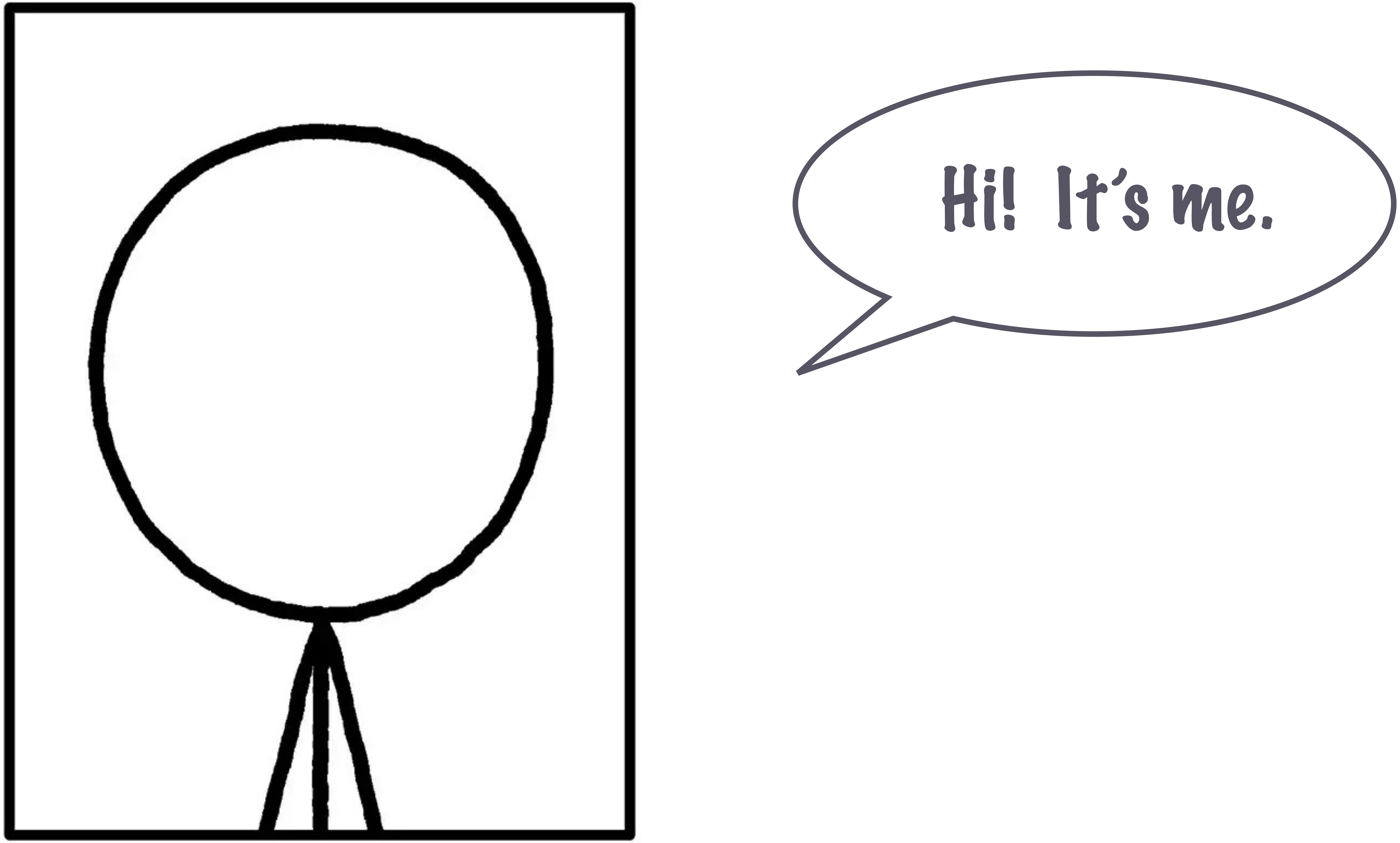
Bayesian data analysis: Theory & practice

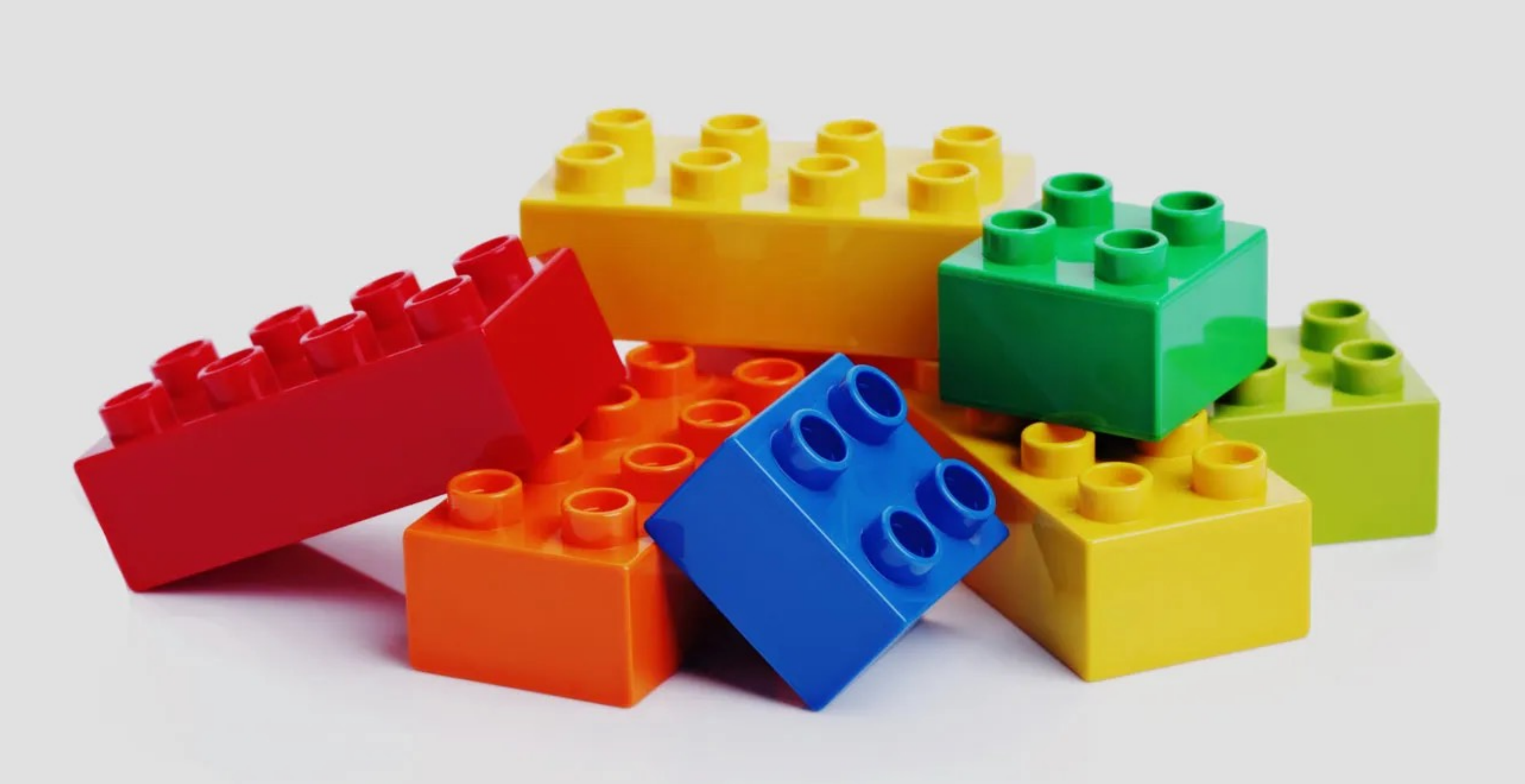
Part 1: Bayesian basics & simple linear regression

Michael Franke


Hi! It's me.

Pros
of BDA

- well-founded & totally general
- easily extensible / customizable
- more informative / insightful
- stimulates view: “models as tools”



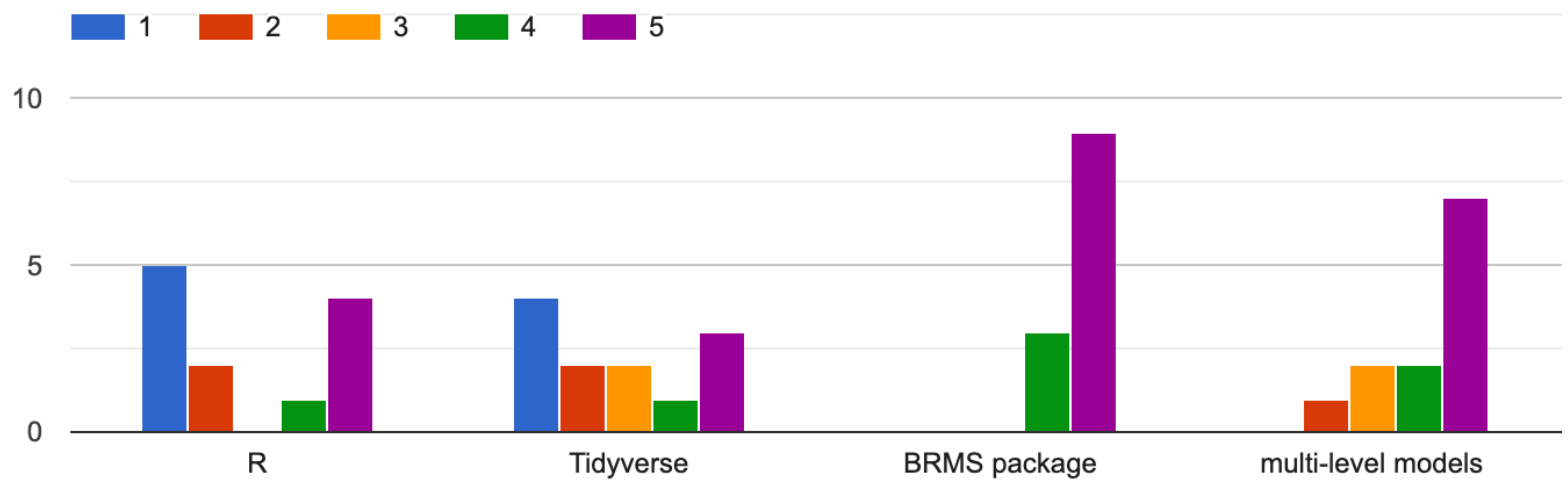
Cons
of BDA

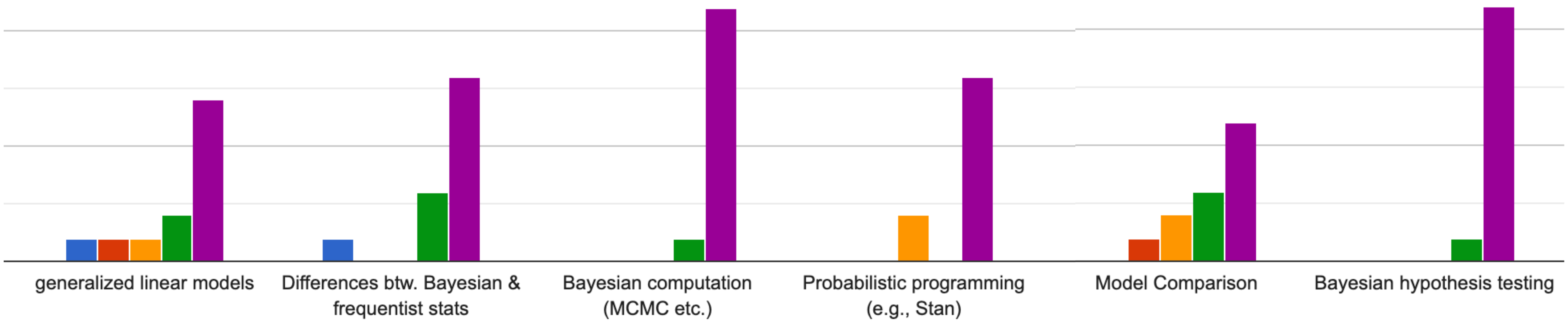
- not yet fully digested by community
- possibly computationally complex
- less ready-made, more hands-on
- requires thinking (wait, that’s a pro!)
  - last two points less valid than 10 years ago

this course

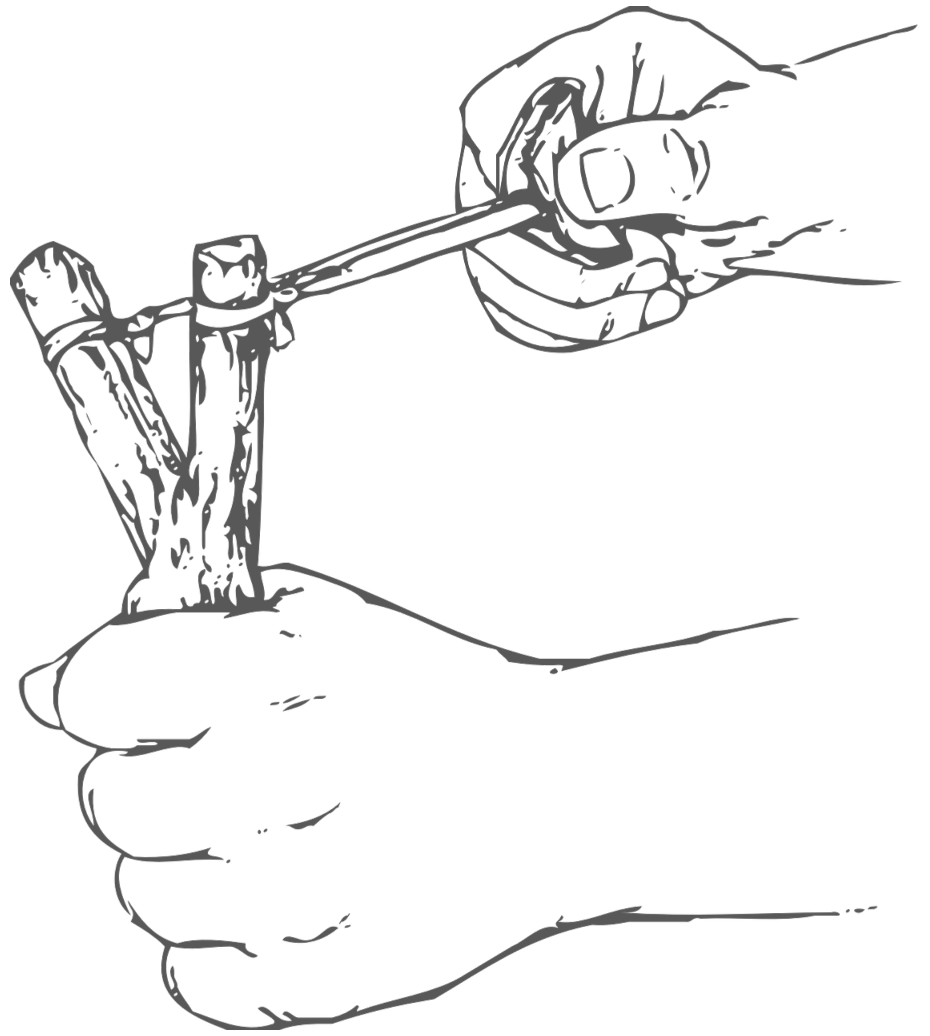
Survey results

what you (seem to) want to learn





Main learning goals
for this course



1. “think (more) computationally Bayesian”
   a. model-centric: explicate data-generating process
   b. sampling-based
2. become (more) comfortable in applying multi-level GLMS
   a. determine the appropriate (kind of) model for a given problem
   b. implement, run and interpret the Bayesian model
   c. draw conclusions regarding evidence for/against research questions
3. understand key concepts of beginners / intermediate BDA
   a. Bayesian computation (MCMC)
   b. priors & predictive functions
   c. model comparison
   d. model checking
   e. Bayesian hypothesis testing

