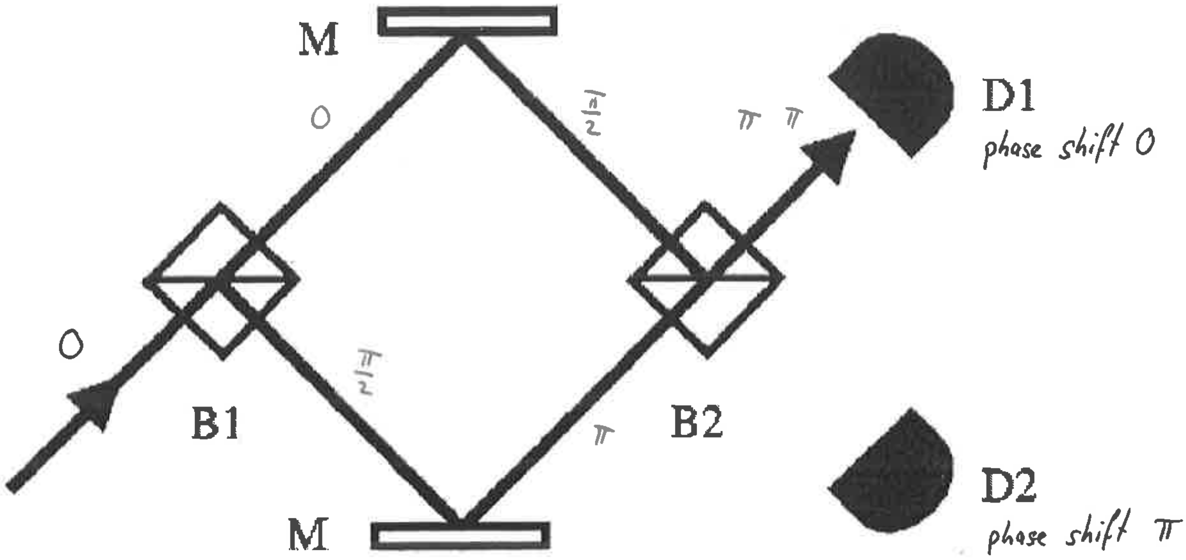
Exercise 6

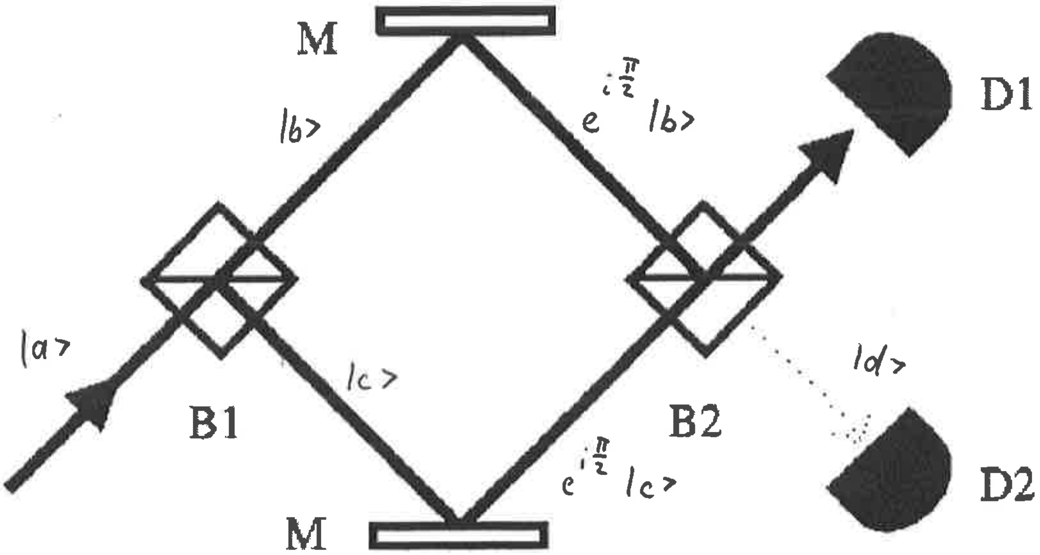
taken from: "Interaction-free measurement", Alan J. DeWeerd, 2001

a)

phase shifts: reflection: $\frac{\pi}{2}$, tranmission: 0

(see "How does a Mach-Zehnder interferometer work?", Zetie, Adams, Tocknell, 2000)





$$|b\rangle = \sqrt{T_1}\,|a\rangle, \qquad |c\rangle = e^{i\frac{\pi}{2}}\sqrt{R_1}\,|a\rangle = i\sqrt{R_1}\,|a\rangle$$

