Bayesian regression modeling: Theory & practice

Part 2: Priors & predictives

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LM notation

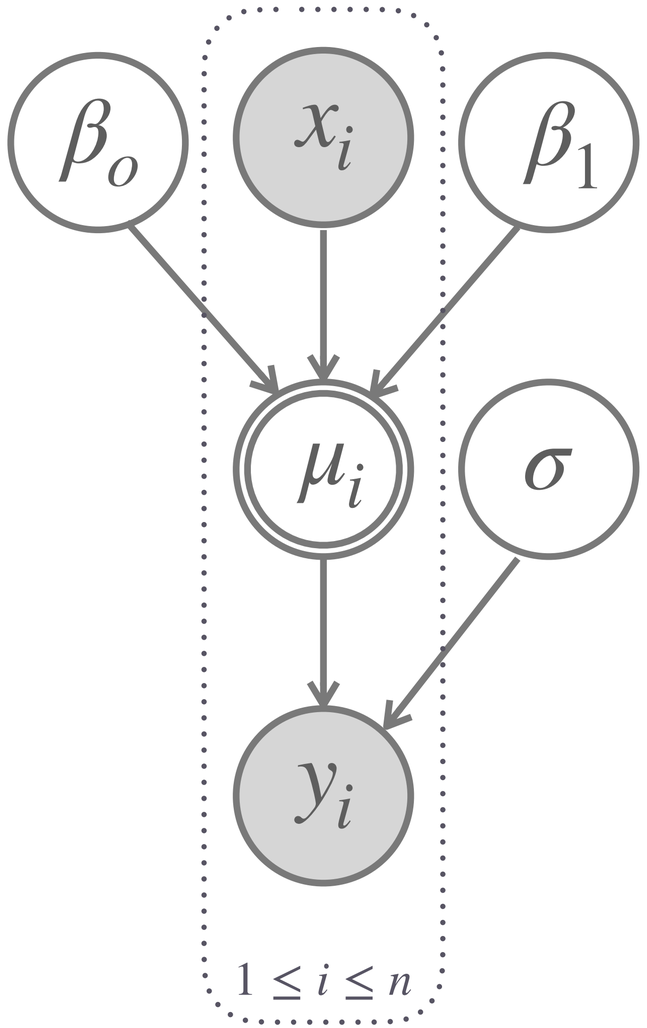
Simple linear regression model

for a single predictor variable

- data: n pairs of numbers $D = \left\{\langle x_1, y_1\rangle, \ldots \langle x_n, y_n\rangle\right\}$
  - x_i is the i-th observation of the **independent / predictor variable**
  - y_i is the i-th observation of the **dependent / response variable**
- parameters:
  - β_0 is the **intercept** parameter
  - β_1 is the **slope** parameter
  - σ is the standard deviation of a normal distribution
- derived variable: [shown in node w/ double lines]
  - μ_i is the linear predictor for observation i
- priors (uninformed):

  $\beta_0, \beta_1 \sim \text{Uniform}(-\infty, \infty)$ $\quad$ $\log(\sigma^2) \sim \text{Uniform}(-\infty, \infty)$
- likelihood:

  $y_i \sim \text{Normal}(\mu_i, \sigma)$ $\quad$ $\mu_i = \beta_0 + x_i \cdot \beta_1$



Linear regression
representation & notation

Data

A	B	C	D
42	4	163	A
19	7	128	B
38	2	99	A
⋮	⋮	⋮	⋮

R model formula

```
A ~ B + C + D
```

Mathematical notation / internal representation

$$\mathbf{y} = \begin{pmatrix} 42 \\ 19 \\ 38 \\ \vdots \end{pmatrix}$$

response variable

$$X = \begin{pmatrix} 1 & 4 & 163 & 0 \\ 1 & 7 & 128 & 1 \\ 1 & 2 & 99 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

design matrix (a.k.a., model matrix)

$$\beta = \begin{pmatrix} 4.2 \\ 5.2 \\ -3.14 \\ \vdots \end{pmatrix}$$

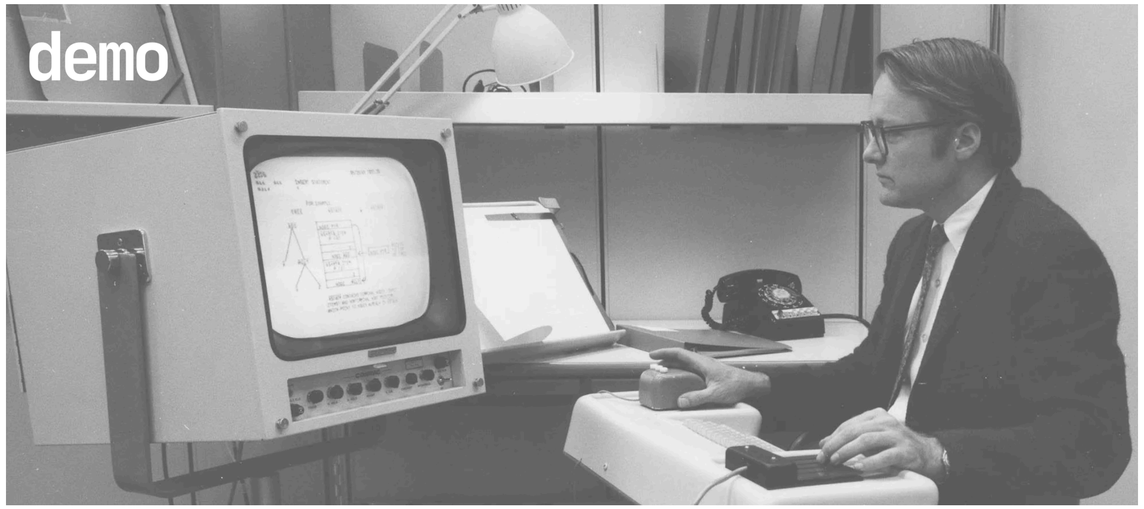
coefficients

$$\eta = X\,\beta$$

linear predictor

$$\mathbf{y} \sim \mathcal{N}(\eta, \sigma I)$$

likelihood function



mini demo on `standata()` (demo 03)

