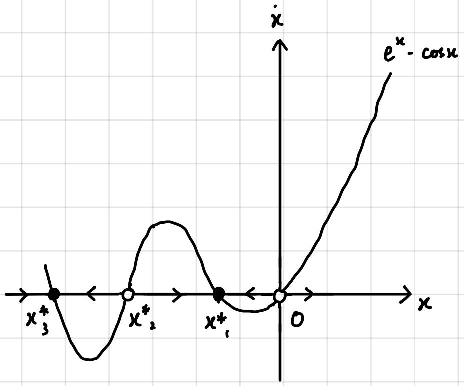
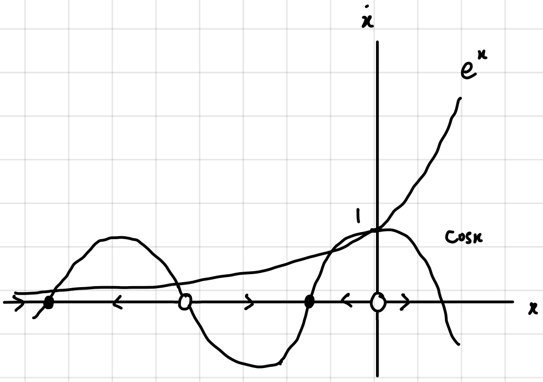
Session 1

1. $\dot{x} = e^{x} - \cos x$





• - stable points (x_1^*, x_3^*)

o - unstable points $(x_2^*, 0)$

where $x_1^* \approx -1.29$, $x_2^* \approx -4.72$, $x_3^* \approx -7.85$

determined by Newton - Raphson method on Matlab setting $e^{x} - \cos x = 0$.

for values of $e^{x} > \cos x$, $f'(x) > 0$

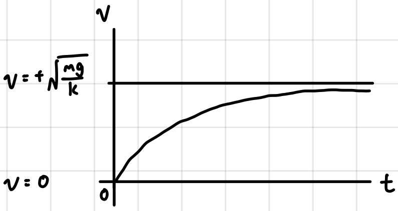
for values of $e^{x} < \cos x$, $f'(x) < 0$

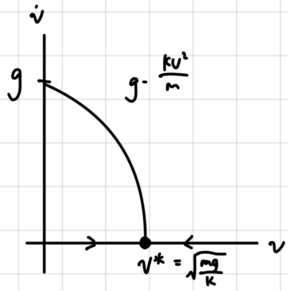
2. $\dot{v} = g - \frac{kv^2}{m}$, $v(0) = 0$

a) To find the fixed points, set $\dot{v} = 0$:

$$\Rightarrow g = \frac{k{v^*}^2}{m}$$

$$v^* = \sqrt{\frac{mg}{k}} \quad (\text{ignore negatives as } v > 0 \ \forall t \in \mathbb{R})$$





$\Rightarrow$ the fixed point $v^* = \sqrt{\frac{mg}{k}}$ is stable, as it is the terminal velocity

2. c) The answers obtained from (a) and (b) are the same, which is consistent with the theory that as $t \to \infty$, the equilibrium terminal velocity should be a fixed stable point.

2.b) $\dot{v} = g - \frac{k}{m} v^2$

$$\dot{v} = \frac{k}{m}\left(\frac{mg}{k} - v^2\right)$$

$$\int \frac{1}{\left(\frac{mg}{k} - v^2\right)} dv = \int \frac{k}{m} dt$$

$$\int \frac{1}{\left(\sqrt{\frac{mg}{k}}\right)^2 - v^2} dv = \frac{k}{m} t + C$$

