# A pendulum theorem†

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We consider $N$ linked pendulums which are inverted and balanced on top of one another, and establish a general theorem which shows how they may be stabilized by small vertical oscillations of the support.

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## 1. Introduction

Stephenson (1908$a$, $b$) showed that it is possible to stabilize a single rigid pendulum in its inverted, or upside-down, equilibrium position by subjecting the pivot to small vertical oscillations of suitably high frequency. He confirmed his theoretical predictions by a practical demonstration of the phenomenon.

While this is a well-known curiosity of classical mechanics it does not seem to be generally known that an inverted *double*, or even *triple* pendulum can be stabilized in the same way. This was, again, first predicted theoretically by Stephenson, in a comparatively overlooked paper of 1909, though the idea has reappeared in a number of subsequent studies (Lowenstern 1932; Hsu 1961; Kalmus 1970; Otterbein 1982; Leiber & Risken 1988).

Here we present a simple but general theorem on the linear stability of an inverted $N$-pendulum *of any kind*. The generality of the theorem is achieved by relating the stability question to just two elementary properties of the system as a whole when it is in its non-inverted, or downward-hanging, state.

## 2. The stability of upside-down pendulums

**Theorem.** *Let $N$ pendulums hang down, one from another, under gravity $g$, each having one degree of freedom, the uppermost being suspended from a pivot point O. Let $\omega_{\max}$ and $\omega_{\min}$ denote the largest and the smallest of the natural frequencies of small oscillation about this equilibrium state.*

*Now turn the whole system upside-down. The resulting configuration of the pendulums can be stabilized (according to linear theory, at least) if we subject the pivot point O to vertical oscillations of suitable amplitude $\epsilon$ and frequency $\omega_0$. When $\omega_0^2 \gg \omega_{\max}^2$ the stability criterion is*

$$\sqrt{2}g/\omega_0\,\omega_{\min} < \epsilon < 0.450g/\omega_{\max}^2. \tag{2.1}$$

*Note.* When several pendulums are involved, $\omega_{\max}^2$ is typically much greater than $\omega_{\min}^2$. The condition $\omega_0^2 \gg \omega_{\max}^2$ is then *necessary* for the stability of the inverted state, so (2.1) then gives the whole stability region in the $\epsilon$–$\omega_0$ plane, as in figure 1.

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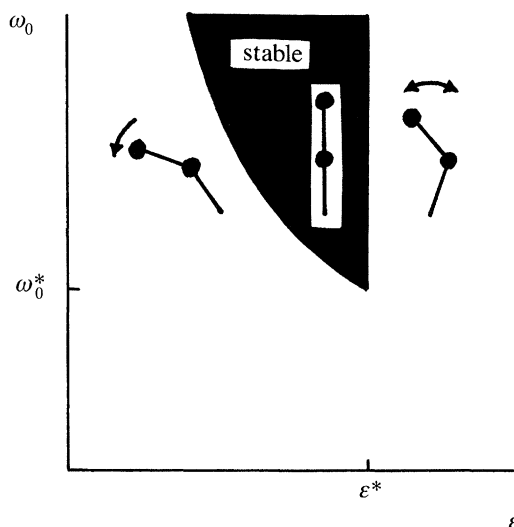


Figure 1. Typical region of linear stability for an upside-down $N$-pendulum, as given by (2.1). Here $\epsilon^* = 0.450\, g/\omega_{\mathrm{max}}^2$ and $\omega_0^* = 3.143\omega_{\mathrm{max}}^2/\omega_{\mathrm{min}}$. The sketches indicate theoretical predictions of the behaviour just outside this stability region. Thus, if at a given frequency $\omega_0$ the amplitude $\epsilon$ is just too large, the system is unstable to rapidly growing buckling oscillations at frequency $\frac{1}{2}\omega_0$. If, on the other hand, $\epsilon$ is a little too small, the pendulums fall over comparatively slowly, keeping to the same side of the vertical.