Linear regression
notation alternatives

Matrix form

$$\eta = X\,\beta$$

linear predictor

$$\mathbf{y} \sim \mathcal{N}(\eta, \sigma I)$$

likelihood function

item-sum form

$$\eta_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \dots$$

linear predictor

$$y_i \sim \mathcal{N}(\eta_i, \sigma)$$

likelihood function

Likelihood, prior & posterior

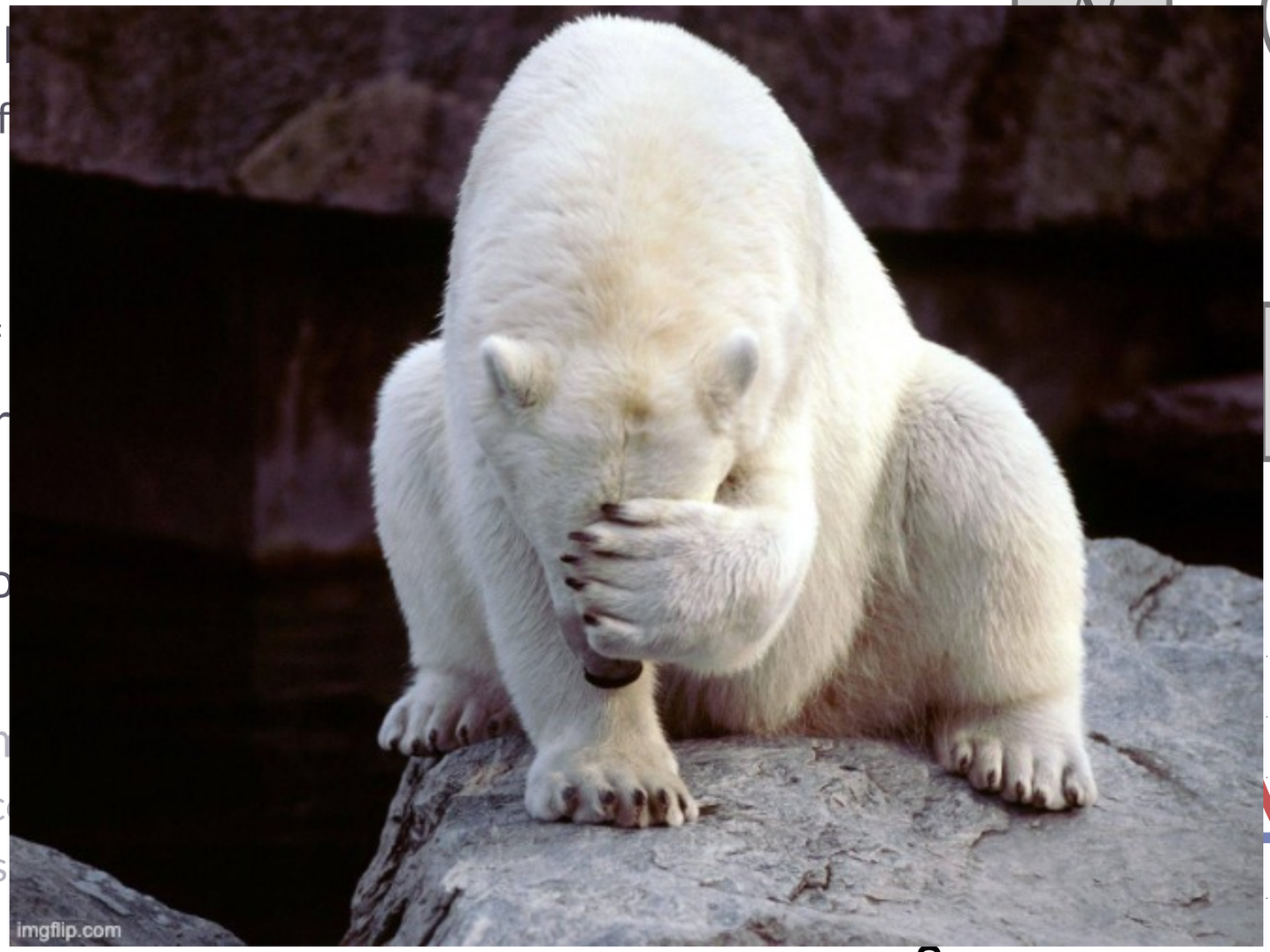
Binomial model

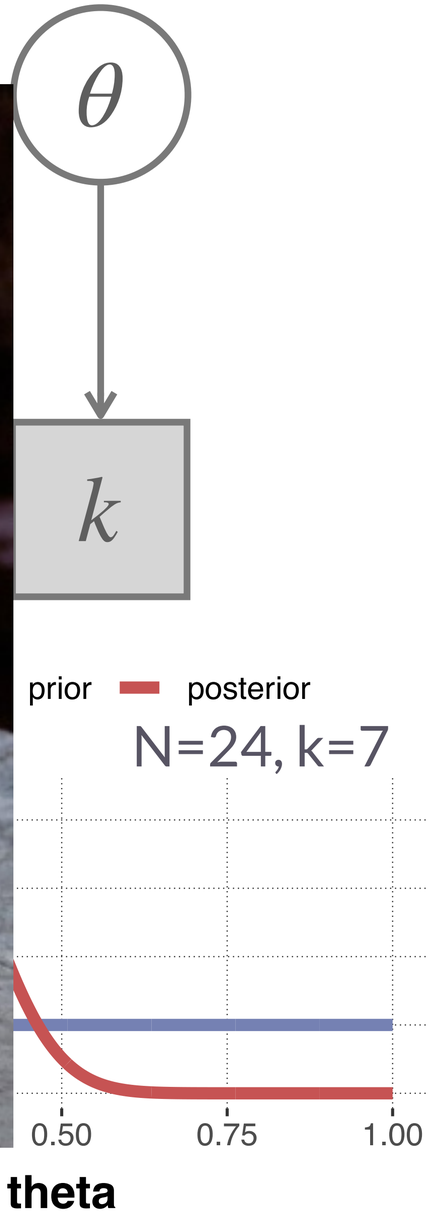
the ‘coin-flip’ model

- data: pair of numb
  - N is the number of
  - k is the number of
- variable:
  - θ is the number of
- uninformed prior

  $\theta \sim \text{Beta}(1,1)$
- likelihood functio

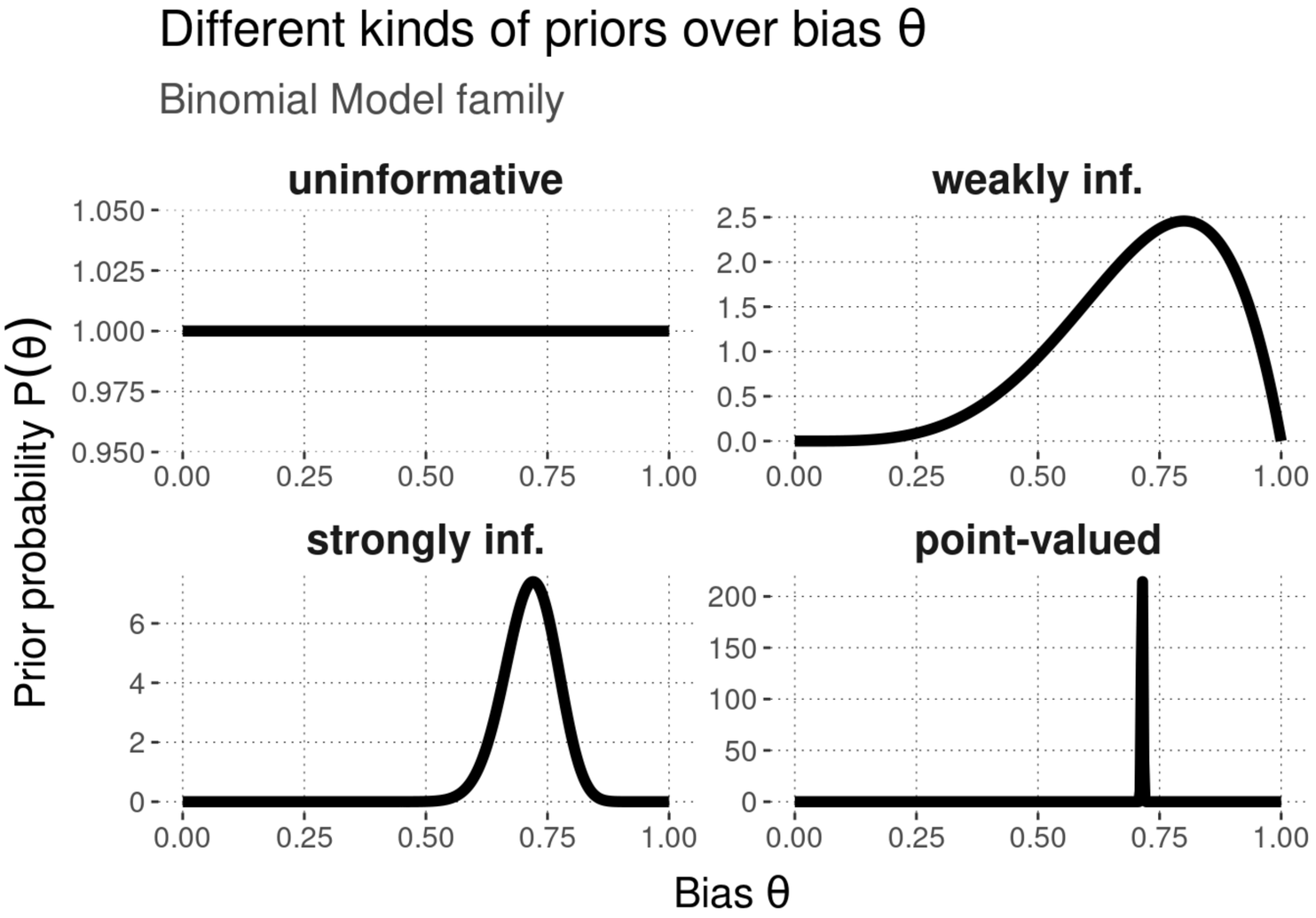
  $k \sim \text{Binomial}(\theta, N)$
- conventions for m
  - circles / squares: c
  - white / gray nodes