Let $v^* = \sqrt{\frac{mg}{k}}$

$$\Rightarrow \int \frac{1}{(v^*)^2 - v^2} dv = \frac{k}{m} t + C$$

Let $v = v^* \tanh(u)$ $\Rightarrow \frac{dv}{du} = v^*(1 - \tanh^2(u))$

$$v^2 = (v^*)^2 \tan^2 h(u)$$

$$(v^*)^2 - v^2 = (v^*)^2 (1 - \tan^2 h(u))$$

$$\therefore \int \frac{1}{(v^*)^2 (1 - \tanh^2(u))} v^* (1 - \tanh^2(u))\, dx = \frac{k}{m} t + C$$

$$\int \frac{1}{v^*} du = \frac{k}{m} t + C$$

$$\frac{u}{v^*} = \frac{k}{m} t + C$$

$$\frac{1}{v^*} \cdot \tanh^{-1}\left(\frac{v}{v^*}\right) = \frac{k}{m} t + C$$

$\because v(0) = 0 \Rightarrow \frac{1}{v^*} \tanh^{-1}(0) = \frac{k}{m}(0) + C$

$$C = 0$$

$$\frac{1}{v^*} \tanh^{-1}\left(\frac{v}{v^*}\right) = \frac{k}{m} t$$

$$v = v^* \tanh\left[\frac{v^* k}{m} t\right]$$

$$\therefore v(t) = \sqrt{\frac{mg}{k}} \tanh\left[\sqrt{\frac{gk}{m}}\, t\right]$$

as $t \to \infty$, $\tanh\left[\sqrt{\frac{gk}{m}}\, t\right] \to 1$

$\Rightarrow$ terminal velocity $v(\infty) \to \sqrt{\frac{mg}{k}}$

3. Normal form of saddle-node bifurcation:

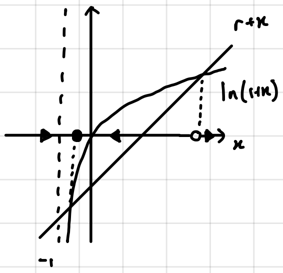
$$\dot{x} = r + x^2$$

$\Rightarrow$ for $\dot{x} = r + x - \ln(1+x)$, it is a saddle-node bifurcation.

set $\dot{x} = 0$:

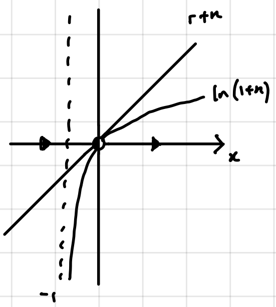
$$r + x - \ln(1+x) = 0$$

$$r + x = \ln(1+x)$$



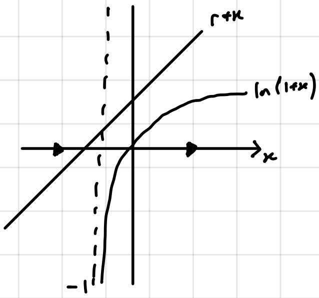
$r<0$

· one stable, one unstable



$r=0$

· one half-stable point



$r>0$

· no critical points

consider when the graphs intersect tangentially,

$\therefore\ r + x = \ln(1+x)$

$$\Rightarrow \frac{d}{dx}(r+x) = \frac{d}{dx}(\ln(1+x))$$

$$1 = \frac{1}{1+x}$$

$$1+x = 1$$

$$x = 0$$

$$r + 0 = \ln(1+0)$$

$$r = 0$$

bifurcation point $r_c = 0$, and the bifurcation occurs at $x=0$.

· Bifurcation diagram of r against x:

$$r = \ln(1+x) - x$$

If we consider the taylor expansion of $\ln(1+x) - x$ near the bifurcation point $x=0$, $r=0$, we have

