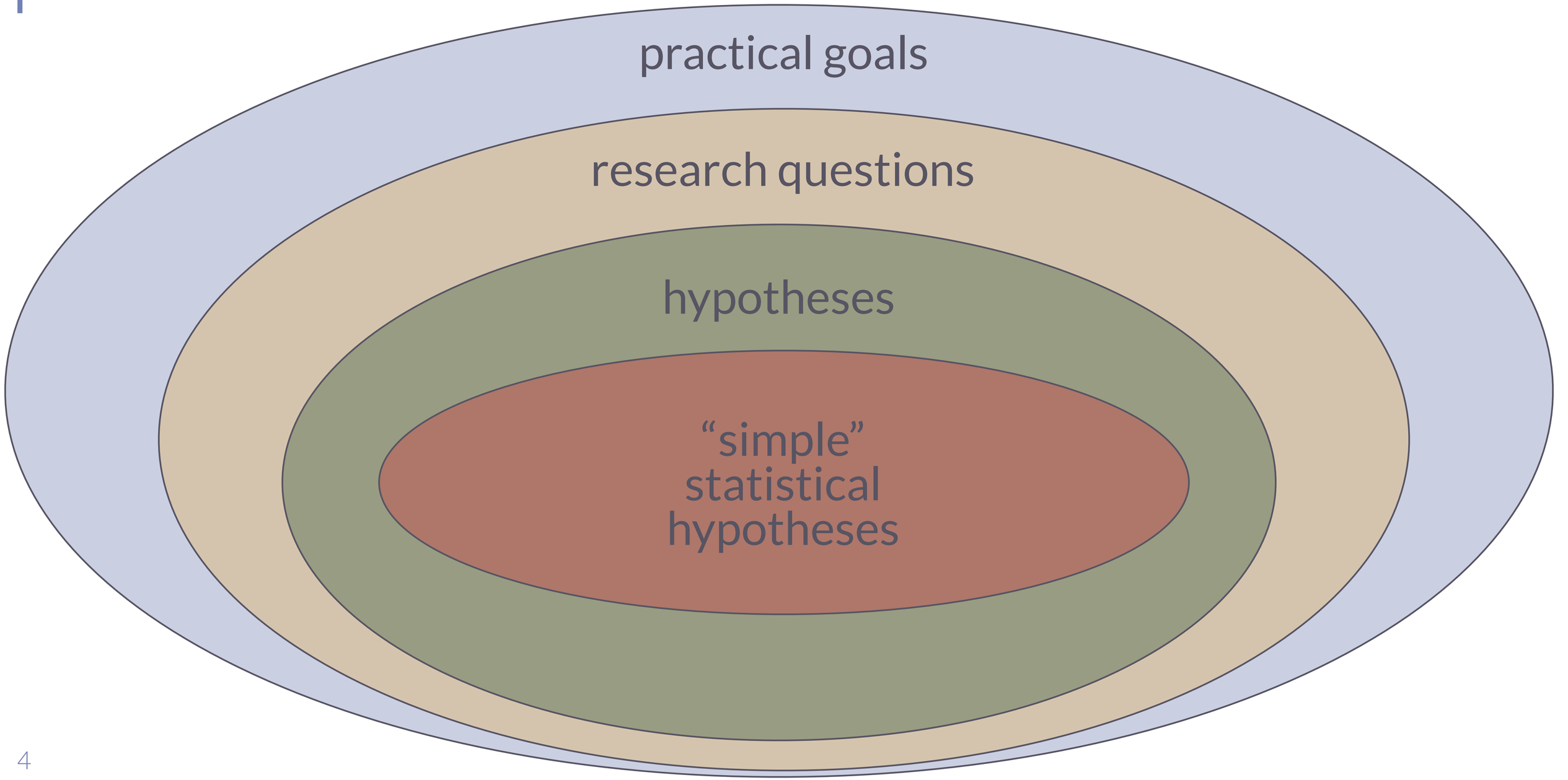
Main learning goals



1. reflect on the relation between research questions / goals and statistical hypotheses
2. explore different methods to address hypotheses in BDA, based on:
    a. parameter estimation
    b. model checking
    c. model comparison
3. understand the conceptual and practical pros and cons of each
4. reflect on terminology for reporting empirical evidence

modeling goals, research questions, & statistical hypotheses

goals, questions, hypotheses



Inference & decision

clearly separate in BDA | often conflated in classical stats

- **inference (belief update)**

$$P(\theta \mid D) \propto P(\theta)\ P(D \mid \theta)$$

- **decision (rational action choice)**

$$\arg\max_a \int P(\theta \mid D)\ U(a, \theta)\ \ \mathbf{d}\theta$$

	$\Pr(t)$	a_{bake}	a_{buy}
t_{foul}	.3	-10	5
t_{good}	.7	10	5

example: decision problem

Lindley (1977)