How this works

- class from 2:00 – 6:00 pm
  - w/ some shorter and one longer break
  - some live demos
- web-book w/ tutorials and exercises
  - live version: here
  - GitHub code: here
- discussion of exercises
  - on Slack (or other means)
  - first part of class
- contribute (if you want)
  - e.g., send pull-requests to **populate the cheat sheet**

Here is ~~the~~ a plan

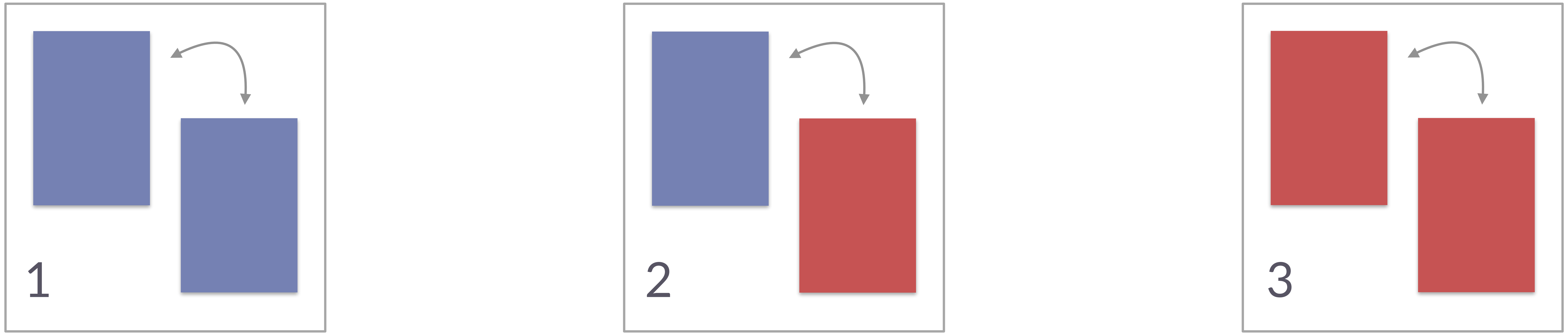
Day 1	Day 2	Day 3	Day 4	Day 5
course structure BDA basics	MCMC	multiple regression (w/ categorical predictors) GLMs	model comparison model criticism	hypothesis testing
data-generating processes in WebPPL simple regression	BRMS	multi-level models	causal inference	🛠️

Bayesian modeling

Three-card problem

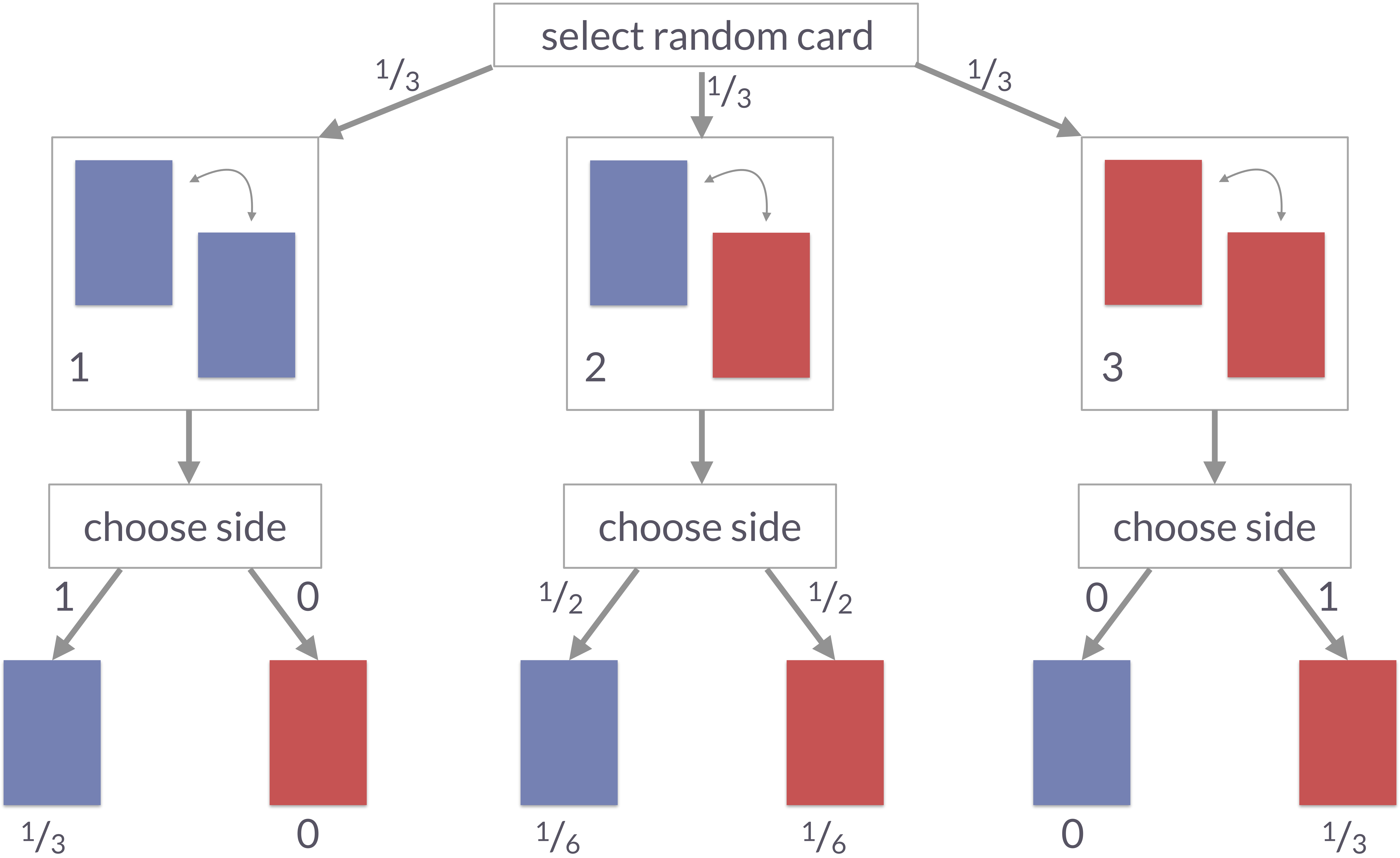
problem statement

- Sample a card (uniformly at random).
- Choose a side of that card to reveal (uniformly at random).
- What's the probability that the side you do not see is **BLUE**, given that the side you see is **BLUE**?



Three-card problem

data-generating process



Conditional probability and Bayes rule

for the three-card problem

- conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- Bayes rule

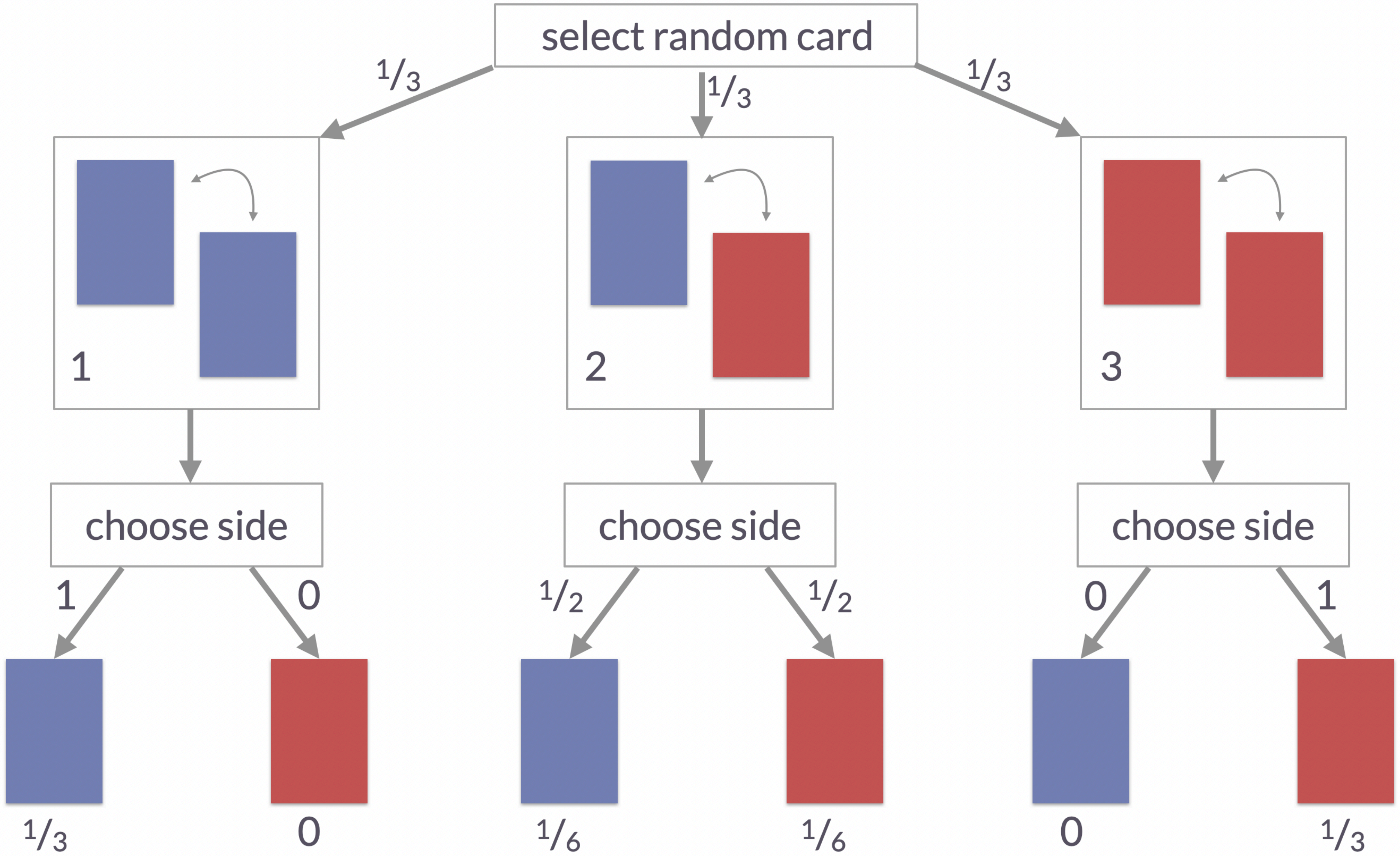
$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- Applied to three-card problem:

$$P(\text{card 1} \mid \text{blue}) = \frac{P(\text{blue} \mid \text{card 1})\ P(\text{card 1})}{P(\text{blue})}$$

$$= \frac{1 \times \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

“reasoning from observed effect to latent cause via a model of the data-generating process”



demo

3-card problem in WebPPL

Bayesian data analysis

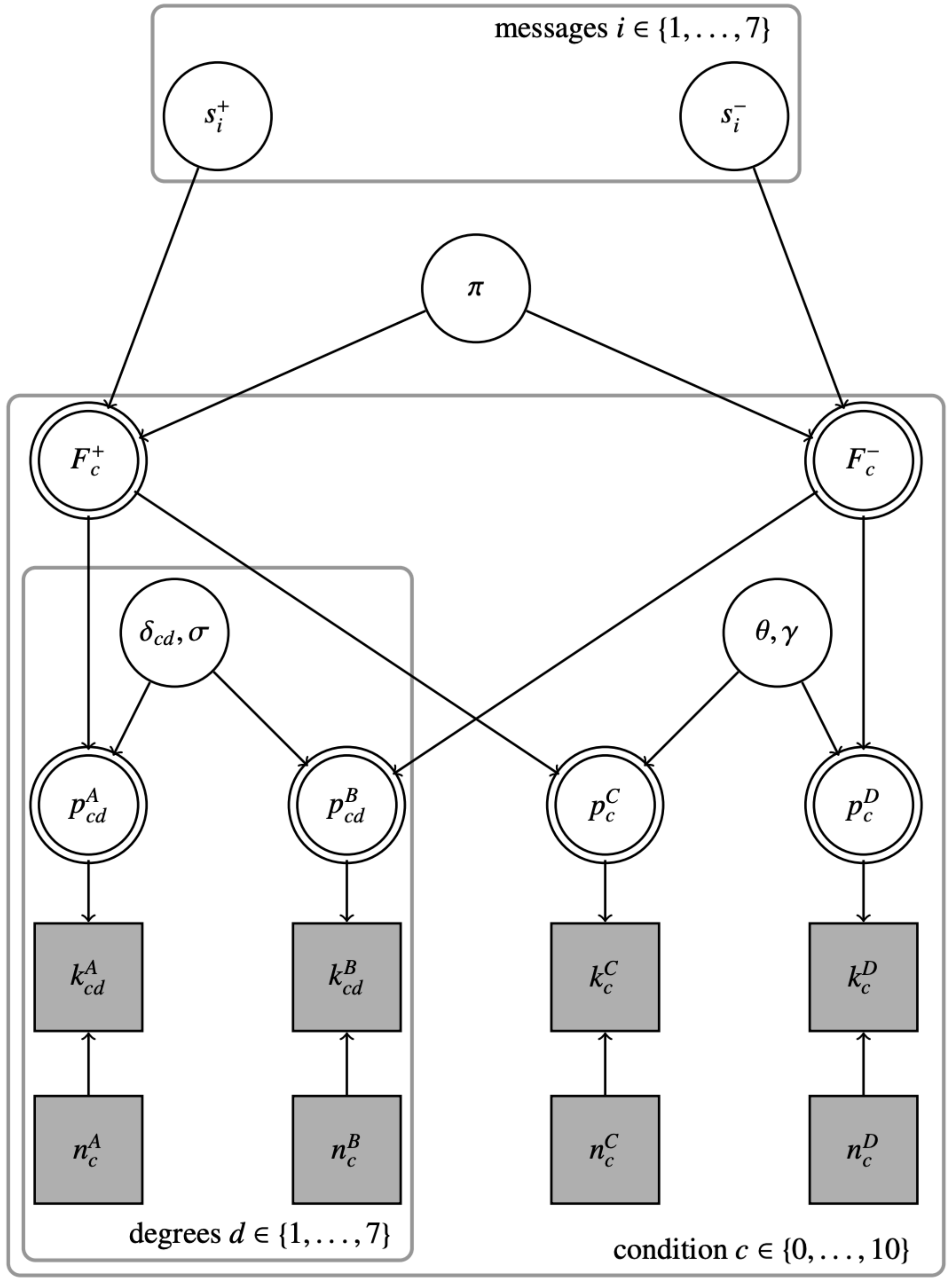
in a nutshel

- BDA is about what we *should* believe given:
  - some observable data, and
  - our model of how this data was generated (a.k.a. **the data-generating process**)
- our best friend will be **Bayes rule**
  - e.g., for **parameter inference**:

$$\underbrace{P(\theta \mid D)}_{\text{posterior}} \propto \underbrace{P(\theta)}_{\text{prior}} \times \underbrace{P(D \mid \theta)}_{\text{likelihood}}$$