Kinds of priors

for a Binomial ('coin flip') model



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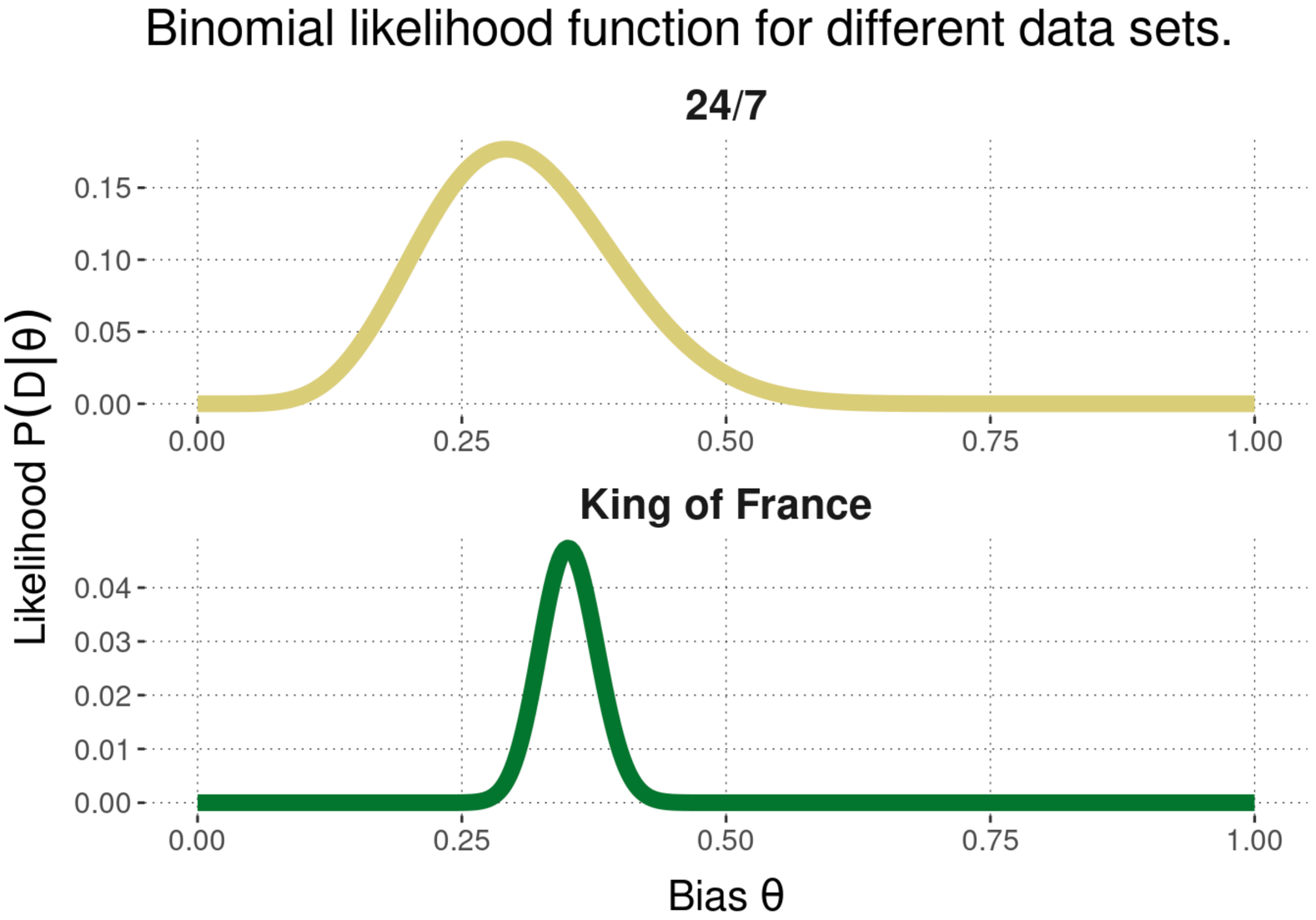
Binomial likelihoods