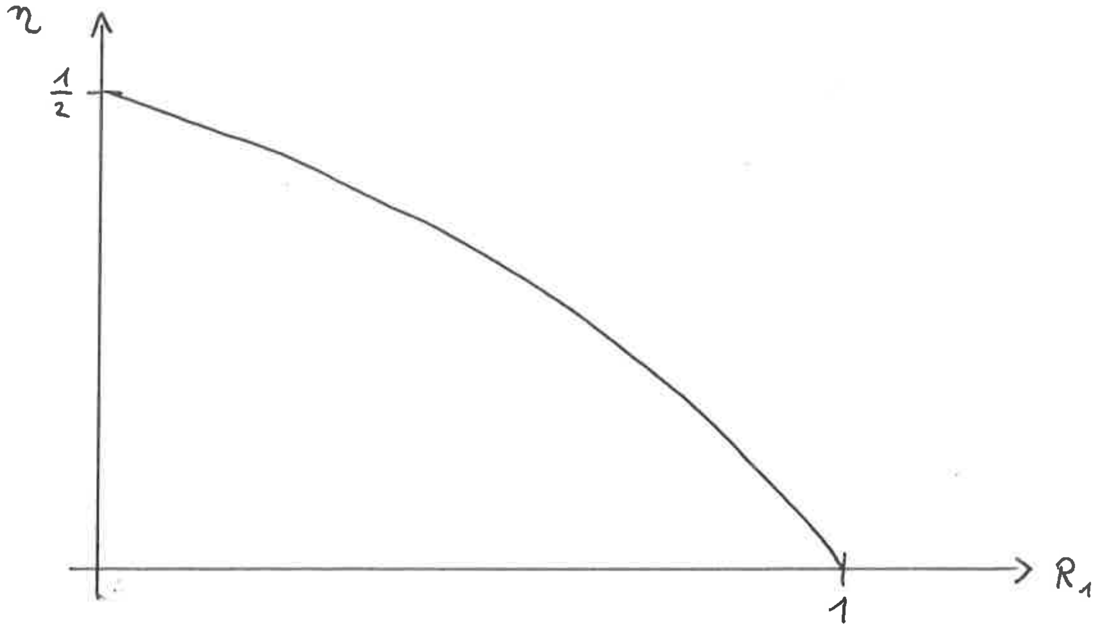
$$|d\rangle = \sqrt{T_2}\,e^{i\frac{\pi}{2}}|b\rangle + i\sqrt{R_2}\,e^{i\frac{\pi}{2}}|c\rangle = i\sqrt{T_1T_2}\,|a\rangle + i^3\sqrt{R_1R_2}\,|a\rangle$$

$$= i\left(\sqrt{T_1T_2} - \sqrt{R_1R_2}\right)|a\rangle \underset{\substack{R_1 = T_2\\ R_2 = T_1}}{=} 0$$

b) probability for absorption : $p_{abs} = R_1$

probability for ifm : $p_{ifm} = T_1 T_2$

$$\eta = \frac{p_{ifm}}{p_{abs} + p_{ifm}} = \frac{T_1 T_2}{R_1 + T_1 T_2} \underset{\substack{T_2 = R_1 \\ R_1 + T_1 = 1}}{=} \frac{T_1}{1 + T_1} = \frac{1 - R_1}{2 - R_1}$$



c) $T = \sin^2\left(\frac{\pi}{2N}\right)$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{BS} \begin{pmatrix} \sqrt{R} \\ i\sqrt{T} \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{BS} \begin{pmatrix} i\sqrt{T} \\ \sqrt{R} \end{pmatrix}$$

$$U = \begin{pmatrix} \sqrt{R} & i\sqrt{T} \\ i\sqrt{T} & \sqrt{R} \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi}{2N}\right) & i\sin\left(\frac{\pi}{2N}\right) \\ i\sin\left(\frac{\pi}{2N}\right) & \cos\left(\frac{\pi}{2N}\right) \end{pmatrix}$$

$$\mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}, \qquad \mathcal{R}(\theta_1 + \theta_2) = \mathcal{R}(\theta_1)\,\mathcal{R}(\theta_2)$$

$$\Rightarrow U^N = \mathcal{R}^N\left(\frac{\pi}{2N}\right) = \mathcal{R}\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$|out\rangle = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix}$$

d) $p_{ifm} = \left[\cos^2\left(\frac{\pi}{2N}\right)\right]^N = \cos^{2N}\vartheta \, , \qquad \vartheta = \frac{\pi}{2N}$

$$p_{abs} = T + RT + R^2T + \dots + R^{N-2}T = \sin^2\vartheta \sum_{k=0}^{N-2} (\cos^2\vartheta)^k$$

$$= \sin^2\vartheta \, \frac{1-(\cos^2\vartheta)^{N-1}}{1-\cos^2\vartheta} = 1 - \cos^{2N-2}\vartheta$$

$\underline{N \gg 1:}$

$$\cos^k\vartheta \approx \left(1 - \frac{\vartheta^2}{2}\right)^k \approx 1 - \frac{k}{2}\vartheta^2 = 1 - \frac{k}{2}\frac{\pi^2}{4N^2}$$

$$p_{ifm} \approx 1 - \frac{\pi^2}{4N}$$

$$p_{abs} \approx 1 - 1 + \frac{1}{2}(2N-2)\frac{\pi^2}{4N^2} = (N-1)\frac{\pi^2}{4N^2} \approx \frac{\pi^2}{4N}$$

$$\Rightarrow \quad \eta = \frac{p_{ifm}}{p_{ifm} + p_{abs}} \approx \frac{1 - \frac{\pi^2}{4N}}{1 - \frac{\pi^2}{4N} + \frac{\pi^2}{4N}} = 1 - \frac{\pi^2}{4N} \xrightarrow{N\to\infty} 1$$