  - or, for **model comparison**:

$$\underbrace{\frac{P(M_1 \mid D)}{P(M_2 \mid D)}}_{\text{posterior odds}} = \underbrace{\frac{P(D \mid M_1)}{P(D \mid M_2)}}_{\text{Bayes factor}} \underbrace{\frac{P(M_1)}{P(M_2)}}_{\text{prior odds}}$$



$$s_i^{+/-} \sim \text{Beta}(1, 1)$$

$$\frac{1}{\pi} \sim \text{Gamma}(0.5, 0.5)$$

$$F_c^{+/-} = \text{F}(c\,;\, \vec{s}^{\,+/-}, \pi)$$

$$\sigma \sim \text{Uniform}(0, 0.4)$$

$$\delta_{d \in \{1,\dots,6\}} \sim \text{Normal}(d/7, 14)$$
$$\delta_0 = -\infty; \quad \delta_7 = \infty$$

$$\theta \sim \text{Normal}(0.5, 0.2)$$

$$\frac{1}{\gamma} \sim \text{Gamma}(1, 1)$$

$$p_{cd}^{A/B} = \int_{\delta_{cd-1}}^{\delta_{cd}} \text{Normal}(x, F_c^{+/-}, \sigma)\, \text{d}x$$

$$p_c^{C/D} = (1 + \exp(-\gamma(F_c^{+/-} - \theta)))^{-1}$$

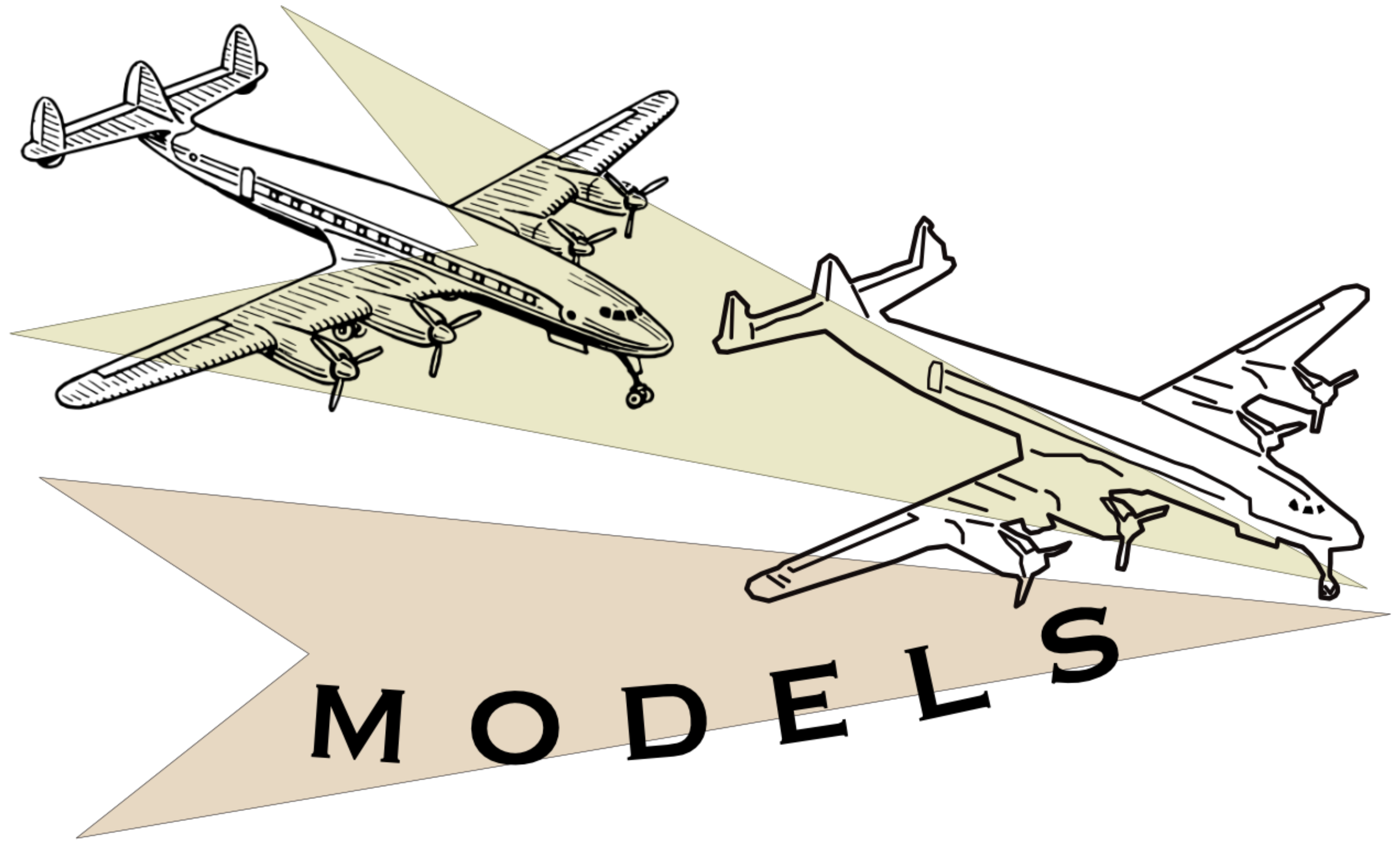
$$k_{cd}^{A/B} \sim \text{Multinomial}(p_{cd}^{A/B}, n_c^{A/B})$$

$$k_c^{C/D} \sim \text{Binomial}(p_c^{C/D}, n_c^{C/D})$$

Statistical models

likelihoods from a data-generating process

- A **statistical model** is a condensed formal representation, following common conventional practices of formalization, of the assumptions we make about what the data is and how it might have been generated by some (usually: stochastic) process.
- "All models are wrong, but some are useful." (Box 1979)
- a **Bayesian statistical model** of stochastic process generating data D consists of:
  - a vector of **parameters** θ
  - a **likelihood function**: $P(D \mid \theta)$
  - a **prior** distribution: $P(\theta)$
- among other things, we can use a model for **inference**:
  - **posterior** distribution: $P(\theta \mid D) \propto P(D \mid \theta)\ P(\theta)$



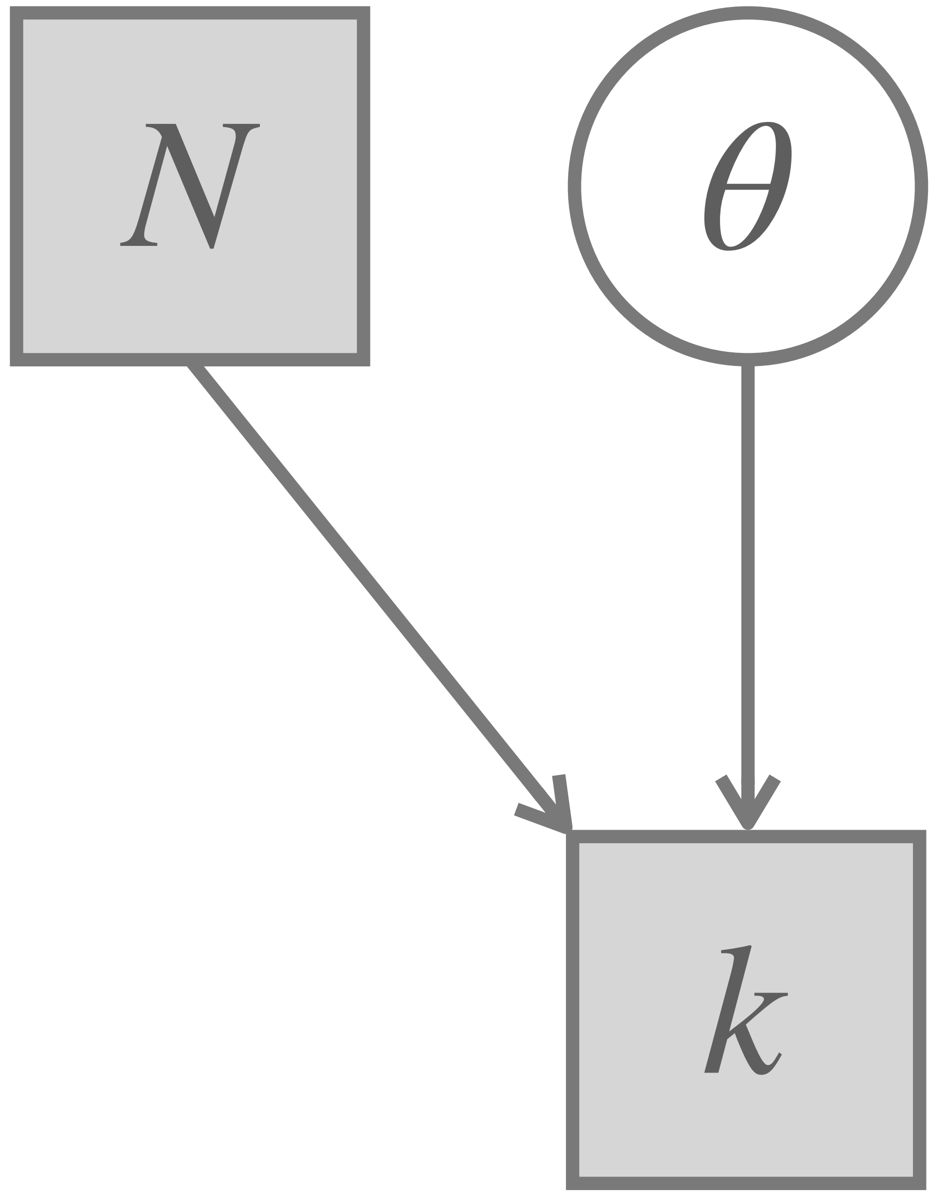
Binomial model

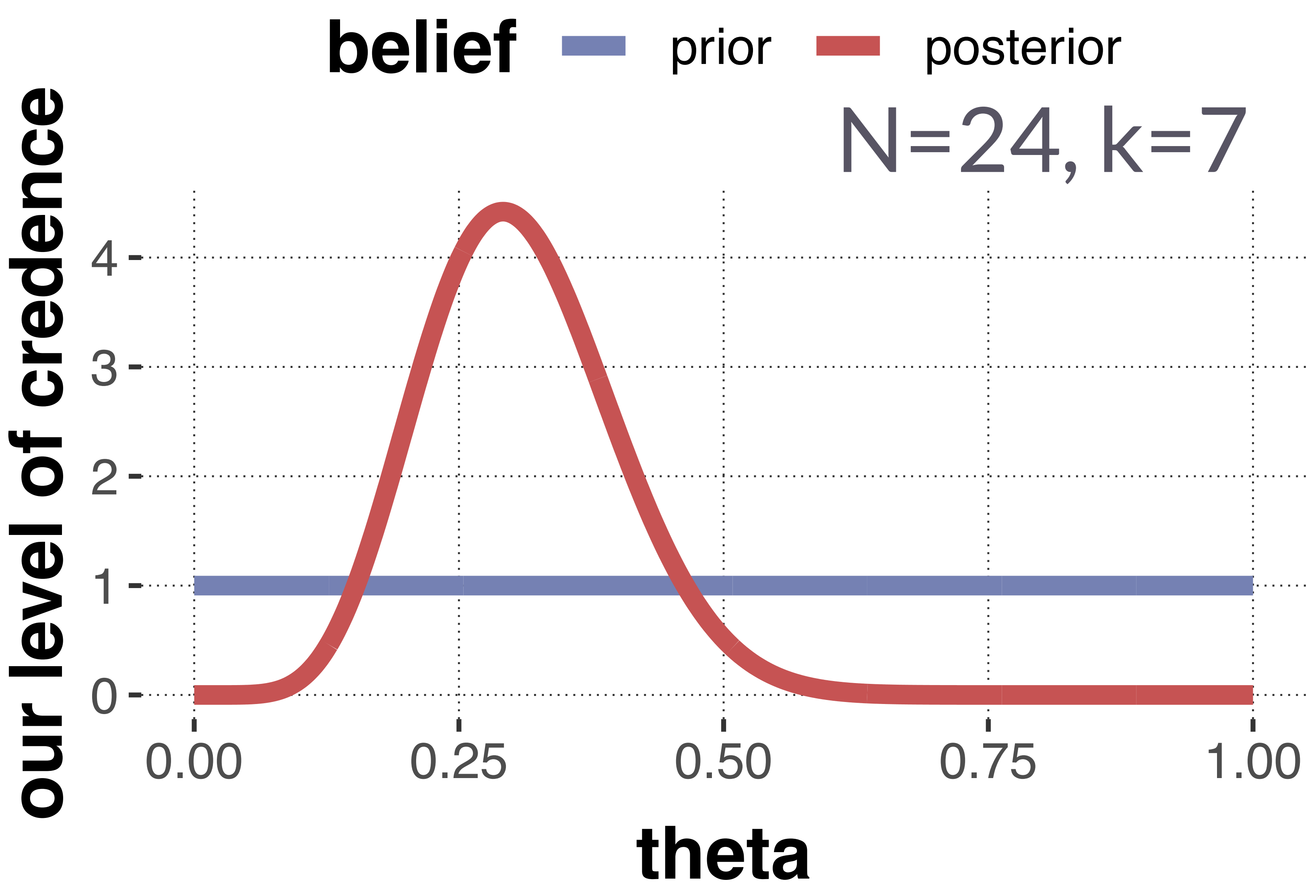
the 'coin-flip' model

- data: pair of numbers $D = \{k, N\}$
  - N is the number of tosses
  - k is the number of heads (successes)
- variable:
  - θ is the number of heads (successes)
- uninformed prior:

  $\theta \sim \text{Beta}(1,1)$
- likelihood function:

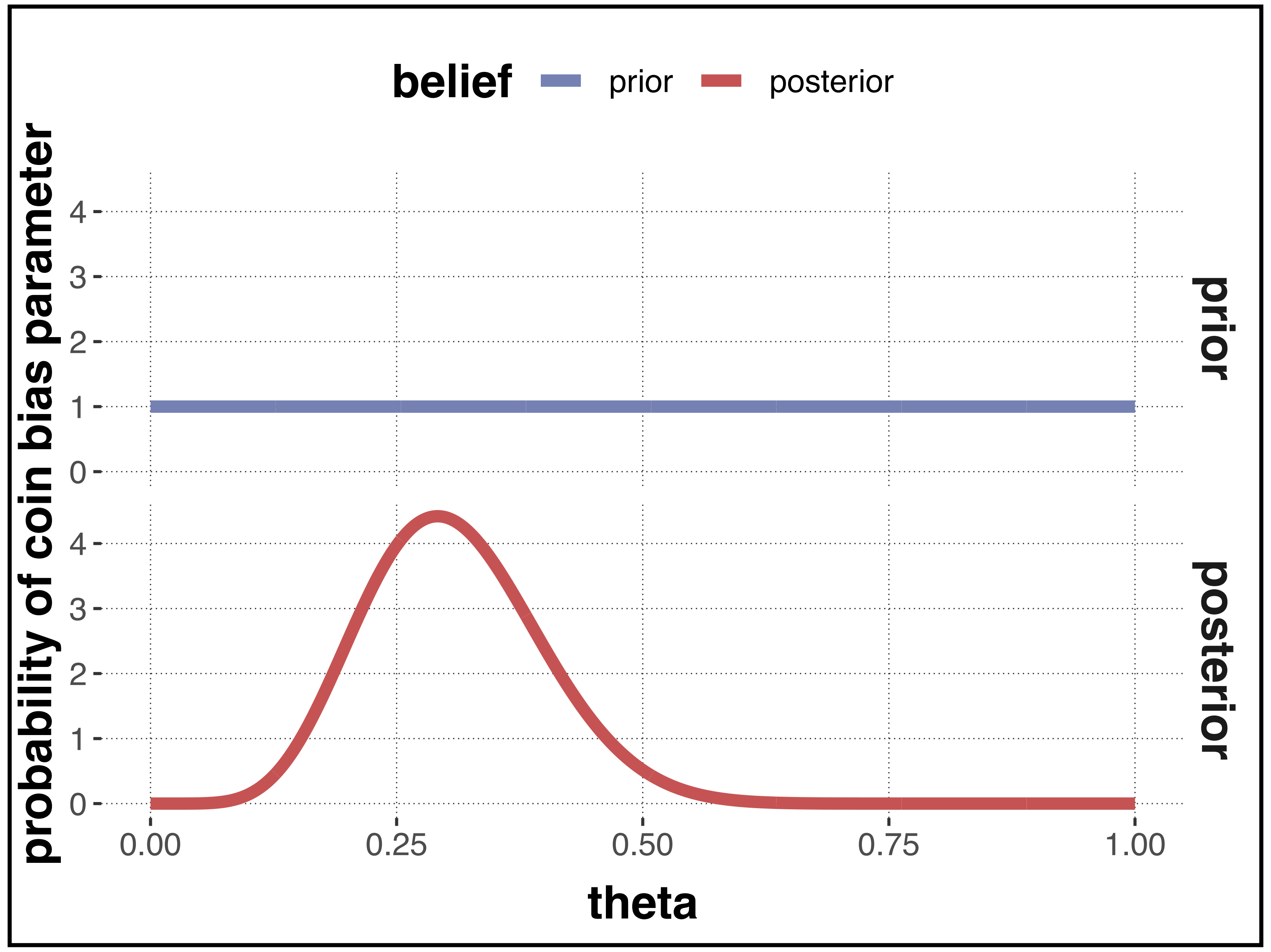
  $k \sim \text{Binomial}(\theta, N)$
- conventions for model graphs:
  - circles / squares: continuous / discrete variables
  - white / gray nodes: latent / observed variables

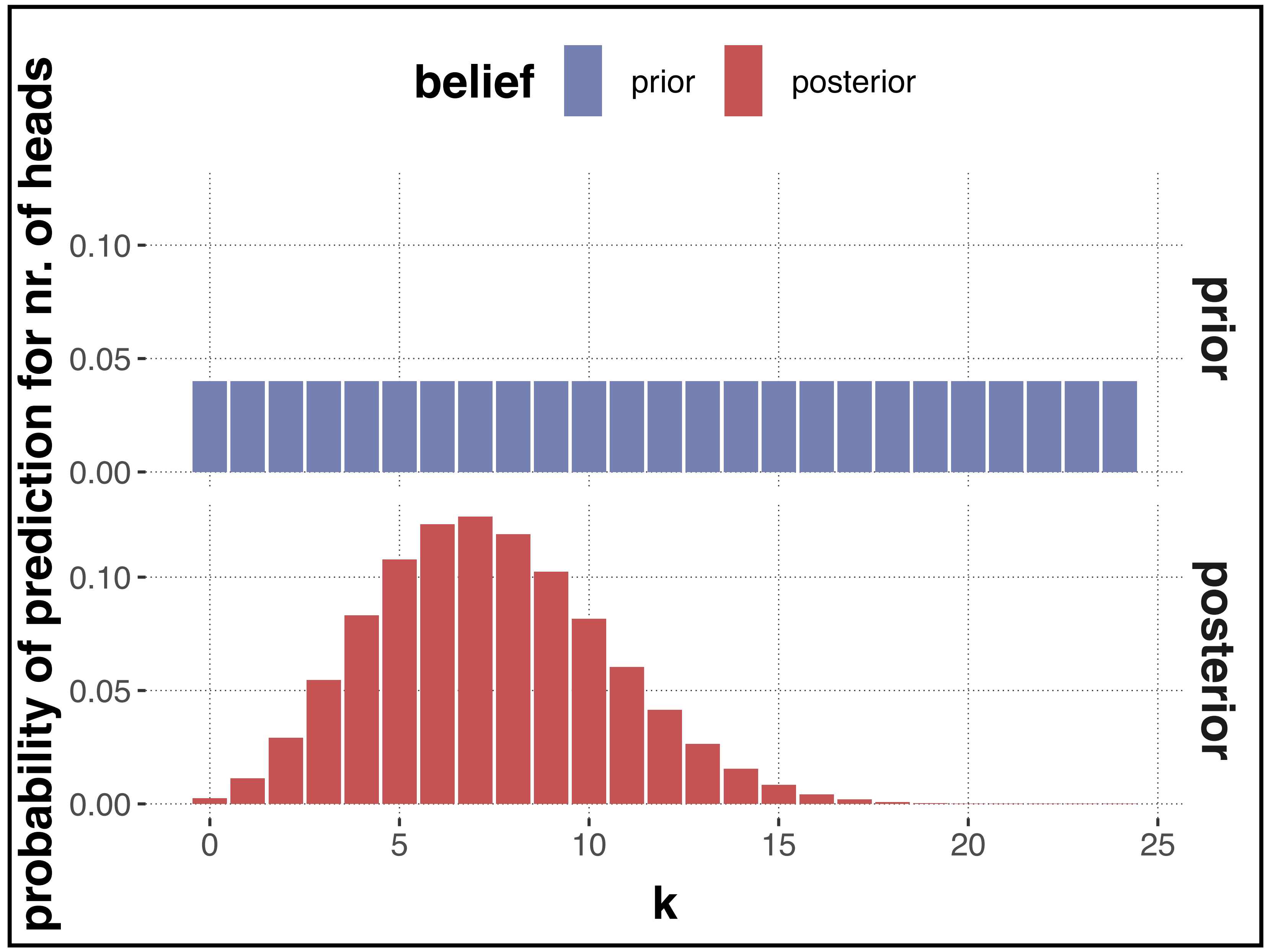




The Big Bayesian 4

- **prior distribution**
  - uncertainty about model parameters *before* seeing the data
- **posterior distribution**
  - uncertainty about model parameters *after* seeing the data
- **prior predictive distribution**
  - distribution over likely future data points before seeing the data
- **posterior predictive distribution**
  - distribution over likely future data points before seeing the data

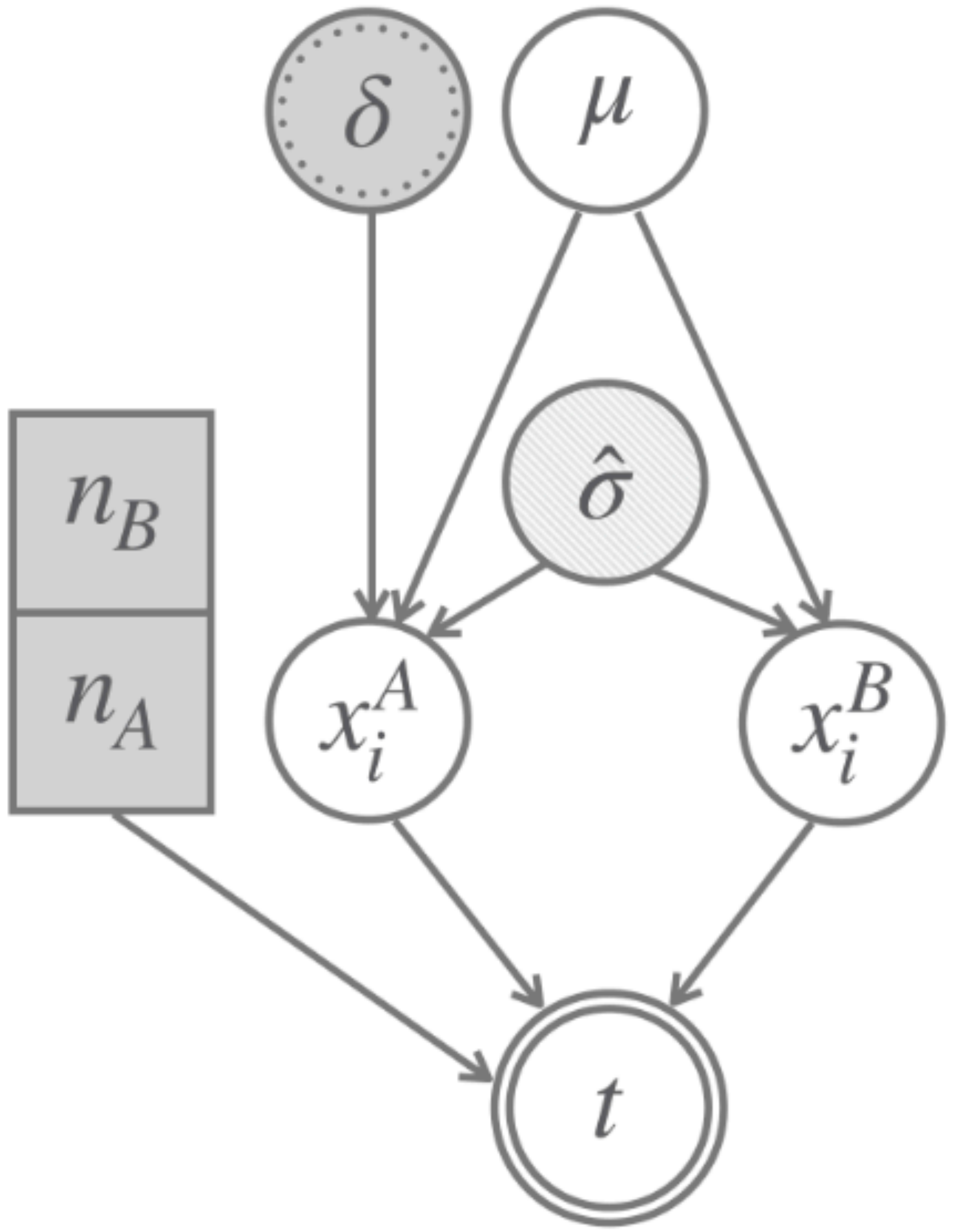




demo

inferring a coin bias in WebPPL

MODELS are everywhere



$$x_i^A \sim \text{Normal}(\mu + \delta, \sigma)$$

$$x_i^B \sim \text{Normal}(\mu, \sigma)$$

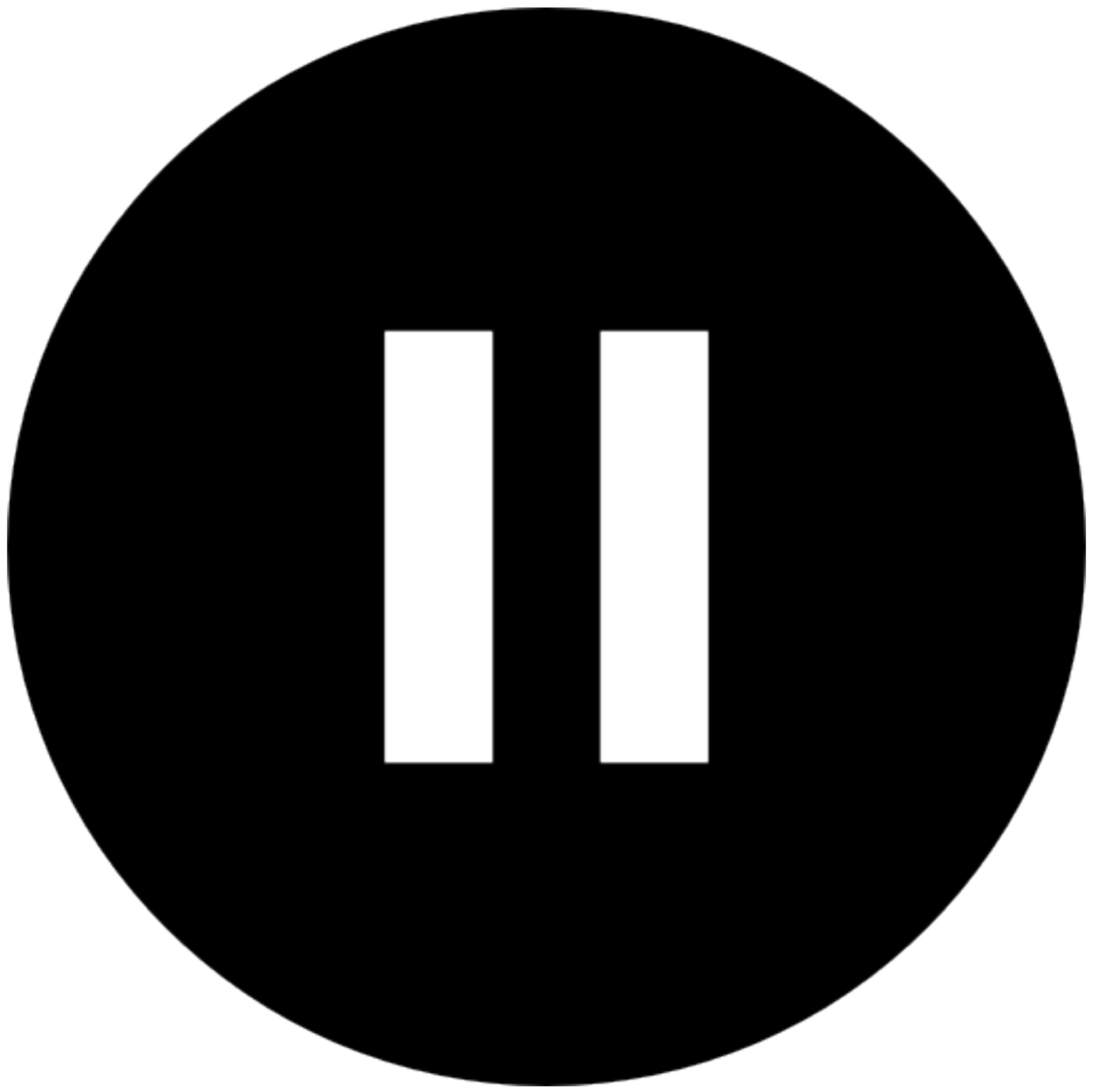
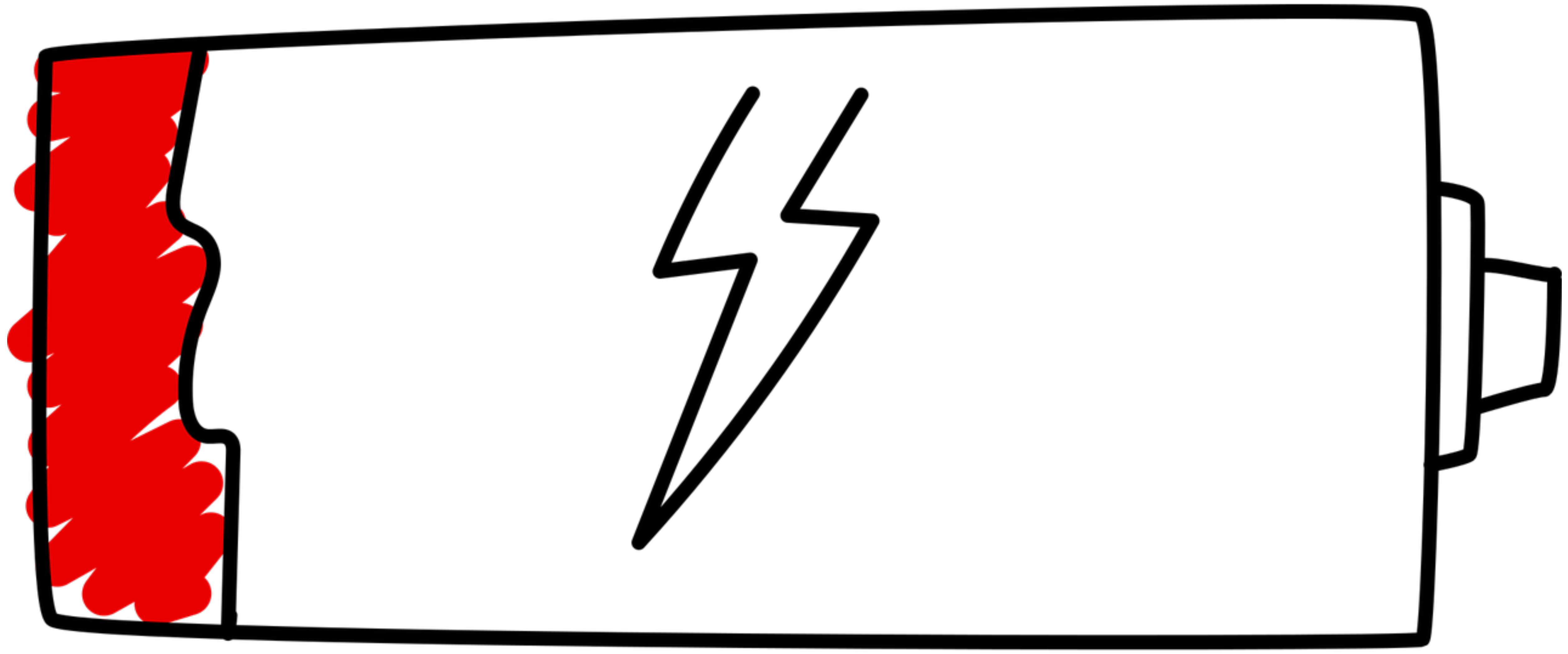
$$\hat{\sigma} = \sqrt{\frac{(n_A - 1)\hat{\sigma}_A^2 + (n_B - 1)\hat{\sigma}_B^2}{n_A + n_B - 2}\left(\frac{1}{n_A} + \frac{1}{n_B}\right)}$$