two data sets



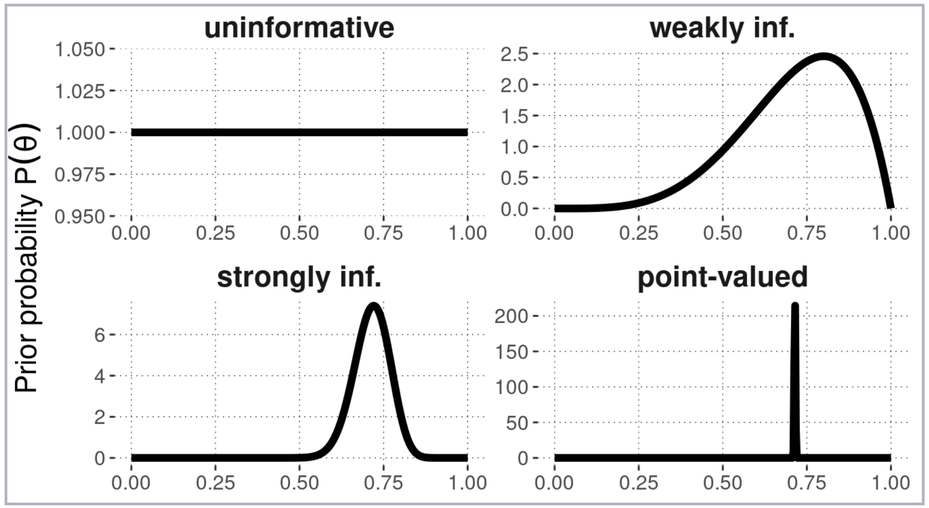
N=24, k=7

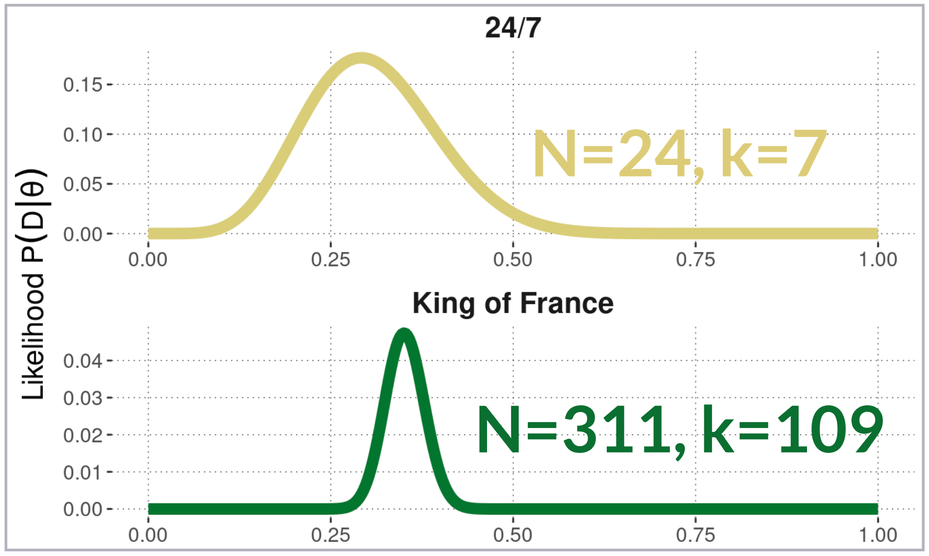
N=311, k=109

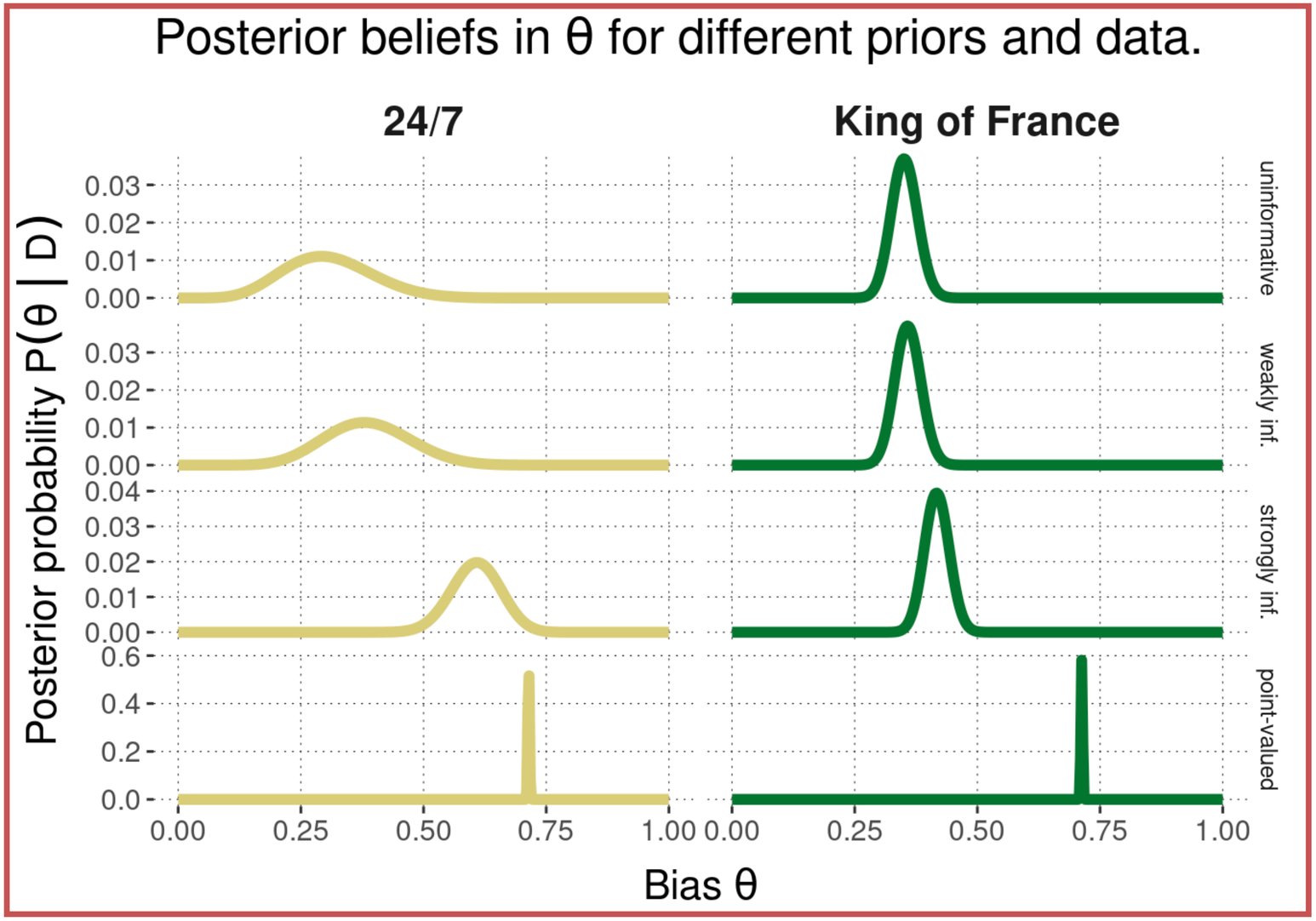
read more here

Posterior distributions

for different priors and likelihoods







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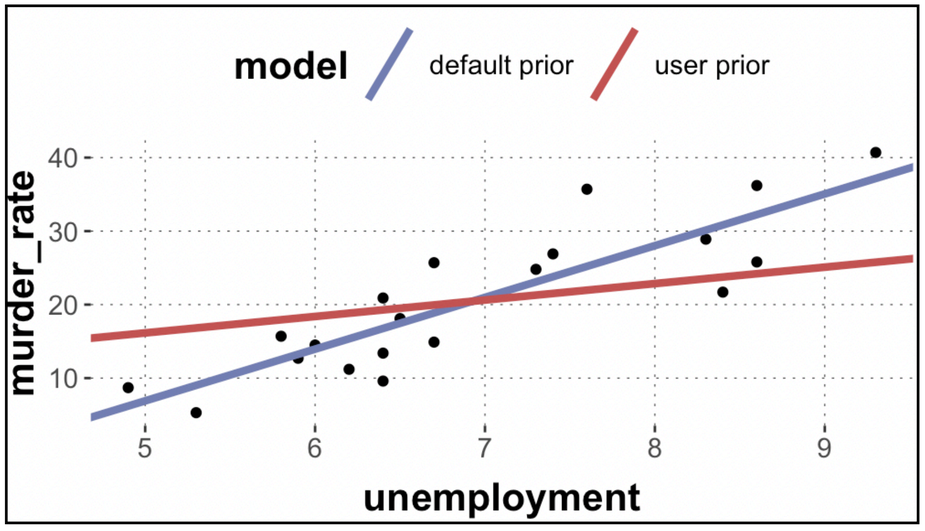
Setting priors in `brms`