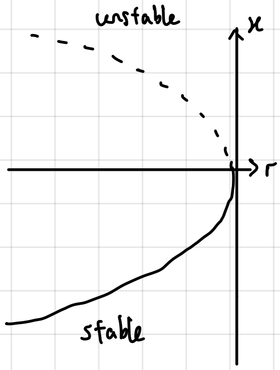
$$r = \left[x - \frac{x^2}{2} + O(x^3)\right] - x$$

$$r \approx -\frac{x^2}{2}$$, which is in the normal form of the saddle-node bifurcation: $x = \pm\sqrt{-2r}$, $r<0$

for $r<0$,

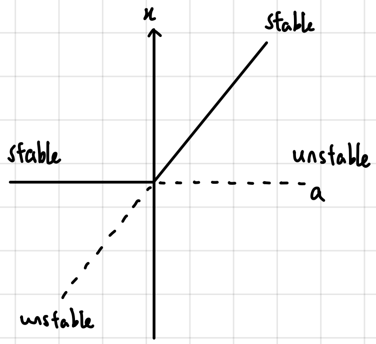
$x>0 \Rightarrow$ unstable fixed point

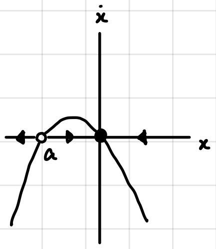
$x<0 \Rightarrow$ stable fixed point



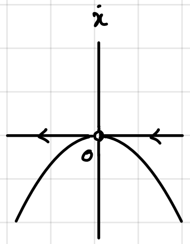
4. $\dot{x} = ax - bx^2$, type of bifurcation is <u>transcritical bifurcation</u>

for $b > 0$:



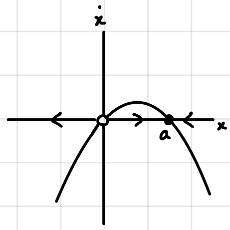


$a < 0 , b > 0$



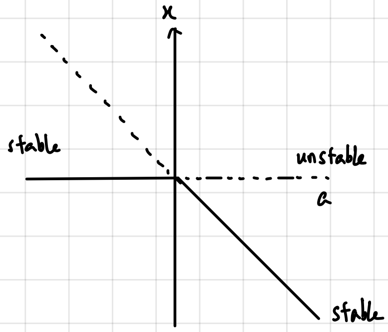
$a = 0 , b > 0$

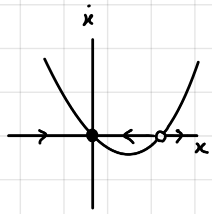
half-stable



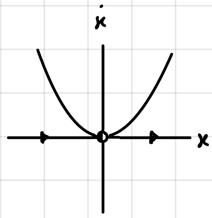
$a > 0 , \ b > 0$

for $b < 0$:



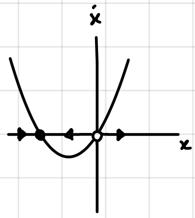


$a < 0 , \ b < 0$



$a = 0 , \ b < 0$

half-stable



$a > 0 , \ b < 0$

Session 2

1. a) $\dot{x} = x - y - x(x^2 + 5y^2)$

$\dot{y} = x + y - y(x^2 + y^2)$

for any system with fixed point at origin $(0,0)$: x and y are deviations from fixed point.

We can linearise the equation by omitting nonlinear terms in x, y:

$\therefore \dot{x} = x - y$, $\dot{y} = x + y$ $\Rightarrow$ Jacobian $A = \begin{pmatrix} \frac{\partial}{\partial x}(x-y) & \frac{\partial}{\partial y}(x-y) \\ \frac{\partial}{\partial x}(x+y) & \frac{\partial}{\partial y}(x+y) \end{pmatrix}_{(x^*,y^*)}$

$= \begin{pmatrix} 1-y & x-1 \\ 1+y & x+1 \end{pmatrix}_{(x^*,y^*)}$

$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\det(A) = \det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, Trace $(A) = 1 + 1$

$= 2$

$= 1 - (-1)$

$= 2$

eigenvalues: $\lambda_{1,2} = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$

$= \dfrac{2 \pm \sqrt{2^2 - 4(2)}}{2}$

$= \dfrac{2 \pm \sqrt{-4}}{2}$

$= \dfrac{2 \pm 2i}{2}$

$= 1 \pm i$

$\because \mathrm{Re}(\lambda) = 1 > 0$, and the eigenvalues are complex conjugates with $\det(A) > 0$, then the fixed point is a spiral with growing oscillations $\Rightarrow$ $(0,0)$ is an unstable spiral.