Excursion: Long-term error control

- **α-error**
  - reject H_0 when it is true

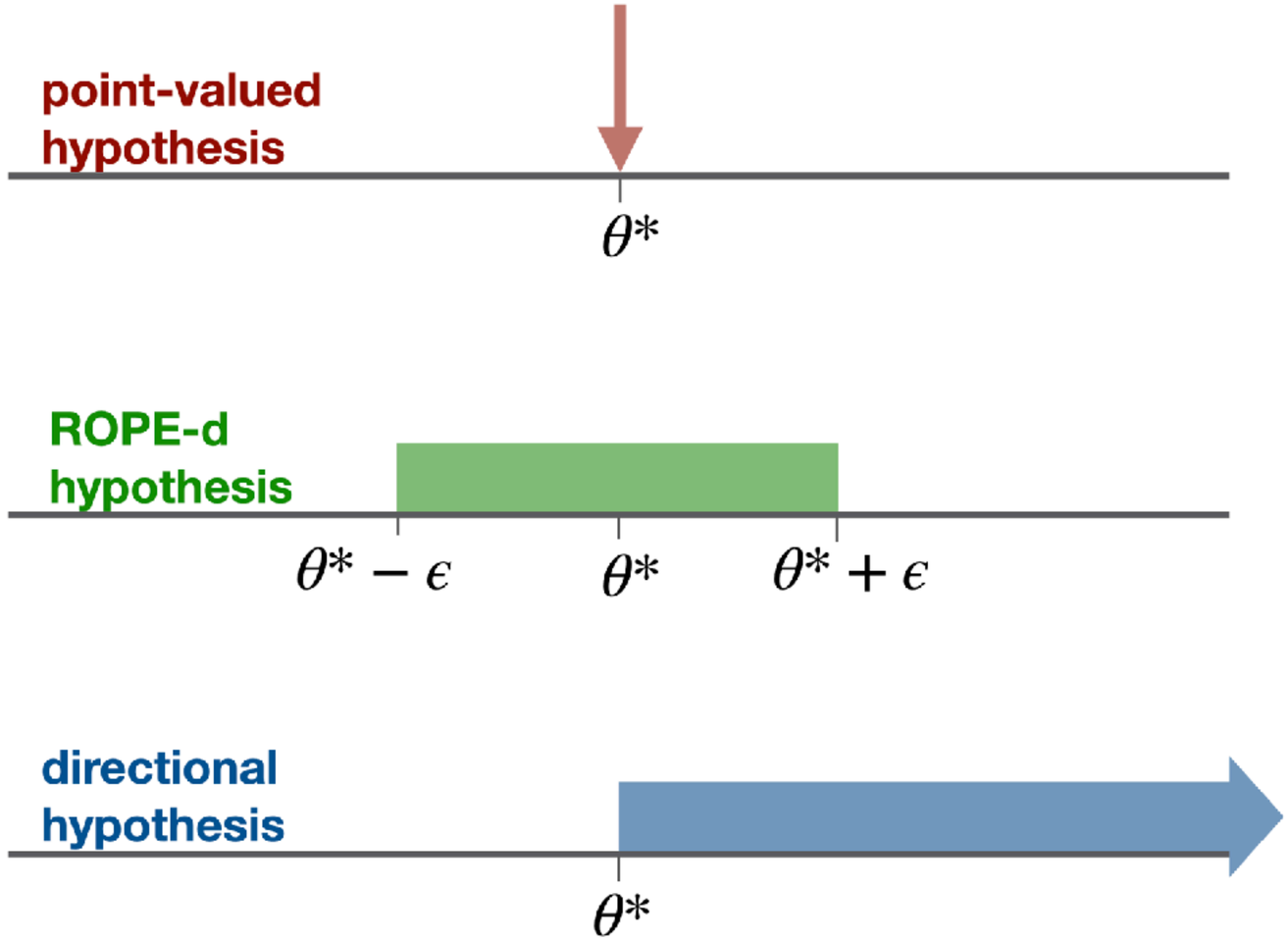
significance threshold

- **β-error**
  - do not reject H_0 when it is false
  - accept H_0 when H_a is true

statistical power

Types of statistical hypotheses

- point-valued $\theta = \theta^* \in \Theta$
- interval-valued $\theta \in I \subseteq \Theta$
  - ROPE-d
  - directional
- distributional $P \in \Delta(\Theta)$



read more here

approaches to testing statistical hypotheses

Three pillars of BDA

1. parameter estimation / inference

$$\underbrace{P(\theta \mid D)}_{\text{posterior}} \propto \underbrace{P(\theta)}_{\text{prior}} \times \underbrace{P(D \mid \theta)}_{\text{likelihood}}$$

2. predictions
   a. prior

$$P(D_{\text{pred}}) = \int P(\theta)\ P(D_{\text{pred}} \mid \theta)\ \mathrm{d}\theta$$

3. model comparison

$$\underbrace{\frac{P(M_1 \mid D)}{P(M_2 \mid D)}}_{\text{posterior odds}} = \underbrace{\frac{P(D \mid M_1)}{P(D \mid M_2)}}_{\text{Bayes factor}} \underbrace{\frac{P(M_1)}{P(M_2)}}_{\text{prior odds}}$$

Three ways to test a hypothesis

$\theta = \theta^*$

1. compute posterior, and check whether
   a. $P(\theta^* \mid D)$ high, and/or
   b. θ^* includes credible interval.

2. fix θ^* and perform prior / posterior predictive check (e.g., w/ likelihood as test statistic)

3. compare models with $\theta = \theta^*$ to another model

estimation-based testing

Estimation-based testing w/ HDIs

- interval-based hypothesis: $\theta \in I$
- posterior credible interval: $[l; u]$
- categorical approach:
  - **reason to accept** hypothesis I if $[l; u]$ is contained in I;
  - **reason to reject** hypothesis I if $[l; u]$ and I have no overlap;
  - **withhold judgement** otherwise.