Setting priors in `brms`

```r
fit_brms_murder <- brm(
  formula = murder_rate ~ unemployment,
  data = aida::data_murder
)

fit_brms_murder_wPrior <- brm(
  formula = murder_rate ~ unemployment,
  data = aida::data_murder,
  prior = prior(normal(0,1), class = "b")
)
```




```
> brms::prior_summary(fit_brms_murder)
                   prior     class         coef group resp dpar nlpar lb ub       source
                  (flat)         b                                                default
                  (flat)         b unemployment                                (vectorized)
 student_t(3, 19.5, 9.7) Intercept                                                default
    student_t(3, 0, 9.7)     sigma                                          0     default
> brms::prior_summary(fit_brms_murder_wPrior)
                   prior     class         coef group resp dpar nlpar lb ub       source
            normal(0, 1)         b                                                  user
            normal(0, 1)         b unemployment                                (vectorized)
 student_t(3, 19.5, 9.7) Intercept                                                default
    student_t(3, 0, 9.7)     sigma                                          0     default
```

```
> tidybayes::summarise_draws(fit_brms_murder)
# A tibble: 5 × 10
  variable          mean median     sd    mad     q5     q95  rhat ess_bulk ess_tail
  <chr>            <dbl>  <dbl>  <dbl>  <dbl>  <dbl>   <dbl> <dbl>    <dbl>    <dbl>
1 b_Intercept    -28.3   -28.3   7.45   6.99   -40.3  -16.0    1.00    2945.    2304.
2 b_unemployment   7.04    7.04  1.06   1.01     5.31   8.76   1.00    2929.    2337.
3 sigma            5.44    5.28  0.994  0.897    4.12   7.26   1.00    2667.    2319.
4 lprior          -6.07   -6.06  0.0816 0.0655  -6.23  -5.98   1.00    2369.    2121.
5 lp__           -66.0   -65.6   1.36   1.10   -68.7  -64.5    1.00    1516.    2070.
> tidybayes::summarise_draws(fit_brms_murder_wPrior)
# A tibble: 5 × 10
  variable          mean median     sd    mad      q5     q95  rhat ess_bulk ess_tail
  <chr>            <dbl>  <dbl>  <dbl>  <dbl>   <dbl>   <dbl> <dbl>    <dbl>    <dbl>
1 b_Intercept      4.96    4.84   7.19   7.28  -6.77    17.1    1.00    2299.    2440.
2 b_unemployment   2.24    2.24   1.01   1.02   0.584    3.89   1.00    2313.    2499.
3 sigma            8.02    7.81   1.65   1.50   5.67    11.1    1.00    2268.    2218.
4 lprior         -10.2    -9.72   2.27   2.06 -14.7     -7.56   1.00    2398.    2541.
5 lp__           -77.8   -77.5    1.28   1.05 -80.2    -76.4    1.00    1881.    2436.
```

read more here

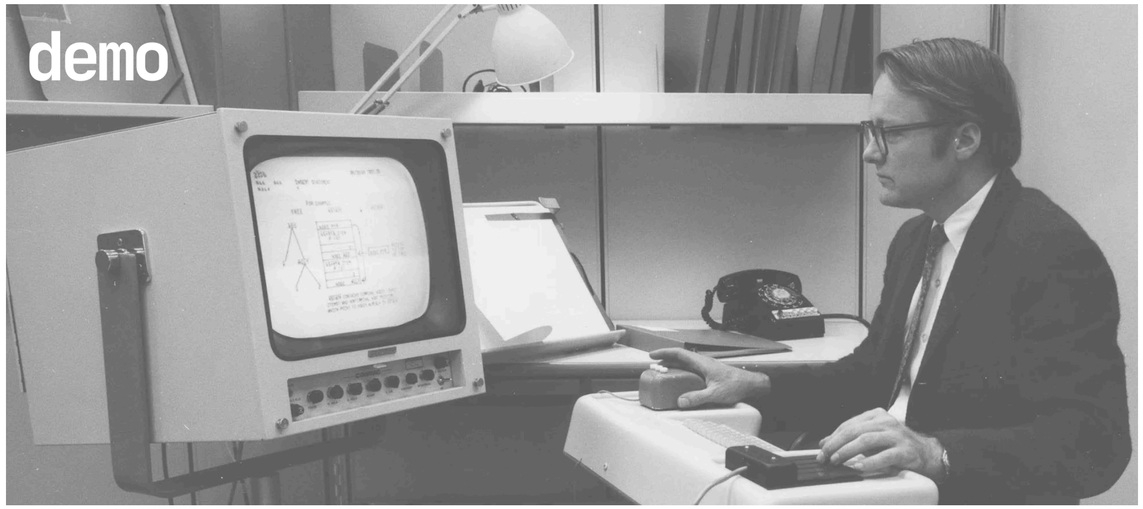
Stan code generated by `brms`

- brms generates Stan code implicitly
- show it using: `brms::stancode(fitted_model)`
- prior information increases the log-score
- priors specified in `brms` must use Stan-like syntax

```
transformed parameters {
  real lprior = 0;  // prior contributions to the log posterior
  lprior += normal_lpdf(b | 0, 1);
  lprior += student_t_lpdf(Intercept | 3, 19.5, 9.7);
  lprior += student_t_lpdf(sigma | 3, 0, 9.7)
    - 1 * student_t_lccdf(0 | 3, 0, 9.7);
}
```

```
// generated with brms 2.18.0
functions {
}
data {
  int<lower=1> N;  // total number of observations
  vector[N] Y;  // response variable
  int<lower=1> K;  // number of population-level effects
  matrix[N, K] X;  // population-level design matrix
  int prior_only;  // should the likelihood be ignored?
}
transformed data {
  int Kc = K - 1;
  matrix[N, Kc] Xc;  // centered version of X without an intercept
  vector[Kc] means_X;  // column means of X before centering
  for (i in 2:K) {
    means_X[i - 1] = mean(X[, i]);
    Xc[, i - 1] = X[, i] - means_X[i - 1];
  }
}
parameters {
  vector[Kc] b;  // population-level effects
  real Intercept;  // temporary intercept for centered predictors
  real<lower=0> sigma;  // dispersion parameter
}
transformed parameters {
  real lprior = 0;  // prior contributions to the log posterior
  lprior += normal_lpdf(b | 0, 1);
  lprior += student_t_lpdf(Intercept | 3, 19.5, 9.7);
  lprior += student_t_lpdf(sigma | 3, 0, 9.7)
    - 1 * student_t_lccdf(0 | 3, 0, 9.7);
}
model {
  // likelihood including constants
  if (!prior_only) {
    target += normal_id_glm_lpdf(Y | Xc, Intercept, b, sigma);
  }
  // priors including constants
  target += lprior;
}
generated quantities {
  // actual population-level intercept
  real b_Intercept = Intercept - dot_product(means_X, b);
}
```

read more here



mini demo on setting priors (demo 04)

Prior & posterior predictions

Three pillars of BDA

1. parameter estimation / inference [which parameter values are credible given data and model?]

$$\underbrace{P(\theta \mid D)}_{\text{posterior}} \propto \underbrace{P(\theta)}_{\text{prior}} \times \underbrace{P(D \mid \theta)}_{\text{likelihood}}$$

---

2. predictions [which future data observations are likely given my model?]

a. prior

$$P(D_{\text{pred}}) = \int P(\theta)\ P(D_{\text{pred}} \mid \theta)\ \mathrm{d}\theta$$

b. posterior

$$P(D_{\text{pred}} \mid D_{\text{obs}}) = \int P(\theta \mid D_{\text{obs}})\ P(D_{\text{pred}} \mid \theta)\ \mathrm{d}\theta$$

---

3. model comparison [which model of two models is more likely to have generated the data?]

$$\underbrace{\frac{P(M_1 \mid D)}{P(M_2 \mid D)}}_{\text{posterior odds}} = \underbrace{\frac{P(D \mid M_1)}{P(D \mid M_2)}}_{\text{Bayes factor}} \underbrace{\frac{P(M_1)}{P(M_2)}}_{\text{prior odds}}$$