b) $x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$x^2 + y^2 = r^2$

$2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2r\dfrac{dr}{dt}$

$x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = r\dfrac{dr}{dt}$

$x\left(x - y - x(x^2 + 5y^2)\right) + y\left(x + y - y(x^2 + y^2)\right) = r\dfrac{dr}{dt}$

$x^2 - xy - x^2(x^2 + 5y^2) + yx + y^2 - y^2(x^2 + y^2) = r\dfrac{dr}{dt}$

$x^2 + y^2 - x^2(x^2 + 5y^2) - y^2(x^2 + y^2) = r\dfrac{dr}{dt}$

$r^2 - x^4 - 5x^2y^2 - x^2y^2 - y^4 = r\dfrac{dr}{dt}$

$r^2 - 6x^2y^2 - x^4 - y^4 = r\dfrac{dr}{dt}$

$r^2 - (x^4 + 6x^2y^2 + y^4) = r\dfrac{dr}{dt}$

$r^2 - \left(r^4\cos^4\theta + r^4\sin^4\theta + 6(r^4\sin^2\theta\cos^2\theta)\right) = r\dfrac{dr}{dt}$

$\dfrac{dr}{dt} = r\left(1 - r^2(\cos^4\theta + \sin^4\theta) - 6r^2\sin^2\theta\cos^2\theta\right)$

$\dfrac{dr}{dt} = r\left(1 - r^2\left(1 - \dfrac{\sin^2 2\theta}{2}\right) - 6r^2(\sin\theta\cos\theta)^2\right)$

$\dfrac{dr}{dt} = r\left(1 - r^2 + r^2\left(\dfrac{\sin^2 2\theta}{2}\right) - 6r^2\left(\dfrac{\sin 2\theta}{2}\right)^2\right)$

$\dfrac{dr}{dt} = r\left[1 - r^2 + r^2\left(\dfrac{\sin^2 2\theta}{2} - \dfrac{6}{4}\sin^2 2\theta\right)\right]$

$\boxed{\dfrac{dr}{dt} = r\left[1 - r^2 - r^2\sin^2(2\theta)\right]}$

$x = r\cos\theta$, $y = r\sin\theta$

$\dfrac{y}{x} = \tan\theta$

$\dfrac{x\left(\frac{dy}{dt}\right) - y\left(\frac{dx}{dt}\right)}{x^2} = \sec^2\theta\dfrac{d\theta}{dt}$

$x\left(\dfrac{dy}{dt}\right) - y\left(\dfrac{dx}{dt}\right) = r^2\cos^2\theta(\sec^2\theta)\dfrac{d\theta}{dt}$

$x\left(\dfrac{dy}{dt}\right) - y\left(\dfrac{dx}{dt}\right) = r^2\dfrac{d\theta}{dt}$

$x\left(x + y - y(x^2 + y^2)\right) - y\left(x - y - x(x^2 + 5y^2)\right) = r^2\dfrac{d\theta}{dt}$

$x^2 + xy - xy(x^2 + y^2) - xy + y^2 + xy(x^2 + 5y^2) = r^2\dfrac{d\theta}{dt}$

$x^2 + y^2 - x^3y - xy^3 + x^3y + 5xy^3 = r^2\dfrac{d\theta}{dt}$

$x^2 + y^2 + 4xy^3 = r^2\dfrac{d\theta}{dt}$

$r^2 + 4(r\cos\theta)(r\sin\theta)^3 = r^2\dfrac{d\theta}{dt}$

$r^2 + 4r^4\cos\theta\sin^3\theta = r^2\dfrac{d\theta}{dt}$

$\boxed{\dfrac{d\theta}{dt} = 1 + 4r^2\cos\theta\sin^3\theta}$

1.c) $\frac{dr}{dt} = r\left[1 - r^2 - r^2 \sin(2\theta)\right]$, $\frac{d\theta}{dt} = 1 + 4r^2 \cos\theta \sin^3\theta$