*Proof.* Consider first small disturbances to the system about the original, i.e. non-inverted equilibrium state, the pivot point O being fixed. Let the natural frequencies be $\omega_i$, and let $X_i$ be the corresponding normal coordinates, each of which will be some linear combination of the (small) angles which the pendulums make with the downward vertical. We then have

$$\ddot{X}_i + \omega_i^2 X_i = 0, \quad i = 1, \dots, N. \tag{2.2}$$

Now, each of the quantities $\omega_i^2$ will be simply proportional to $g$; this follows from the standard theory of small oscillations based on Lagrange's equations of motion (see, for example, Landau & Lifschitz 1976), given that the uniform gravitational field $g$ is the sole source of potential energy in this problem. If we tackle the stability of the inverted state by changing the sign of $g$ we then find that small disturbances are governed instead by

$$\ddot{X}_i - \omega_i^2 X_i = 0, \quad i = 1, \dots, N. \tag{2.3}$$

Suppose now that the pivot point O oscillates up and down so that its coordinate in the direction of the upward vertical is $h = -\epsilon \cos \omega_0 t$. We may allow for this simply by replacing $g$ in the above argument by apparent gravity $g + \ddot{h} = g + \epsilon \omega_0^2 \cos \omega_0 t$. In this way we find that

$$\ddot{X}_i - \omega_i^2 (1 + (\epsilon \omega_0^2/g) \cos \omega_0 t) X_i = 0, \quad i = 1, \dots, N.$$

On introducing the scaling $T = \omega_0 t$ we therefore obtain $N$ uncoupled Mathieu equations:

$$\mathrm{d}^2 X_i/\mathrm{d}T^2 + (\alpha_i + \beta_i \cos T) X_i = 0, \quad i = 1, \dots, N, \tag{2.4}$$

where
$$\alpha_i = -\omega_i^2/\omega_0^2, \quad \beta_i = -\omega_i^2 \epsilon/g. \tag{2.5}$$
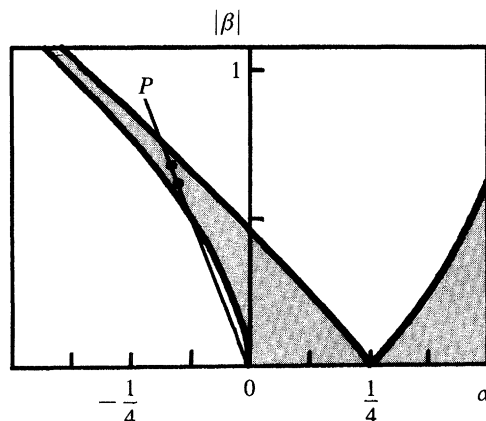
Figure 2. Stable (shaded) and unstable regions for the $X = 0$ solution of Mathieu's equation $d^2X/dT^2 + (\alpha + \beta \cos T)X = 0$. The points (2.6) lie on the straight line $OP$, and for the inverted state to be stable they must all lie within the shaded region with $\alpha < 0$, in the manner indicated (for $N = 2$).

Now, the stability regions of Mathieu's equation are well known and indicated in figure 2. We achieve stability of our inverted system if, by choosing $\epsilon$ and $\omega_0$ appropriately, we can place all the points

$$\omega_i^2(-1/\omega_0^2, \epsilon/g), \quad i = 1, \ldots, N \tag{2.6}$$

in the shaded part of the region $\alpha < 0$.

To see that this can indeed be done it is simplest to pass directly to the high-frequency limit, i.e. $\omega_0^2 \gg \omega_{\max}^2$, so that $\omega_i^2/\omega_0^2$ is small for all $i = 1, \ldots, N$. Now, when $|\alpha|$ is small the stability boundary which passes through the origin is given asymptotically by $\alpha = -\frac{1}{2}\beta^2$ (see, for example, Jordan & Smith 1987, p. 257). Moreover, the upper boundary to the shaded region in $\alpha < 0$ is known to meet the $\alpha = 0$ axis at $|\beta| = 0.450$. Thus all the points (2.6) will be in the stable region if

$$(2\omega_i^2/\omega_0^2)^{\frac{1}{2}} < \omega_i^2 \epsilon/g < 0.450, \quad i = 1, \ldots, N,$$

i.e. if
$$\sqrt{2}g/\omega_i\omega_0 < \epsilon < 0.450g/\omega_i^2 \tag{2.7}$$

for all $i = 1, \ldots, N$. This is ensured by (2.1).

