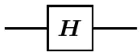
Exercise 6 – Quantum Gates, Superdense Coding

Problem 1

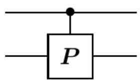
We denote the *Hadamard gate* as



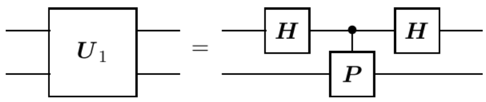
and the two-qubit *controlled phase gate* $\Lambda(\mathbf{P})$ acts in the canonical basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ as the diagonal 4×4 matrix

$$\Lambda(\mathbf{P}) = \mathrm{diag}(1, 1, 1, i) \tag{1}$$

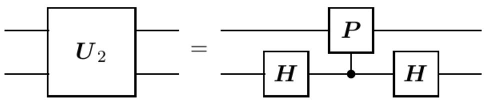
and can be denoted by



a) Consider the two-qubit unitary transformations U_1 and U_2 defined by the quantum circuits



and



Let $|ab\rangle$ denote the element of the standard basis where a labels the upper qubit in the circuit diagram and b labels the lower qubit. Show that U_1 and U_2 both act trivially on the states

$$|00\rangle, \quad \frac{1}{\sqrt{3}}\left(|01\rangle + |10\rangle + |11\rangle\right) \tag{2}$$

b) Thus U_1 and U_2 act nontrivially only in the two-dimensional space spanned by

$$\left\{ |\phi_1\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right), \quad |\phi_2\rangle = \frac{1}{\sqrt{6}}\left(|01\rangle + |10\rangle - 2\,|11\rangle\right) \right\}. \tag{3}$$

Find U_1 and U_2 in this basis.

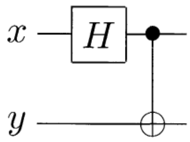


Figure 1: Quantum circuit

Problem 2

What is the effect of the quantum circuit depicted in Fig. 1 on the initial states $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$?

Problem 3

Suppose Alice wants to send Bob two bits of classical information. Initially, they share an entangled qubit in the state

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right). \tag{4}$$

Alice performs any operation on her qubit and then sends it to Bob. How can she send two bits of classical information to Bob?