Exercise 4 – Kronecker Product, Walsh-Hadamard Transform, Quantum Teleportation

Problem 1

Let $A \equiv (a_{ij})_{ij}$ be an $m \times n$ matrix and B an $r \times s$ matrix. The Kronecker product of A and B is defined as the $(m \cdot r) \times (n \cdot s)$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}. \tag{1}$$

a) The states

$$|\phi_1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\phi_2\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

form a basis of $\mathbb{C}^2$. Calculate

$$|\phi_1\rangle \otimes |\phi_1\rangle\,, \quad |\phi_1\rangle \otimes |\phi_2\rangle\,, \quad |\phi_2\rangle \otimes |\phi_1\rangle\,, \quad |\phi_2\rangle \otimes |\phi_2\rangle\,. \tag{3}$$

Form these vectors also a basis of $\mathbb{C}^4$?

b) Consider the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

Find $\sigma_x \otimes \sigma_z$ and $\sigma_z \otimes \sigma_x$.

Problem 2

The single-bit *Walsh-Hadamard transform* is the unitary map W_1 given by

$$W_1\,|0\rangle = \frac{1}{\sqrt{2}}\,(|0\rangle + |1\rangle)\,, \qquad W_1\,|1\rangle = \frac{1}{\sqrt{2}}\,(|0\rangle - |1\rangle) \tag{5}$$

The n-bit Walsh-Hadamard transform W_n is defined as

$$W_n \equiv W_1 \otimes W_1 \otimes \cdots \otimes W_1 \qquad (n\text{-times}). \tag{6}$$

a) Find W_2 using the canonical basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{7}$$

b) Find the inverse of W_2.

c) Find $W_2(|00\rangle)$.

d) Find $W_n(|00\ldots 0\rangle)$.

Problem 3

Suppose Alice has a pure state $|\psi\rangle_S$ of a system S and wants to send it to Bob. Using a shared entangled state, she can "teleport" the state $|\psi\rangle_S$ without having to physically move the state over.
We have three systems $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$. The initial state is given by

$$|\psi\rangle_{\text{in}} = |\psi\rangle_S \otimes \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right), \tag{8}$$

i.e. Alice and Bob share a fully entangled Bell state, and the total state is separable with respect to S. We may write $|\psi\rangle_S = \alpha\,|0\rangle + \beta\,|1\rangle$.

a) In a first step, Alice will measure systems S and A jointly in the Bell basis,

$$\left\{ \begin{array}{cc} \frac{1}{\sqrt{2}}\left(|0_S 0_A\rangle + |1_S 1_A\rangle\right) & \frac{1}{\sqrt{2}}\left(|0_S 0_A\rangle - |1_S 1_A\rangle\right) \\ \frac{1}{\sqrt{2}}\left(|0_S 1_A\rangle + |1_S 0_A\rangle\right) & \frac{1}{\sqrt{2}}\left(|0_S 1_A\rangle - |1_S 0_A\rangle\right) \end{array} \right\}. \tag{9}$$

What is the reduced state of Bob's system for each of the possible outcomes? What are the probabilities for the possible outcomes?

b) Alice then (classically) communicates the result of her measurement to Bob. What operations can Bob apply to his system to recover $|\psi\rangle_S$?

c) Suppose that Alice does not manage to tell Bob the outcome of her measurement. Show that in this case he does not have any information about his reduced state and therefore does not know which operation to apply in order to obtain $|\psi\rangle_S$.