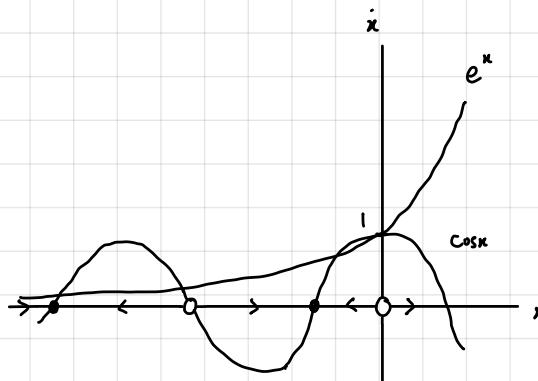
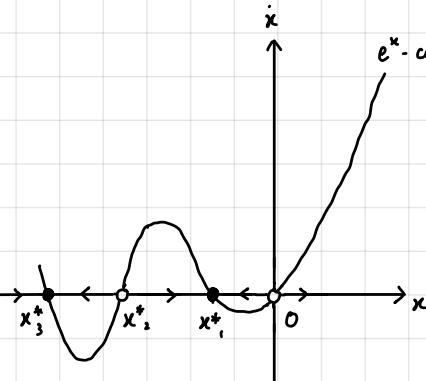


Session 1

$$1. \dot{x} = e^x - \cos x$$



• - stable points (x_1^*, x_2^*)

for values of $e^x > \cos x$, $f'(x) > 0$

○ - unstable points $(x_2^*, 0)$

for values of $e^x < \cos x$, $f'(x) < 0$

where $x_1^* \approx -1.29$, $x_2^* \approx -4.72$, $x_3^* \approx -7.85$

determined by Newton-Raphson method or

Matlab setting $e^x - \cos x = 0$.

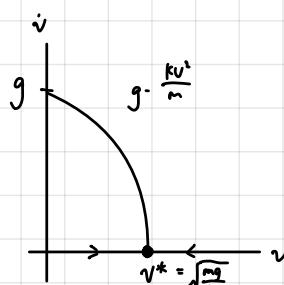
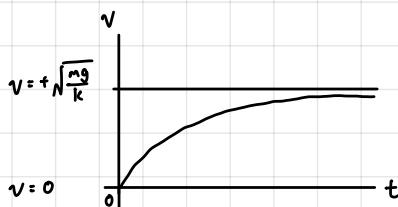
$$2.b) \dot{v} = g - \frac{k}{m} v^2$$

$$2. \dot{v} = g - \frac{kv^2}{m}, \quad v(0) = 0$$

a) To find the fixed points, set $\dot{v} = 0$:

$$\Rightarrow g = \frac{kv^2}{m}$$

$$v^* = \sqrt{\frac{mg}{k}} \quad (\text{ignore negatives as } v > 0 \quad \forall t \in \mathbb{R})$$



\Rightarrow the fixed point $v^* = \sqrt{\frac{mg}{k}}$ is stable, as it is the terminal velocity

2. c) The answers obtained from (a) and (b) are the same,

which is consistent with the theory that as $t \rightarrow \infty$,

the equilibrium terminal velocity should be a fixed

stable point.

$$\dot{v} = \frac{k}{m} (\frac{mg}{k} - v^2)$$

$$\int \frac{1}{(\frac{mg}{k} - v^2)} dv = \int \frac{k}{m} dt$$

$$\int \frac{1}{(\frac{mg}{k})^2 - v^2} dv = \frac{k}{m} t + C$$

$$\text{Let } v^* = \sqrt{\frac{mg}{k}}$$

$$\Rightarrow \int \frac{1}{(v^*)^2 - v^2} dv = \frac{k}{m} t + C$$

$$\text{Let } v = v^* \tanh(u) \Rightarrow \frac{dv}{du} = v^* (1 - \tanh^2(u))$$

$$v^2 = (v^*)^2 \tanh^2(u)$$

$$(v^*)^2 - v^2 = (v^*)^2 (1 - \tanh^2(u))$$

$$\therefore \int \frac{1}{(v^*)^2 (1 - \tanh^2(u))} v^* (1 - \tanh^2(u)) du = \frac{k}{m} t + C$$