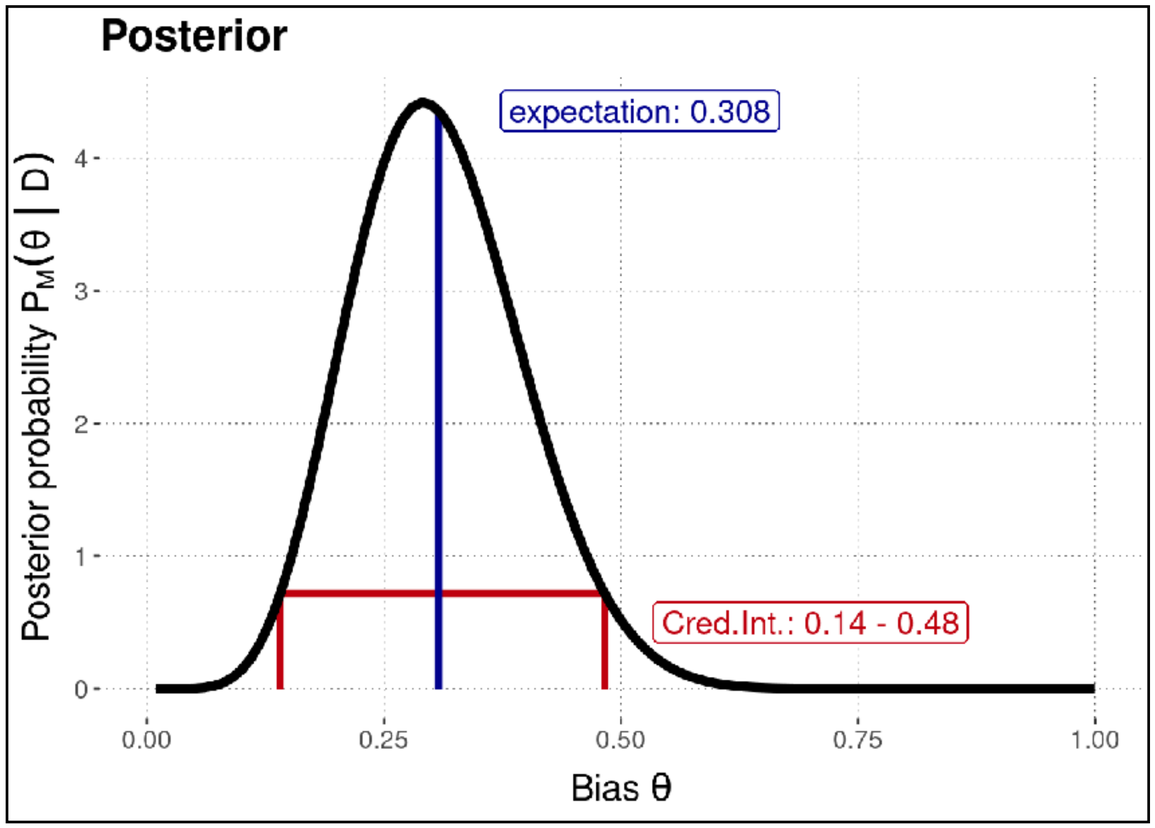
text for preregistration / methods section

> We test hypothesis H by comparing a ROPE of $\pm$ 0.01 around the critical value $\theta = 0.5$ against a 95% credible interval of the posterior. We speak of *suggestive evidence in favor* of H, if the CredInt lies entirely inside the ROPE. […]

read more here

Estimation-based testing with HDIs

example



text for results / discussion section

The 95% credible interval of the relevant parameter for hypothesis H is about [0.14; 0.48]. The defined ROPE [0.49; 0.51] lies entirely outside of that interval, so that, in line with preregistered / initially stated criteria, we interpret this as suggestive evidence *against* H.

Estimation-based testing w/ posterior probability

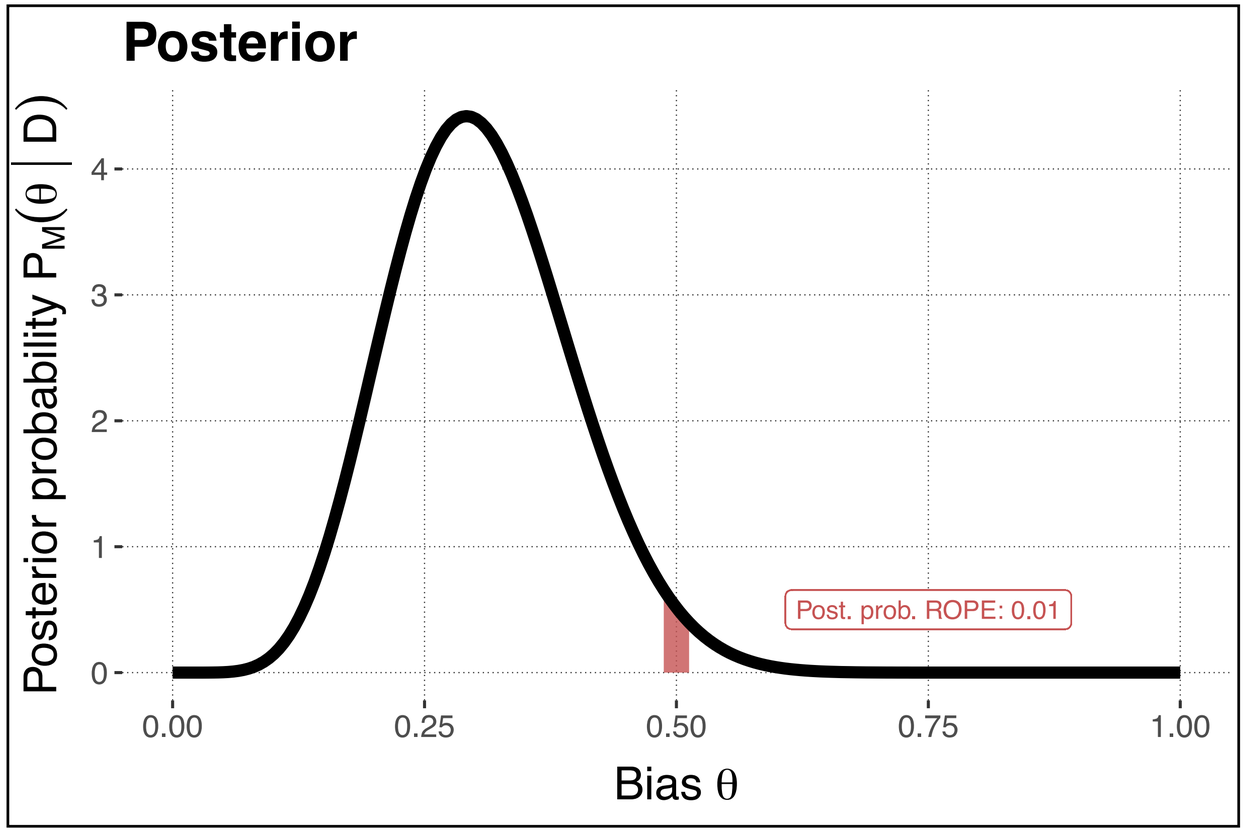
- interval-based hypothesis: $\theta \in I$
- **posterior probability** of I: $P(\theta \in I \mid D)$
- categorical approach:
  - **reason to accept** hypothesis I if $P(\theta \in I \mid D)$ is high;
  - **reason to reject** hypothesis I if $P(\theta \in I \mid D)$ is low;
  - **withhold judgement** otherwise.