$$t = \left((\bar{x}_A - \bar{x}_B) - \delta\right) \cdot \frac{1}{\hat{\sigma}}$$

Sampling distribution:
$t \sim \text{Student-t}(\nu = n_A + n_B - 2)$

model of the data-generating process buried inside a two-sample t-test

read more here

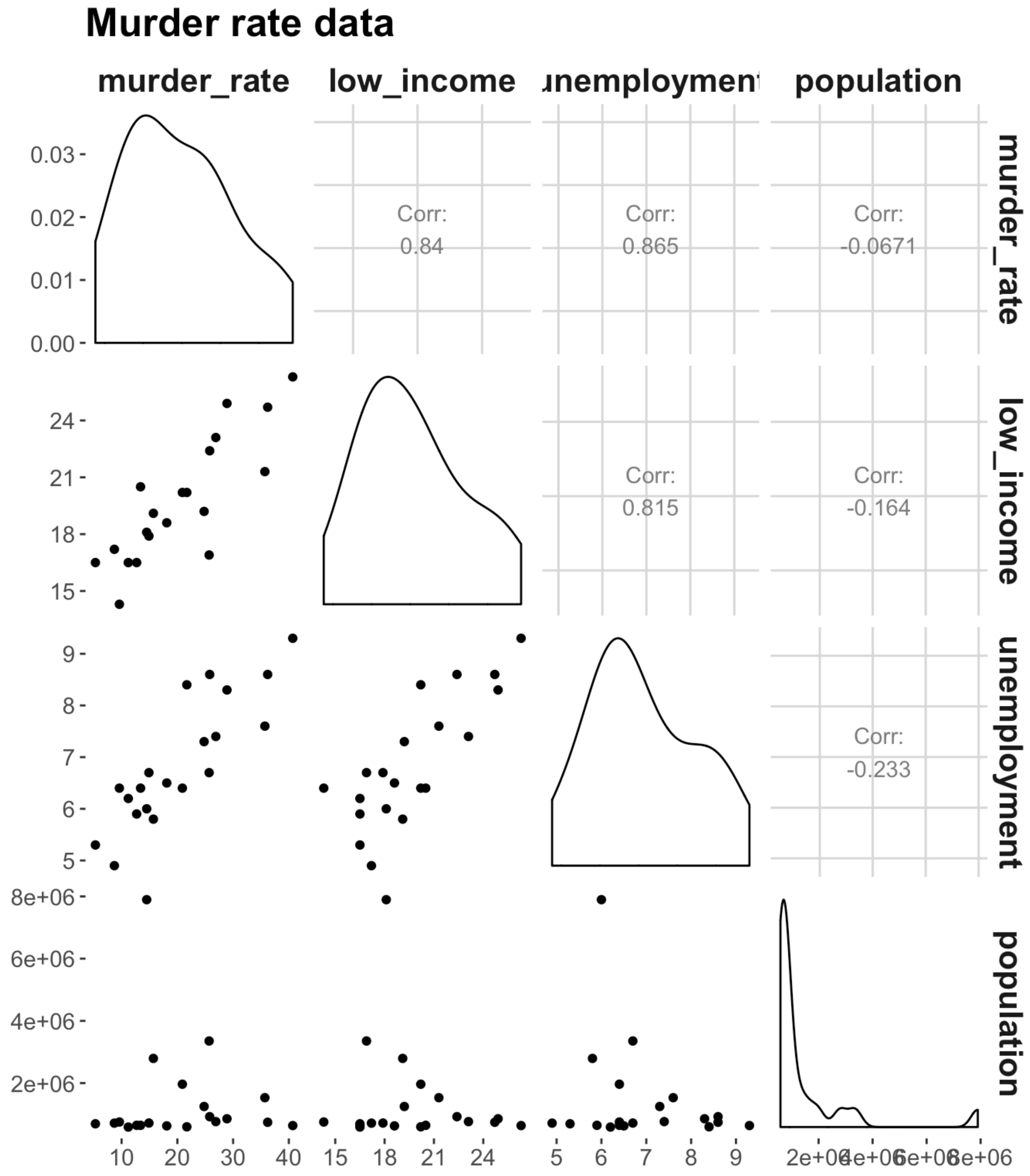
Simple linear regression

likelihood & Bayesian posterior

Murder data

annual murder rate, average income, unemployment rates and population

```
## # A tibble: 20 x 4
##    murder_rate low_income unemployment population
##          <dbl>      <dbl>        <dbl>      <dbl>
##  1        11.2       16.5          6.2     587000
##  2        13.4       20.5          6.4     643000
##  3        40.7       26.3          9.3     635000
##  4         5.3       16.5          5.3     692000
##  5        24.8       19.2          7.3    1248000
##  6        12.7       16.5          5.9     643000
##  7        20.9       20.2          6.4    1964000
##  8        35.7       21.3          7.6    1531000
##  9         8.7       17.2          4.9     713000
## 10         9.6       14.3          6.4     749000
## 11        14.5       18.1          6      7895000
## 12        26.9       23.1          7.4     762000
## 13        15.7       19.1          5.8    2793000
## 14        36.2       24.7          8.6     741000
## 15        18.1       18.6          6.5     625000
## 16        28.9       24.9          8.3     854000
## 17        14.9       17.9          6.7     716000
## 18        25.8       22.4          8.6     921000
## 19        21.7       20.2          8.4     595000
## 20        25.7       16.9          6.7    3353000
```



annual murders per million inhabitants

percentage inhabitants with low income

percentage inhabitants who are unemployed

total population

Predicting murder rate

no information at all

```
## # A tibble: 20 x 4
##    murder_rate low_income unemployment population
##          <dbl>      <dbl>        <dbl>      <dbl>
##  1        11.2       16.5          6.2     587000
##  2        13.4       20.5          6.4     643000
##  3        40.7       26.3          9.3     635000
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## 14        36.2       24.7          8.6     741000
## 15        18.1       18.6          6.5     625000
## 16        28.9       24.9          8.3     854000
## 17        14.9       17.9          6.7     716000
## 18        25.8       22.4          8.6     921000
## 19        21.7       20.2          8.4     595000
## 20        25.7       16.9          6.7    3353000
```
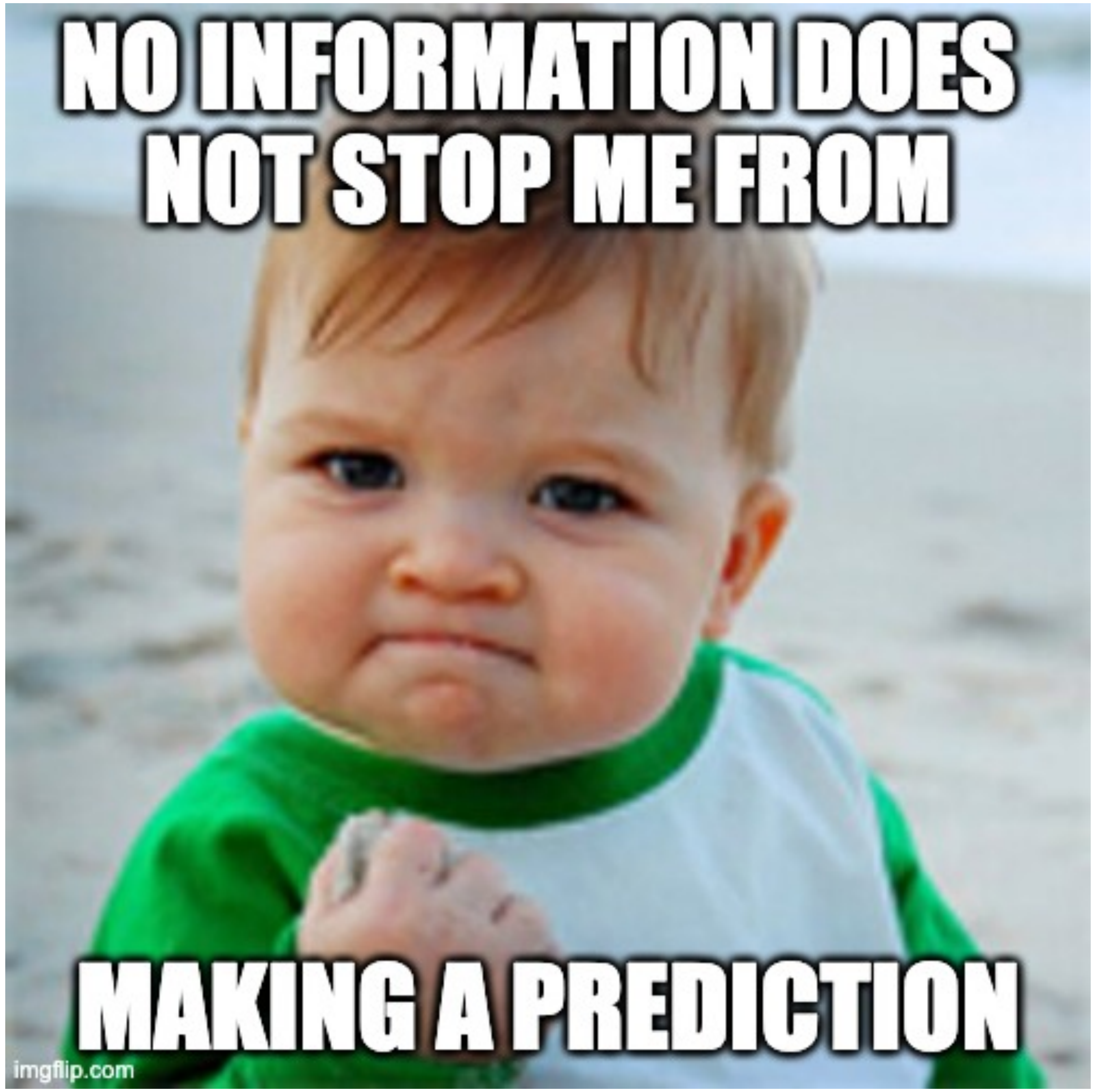
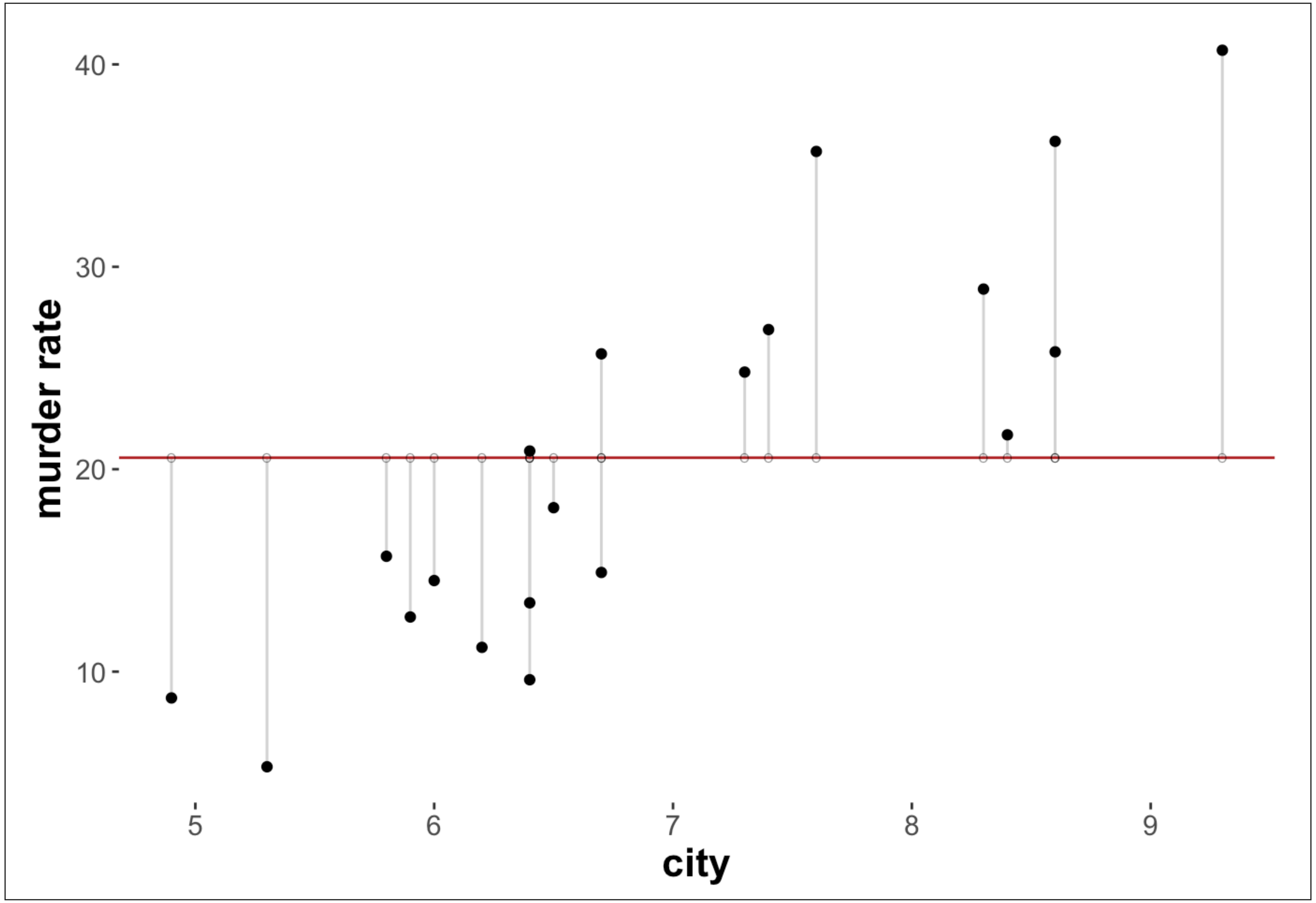



