Exercise 2 – Qubits

Problem 1

We denote two orthonormal states of a single qubit as $\{|0\rangle, |1\rangle\}$ where

$$\langle 0|0\rangle = \langle 1|1\rangle = 1, \qquad \langle 0|1\rangle = \langle 1|0\rangle = 0. \tag{1}$$

Any state $|\Psi\rangle$ of this system can be written as a superposition

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \quad |\alpha|^2 + |\beta|^2 = 1, \qquad \alpha, \beta \in \mathbb{C}. \tag{2}$$

a) Find a parameter representation for $|\Psi\rangle$ if the underlying field is (i) the set of real numbers and (ii) the set of complex numbers.

b) Consider the normalized states

$$|\Psi_1\rangle = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \qquad |\Psi_2\rangle = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}. \tag{3}$$

Find the condition on θ_1 and θ_2 such that $|\Psi_1\rangle + |\Psi_2\rangle$ is normalized.

c) Let

$$A = |0\rangle\,\langle 0| + |1\rangle\,\langle 1|\,. \tag{4}$$

Calculate A for

(i) $|0\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$

(ii) $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(iii) $|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Problem 2

The *Walsh-Hadamard* transform is defined as

$$|0\rangle \to \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \qquad |1\rangle \to \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \tag{5}$$

a) Find the unitary operator U_{H} which implements the *Walsh-Hadamard* transform with respect to the basis $\{|0\rangle, |1\rangle\}$.

b) Find the inverse of the operator U_{H}.

c) Find the matrix representation of U_{H} for the basis

$$|0\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}. \tag{6}$$

Problem 3

Let $|\Psi\rangle = \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}$ where $\phi, \theta \in \mathbb{R}$.

a) Find $\rho = |\Psi\rangle \langle\Psi|$.

b) Find $\operatorname{tr}\rho$.

c) Find $\operatorname{tr}\rho^2$.

Problem 4

Given the Hamilton operator

$$H = \hbar\omega\sigma_x, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7}$$

a) Find the eigenenergies and the eigenstates of the Hamiltonian.

b) Find the solution $|\Psi(t)\rangle$ for the time-dependent Schrödinger equation

$$i\hbar\partial_t |\Psi\rangle = H |\Psi\rangle \tag{8}$$

with the initial conditions $|\Psi(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

c) Find and discuss the probability $|\langle\Psi(t=0)|\Psi(t)\rangle|^2$.

Problem 5

A system of n-qubits represents a finite-dimensional Hilbert space over the complex numbers of dimension 2^n. A state $|\Psi\rangle$ is a superposition of the basic states

$$|\Psi\rangle = \sum_{j_1,j_2,\dots,j_n=0}^{1} c_{j_1,j_2,\dots,j_n} |j_1\rangle \otimes |j_2\rangle \otimes \cdots \otimes |j_n\rangle = \sum_{j_1,j_2,\dots,j_n=0}^{1} c_{j_1,j_2,\dots,j_n} |j_1 j_2 \dots j_n\rangle\,. \tag{9}$$

Can the state

$$|\Psi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \tag{10}$$

be written as a product state, i.e. in the form of $|\Psi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle$?