text for preregistration / methods section

We test hypothesis H by considering a ROPE of $\pm$ 0.01 around the critical value $\theta = 0.5$. We speak of *suggestive evidence in favor* of H, if the posterior probability of the ROPE is at least 0.98. […]

Estimation-based testing w/ posterior probability

example



text for results / discussion section

The posterior probability of hypothesis H is about 0.01. In line with preregistered / initially stated criteria, we interpret this as suggestive evidence *against* H.

demo

Bayesian hypothesis testing

sensitivity analysis

Sensitivity analysis

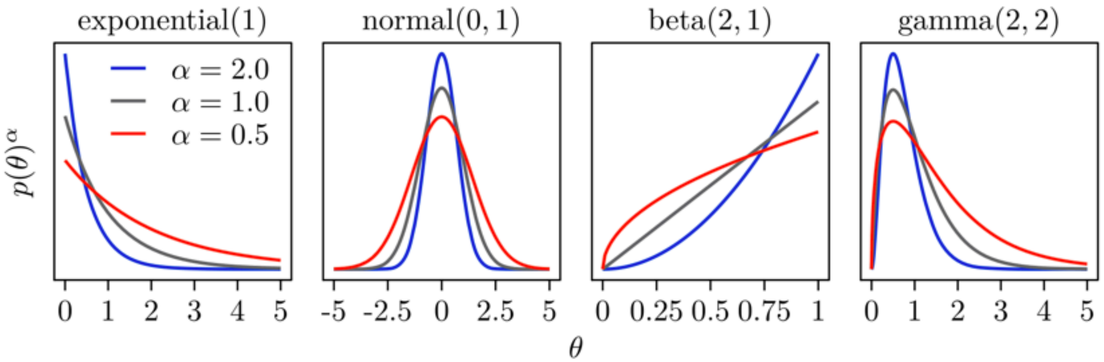
How much do results rely on the modelers' choices?

- prior sensitivity
  - on posterior
  - on Bayes factors
  - on cross-validation
  - …
- ROPE sensitivity
  - on categorical decision-making
  - on Bayes factors
  - …
- …

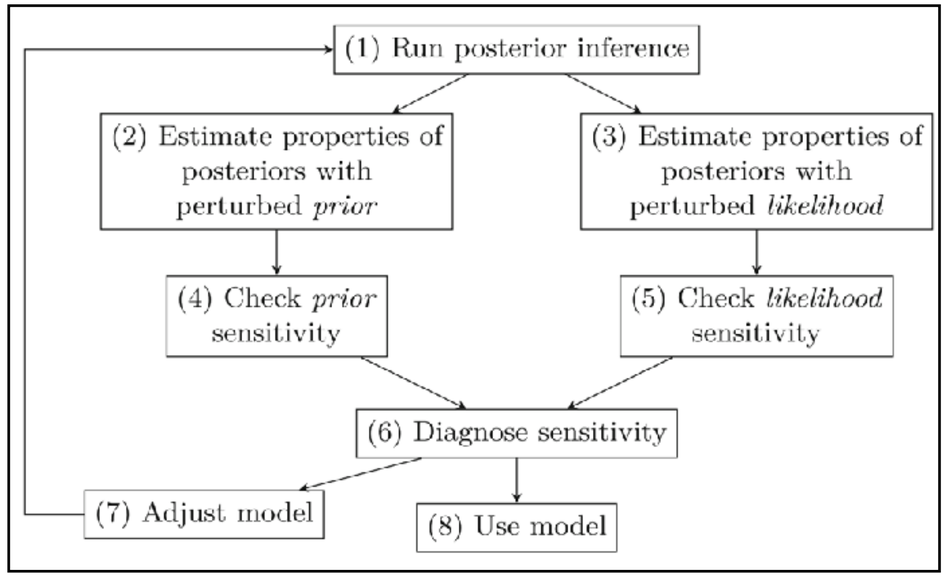
just rerun everything with different assumptions & modeling decision 😅

The `priorsense` package

- efficient computation of effect of power-scaling of prior and likelihood on the posterior distribution based on importance sampling (from MCMC samples)
  - power scaling: $g(x) \propto f(x)^{\alpha}$



- compare cumulative Jensen-Shannon divergence of posterior before and after power-scaling
  - recommended threshold: $D_{CJS} \geq 0.05$



	Prior sensitivity	
	No	Yes
No		Likelihood noninformativity
Yes	Likelihood domination	Prior-data conflict

Kallioinen et al. (2023)

demo

Bayesian hypothesis testing

hypothesis testing with Bayesian p-values

Bayesian predictive *p*-values

a generalization

- fix a model with $P(D \mid \theta)$ and $P(\theta)$
  - latter can be prior or posterior
  - gives prior / posterior predictive *p*-values
- $P_M(D)$ is the predictive distribution for model M
- Bayesian predictive *p*-value for observed data d_{obs}:

$$p(d_{\text{obs}}) = P_M\left(D \in \{d \mid P_M(d) \leq P_M(d_{\text{obs}})\}\right)$$