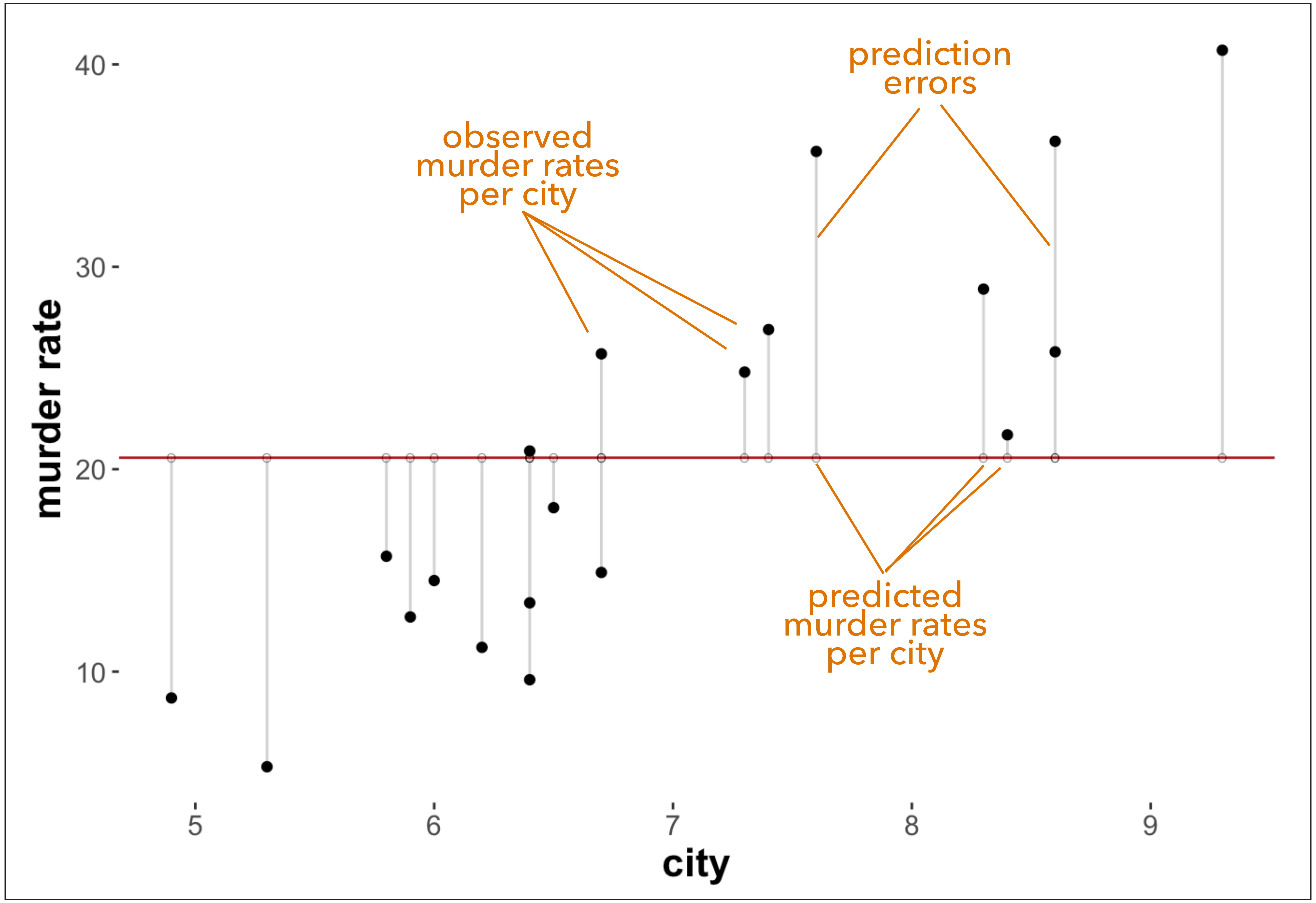
Predicting murder rate

by empirical mean



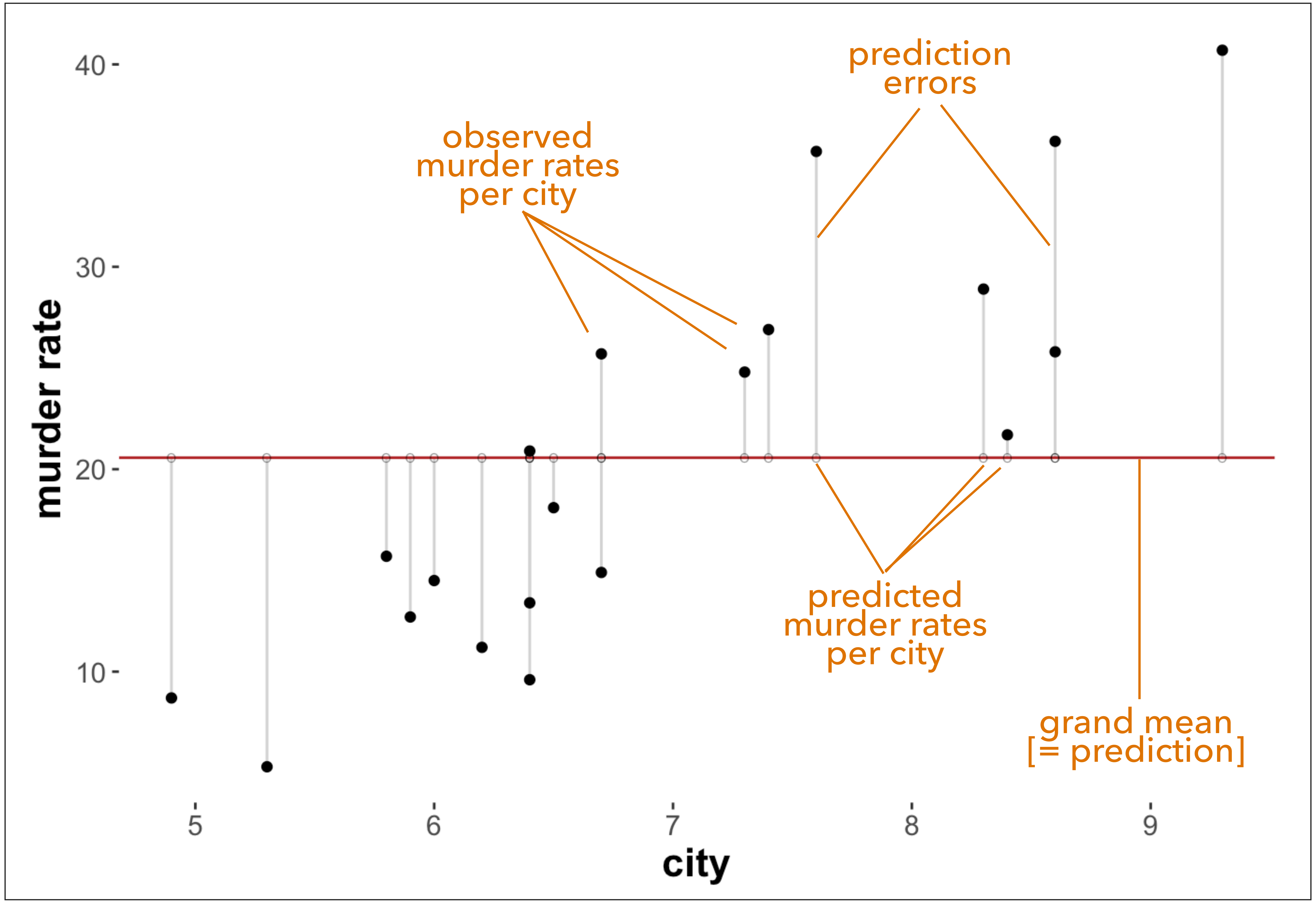
Predicting murder rate

by grand mean



Predicting murder rate

by grand mean



$$\text{TSS} = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

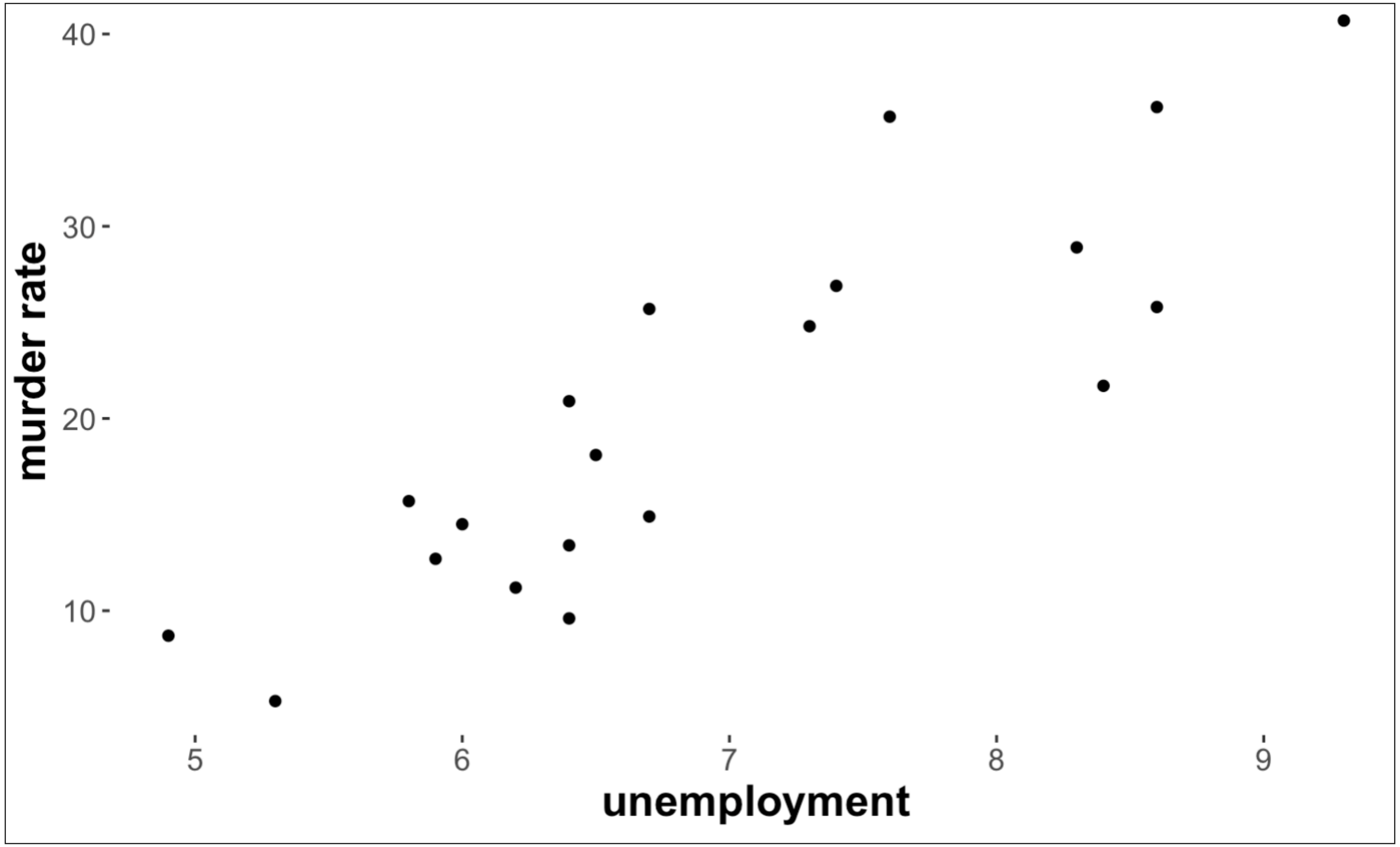
[total sum of squares]

```r
y <- murder_data %>% pull(murder_rate)
n <- length(y)
tss_simple <- sum((y - mean(y))^2)
tss_simple
```

```
## [1] 1855.202
```

Predicting murder rate based on unemployment rate

some wild linear guessing



We are to predict the murder rate y_i of a randomly drawn city i. **We know that city's unemployment rate, x_i, but nothing more.**

