Exercise 5 – Density Operator

The *density operator* ϱ is defined as

$$\varrho \equiv \sum_j p_j \left|\Psi_j\right\rangle \left\langle\Psi_j\right| , \qquad \sum_j p_j = 1 \,. \tag{1}$$

The system may be found in the state $\left|\Psi_j\right\rangle$ with probability p_j. If the state of the system is known exactly, i.e., $\varrho = |\Psi\rangle\langle\Psi|$, the state is called *pure*, otherwise *mixed.*
Suppose we have a system ϱ^{AB} which consists of two subsystems A and B. The *reduced density operator* is defined by

$$\varrho^A \equiv \mathrm{Tr}_B \left(\varrho^{AB}\right) , \tag{2}$$

where

$$\mathrm{Tr}_B \left(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|\right) \equiv |a_1\rangle\langle a_2| \,\mathrm{Tr}\left(|b_1\rangle\langle b_2|\right) , \tag{3}$$

and $|a_1\rangle , |a_2\rangle$ are states of A, and $|b_1\rangle , |b_2\rangle$ are states of B.

Problem 1

Proof that

a) $\mathrm{Tr}\,\varrho = 1$,

b) ϱ is Hermitian,

c) ϱ is semi-positive,

d) $\mathrm{Tr}\,\varrho^2 \leq 1$ and $\mathrm{Tr}\,\varrho^2 = 1$ if and only if ϱ is a pure state.

Problem 2

a) Show that the density operator ϱ fulfills the *von Neumann* equation

$$\dot{\varrho} = -\frac{i}{\hbar} \left[H, \varrho\right] \,. \tag{4}$$

b) Show that an arbitrary density matrix for a mixed state qubit may be written as

$$\varrho = \frac{1}{2} \left(\mathbb{1} + \vec{r} \cdot \vec{\varrho}\right) \,. \tag{5}$$

For what $\vec{r}$ is ϱ pure?

Problem 3

Consider the state

$$\varrho^{12} = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right) \tag{6}$$

a) Show that ϱ^{12} is a pure state.

b) Calculate ϱ^1 and show that ϱ^1 is a mixed state.

Problem 4

Suppose a composite system of A and B is in the state $|a\rangle\,|b\rangle$, where $|a\rangle$ is a pure state of system A, and $|b\rangle$ is a pure state of system B. Show that the reduced density operator of system A alone is a pure state.

Problem 5

In exercise 5 the initial state was given by

$$|\Psi\rangle_{\text{in}} = \frac{1}{\sqrt{2}}\left(\alpha\,|0_S 0_A 0_B\rangle + \alpha\,|0_S 1_A 1_B\rangle + \beta\,|1_S 0_A 0_B\rangle + \beta\,|1_S 1_A 1_B\rangle\right) . \tag{7}$$

Calculate the reduced density operator ϱ^B and show that it is a mixed state.