While this completes the proof, it is worth considering a more general and geometric approach to the matter. All the points (2.6) lie on a line through the origin with slope $-\epsilon\omega_0^2/g$. Imagine, then, that we attach an elastic band $OP$ to the origin in figure 2, stretch it out, and mark points along it so that their distances from the origin are in the proportion $\omega_1^2 : \omega_2^2 : \ldots : \omega_N^2$. By choosing $\epsilon$ and $\omega_0$ we may vary the length of the band, and its slope $-\epsilon\omega_0^2/g$, at will.

Now, if $\omega_{\max}$ and $\omega_{\min}$ are almost equal, so that the marked points are tightly clustered along the band, we may evidently steer those points into the stable region in figure 2 even if $|\alpha|$ is not particularly small, so that $\omega_0^2$ is not large compared with $\omega_{\max}^2$. If, on the other hand, $\omega_{\max}$ is substantially greater than $\omega_{\min}$ – as will typically be the case – we may still steer all the points into the stable region, but only by moving the band very close to the $\alpha = 0$ axis in figure 2 and by shrinking it accordingly. The stability criterion (2.1) will then apply, and the region of stability in the $\epsilon$–$\omega_0$ plane will be as indicated in figure 1.

## 3. Examples

### (a) Stephenson's system inverted

Stephenson's original double-pendulum analysis was for two identical rods of uniform density and equal length $l$. An elementary exercise shows that the natural frequencies are then

$$\omega_{\max, \min} = (3 \pm 6/\sqrt{7})^{\frac{1}{2}} (g/l)^{\frac{1}{2}}, \qquad (3.1)$$

and (2.1) becomes

$$(1.653/\omega_0) (g/l)^{\frac{1}{2}} < \epsilon/l < 0.085. \qquad (3.2)$$

The first of these inequalities may be written as

$$(\tfrac{1}{2}\epsilon\omega_0)^2 > 0.683gl, \qquad (3.3)$$

which is Stephenson's original criterion, derived by a different method.

His formula for the inverted triple pendulum appears, unfortunately, to be in error; the theorem of §2 gives

$$(1.738/\omega_0) (g/l)^{\frac{1}{2}} < \epsilon/l < 0.03 \qquad (3.4)$$

as the high-frequency stability criterion for an inverted triple pendulum consisting of three equal heavy rods.

### (b) The point-mass double pendulum

Consider now a double pendulum consisting of two point masses $m_1$, $m_2$ and two light rods of equal length $l$ (as sketched in figure 1). The normal modes of small oscillation about the lower, inherently stable equilibrium position (with O fixed) have frequencies

$$\omega_{\max} = \frac{(g/l)^{\frac{1}{2}}}{(1 - m^{\frac{1}{2}})^{\frac{1}{2}}} \quad \text{and} \quad \omega_{\min} = \frac{(g/l)^{\frac{1}{2}}}{(1 + m^{\frac{1}{2}})^{\frac{1}{2}}}, \qquad (3.5)$$

where

$$m = m_2/(m_1 + m_2), \qquad (3.6)$$

$m_1$ denoting the point mass which is attached directly to O. The high-frequency criterion (2.1) for the stability of the inverted equilibrium position then reduces to

$$\frac{\sqrt{2}(1 + m^{\frac{1}{2}})^{\frac{1}{2}}}{\omega_0} \left(\frac{g}{l}\right)^{\frac{1}{2}} < \frac{\epsilon}{l} < 0.450(1 - m^{\frac{1}{2}}). \qquad (3.7)$$

For comparison, we show in figure 3 the stability boundaries for this inverted double pendulum obtained by direct numerical integration of the equations of motion, linearized about the upward vertical. At high frequencies $\omega_0$ the stable range of $\epsilon/l$ is given well by (3.7), but if the upper mass $m_2$ is much smaller than the lower mass $m_1$ it is possible to stabilize the system with a comparatively small driving frequency $\omega_0$ and (3.7) does not then apply. This case corresponds, of course, to $\omega_{\max}$ and $\omega_{\min}$ being almost equal (see (3.5) and the end of §2); indeed, when $m = 0$ the two coincide and (3.7) reduces to the stability criterion for an inverted single pendulum.

