Exercise 8

Problem 1 – Z-Y decomposition of a single qubit

The Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

We will show that any unitary gate on a single qubit can be implemented using only Z and Y rotations.

a) Show that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$.

b) Show that, if A is a matrix such that $A^2 = \mathbb{1}$, then, for any real number x,

$$e^{ixA} = \cos x \mathbb{1} + i \sin x A. \tag{2}$$

c) Use the previous step to show that

$$R_y(\theta) \equiv e^{-i\frac{\theta}{2}\sigma_y} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{3}$$

$$R_z(\theta) \equiv e^{-i\frac{\theta}{2}\sigma_z} = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix} \tag{4}$$

d) Show that a unitary 2×2 matrix can be written as

$$U = \begin{pmatrix} e^{i(\alpha-\beta-\delta)}\cos\gamma & -e^{i(\alpha-\beta+\delta)}\sin\gamma \\ e^{i(\alpha+\beta-\delta)}\sin\gamma & e^{i(\alpha+\beta+\delta)}\cos\gamma \end{pmatrix} \tag{5}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers.

e) Use the results above to show that U can be implemented as

$$U = e^{i\alpha} R_z(2\beta) R_y(2\gamma) R_z(2\delta). \tag{6}$$