1) $\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$, $E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$, $E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}$, $0 < p < 1$

i) $$\mathcal{E}(\rho) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \underbrace{\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}}_{= \begin{pmatrix} a & b\sqrt{1-p} \\ b^* & c\sqrt{1-p} \end{pmatrix}} + \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix} \underbrace{\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}}_{= \begin{pmatrix} 0 & \sqrt{p}\, b \\ 0 & \sqrt{p}\, c \end{pmatrix}} = \begin{pmatrix} a & b\sqrt{1-p} \\ b^*\sqrt{1-p} & c(1-p) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & pc \end{pmatrix}$$

$$= \begin{pmatrix} a & b\sqrt{1-p} \\ b^*\sqrt{1-p} & c \end{pmatrix}$$

$\mathcal{E}(\rho)$ is ~~unitary~~ Hermitian and trace-preserving, $\operatorname{tr} \rho = \operatorname{tr} \mathcal{E}(\rho)$

ii) $$\mathcal{E}^N(\rho) = \begin{pmatrix} a & b\sqrt{1-p}^{\,N} \\ b^*\sqrt{1-p}^{\,N} & c \end{pmatrix} \overset{N \to \infty}{=} \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

2) $$i\hbar\, \partial_t |\psi\rangle = H |\psi\rangle$$

$$\partial_t |\psi\rangle = -\frac{i}{\hbar} H |\psi\rangle$$

$$\partial_t \langle\psi| = \frac{i}{\hbar} \langle\psi| H$$

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$$

$$\dot{\rho} = \partial_t \rho = \sum_j \left(p_j |\dot{\psi}_j\rangle\langle\psi_j| + p_j |\psi_j\rangle\langle\dot{\psi}_j| \right)$$

$$= \sum_j p_j \left(-\frac{i}{\hbar} H |\psi_j\rangle\langle\psi_j| + \frac{i}{\hbar} |\psi_j\rangle\langle\psi_j| H \right)$$

$$= -\frac{i}{\hbar} \sum_j p_j \left(H |\psi_j\rangle\langle\psi_j| - |\psi_j\rangle\langle\psi_j| H \right) = -\frac{i}{\hbar} [H, \rho]$$

3)

i) $\sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\quad \sigma_- = \frac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\quad \sigma_+^{\dagger} = \sigma_-$, $\sigma_-^{\dagger} = \sigma_+$

$$\sigma_+\sigma_- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_-\sigma_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[\sigma_+, \sigma_-] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z, \qquad \{\sigma_+, \sigma_-\} = \mathbb{1}$$

ii)
$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \gamma_+\left(\sigma_+\rho\sigma_+^{\dagger} - \frac{1}{2}\rho\sigma_+^{\dagger}\sigma_+ - \frac{1}{2}\sigma_+^{\dagger}\sigma_+\rho\right) + \gamma_-\left(\sigma_-\rho\sigma_-^{\dagger} - \frac{1}{2}\rho\sigma_-^{\dagger}\sigma_- - \frac{1}{2}\sigma_-^{\dagger}\sigma_-\rho\right)$$

$$= \frac{-i\omega}{2}\left(\sigma_x\rho - \rho\sigma_x\right) + \gamma_+\left(\sigma_+\rho\sigma_+^{\dagger} - \frac{1}{2}\rho\sigma_+^{\dagger}\sigma_+ - \frac{1}{2}\sigma_+^{\dagger}\sigma_+\rho\right) + \gamma_-\left(\sigma_-\rho\sigma_-^{\dagger} - \frac{1}{2}\rho\sigma_-\sigma_-^{\dagger} - \frac{1}{2}\sigma_-\sigma_-^{\dagger}\rho\right)$$

$$= -\frac{i\omega}{2}\left(\sigma_x\rho - \rho\sigma_x\right) + \gamma_+\left(\sigma_+\rho\sigma_- - \frac{1}{2}\rho\sigma_-\sigma_+ - \frac{1}{2}\sigma_-\sigma_+\rho\right) + \gamma_-\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\rho\sigma_-\sigma_+ - \frac{1}{2}\sigma_-\sigma_+\rho\right)$$