More typically, rather high frequencies $\omega_0$ are needed to give stability, and (2.1) then provides a good approximation to the whole stability region in the $\epsilon$–$\omega_0$ plane. The case of equal masses ($m = 0.5$) is a good example:

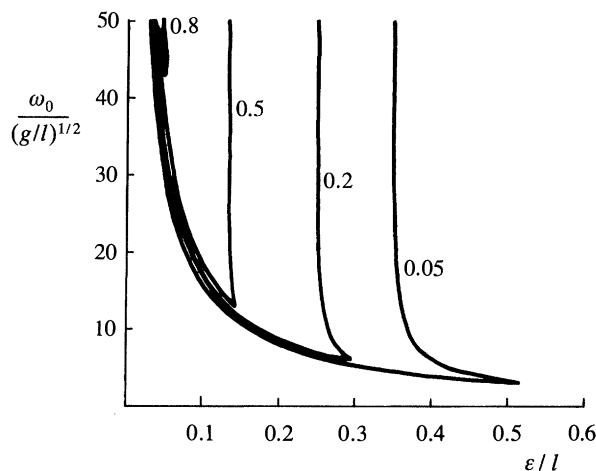$$(1.85/\omega_0) (g/l)^{\frac{1}{2}} < \epsilon/l < 0.132, \qquad (3.8)$$

Figure 3. Computed linear stability boundaries for the inverted point-mass double pendulum for four values of $m = m_2/(m_1 + m_2)$.

for the right-hand stability boundary in figure 3 is indeed almost a straight line parallel to the $\omega_0$-axis, save at the very lowest frequencies giving stability.

The larger the value of $m$, clearly, the smaller we have to take $\epsilon$ and the higher we have to take $\omega_0$, but there is in theory no upper limit to the mass ratio $m_2/m_1$ which can be stabilized by vertical oscillations of the support.

### (c) An inverted N-pendulum

We now apply the theorem of §2 to a pendulum with $N$ equal point masses connected by $N$ equal light rods of total length $L = Nl$.

The natural frequencies of oscillation about the lower equilibrium position are

$$\omega_i = \xi_i^{\frac{1}{2}}(g/l)^{\frac{1}{4}}, \quad i = 1, \ldots, N, \tag{3.9}$$

where $\xi_i$ denotes the $i$th zero of the Laguerre polynomial $L_N(x)$; this was shown by Daniel Bernoulli and by Euler in the late 1730s (see Cannon & Dostrovsky 1981).

In the case $N = 3$ this gives

$$\omega_1 = 0.64(g/l)^{\frac{1}{2}}, \quad \omega_2 = 1.51(g/l)^{\frac{1}{2}}, \quad \omega_3 = 2.51(g/l)^{\frac{1}{2}}, \tag{3.10}$$

and on taking the greatest and least of these we find that the criterion (2.1) becomes

$$(2.21/\omega_0)(g/l)^{\frac{1}{2}} < \epsilon/l < 0.07. \tag{3.11}$$

For comparison, figure 4 shows the stability boundary obtained by numerical integration of the linearized equations of motion.