Predictions of a (generalized) LM

1. **data prediction**

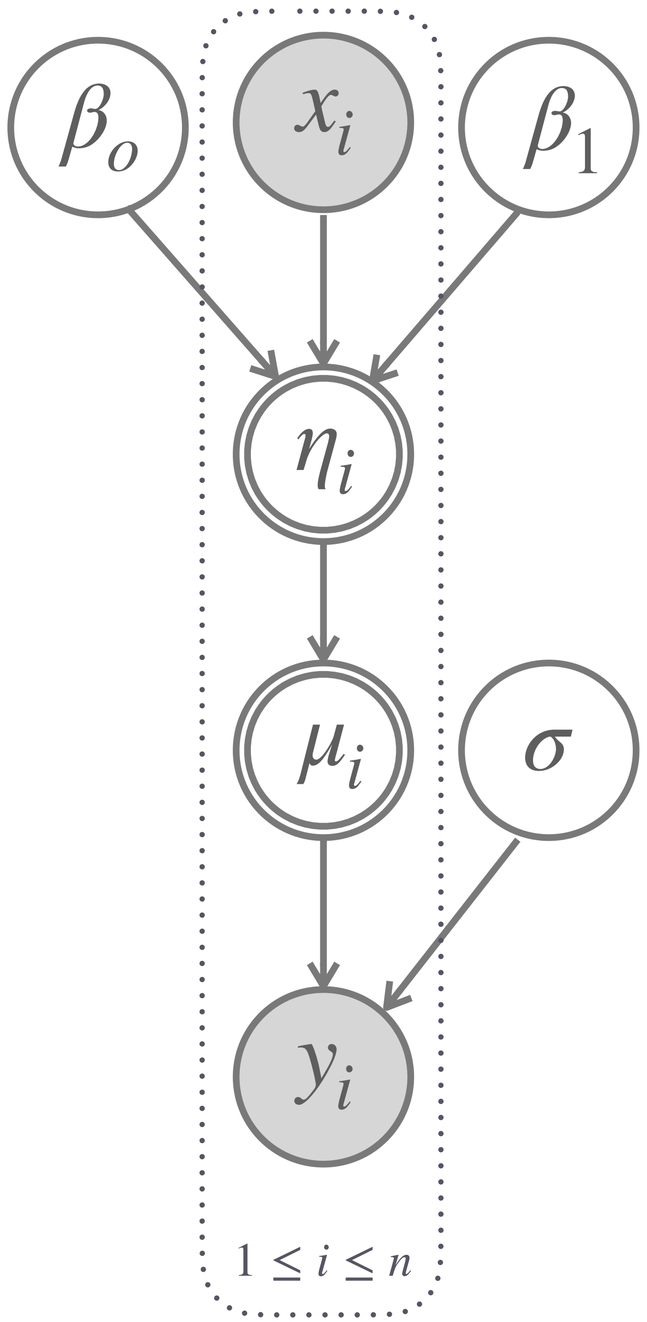
   predict (hypothetical) data points y

2. **central tendency**

   predict expected value of (hypothetical) data points μ

3. **linear predictor**

   predict η



Prior and posterior (data) predictions

Monte Carlo sampling

- fix a model with $P(D \mid X, \theta)$ and $P(\theta)$
  - latter can be prior or post. conditioned on D'
- sample from the predictive distribution by:

  (i) sampling a vector of parameters $\theta^* \sim P(\theta)$

  (ii) sampling "fake" data D^* from the likelihood function, conditioned on the sampled θ^* (given the relevant predictor values X) :

  $D^* \sim P(D \mid X, \theta^*)$
- Monte Carlo sampling:
  - by taking many samples and "aggregating", we approximate the integrals
    - "aggregating" means that (i) we usually don't care for just one sample, but the distributional information in a lot of samples, and (ii) we might want to focus on particular aspects of each sampled "fake" data (e.g., a particular **summary statistic**)

prior predictive

$$P(D_{\text{pred}}) = \int P(\theta)\ P(D_{\text{pred}} \mid \theta)\ \mathrm{d}\theta$$

posterior predictive

$$P(D_{\text{pred}} \mid D_{\text{obs}}) = \int P(\theta \mid D_{\text{obs}})\ P(D_{\text{pred}} \mid \theta)\ \mathrm{d}\theta$$

predicted linear predictor

- fix a linear model with $P(\theta)$
  - $P(\theta)$ can be prior or posterior
- sample a linear predictor $\mu^* \sim P(\mu^* \mid X, \theta)$
  - X is a matrix of independent variables

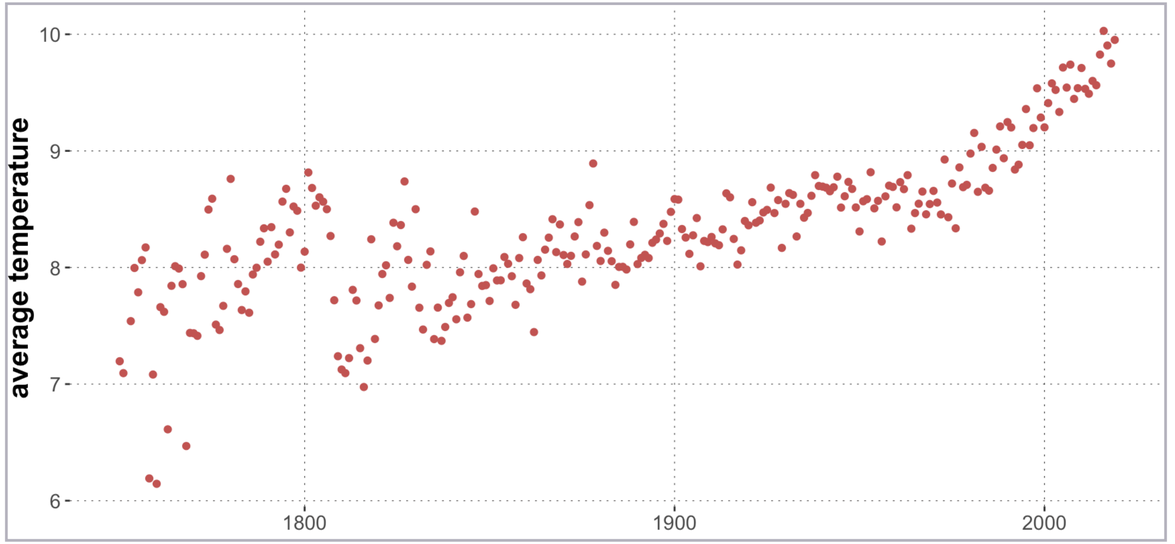
Example: World-temperature data

data, model and posterior predictions

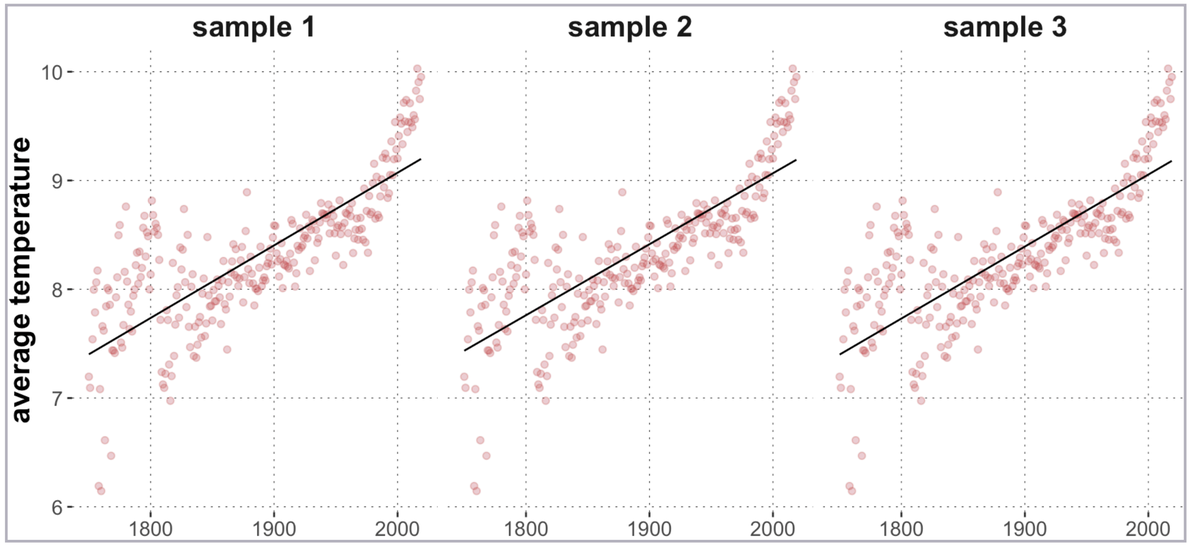
- data:
  - average world temperature 1750-2019
- model:

```
fit_worldTemp <- brm(
  avg_temp ~ year,
  data = aida::data_WorldTemp,
  prior = prior(student_t(1,0,5), class = "b")
  )
```