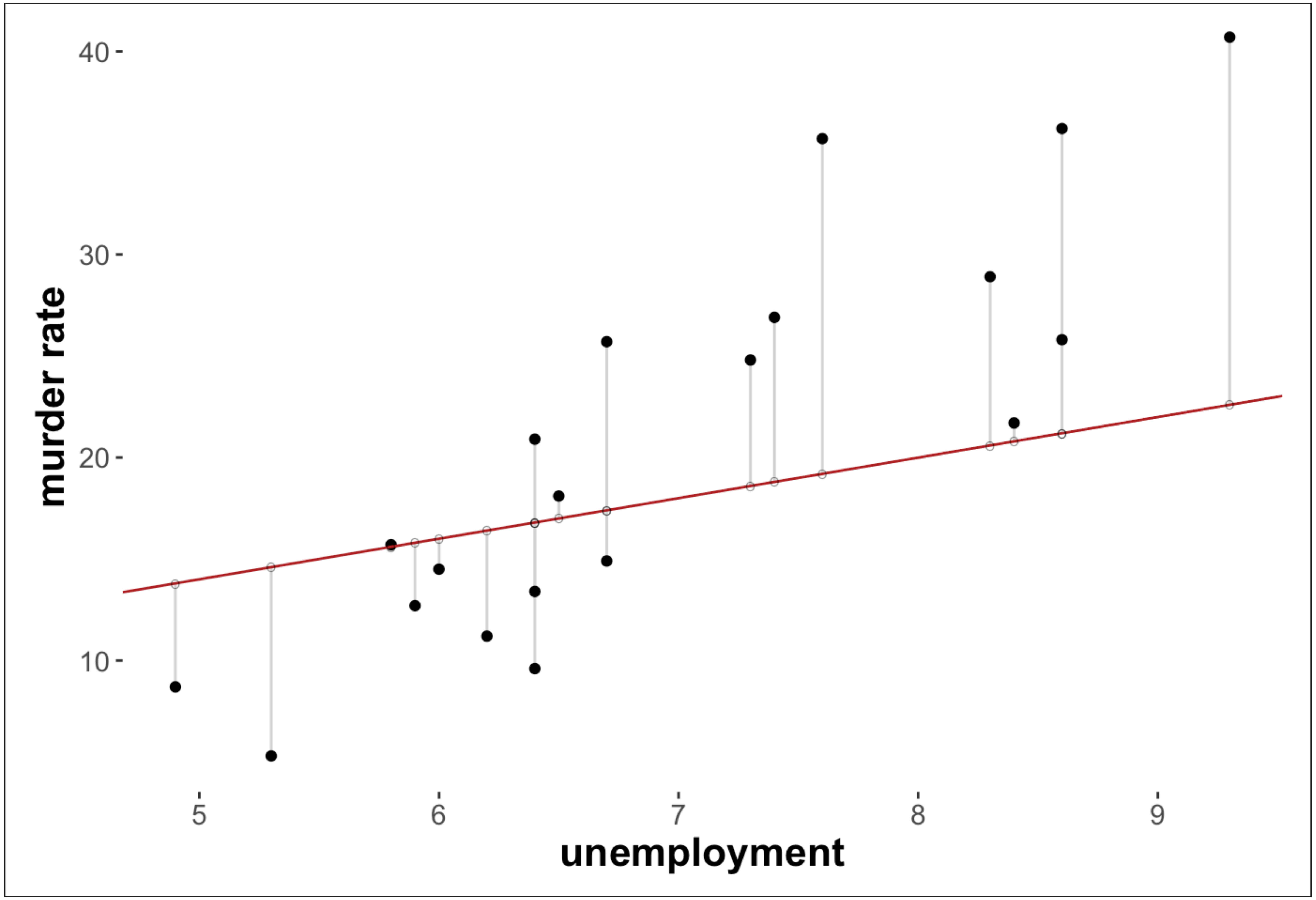
Let's just assume the following **linear relationship** to make a prediction b/c why not?!?

$$\hat{y}_i = 4 + 2x_i$$

How good is this prediction?

How good is any given prediction?

quantifying distance or likelihood



Distance-based approach

Residual Sum-of-Squares:

$$\text{RSS} = \sum_{i=1}^{n} \left(\hat{y}_i - y_i\right)^2$$

- no predictions about spread around linear predictor

Likelihood-based approach:

Normal likelihood:

$$\text{LH} = \prod_{i=1}^{n} \mathcal{N}\left(y_i \mid \mu = \hat{y}_i, \sigma\right)$$

- fully predictive

Likelihood-based simple linear regression

- likelihood:

  $$y_i \sim \text{Normal}(\mu_i, \sigma)$$

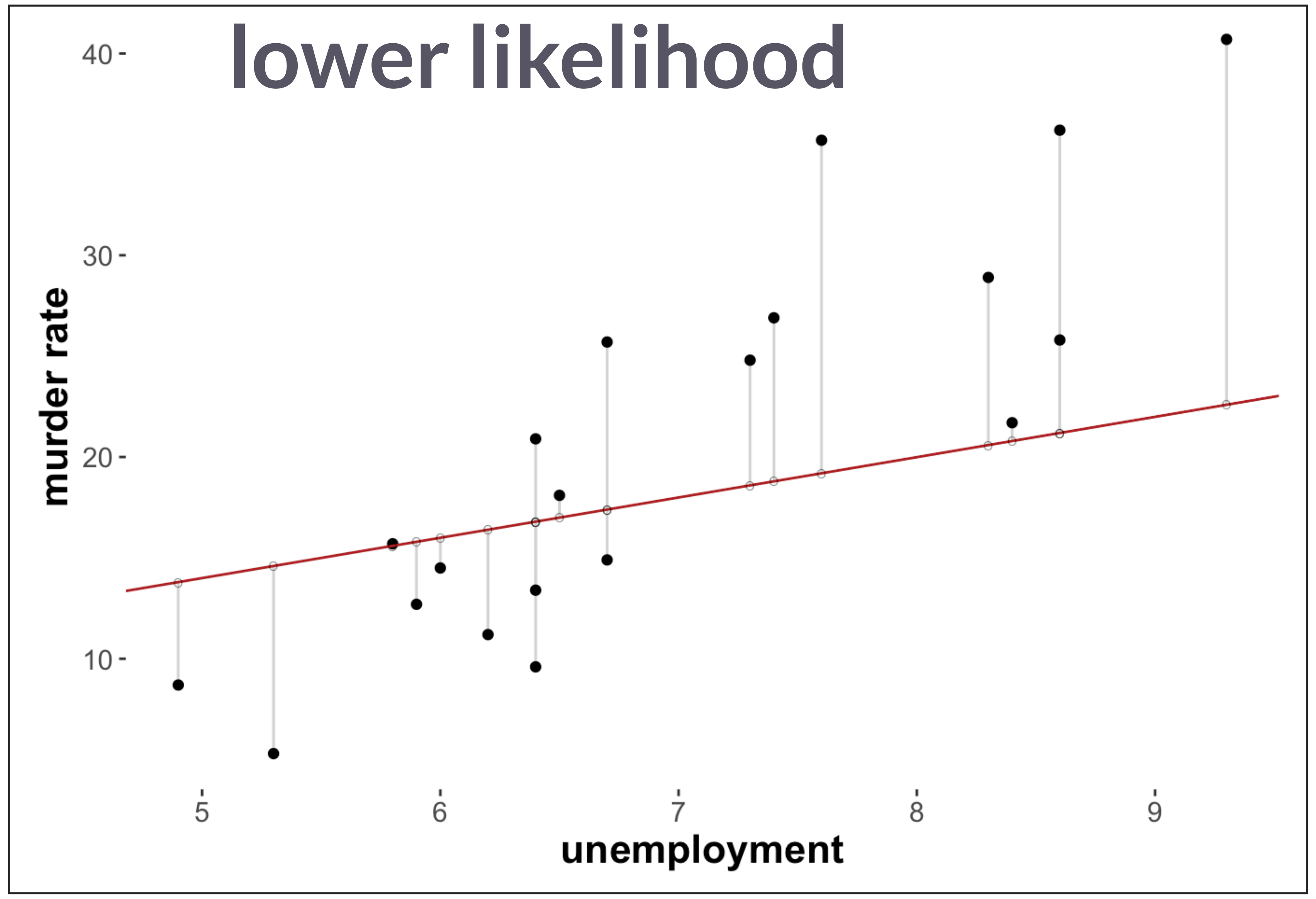
  $$\mu_i = \beta_0 + x_1 \cdot \beta_1$$

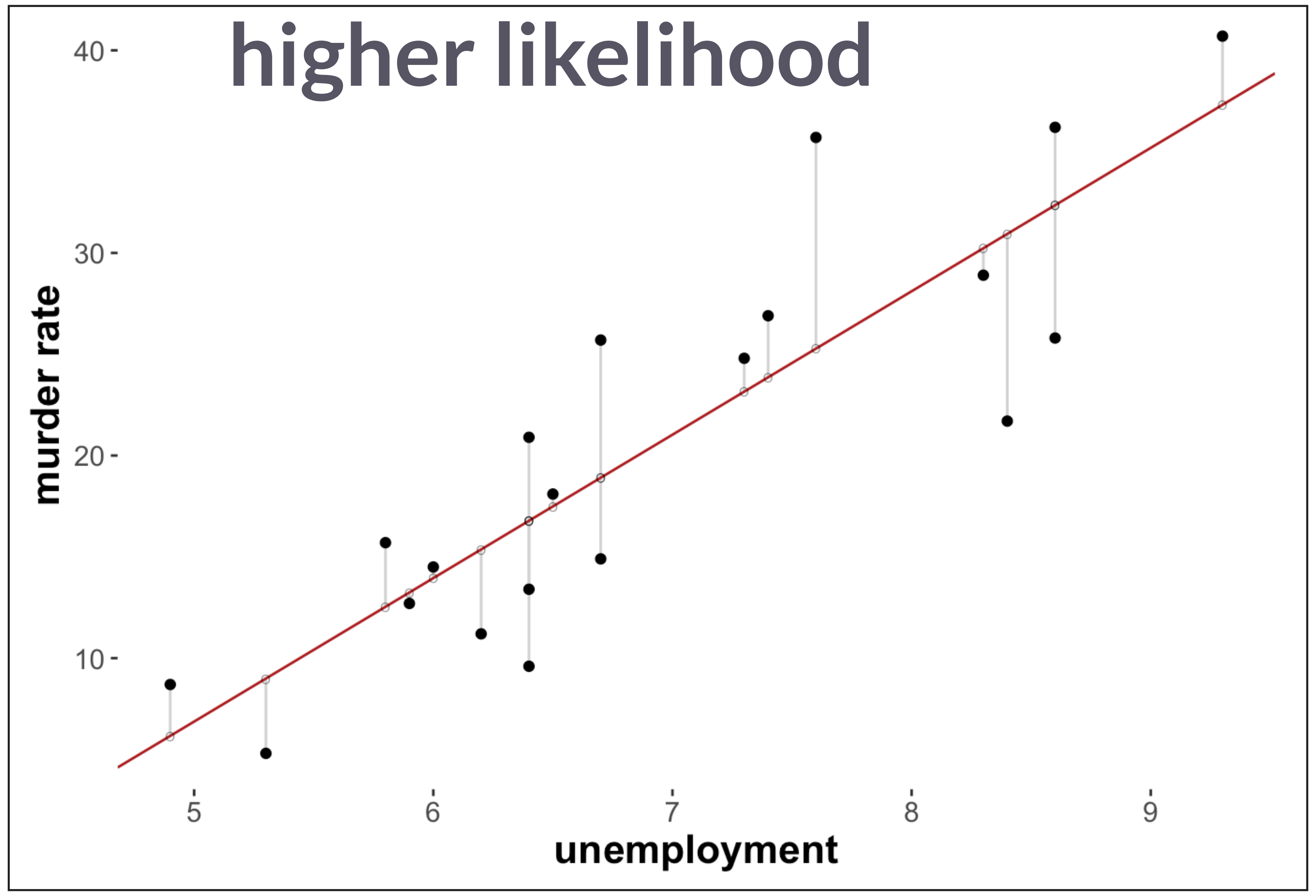
- differential likelihood:
  - parameter triples $\langle \beta_0, \beta_1, \sigma \rangle$ can be better or worse
  - higher vs. lower likelihood $P(D \mid \beta_0, \beta_1, \sigma)$
- maximum-likelihood solution:

  $$\arg\max_{\beta_0, \beta_1, \sigma} P(D \mid \beta_0, \beta_1, \sigma)$$

  - standard (frequentist) solution
  - MLE corresponds to MAP for “flat” priors
- Bayesian approach: full posterior distribution

  $$P(\beta_0, \beta_1, \sigma \mid D) \propto P(D \mid \beta_0, \beta_1, \sigma)\ P(\beta_0, \beta_1, \sigma)$$





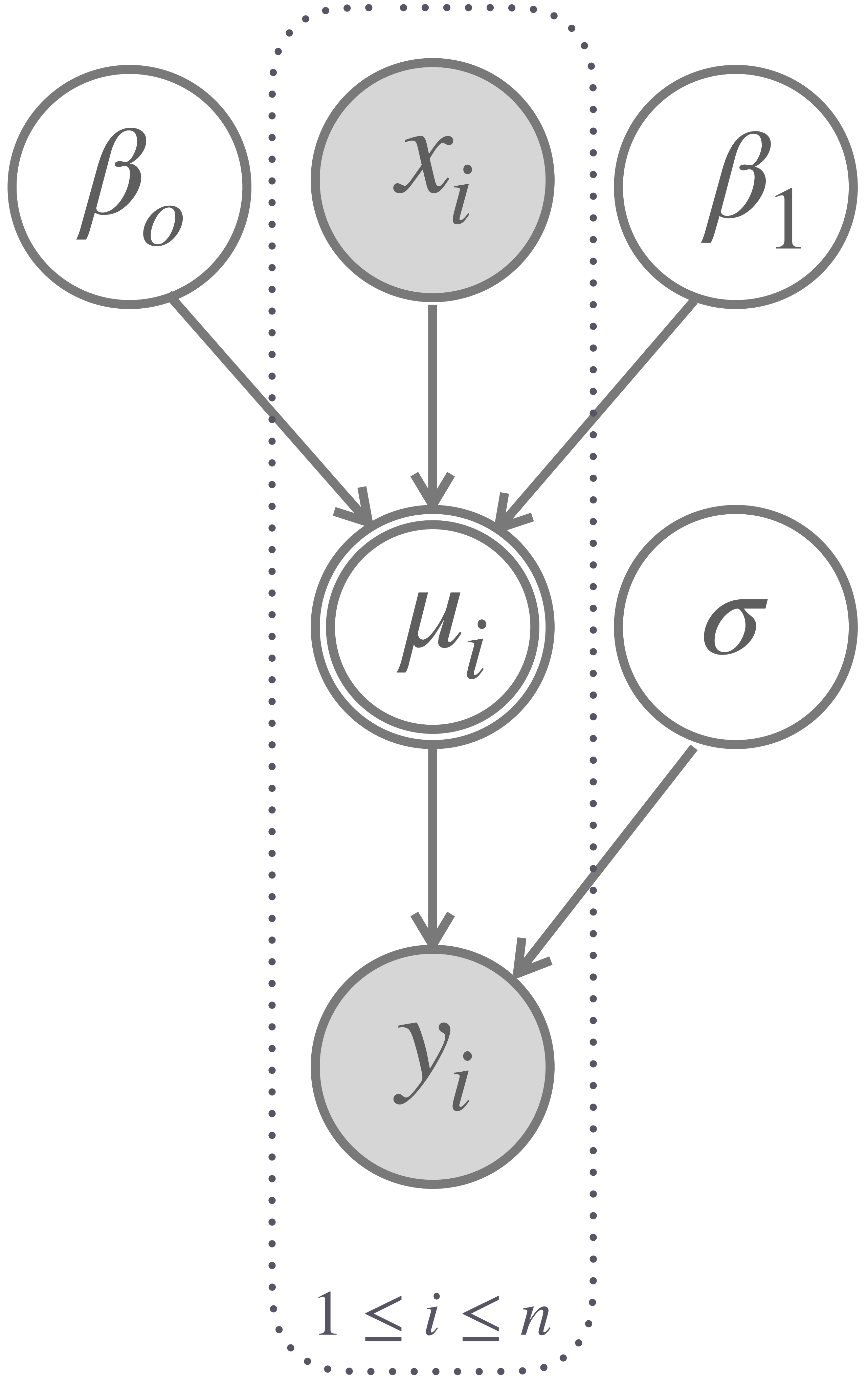
Simple linear regression model

for a single predictor variable

- data: n pairs of numbers $D = \{\langle x_1, y_1 \rangle, \ldots \langle x_n, y_n \rangle\}$
  - x_i is the i-th observation of the **independent / predictor variable**
  - y_i is the i-th observation of the **dependent / response variable**
- parameters:
  - β_0 is the **intercept** parameter
  - β_1 is the **slope** parameter
  - σ is the standard deviation of a normal distribution
- derived variable: [shown in node w/ double lines]
  - μ_i is the linear predictor for observation i
- priors (uninformed):

