# Exercise 10

### Problem 1 – Von-Neumann equation

Show that the derivation of the density operator $\varrho$ in the Schrödinger picture is given by the Von-Neumann equation

$$\frac{\partial \varrho}{\partial t} = -\frac{i}{\hbar}[H, \varrho]\,. \tag{1}$$

### Problem 2 – Lindblad equation

A master equation in Lindblad form

$$\frac{\partial \varrho}{\partial t} = -\frac{i}{\hbar}[H, \varrho] + \sum_{j=1}^{N^2-1} \gamma_j \left( A_j \varrho A_j^\dagger - \frac{1}{2}\varrho A_j^\dagger A_j - \frac{1}{2} A_j^\dagger A_j \varrho \right), \qquad \gamma_j \geq 0 \tag{2}$$

is the most general type of a Markovian and time-homogeneous master equation describing non-unitary evolution of the density matrix $\varrho$ that is trace-preserving and completely positive for any initial condition. The operators $A_j$ are called jump operators and are operators of the Hilbert space.

We consider the Hamiltonian

$$H = \frac{\hbar\omega}{2}\sigma_x \tag{3}$$

with jump operators

$$\sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y)\,, \qquad \sigma_- = \frac{1}{2}(\sigma_x - i\sigma_y) \tag{4}$$

and associated rates $\gamma_+$ and $\gamma_-$.

i) Calculate $\sigma_+\sigma_-$ and $\sigma_-\sigma_+$, the commutator $[\sigma_+, \sigma_-]$, and the anti-commutator $\{\sigma_+, \sigma_-\}$.

ii) The Kronecker product can be used to get a convenient representation for some matrix equations,

$$A\varrho B = C \quad \Leftrightarrow \quad (B^T \otimes A)\,\mathrm{vec}(\varrho) = \mathrm{vec}(A\varrho B) = \mathrm{vec}(\varrho) \equiv \vec{\varrho}. \tag{5}$$

Here, $\mathrm{vec}(\varrho)$ denotes the vectorization of the matrix $\varrho$ formed by stacking the columns of $\varrho$ into a single column vector,

$$\mathrm{vec}\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} = \begin{pmatrix} a \\ b^* \\ b \\ c \end{pmatrix}. \tag{6}$$

Calculate the $4 \times 4$ matrix $\mathcal{L}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\varrho} = \mathcal{L}\vec{\varrho}. \tag{7}$$

iii) Show that the Lindblad equation for this system is trace preserving.

iv) Calculate the steady state of the system for the rates given by $\gamma_+ = \gamma(\nu - 1)$ and $\gamma_- = \gamma\nu$. What is the interpretation of $\gamma$ and $\nu$?

**Problem 3 – Phase-damping channel**

Define a quantum operation $\mathcal{E}$ on a qubit state as

$$\mathcal{E}(\varrho) = E_0 \varrho E_0^\dagger + E_1 \varrho E_1^\dagger \tag{8}$$

with

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}, \tag{9}$$

and $0 < p < 1$.

i) Prove that $\mathcal{E}$ maps a density matrix to a density matrix.

ii) Show that for any $\varrho = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$

$$\lim_{N \to \infty} \mathcal{E}^N(\varrho) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}. \tag{10}$$

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Pauli matrices:

$$\sigma_0 = \mathbb{1}, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$