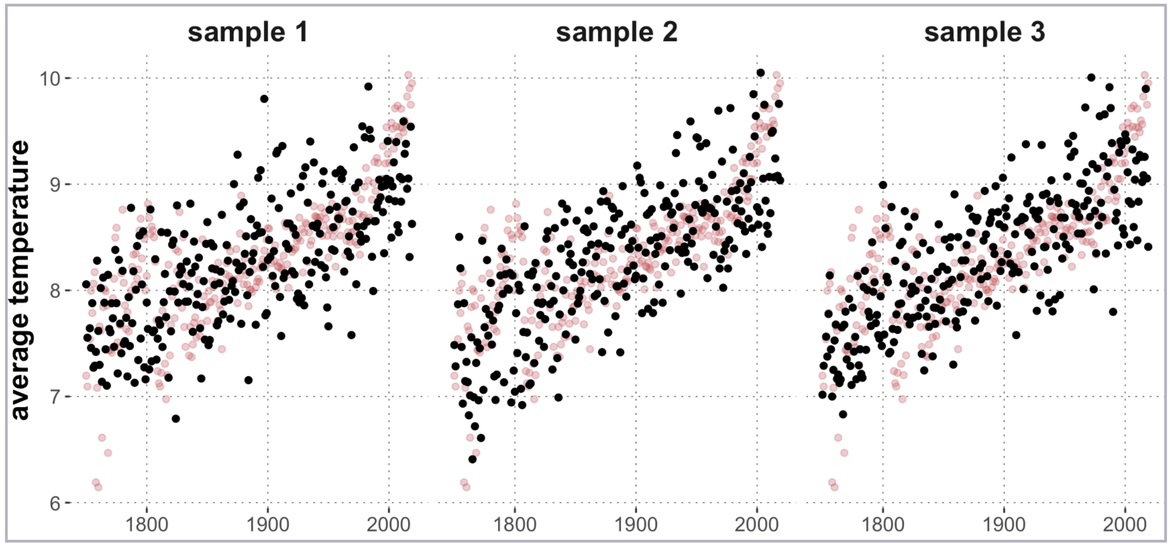
Data



Posterior samples for the central tendency



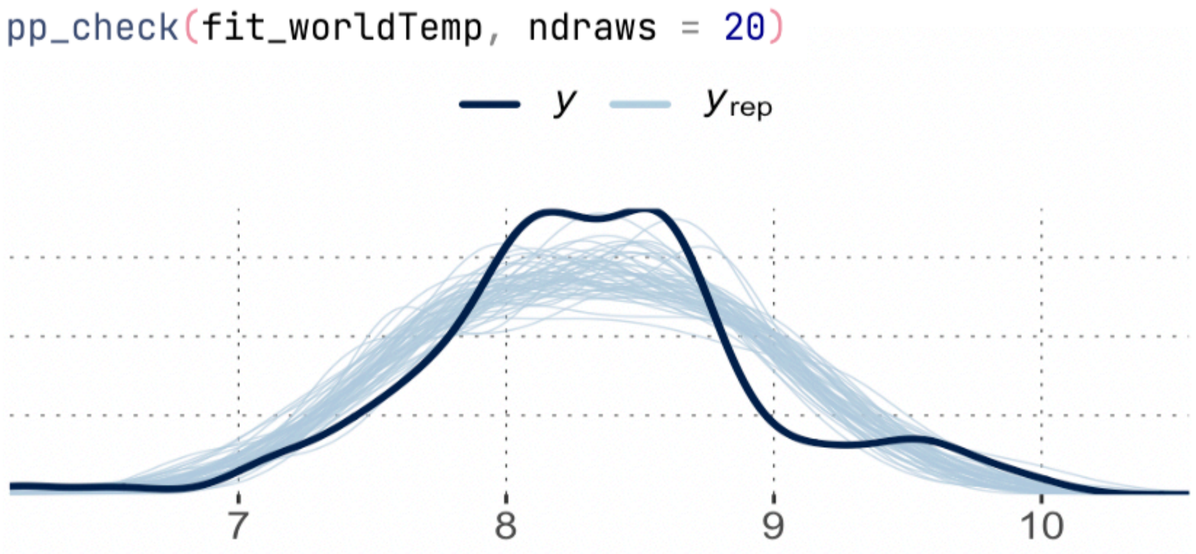
Samples from the posterior predictive (data)



Visual posterior predictive checks

for world-temperature data

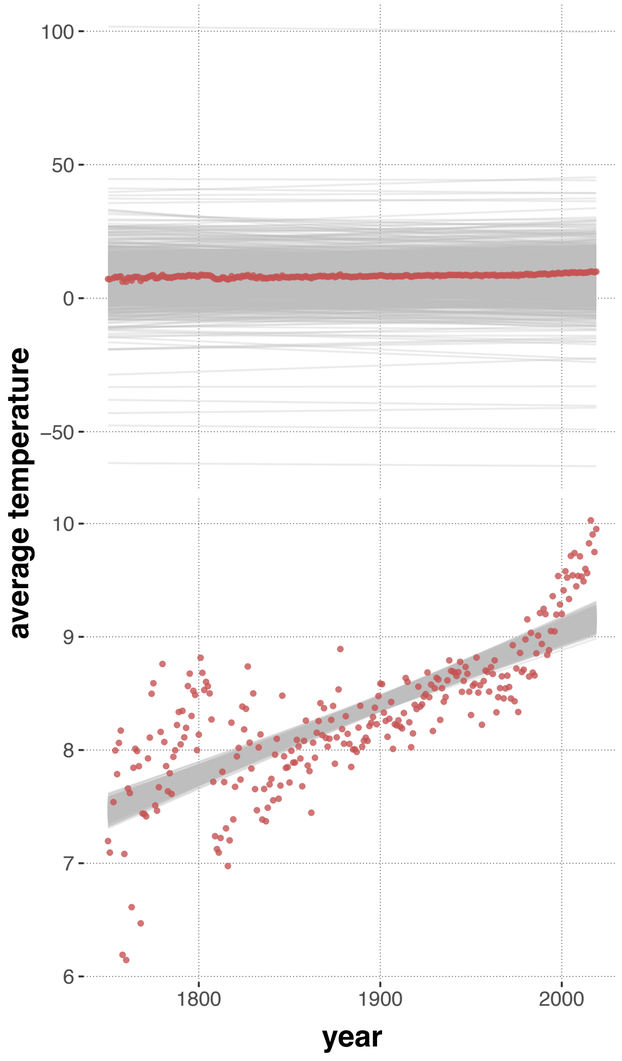
- black line:
  - distribution of observed temperature
- each of the 50 blue lines:
  - distribution of temperatures predicted for same years given a sample from the posterior



