Exercise 1

Problem 1

a) i) $|\psi\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

ii) $|\psi\rangle = \begin{pmatrix} e^{i\varphi}\cos\theta \\ \sin\theta \end{pmatrix}$

b) $$\left\| |\psi_1\rangle + |\psi_2\rangle \right\| = \left\| \begin{pmatrix} \cos\theta_1 + \cos\theta_2 \\ \sin\theta_1 + \sin\theta_2 \end{pmatrix} \right\| = \cos^2\theta_1 + 2\cos\theta_1\cos\theta_2 + \cos^2\theta_2 + \sin^2\theta_1 + 2\sin\theta_1\sin\theta_2 + \sin^2\theta_2$$

$$= 2 + 2(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2) = 2 + 2\cos(\theta_1 - \theta_2) \overset{!}{=} 1$$

$$\Rightarrow \cos(\theta_1 - \theta_2) = -\frac{1}{2} \Rightarrow \theta_1 - \theta_2 = \pm\frac{2\pi}{3}$$

c) $A = |0\rangle\langle 0| + |1\rangle\langle 1|$

i) $$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\ 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0\ 1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

ii) $$A = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1\ 1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}(1\ -1) \right] = \frac{1}{2}\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right] = \mathbb{1}$$

iii) $$A = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}(\cos\theta\ \sin\theta) + \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}(\sin\theta\ -\cos\theta)$$

$$= \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} + \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix} = \mathbb{1}$$

Problem 2

a) $U_H = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle \right)\langle 0| + \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle \right)\langle 1|$

b) U_H is unitary, so

$$U_H^{-1} = U_H = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1| \right) = U_H$$

c) $$U_H = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} (1 \quad 0) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (0 \quad 1) \right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right]$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

d) $$U_H = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix} 2 \\ 0 \end{pmatrix} (1 \quad 1) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} (1 \quad -1) \right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix} \right]$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Problem 3

a) $$\rho = \underbrace{\begin{pmatrix} e^{i\varphi}\cos\theta \\ \sin\theta \end{pmatrix}}_{= |\psi\rangle} \underbrace{\left(e^{-i\varphi}\cos\theta \quad \sin\theta \right)}_{= \langle\psi|} = \begin{pmatrix} \cos^2\theta & e^{i\varphi}\sin\theta\cos\theta \\ e^{-i\varphi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

b) $\mathrm{Tr}\,\rho = \sin^2\theta + \cos^2\theta = 1$

c) $\rho^2 = \left(|\psi\rangle\langle\psi| \right)^2 = |\psi\rangle \underbrace{\langle\psi|\psi\rangle}_{=1} \langle\psi| = |\psi\rangle\langle\psi| = \rho$

Problem 4

$$H = \hbar\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

a) $E_\pm = \pm \hbar\omega$, $\quad |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $$|\psi(t)\rangle = \alpha e^{-E_+ t/\hbar} |+\rangle + \beta e^{-E_- t/\hbar} |-\rangle$$

$$= \alpha e^{-i\omega t} |+\rangle + \beta e^{i\omega t} |-\rangle$$

$$= \frac{\alpha}{\sqrt{2}} e^{-i\omega t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{\beta}{\sqrt{2}} e^{i\omega t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[\alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \overset{!}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \alpha = \beta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow |\psi(t)\rangle = \frac{1}{2} \begin{pmatrix} e^{-i\omega t} + e^{i\omega t} \\ -e^{-i\omega t} + e^{i\omega t} \end{pmatrix} = \begin{pmatrix} \cos\omega t \\ -i \sin\omega t \end{pmatrix}$$

c) $$|\langle\psi(t=0)|\psi(t)\rangle|^2 = \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\omega t \\ -i\sin\omega t \end{pmatrix} \right|^2 = \cos^2 \omega t$$

Rabi cycle