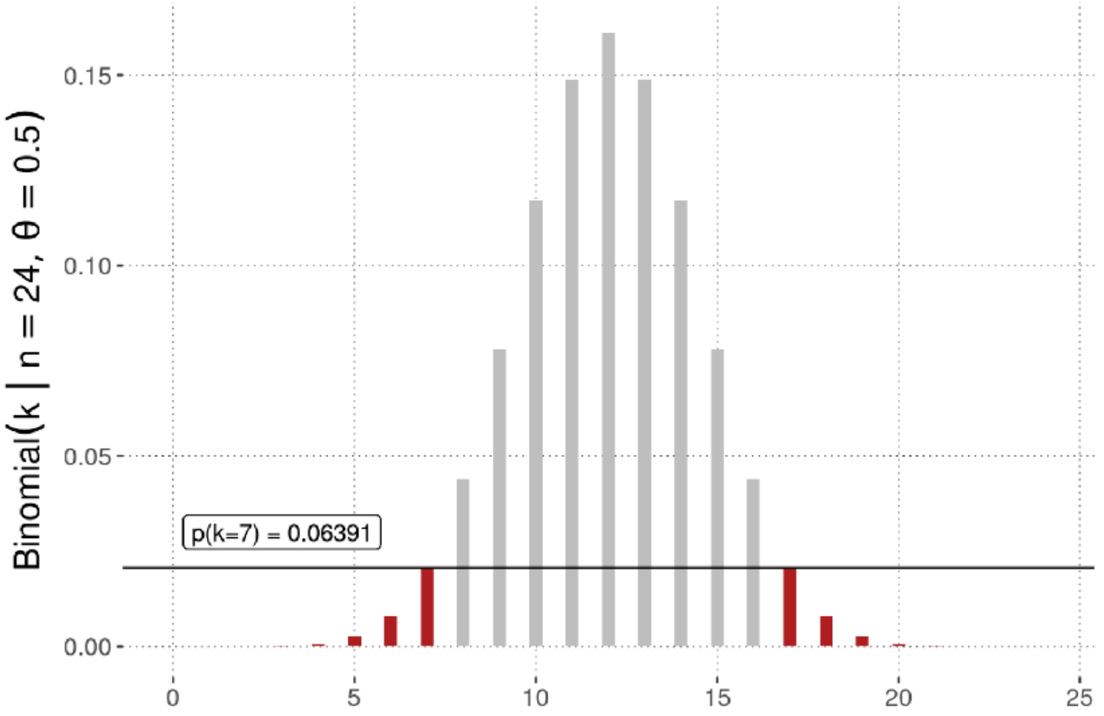
- approximated by sampling:

$$p(d_{\text{obs}}) \approx \frac{1}{n} \sum_{i=1}^{n} \left[P_M(d_i) \leq P_M(d_{\text{obs}})\right]$$

where $d_i \sim P_M(D)$ is a sample from the predictive distribution

Recap: frequentist *p*-values

$$p\left(D_{\text{obs}}\right) = P\left(T^{|H_0} \succeq^{H_{0,a}} t\left(D_{\text{obs}}\right)\right)$$



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demo

Bayesian hypothesis testing

Bayes factors for nested models

- Savage-Dickey method
- encompassing priors

Nested models

- suppose that there are n continuous parameters of interest $\theta = \langle \theta_1, \ldots, \theta_n \rangle$
- M_1 is a model defined by $P(\theta \mid M_1)$ & $P(D \mid \theta, M_1)$
- M_0 is **properly nested** under M_1 if:
  - M_0 assigns fixed values to some parameters $\theta_i = x_i, \ldots, \theta_n = x_n$
  - $\lim_{\theta_i \to x_i, \ldots, \theta_n \to x_n} P(\theta_1, \ldots, \theta_{i-1} \mid \theta_i, \ldots, \theta_n, M_1) = P(\theta_1, \ldots, \theta_{i-1} \mid M_0)$
  - $P(D \mid \theta_1, \ldots, \theta_{i-1}, M_0) = P(D \mid \theta_1, \ldots, \theta_{i-1}, \theta_i = x_i, \ldots, \theta_n = x_n, M_1)$