i) $\forall r < r_1 = \frac{1}{\sqrt{2}}$:

$$\frac{dr}{dt} = r^3\left[\frac{1}{r^2} - 1 - \sin^2(2\theta)\right]$$

$$= r - r^3 - r^3\sin^2(2\theta)$$

$\because$ $0 < r_1 < 1$ and $r^3 << r$ $\forall r = r_1$,

$0 \leq \sin^2(2\theta) \leq 1$, then $r - r^3 - r^3 \sin(2\theta) > 0$

$\Rightarrow \frac{dr}{dt} > 0$ $\forall r < r_1 = \frac{1}{\sqrt{2}}$

at $r = \frac{1}{\sqrt{2}}$, $\frac{dr}{dt} = \frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}}\right)^3 - \left(\frac{1}{\sqrt{2}}\right)^3 \sin^2(2\theta)$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{8}} - \frac{1}{\sqrt{8}}\sin^2(2\theta)$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}\sin^2(2\theta)$$

$$= \frac{\sqrt{2}}{4}\left(1 - \sin^2(2\theta)\right)$$

$\because$ $0 \leq \sin^2(2\theta) \leq 1$, then $\frac{dr}{dt} > 0$ for $r = \frac{1}{\sqrt{2}}$

$\Rightarrow \frac{dr}{dt} > 0$ $\forall r \leq r_1 = \frac{1}{\sqrt{2}}$

ii) $\forall r \geq r_2 = 1$:

$$\frac{dr}{dt} = r - r^3 - r^3\sin^2(2\theta)$$

$r = 1$,

$\frac{dr}{dt} = 1 - 1 - \sin^2(2\theta)$

$= -\sin^2(2\theta)$

$\because$ $0 \leq \sin^2(2\theta) \leq 1$, then

$\frac{dr}{dt} < 0$ for $r = 1$

for $r > 1$:

$r^3 >> r$,

$\frac{dr}{dt} = r - r^3\left(1 + \sin^2(2\theta)\right)$

$\because$ $0 \leq \sin^2(2\theta) \leq 1$,

then $r - r^3\left(1 + \sin^2(2\theta)\right) < 0$

$\Rightarrow \frac{dr}{dt} < 0$ $\forall r > 1$

$\therefore \frac{dr}{dt} < 0$ $\forall r \geq r_2 = 1$

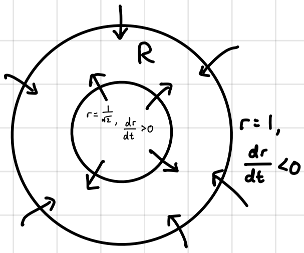
iii) $\frac{\sqrt{2}}{2} \leq r \leq 1$: Consider two concentric circles $r_1 = \frac{1}{\sqrt{2}}$, $r_2 = 1$,

From part i and ii, we know that $\frac{dr}{dt} > 0$

for $r \leq r_1 = \frac{1}{\sqrt{2}}$, and $\frac{dr}{dt} < 0$ for $r \geq r_2 = 1$,

$\Rightarrow \frac{dr}{dt} > 0$ on inner circle and $\frac{dr}{dt} < 0$ for outer circle.

$\therefore$ We have the annulus $0 < \frac{1}{\sqrt{2}} \leq r \leq 1$ which is our bounded region R.



$\Rightarrow$ in this trapping region R, we have trajectories outside the region pointing inwards (from part ii), and trajectories inside the region on the inner circle pointing outwards towards R (from part i), thus all trajectories in R are confined.

(satisfies condition (1) and (4) of Poincaré-Bendixson theorem).