  $\beta_0, \beta_1 \sim \text{Uniform}(-\infty, \infty) \qquad \log(\sigma^2) \sim \text{Uniform}(-\infty, \infty)$
- likelihood:

  $y_i \sim \text{Normal}(\mu_i, \sigma) \qquad \mu_i = \beta_0 + x_1 \cdot \beta_1$



demo

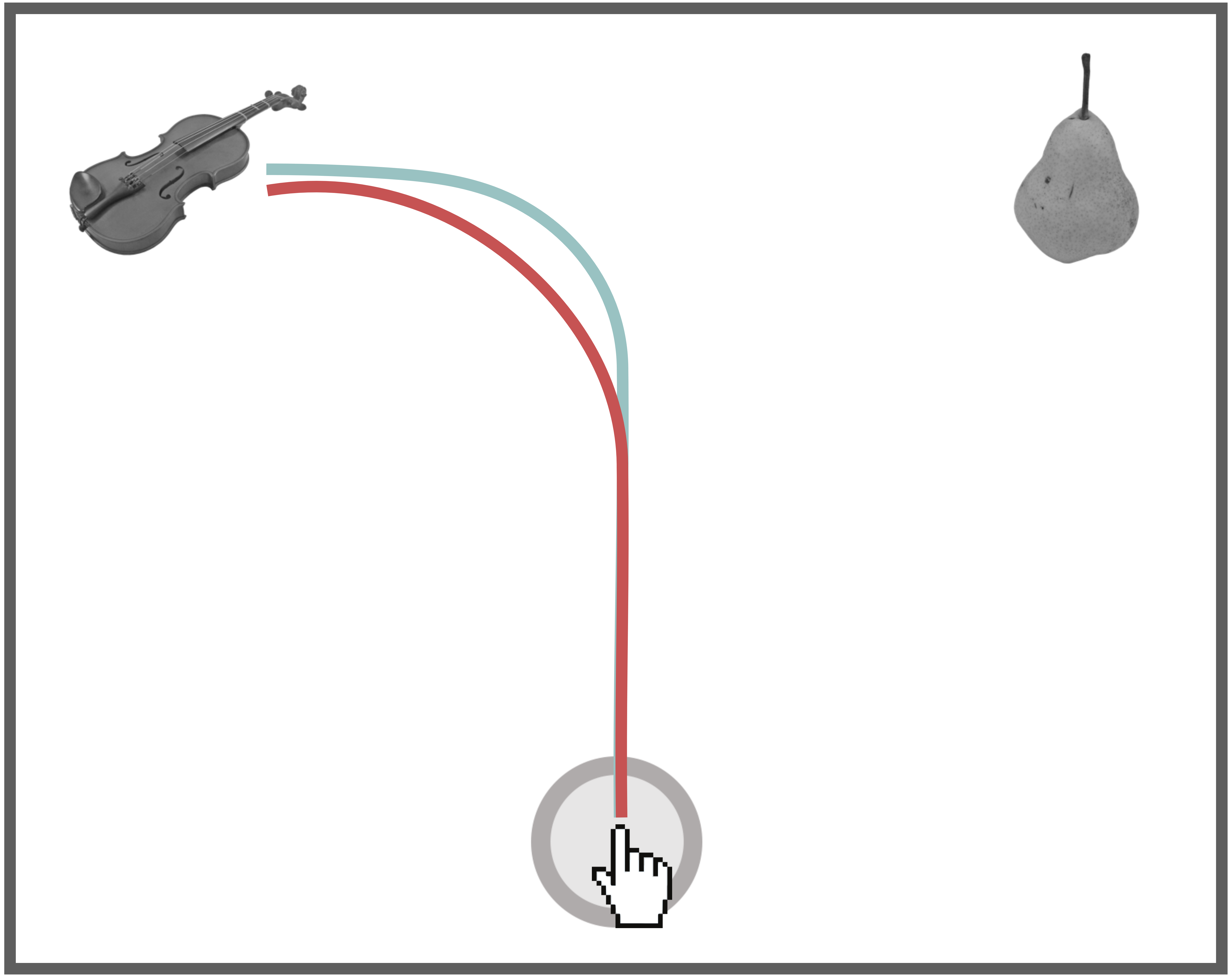
simple linear regression in WebPPL

Mouse-tracking data on typicality in category decisions

Mouse-tracking

Hand-movement during decision making

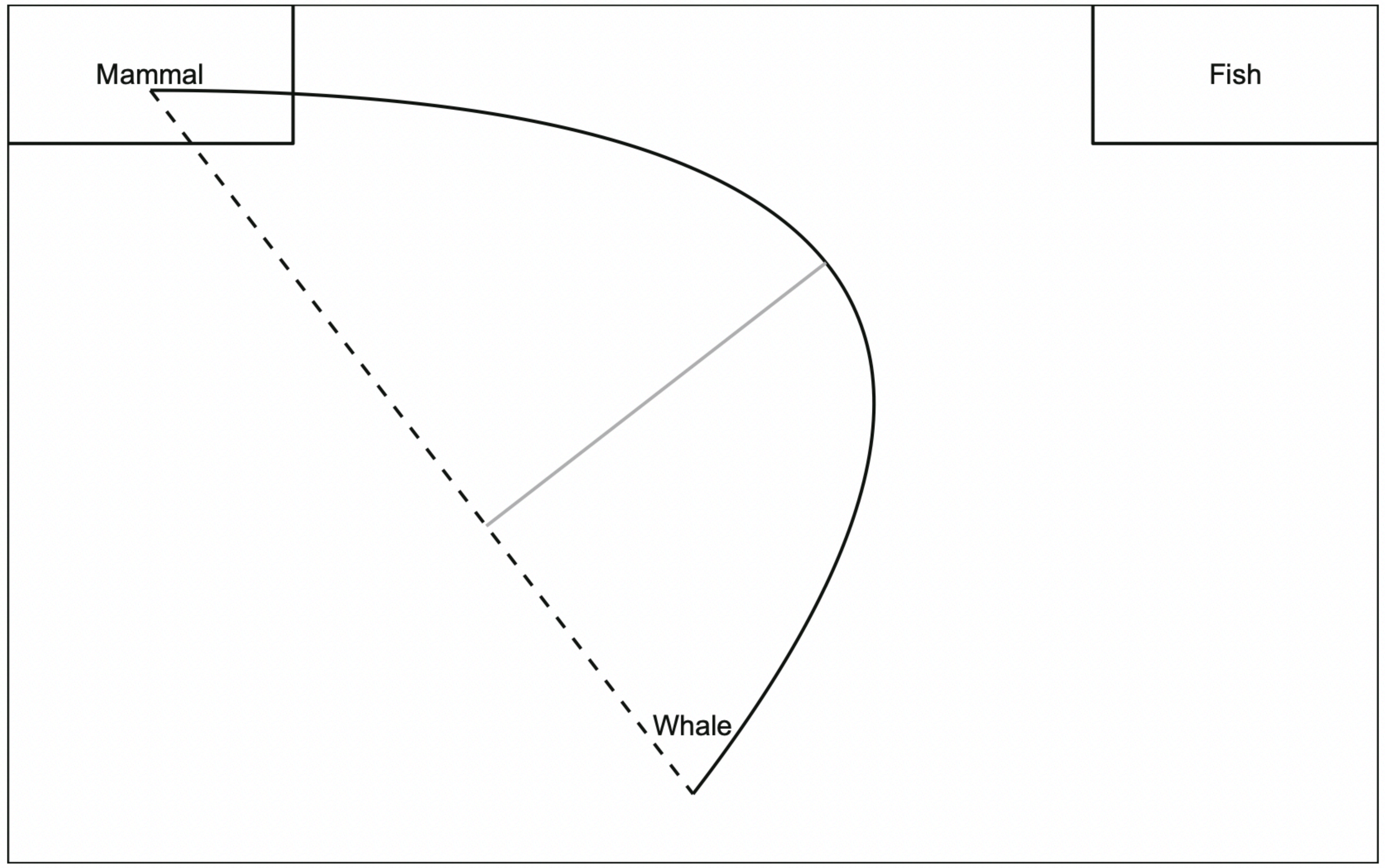
- general idea: motor-execution provides information about the ongoing decision process
  - uncertainty
  - gradual evidence accumulation
  - change-of-mind
  - time-point of decision
  - ...
- many subtle design decisions
  - click vs touch
  - move horizontally or vertically
  - ...

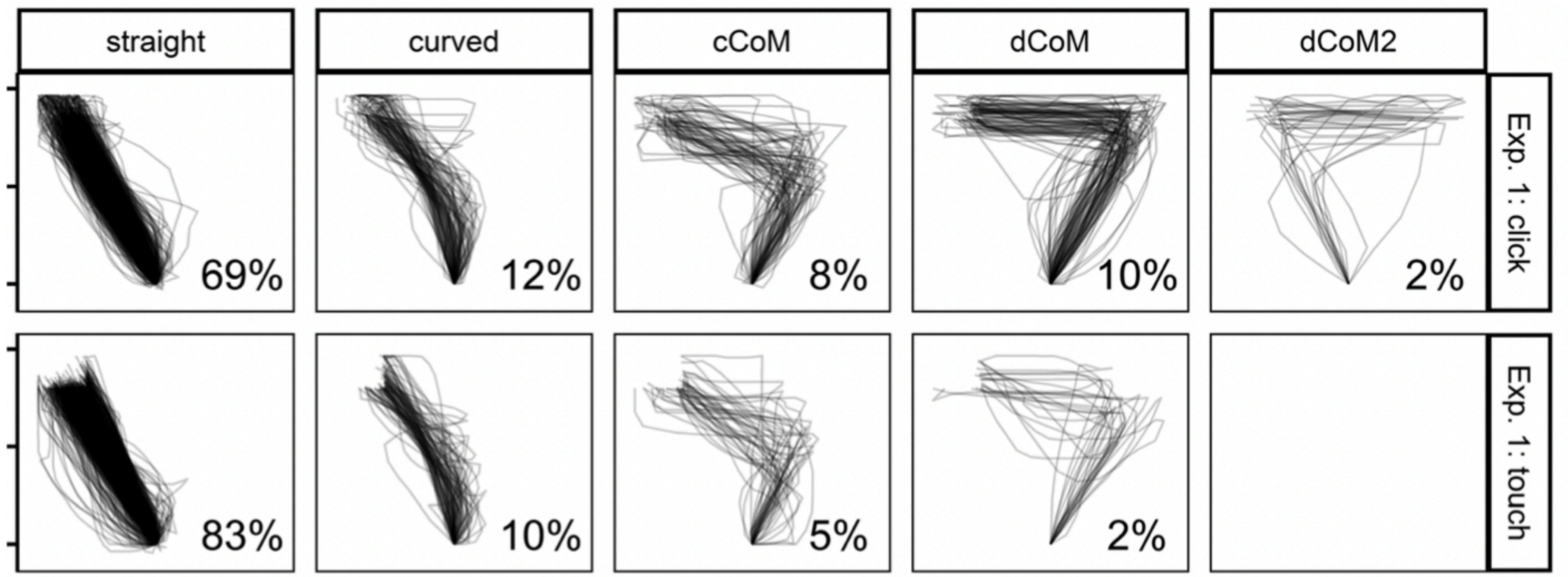


Mouse-tracking

common measures of mouse-trajectories

- raw data are lists of triples
  - (time, x-position, y-position)
- commonly used measures
  - area-under the curve (AUC)
    - area between the mouse trajectory and a straight line from start to selected option
  - maximal deviation (MAD)
    - maximum distance between trajectory and straight line from start to selected option
  - correctness
    - whether choice of option was correct or not
  - reaction time (RT)
    - how long did the movement last in total
  - type of trajectory
    - result of clustering analysis based on shape of the trajectories (usually some 3-5 categories)
  - x-flips
    - number of times the trajectory crossed the vertical middle line (at x = 0)





Running example

category recognition for typical vs atypical exemplars

- materials & procedure
  - participants read an animal name (e.g. 'dolphin')
  - they choose the true category the animal belongs to (e.g., 'fish' or 'mammal')
  - some trigger words are typical others atypical representatives of the true category
- methodological investigation:
  - two groups: **click vs touch** to select category
- **hypothesis:** typical exemplars are easier to categorize than atypical ones
  - fewer mistakes
  - smaller RTs, AUC, MAD
  - less x-flips
  - less "change-of-mind" curve types
- **research question (methods):** any differences between click & touch selection?

variables used in the data set

`trial_id` = unique id for individual trials
`MAD` = maximal deviation into competitor space
`AUC` = area under the curve
`xpos_flips` = the amount of horizontal direction changes
`RT` = reaction time in ms
`prototype_label` = different categories of prototypical movement strategies
`subject_id` = unique id for individual participants
`group` = groups differ in the response design (click vs. touch)
`condition` = category membership (Typical vs. Atypical)
`exemplar` = the concrete animal
`category_left` = the category displayed on the left
`category_right` = the category displayed on the right
`category_correct` = the category that is correct
`response` = the selected category
`correct` = whether or not the `response` matches `category_correct`

Kieslich et al.'s (2019) replication of Dale et al.'s (2003) experiment

outlook

Three pillars of BDA

1. parameter estimation / inference [which parameter values are credible given data and model?]

$$\underbrace{P(\theta \mid D)}_{\text{posterior}} \propto \underbrace{P(\theta)}_{\text{prior}} \times \underbrace{P(D \mid \theta)}_{\text{likelihood}}$$

2. predictions [which future data observations are likely given my model?]

a. prior

$$P(D_{\text{pred}}) = \int P(\theta)\ P(D_{\text{pred}} \mid \theta)\ \mathrm{d}\theta$$

b. posterior

$$P(D_{\text{pred}} \mid D_{\text{obs}}) = \int P(\theta \mid D_{\text{obs}})\ P(D_{\text{pred}} \mid \theta)\ \mathrm{d}\theta$$

3. model comparison [which model of two models is more likely to have generated the data?]

$$\underbrace{\frac{P(M_1 \mid D)}{P(M_2 \mid D)}}_{\text{posterior odds}} = \underbrace{\frac{P(D \mid M_1)}{P(D \mid M_2)}}_{\text{Bayes factor}}\ \underbrace{\frac{P(M_1)}{P(M_2)}}_{\text{prior odds}}$$

Roadmap “beyond vanilla”

common extensions of linear regression modeling