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Savage-Dickey method

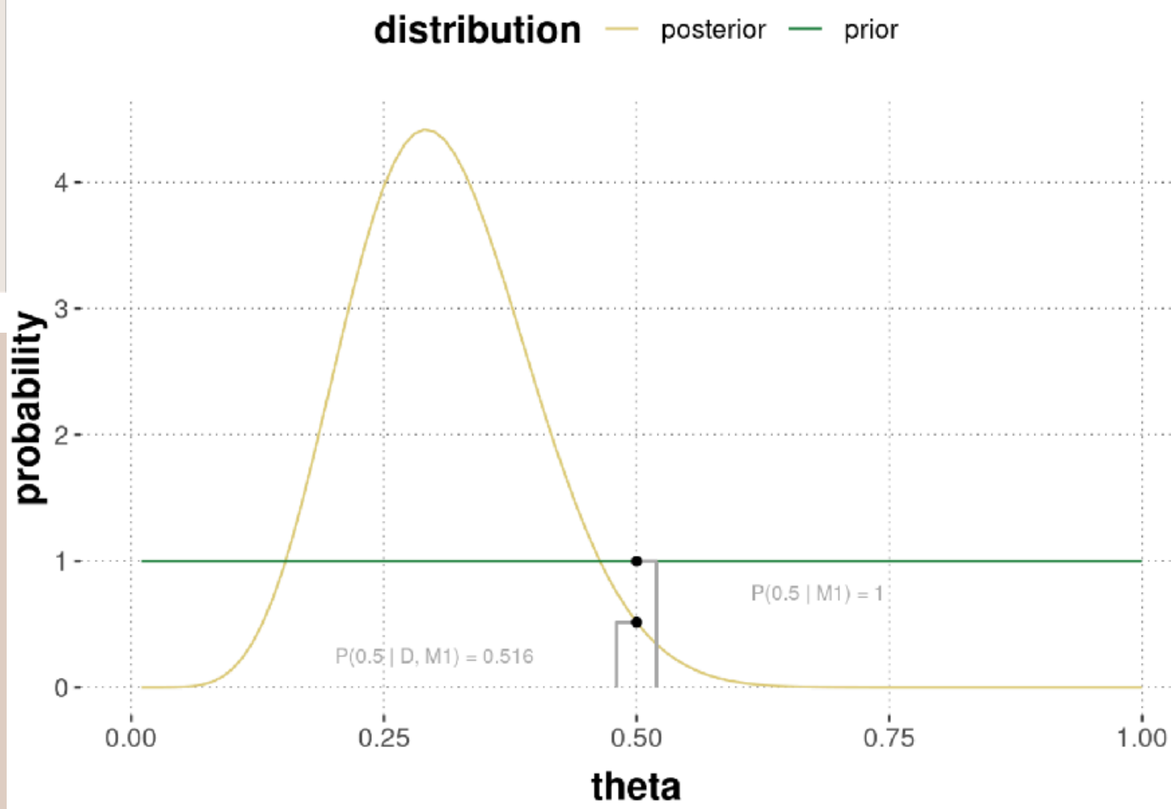
Theorem 11.1 (Savage-Dickey Bayes factors for nested models) *Let M_0 be properly nested under M_1 s.t. M_0 fixes $\theta_i = x_i, \ldots, \theta_n = x_n$. The Bayes factor BF_{01} in favor of M_0 over M_1 is then given by the ratio of posterior probability to prior probability of the parameters $\theta_i = x_i, \ldots, \theta_n = x_n$ from the point of view of the nesting model M_1:*

$$\mathrm{BF}_{01} = \frac{P(\theta_i = x_i, \ldots, \theta_n = x_n \mid D, M_1)}{P(\theta_i = x_i, \ldots, \theta_n = x_n \mid M_1)}$$

Proof. Let's assume that M_0 has parameters $\theta = \langle \phi, \psi \rangle$ with $\phi = \phi_0$, and that M_1 has parameters $\theta = \langle \phi, \psi \rangle$ with ϕ free to vary. If M_0 is properly nested under M_1, we know that $\lim_{\phi \to \phi_0} P(\psi \mid \phi, M_1) = P(\psi \mid M_0)$. We can then rewrite the marginal likelihood under M_0 as follows:

$$\begin{aligned}
P(D \mid M_0) &= \int P(D \mid \psi, M_0) P(\psi \mid M_0) \, \mathrm{d}\psi && \text{[marginalization]} \\
&= \int P(D \mid \psi, \phi = \phi_0, M_1) P(\psi \mid \phi = \phi_0, M_1) \, \mathrm{d}\psi && \text{[assumption of nesting]} \\
&= P(D \mid \phi = \phi_0, M_1) && \text{[marginalization]} \\
&= \frac{P(\phi = \phi_0 \mid D, M_1) P(D \mid M_1)}{P(\phi = \phi_0 \mid M_1)} && \text{[Bayes rule]}
\end{aligned}$$

The result follows if we divide by $P(D \mid M_1)$ on both sides of the equation. □



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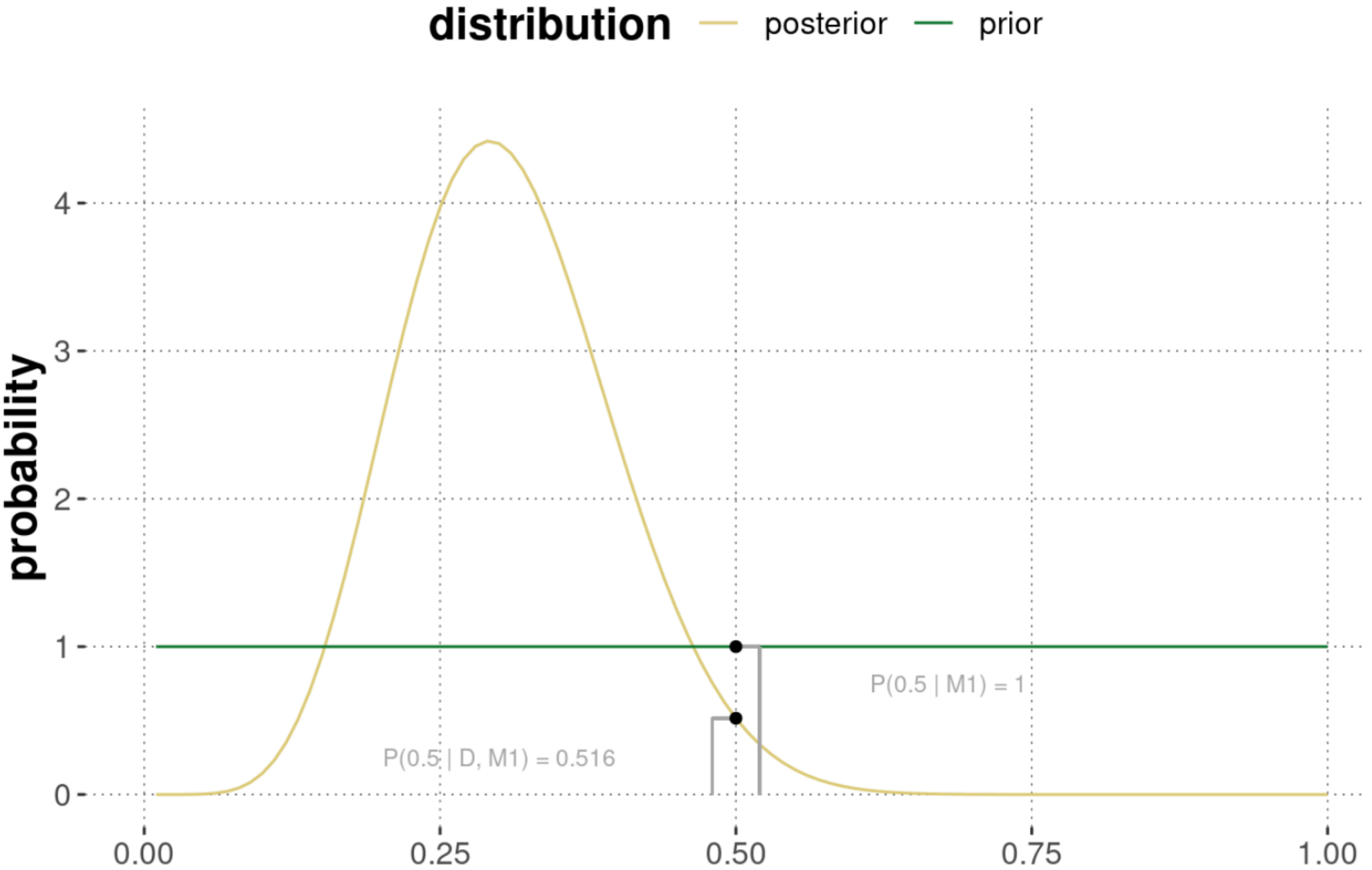
Savage-Dickey method