$$\dot{\vec{\rho}} = \Big\{-\frac{i\omega}{2}\left(\mathbb{1}\otimes\sigma_x - \sigma_x\otimes\mathbb{1}\right) + \gamma_+\left(\sigma_+\otimes\sigma_+ - \frac{1}{2}(\sigma_-\sigma_+)\otimes\mathbb{1} - \frac{1}{2}\mathbb{1}\otimes(\sigma_-\sigma_+)\right)$$
$$+ \gamma_-\left(\sigma_-\otimes\sigma_- - \frac{1}{2}(\sigma_+\sigma_-)\otimes\mathbb{1} - \frac{1}{2}\mathbb{1}\otimes(\sigma_+\sigma_-)\right)\Big\}\vec{\rho}$$

$$= \underbrace{\left[\frac{i\omega}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} -\gamma_- & 0 & 0 & \gamma_+ \\ 0 & -\frac{1}{2}(\gamma_+ + \gamma_-) & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(\gamma_- + \gamma_+) & 0 \\ \gamma_- & 0 & 0 & -\gamma_+ \end{pmatrix}\right]}_{=\mathcal{L}} \vec{\rho}$$

$$\mathcal{L} = \frac{1}{2}\begin{pmatrix} -2\gamma_- & -i\omega & i\omega & 2\gamma_+ \\ -i\omega & -\gamma_+ - \gamma_- & 0 & i\omega \\ i\omega & 0 & -\gamma_+ - \gamma_- & -i\omega \\ 2\gamma_- & i\omega & -i\omega & -2\gamma_+ \end{pmatrix}$$

iii) $\vec{\rho} = \begin{pmatrix} a \\ b^* \\ b \\ c \end{pmatrix}$

$$\dot{\vec{\rho}} = \mathcal{L}\,\vec{\rho} = \frac{1}{2}\begin{pmatrix} -2\gamma_- & -i\omega & i\omega & 2\gamma_+ \\ -i\omega & -\gamma_- -\gamma_+ & 0 & i\omega \\ i\omega & 0 & -\gamma_- -\gamma_+ & -i\omega \\ 2\gamma_- & i\omega & -i\omega & -2\gamma_+ \end{pmatrix}\begin{pmatrix} a \\ b^* \\ b \\ c \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} -\gamma_- a - \frac{i\omega}{2} b^* + \frac{i\omega}{2} b + \gamma_+ c \\ -\frac{i\omega}{2} a - \frac{\gamma_- + \gamma_+}{2} b^* + \frac{i\omega}{2} c \\ \frac{i\omega}{2} a - \frac{\gamma_- + \gamma_+}{2} b - \frac{i\omega}{2} c \\ \gamma_- a + \frac{i\omega}{2} b^* - \frac{i\omega}{2} b - \gamma_+ c \end{pmatrix} \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix}$$

$$\Rightarrow \operatorname{tr} \dot{\vec{\rho}} = 0 \quad \Leftrightarrow \quad \operatorname{tr} \dot{\rho} = \frac{d}{dt} \operatorname{tr} \rho = 0$$

iv) $c = 1 - a$, $\quad \mathcal{L}\, \vec{\rho}_{ss} \overset{!}{=} 0$

from (2) $\quad \frac{i\omega}{2}(c-a) - \frac{\gamma_- + \gamma_+}{2} b^* = 0 \quad \Rightarrow \quad b = \frac{i\omega(2a-1)}{\gamma_- + \gamma_+}$

from (1) $\quad -\gamma_- a + \frac{i\omega}{2}(b - b^*) + \gamma_+ c = 0$

$$\Rightarrow a = \frac{\omega^2 + \gamma_+^2 + \gamma_+\gamma_-}{(\gamma_+ + \gamma_-)^2 + 2\omega^2}$$

$$\Rightarrow \rho_{ss} = \frac{1}{2\omega^2 + (\gamma_+ + \gamma_-)} \begin{pmatrix} \omega^2 + \gamma_+^2 + \gamma_+\gamma_- & i\omega(\gamma_+ - \gamma_-) \\ -i\omega(\gamma_+ - \gamma_-) & \omega^2 + \gamma_-^2 + \gamma_-\gamma_+ \end{pmatrix}$$

$$= \frac{1}{2\omega^2 + \gamma^2(2\upsilon - 1)^2} \begin{pmatrix} \omega^2 + \gamma^2(2\upsilon^2 - 3\upsilon + 1) & i\gamma/2 \\ -i\gamma/2 & \omega^2 + \gamma^2(2\upsilon^2 - \upsilon) \end{pmatrix}$$

γ: dissipation, $\quad \upsilon$: temperature, $\quad \upsilon = 1 \Leftrightarrow T = 0$

$\upsilon = \infty \Leftrightarrow T = \infty$