When $N$ is large we can use the asymptotic expression $\xi_i \sim \chi_i^2/4N$, where $\chi_i$ denotes the $i$th zero of the Bessel function $J_0(x)$. With $\chi_1 \doteq 2.4$ and $\chi_N \doteq (N-\frac{1}{4})\pi$ we then have

$$\omega_{\min} \doteq 1.2(g/L)^{\frac{1}{2}}, \quad \omega_{\max} \doteq \tfrac{1}{2}(N-\tfrac{1}{4})\pi(g/L)^{\frac{1}{2}},$$

and therefore

$$\frac{1.18}{\omega_0}\left(\frac{g}{L}\right)^{\frac{1}{2}} < \frac{\epsilon}{L} < \frac{0.182}{(N-\frac{1}{4})^2} \tag{3.12}$$

as the criterion for the stability of this particular inverted $N$-pendulum, when $N$ is large.
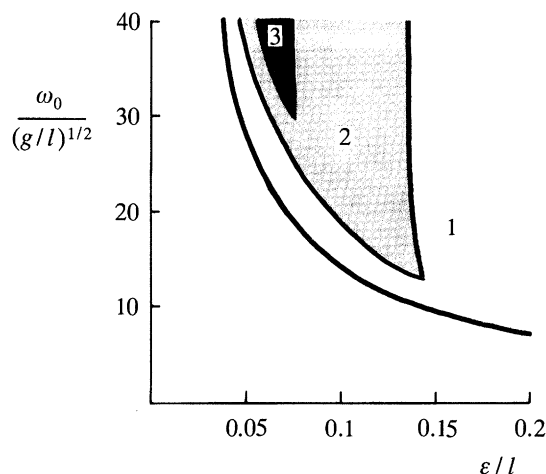
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Figure 4. Computed regions of linear stability for the upside-down $N$-pendulum, for $N = 1, 2, 3$, in the case when the various (point) masses are equal.

Suppose finally that $L$ is fixed. Then, the larger the value of $N$, the smaller we must take $\epsilon$, and, consequently, the larger we must take $\omega_0$ if stability is to be achieved. The limit $N \to \infty$ for fixed $L$ corresponds to a perfectly flexible string, and was of major interest in the original investigations of Bernoulli and Euler. The region of stability of the *inverted* state vanishes altogether in this limit, so we cannot perform the Indian rope trick.

## 4. Concluding remarks

While the theorem of this paper concerns linear stability, numerical integrations of the full equations of motion have also been carried out, together with laboratory experiments (Acheson & Mullin 1993). These further studies show the stability of these inverted states to be remarkably robust, at least in the cases $N = 1, 2, 3$. In particular, we have seen an inverted triple pendulum recover from very severe initial disturbances and gradually wobble back to the upward vertical.

## References

Acheson, D. J. & Mullin, T. 1993 On upside-down pendulums. *Nature, Lond.* (Submitted.)

Cannon, J. T. & Dostrovsky, S. 1981 *The evolution of dynamics: vibration theory from 1687 to 1742.* Springer-Verlag.

Hsu, C. S. 1961 On a restricted class of coupled Hill's equations and some applications. *J. appl. Mech.* **28**, 551–556.

Jordan, D. W. & Smith, P. 1987 *Nonlinear ordinary differential equations*, 2nd edn. Oxford: Clarendon Press.

Kalmus, H. P. 1970 The inverted pendulum. *Am. J. Phys.* **38**, 874–878.

Landau, L. D. & Lifshitz, E. M. 1976 *Mechanics*, 3rd edn. Pergamon.

Leiber, T. & Risken, H. 1988 Stability of parametrically excited dissipative systems. *Phys. Lett.* A **129**, 214–218.

Lowenstern, E. R. 1932 The stabilizing effect of imposed oscillations of high frequency on a dynamical system. *Phil. Mag.* **8**, 458–486.

Otterbein, S. 1982 Stabilisierung des $n$-Pendels und der Indische Seiltrick. *Arch. ration. Mech. Analysis* **78**, 381–393.

Stephenson, A. 1908*a* On a new type of dynamical stability. *Mem. Proc. Manch. Lit. Phil. Soc.* **52** (8), 1–10.

Stephenson, A. 1908*b* On induced stability. *Phil. Mag.* **15**, 233–236.

Stephenson, A. 1909 On induced stability. *Phil. Mag.* **17**, 765–766.