Exercise 5)

Problem 1)

i) $\mathrm{tr}\,\varrho = \mathrm{tr}\left[\sum_j p_j |\psi_j\rangle\langle\psi_j|\right] = \sum_j p_j\, \mathrm{tr}\left[|\psi_j\rangle\langle\psi_j|\right]$

$= \sum_j p_j = 1$

trace of projection operator is dimension of target space

ii) $\varrho^\dagger = \left[\sum_j p_j |\psi_j\rangle\langle\psi_j|\right] = \sum_j p_j \left[|\psi_j\rangle\langle\psi_j|\right]^\dagger$

$= \sum_j p_j \left[\langle\psi_j|\right]^\dagger \left[|\psi_j\rangle\right]^\dagger = \sum_j p_j |\psi_j\rangle\langle\psi_j| = \varrho$

iii) Suppose $|\varphi\rangle$ is an arbitrary state vector

$\langle\varphi|\varrho|\varphi\rangle = \sum_j p_j \langle\varphi|\psi_j\rangle\langle\psi_j|\varphi\rangle = \sum_j p_j \langle\varphi|\psi_j\rangle\langle\varphi|\psi_j\rangle^*$

$= \sum_j p_i |\langle\varphi|\psi_j\rangle|^2 \geq 0$

iv) ϱ is Hermitian and thus diagonalizable, $\varrho = S D S^{-1}$, $D = \mathrm{diag}(\lambda_1, .., \lambda_n)$

$0 \leq \lambda_j \leq 1$

$\mathrm{tr}\,\varrho^2 = \mathrm{tr}(\underbrace{SS^{-1}}_{=\mathbb{1}}\varrho)^2 = \mathrm{tr}(\underbrace{S^{-1}\varrho S}_{=D})^2 = \mathrm{tr}\,D^2 = \sum_j \lambda_j^2$

$\leq \sum_j \lambda_j = 1$

"=" if pure state.

Problem 2)

a) Schrödinger equation: $i\hbar |\dot{\psi}\rangle = H|\psi\rangle$

$$\Rightarrow \quad |\dot{\psi}\rangle = -\frac{i}{\hbar} H|\psi\rangle , \qquad \langle\dot{\psi}| = \frac{i}{\hbar} \langle\psi| H$$

$$\dot{\rho} = \frac{d}{dt}\left[\sum_j p_j |\psi_j\rangle\langle\psi_j| \right] = \sum_j p_j \left[|\dot{\psi}_j\rangle\langle\psi_j| + |\psi_j\rangle\langle\dot{\psi}_j| \right]$$

$$= \sum_j p_j \left[-\frac{i}{\hbar} H |\psi_j\rangle\langle\psi_j| + \frac{i}{\hbar} |\psi_j\rangle\langle\psi_j| H \right]$$

$$= -\frac{i}{\hbar}\left[H \sum_j p_j |\psi_j\rangle\langle\psi_j| - \left(\sum_j p_j |\psi_j\rangle\langle\psi_j| \right) H \right]$$

$$= -\frac{i}{\hbar}\left(H\rho - \rho H \right) = -\frac{i}{\hbar}[H, \rho]$$

b)

$$\rho = \frac{1}{2}\left[\mathbb{1} + \alpha\sigma_x + \beta\sigma_y + \gamma\sigma_z \right] = \frac{1}{2}\begin{pmatrix} 1+\gamma & \alpha - i\beta \\ \alpha + i\beta & 1-\gamma \end{pmatrix}$$

$\mathrm{tr}\,\rho = 1$ ✓

ρ Hermitian ✓ $\qquad \vec{r} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$

ρ semi-positive ✓ $\left(\lambda_{1,2} = \frac{1}{2}(1 \pm |\vec{r}|) \right)$

ρ pure $\Leftrightarrow |\vec{r}| = 1$

Problem 3)

a) $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$\rho = |\psi\rangle\langle\psi|, \quad \rho^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho$

b) $\rho^1 = tr_2\, \rho = tr_2 \left[\frac{1}{2} \left(\underline{|00\rangle\langle 00|} + |00\rangle\langle 11| + |11\rangle\langle 00| + \underline{|11\rangle\langle 11|} \right) \right]$

$= \frac{1}{2} \left(|0\rangle\langle 0| + |1\rangle\langle 1| \right)$

Problem 4)

$\rho = |ab\rangle\langle ab|$

$\rho^A = tr_B\, \rho = tr_B\, |ab\rangle\langle ab| = |a\rangle\langle a|$

Problem 5)

$$\rho = \frac{1}{2} \Big[\underline{\alpha^2 |000\rangle\langle 000|} + \alpha^2 |000\rangle\langle 011| + \alpha\beta |000\rangle\langle 100| + \alpha\beta |000\rangle\langle 111|$$

$$+ \alpha^2 |011\rangle\langle 000| + \underline{\alpha^2 |011\rangle\langle 011|} + \alpha\beta |011\rangle\langle 100| + \alpha\beta |011\rangle\langle 111|$$

$$+ \alpha\beta |100\rangle\langle 000| + \alpha\beta |100\rangle\langle 011| + \underline{\beta^2 |100\rangle\langle 100|} + \beta^2 |100\rangle\langle 111|$$

$$+ \alpha\beta |111\rangle\langle 000| + \alpha\beta |111\rangle\langle 011| + \beta^2 |111\rangle\langle 100| + \underline{\beta^2 |111\rangle\langle 111|} \Big]$$

$$\rho^A = tr_{AS}\, \rho = \frac{1}{2} \left[\alpha^2 |0_B\rangle\langle 0_B| + \alpha^2 |1_B\rangle\langle 1_B| + \beta^2 |0_B\rangle\langle 0_B| + \beta^2 |1_B\rangle\langle 1_B| \right]$$

$$= \frac{1}{2} \left[|0_B\rangle\langle 0_B| + |1_B\rangle\langle 1_B| \right] \qquad (\rho^A)^2 = \frac{1}{4} \left[|0_B\rangle\langle 0_B| + |1_B\rangle\langle 1_B| \right]$$