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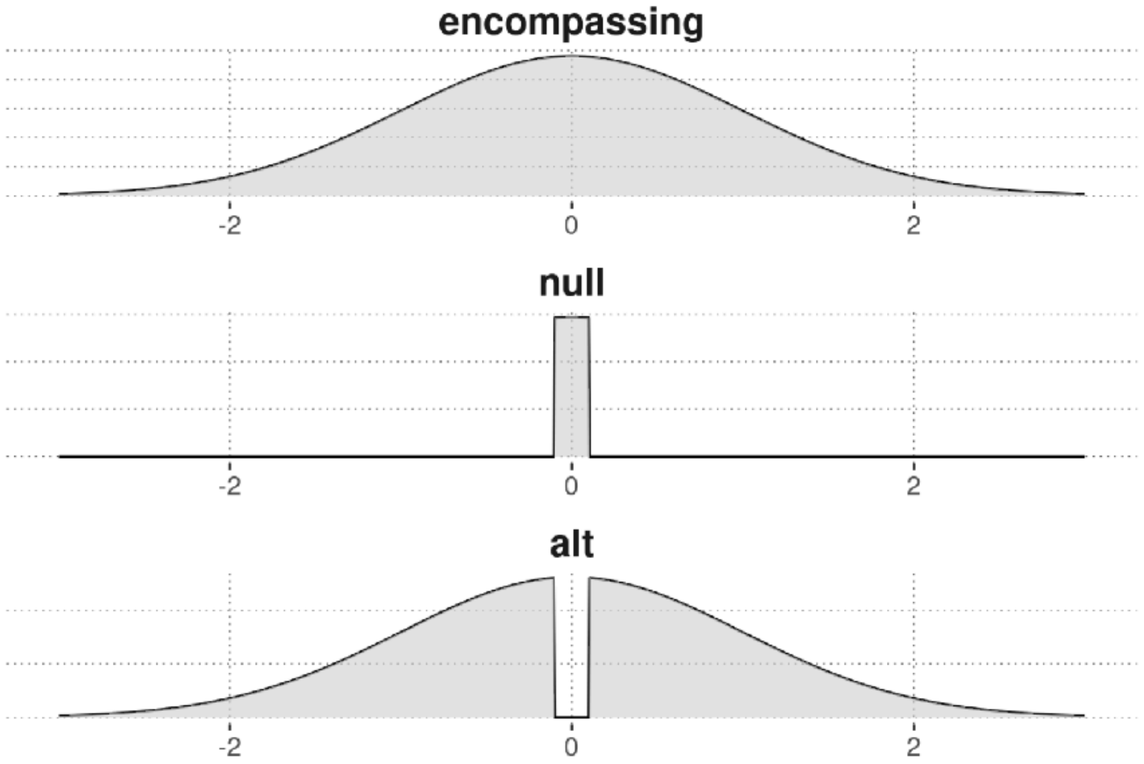
Encompassing model

- target hypothesis is interval-based: $H_0: \theta \in I_0$
  - let I_1 be the complement of I_0
- an **encompassing model** M_e consists of:
  - likelihood $P(D \mid \omega, \theta, M_e)$
  - prior $P(\omega, \theta \mid M_e)$
- the **encompassed models** M_0 and M_1 share the likelihood function with M_e and have priors:
  - $P(\omega, \theta \mid M_i) = P(\omega, \theta \mid I_i, M_e)$



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generalized Savage-Dickey method

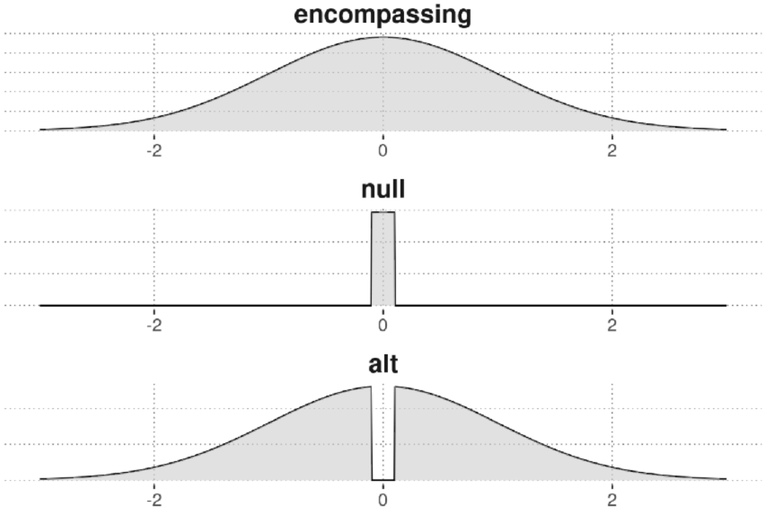
for encompassing models

Theorem 11.2 *The Bayes Factor in favor of nested model M_i over encompassing model M_e is:*

$$\text{BF}_{ie} = \frac{P(\theta \in I_i \mid D, M_e)}{P(\theta \in I_i \mid M_e)}$$

