## Exercise 8)

1) a) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$

b) $$e^{ixA} = \sum_{k=0}^{\infty} \frac{1}{k!} (ixA)^k = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}}_{=\cos x} \underbrace{A^{2k}}_{=\mathbb{1}} + i \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}}_{=\sin x} A \underbrace{A^{2k}}_{=\mathbb{1}}$$

$$= \cos x \, \mathbb{1} + i \sin x \, A$$

c) $$R_y(\theta) = e^{-i\frac{\theta}{2}\sigma_y} = \cos\left(-\frac{\theta}{2}\right)\mathbb{1} + i\sigma_y \sin\left(-\frac{\theta}{2}\right) = \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sin\left(\frac{\theta}{2}\right)$$

$$= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$R_z(\theta) = e^{-i\frac{\theta}{2}\sigma_z} = \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sin\left(\frac{\theta}{2}\right) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right) & 0 \\ 0 & \cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$$

d) $$UU^{+} = e^{i\alpha}\begin{pmatrix} e^{i(-\beta-\delta)}\cos\gamma & -e^{i(-\beta+\delta)}\sin\gamma \\ e^{i(\beta-\delta)}\sin\gamma & e^{i(\beta+\delta)}\cos\gamma \end{pmatrix} e^{-i\alpha}\begin{pmatrix} e^{-i(-\beta-\gamma)}\cos\gamma & e^{-i(\beta-\gamma)}\sin\gamma \\ -e^{-i(-\beta+\delta)}\sin\gamma & e^{-i(\beta+\delta)}\cos\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\gamma + \sin^2\gamma & e^{-2i\beta}(\sin\gamma\cos\gamma - \cos\gamma\sin\gamma) \\ e^{2i\beta}(\sin\gamma\cos\gamma - \cos\gamma\sin\gamma) & \sin^2\gamma + \cos^2\gamma \end{pmatrix} = \mathbb{1} \quad \checkmark$$

e) $$U = e^{i\alpha} R_z(2\beta) R_y(2\gamma) R_z(2\delta) = e^{i\alpha}\begin{pmatrix} e^{-i\beta} & 0 \\ 0 & e^{i\beta} \end{pmatrix}\begin{pmatrix} e^{-i\delta}\cos\gamma & -e^{i\delta}\sin\gamma \\ e^{-i\delta}\sin\gamma & e^{i\delta}\cos\gamma \end{pmatrix}$$

$$= e^{i\alpha}\begin{pmatrix} e^{i(-\beta-\delta)}\cos\gamma & -e^{i(-\beta+\delta)}\sin\gamma \\ e^{i(\beta-\delta)}\sin\gamma & e^{i(\beta+\delta)}\cos\gamma \end{pmatrix} \quad \checkmark$$

cf. Bloch sphere

2)

a) $$UU^{+} = \frac{1}{2}\begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & i & -1 \\ 1 & -i & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 1 \\ -i & 0 & 0 & i \\ 0 & -i & -i & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \mathbb{1} \quad \checkmark$$

b) $$U_1 = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \end{pmatrix}$$ (Hadamard, $\frac{\sigma_x + \sigma_z}{\sqrt{2}}$)

$$U_1 U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & i/\sqrt{2} & -1/\sqrt{2} \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$ ($\sigma_x$ Matrix)

$$U_2 U_1 U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$U_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -i/\sqrt{2} & -i/\sqrt{2} \\ 0 & 0 & i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}$$ (phase factor + Hadamard)

$$U_3 U_2 U_1 U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$U_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$ (phase factors)

$$U_4 U_3 U_2 U_1 U = \mathbb{1}$$

c) $$U = (U_4 U_3 U_2 U_1)^{-1}\mathbb{1} = (U_4 U_3 U_2 U_1)^{+}$$
$$= U_1^{+} U_2^{+} U_3^{+} U_4^{+}$$