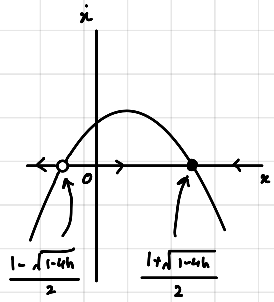
$\because$ $\frac{dr}{dt} = r\left[1 - r^2 - r^2\sin^2(2\theta)\right]$ is continuously differentiable and the only fixed point is at the origin and unstable, which is outside of our trapping region R $\frac{1}{\sqrt{2}} \leq r \leq 1$, (satisfies (2) and (3) of Poincaré-Bendixson theorem)

$\Rightarrow$ By the Poincaré-Bendixson theorem, there exists a closed limiting cycle for $\frac{1}{\sqrt{2}} \leq r \leq 1$ which does not contain any fixed points.

2. $\dot{x} = x(1-x) - h \quad , \quad h \geq 0$

To find bifurcation values, $\dot{x} = 0$:

$$x(1-x) - h = 0$$
$$x - x^2 - h = 0$$
$$x^2 - x + h = 0$$
$$x = \frac{1 \pm \sqrt{1^2 - 4(1)(h)}}{2}$$
$$x^* = \frac{1 \pm \sqrt{1-4h}}{2}$$



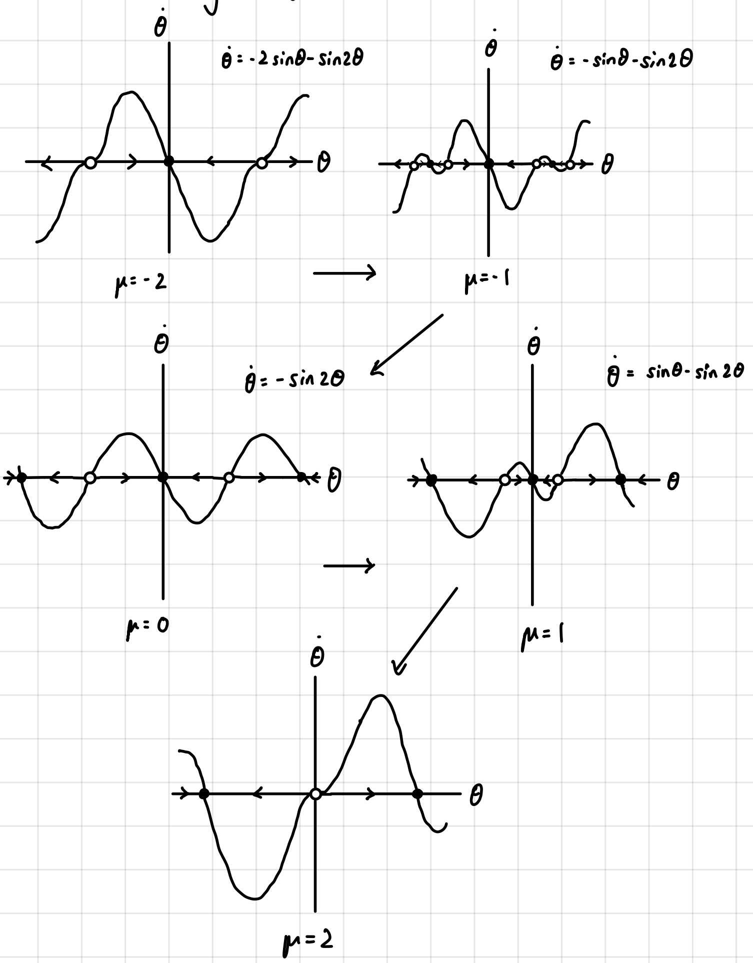
$\Rightarrow$ saddle-node bifurcation, as if we vary h, then the graph translates vertically and the fixed point changes from one stable and one unstable, for $h \geq 0$ to one half stable point at $h = \frac{1}{4}$, and then no fixed points for $h < 0$.