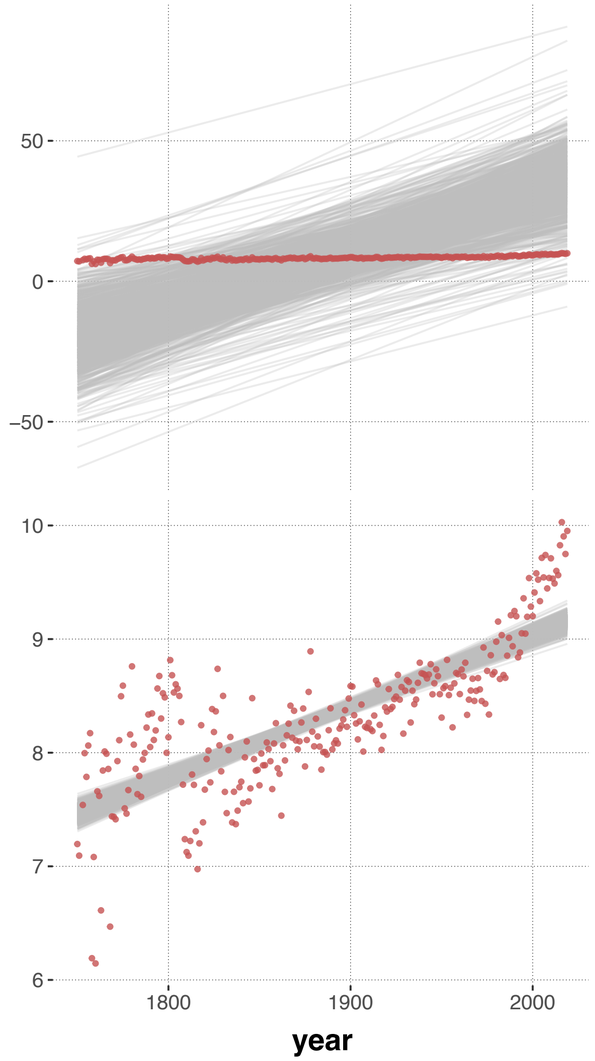
Exploring prior and posterior prediction

Prior & posterior samples of linear predictor value (μ)

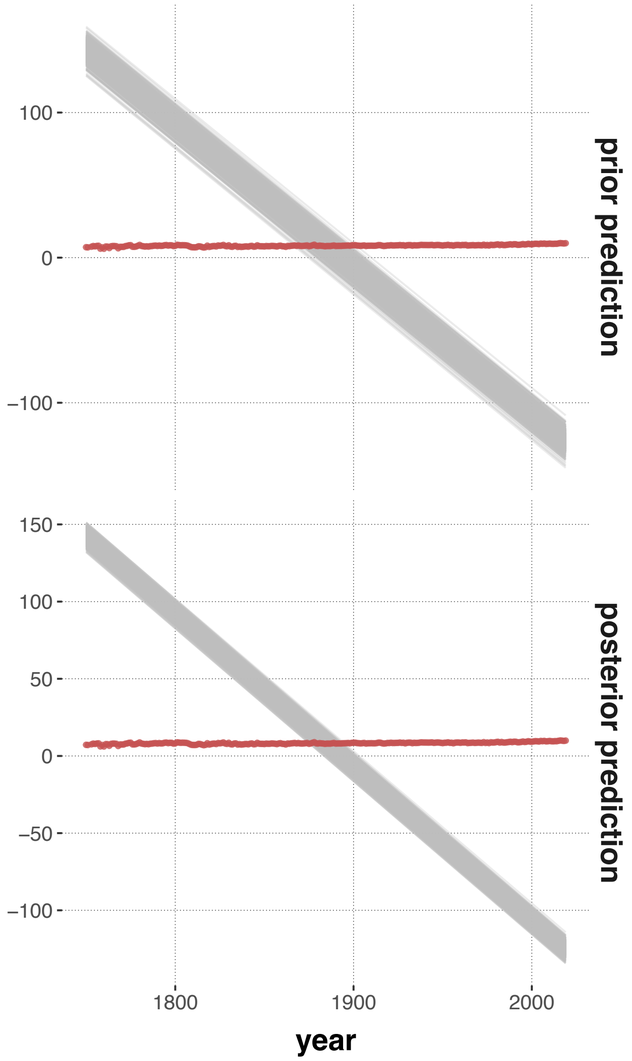
$\beta_{\text{year}} \sim \text{Normal}(0, 0.02)$



$\beta_{\text{year}} \sim \text{Normal}(0.2, 0.05)$



$\beta_{\text{year}} \sim \text{Normal}(-1, 0.005)$



Bayesian predictive *p*-values

a generalization

- fix a model with $P(D \mid \theta)$ and $P(\theta)$
  - latter can be prior or posterior
  - gives prior / posterior predictive *p*-values
- $P_M(D)$ is the predictive distribution for model M
- Bayesian predictive *p*-value for observed data d_{obs}:

$$p(d_{\text{obs}}) = P_M\left(D \in \{d \mid P_M(d) \leq P_M(d_{\text{obs}})\}\right)$$

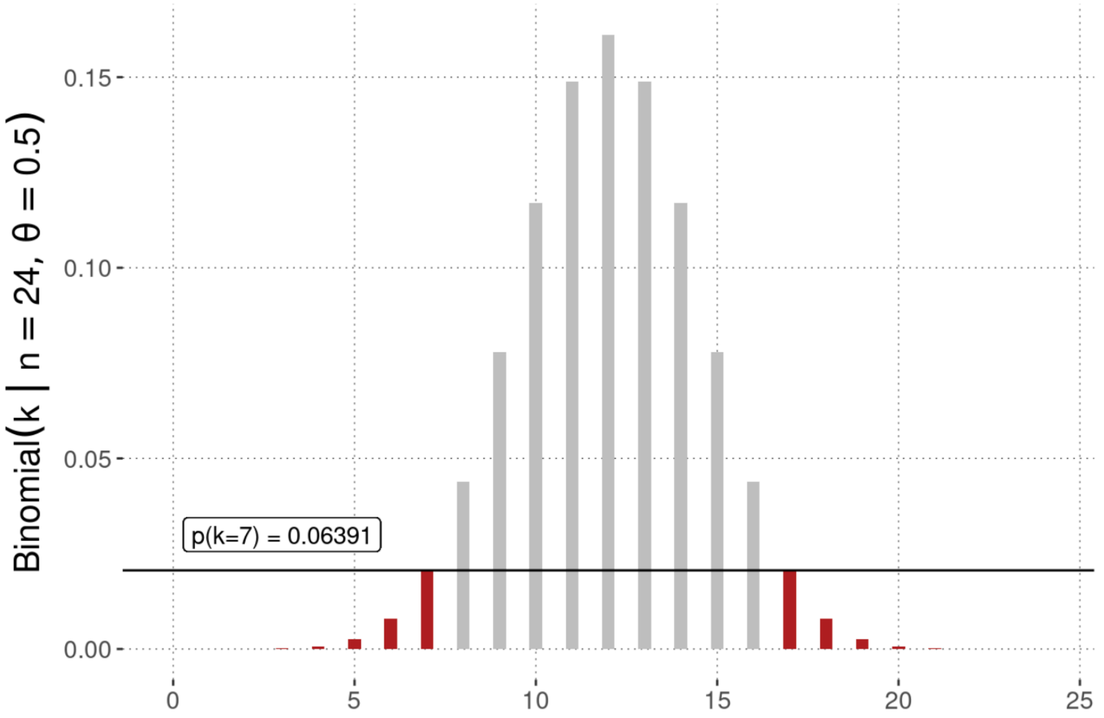
- approximated by sampling:

$$p(d_{\text{obs}}) \approx \frac{1}{n} \sum_{i=1}^{n} \left[P_M(d_i) \leq P_M(d_{\text{obs}})\right]$$

where $d_i \sim P_M(D)$ is a sample from the predictive distribution

Recap: frequentist *p*-values

$$p\left(D_{\text{obs}}\right) = P\left(T^{|H_0} \succeq^{H_{0,a}} t\left(D_{\text{obs}}\right)\right)$$



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demo

assessing predictive samples in BRMS (demo 05)