Theorem 11.3 *The Bayes Factor in favor of model M_0 over alternative model M_1 is:*

$$\text{BF}_{01} = \frac{P(\theta \in I_0 \mid D, M_e)}{P(\theta \in I_1 \mid D, M_e)} \frac{P(\theta \in I_1 \mid M_e)}{P(\theta \in I_0 \mid M_e)}$$



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LOO-based testing

demo

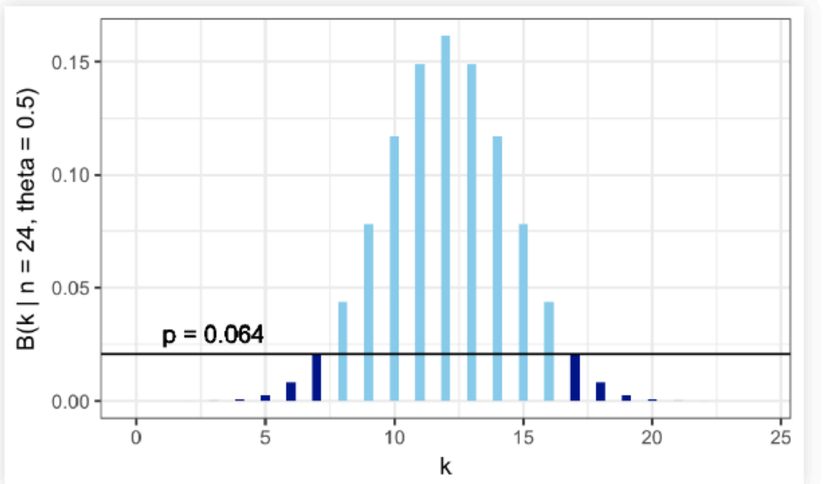
Bayesian hypothesis testing

Comparison

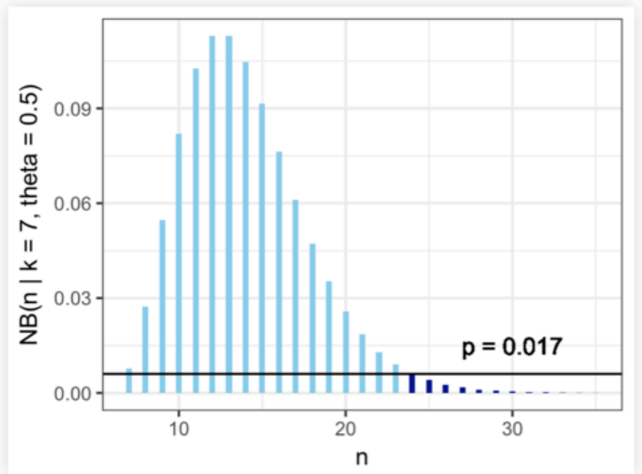
p-problems

- no evidence *for* the tested hypothesis
- dependence on sampling distribution

$$B(k; n = 24, \theta = 0.5) = \binom{n}{k} \theta^k (1-\theta)^{n-k}$$



$$NB(n; k = 7, \theta = 0.5) = \frac{k}{n}\binom{n}{k} \theta^k (1-\theta)^{n-k}$$



Jeffrey-Lindley paradox

clairvoyance?

data

```
k = 49581
N = 98451
```

frequentist p-value

```
binom.test(k, N)$p.value
```

```
## [1] 0.02364686
```

BF w/ Savage-Dickey
(alternative: flat prior)

```
dbeta(0.5, k + 1, N - k + 1)
```

```
## [1] 19.21139
```

Comparison of approaches

approach	method	computation	interpretation	measure	pro	con
estimation	Cred. Interval	easy	easy	reasonable to believe (categ.)	easy	dep. on prior
estimation	posterior prob.	easy	easy	level of credence (quant)	easy	dep. on prior
criticism	p-values	hard	hard	surprise (quant)	actual test	dep. on sampling distribution, only evidence *against* H
comparison	Bayes factors	hard	medium	relative strength of evidence (quant)	intuitive	dep. on alternative model & priors
comparison	LOO	medium	hard	post. predictive accuracy (quant)	cool and coming	unclear if actually a test

best practices

Bayesian analysis reporting guidelines

BARG Kruschke (2023)

- BARG information should always be included completely (but need not go in into the main text)

- **preamble**
  - motivation for BDA & goals of analysis

1. **explain the model**
   - data structure, likelihood function, model parameters, prior distribution (w/ prior predictive checks), formal / computational specification of whole model (maybe graphical representation)
2. **report details of the computation**
   - software used (package versions), for all parameters: MCMC convergence stats (R-hat) & ESS
4. **describe the posterior distribution**
   - for each parameter: summary of posterior samples (mean & 95% CredInt), (maybe graphical representation, densities)
5. **report decisions & decision criteria**
   - motivate decision & criteria (if applicable): explicate loss function, ROPE limits, BF thresholds, model posterior thresholds
6. **report sensitivity analysis**
   - characterize the range of prior assumptions that support the same result as reported as main conclusion
7. **make it reproducible**
   - share code, data, software information, all auxiliary files; use seeds and/or share MCMC chains

Bayesian workflow

- further recommendations for robust Bayesian workflow
  - Gelman et al. (2020)
  - Schad et al. (2021)