$\therefore$ bifurcation occurs at $h = \frac{1}{4}$ and

$$x = \frac{1 \pm \sqrt{1 - 4(\frac{1}{4})}}{2}$$
$$x = \frac{1}{2}$$

3. $\dot{\theta} = \mu \sin\theta - \sin 2\theta$

Plotting $\dot{\theta}$ against θ, we have:



$\because$ The origin and the nearest two other fixed points switch stability from $\mu = -2$ to $\mu = 2$, then the bifurcation that occurs is a subcritical pitchfork bifurcation.

4. $\frac{dR}{dt} = aR + bJ$, $\frac{dJ}{dt} = bR + aJ$

Let matrix A be $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

$\text{Trace}(A) = \tau = 2a$, $\det(A) = \Delta = a^2 - b^2$, $\tau^2 - 4\Delta = 4b^2$

$$\det(A - \lambda I) = \det\begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix}$$
$$= (a-\lambda)^2 - b^2$$
$$= a^2 - 2a\lambda + \lambda^2 - b^2$$
$$= \lambda^2 - 2a\lambda + a^2 - b^2$$

$$\Rightarrow \lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2-b^2)}}{2}$$
$$\lambda = a \pm \tfrac{1}{2}\sqrt{4b^2}$$
$$\lambda = a \pm b \quad \text{(eigenvalues)}$$

$\begin{pmatrix} \dot{R} \\ \dot{J} \end{pmatrix} = A \begin{pmatrix} R \\ J \end{pmatrix}$ - To find eigenvectors:

$$\begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\lambda = \lambda_1 = a+b$

$$\Rightarrow \begin{pmatrix} a-a-b & b \\ b & a-a-b \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -b & b \\ b & -b \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} -bv_1 + bv_2 = 0 \\ bv_1 - bv_2 = 0 \end{matrix}\right\} \begin{matrix} b(-v_1 + v_2) = 0 \\ b(v_1 - v_2) = 0 \end{matrix}$$

$\therefore$ the eigenvector corresponding to $\lambda_1 = a+b$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

, For $\lambda = \lambda_2 = a-b$:

$$\begin{pmatrix} a-a+b & b \\ b & a-a+b \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} b & b \\ b & b \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} bv_1 + bv_2 = 0 \\ bv_1 + bv_2 = 0 \end{matrix}\right\} \begin{matrix} b(v_1 + v_2) = 0 \\ b(v_1 + v_2) = 0 \end{matrix}$$

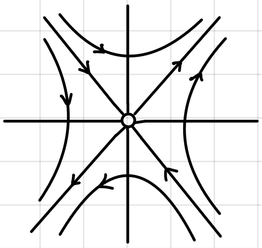
$\therefore$ the eigenvector corresponding to $\lambda_2 = a-b$ is : $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

1. For $a < b$, $0 < a < b$

$\lambda_1 = a+b > 0$

$\lambda_2 = a-b < 0$

$\therefore \lambda_1 > 0 > \lambda_2$



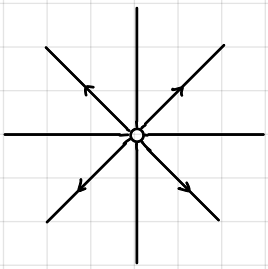
All trajectories approach the line $R = J$, and the fixed point $(0,0)$ is a saddle point

2. For $a < b < 0$,

$\lambda_1 = a+b < 0$

$\lambda_2 = a-b < 0$

$\therefore \lambda_1 < \lambda_2 < 0$



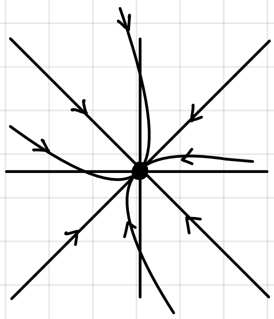
All trajectories lead out to infinity from the unstable node $(0,0)$. $\Rightarrow$ source node

3. For $0 < b < a$

$\lambda_1 = a+b > 0$

$\lambda_2 = a-b > 0$

$\lambda_1 > \lambda_2 > 0$



The fixed point $(0,0)$ is a stable node, all trajectories lead to $(0,0)$