$$\int \frac{1}{v^*} du = \frac{k}{m} t + C$$

$$\frac{u}{v^*} = \frac{k}{m} t + C$$

$$\frac{1}{v^*} \cdot \tanh^{-1}\left(\frac{v}{v^*}\right) = \frac{k}{m} t + C$$

$$\therefore v(0) = 0 \Rightarrow \frac{1}{v^*} \tanh^{-1}(0) = \frac{k}{m}(0) + C$$

$$C = 0$$

$$\frac{1}{v^*} \tanh^{-1}\left(\frac{v}{v^*}\right) = \frac{k}{m} t$$

$$v = v^* \tanh\left[\frac{v^* k}{m} t\right]$$

$$\therefore v(t) = \sqrt{\frac{mg}{k}} \tanh\left[\sqrt{\frac{gk}{m}} t\right]$$

as $t \rightarrow \infty$, $\tanh\left[\sqrt{\frac{gk}{m}} t\right] \rightarrow 1$

\Rightarrow terminal velocity $v(\infty) \rightarrow \sqrt{\frac{mg}{k}}$

3. Normal form of saddle-node bifurcation:

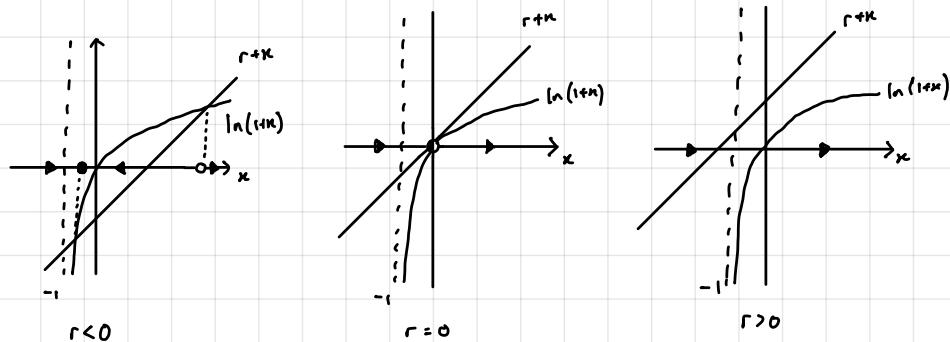
$$\dot{x} = r + x^2$$

\Rightarrow for $\dot{x} = r + x - \ln(1+x)$, it is a saddle-node bifurcation.

set $\dot{x} = 0$:

$$r + x - \ln(1+x) = 0$$

$$r + x = \ln(1+x)$$



• one stable, one unstable

• one half-stable point

• no critical points

Consider when the graphs intersect tangentially,

$$\therefore r + x = \ln(1+x)$$

$$\Rightarrow \frac{d}{dx}(r+x) = \frac{d}{dx}(\ln(1+x))$$

$$1 = \frac{1}{1+x}$$

$$1+x = 1$$

$$x = 0$$

$$r+0 = \ln(1+0)$$

$$r = 0$$

bifurcation point $r_c = 0$, and the bifurcation occurs at $x=0$.

Bifurcation diagram of r against x :

$$r = \ln(1+x) - x$$

If we consider the Taylor expansion of $\ln(1+x) - x$ near the bifurcation point $x=0, r=0$, we have

$$r = \left[x - \frac{x^2}{2} + O(x^3) \right] - x$$

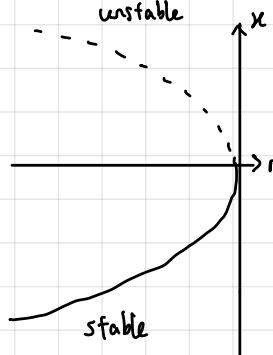
$r \approx -\frac{x^2}{2}$, which is in the normal form of the saddle-node bifurcation: $x = \pm \sqrt{-2r}, r < 0$

for $r < 0$,

$x > 0 \Rightarrow$ unstable fixed point

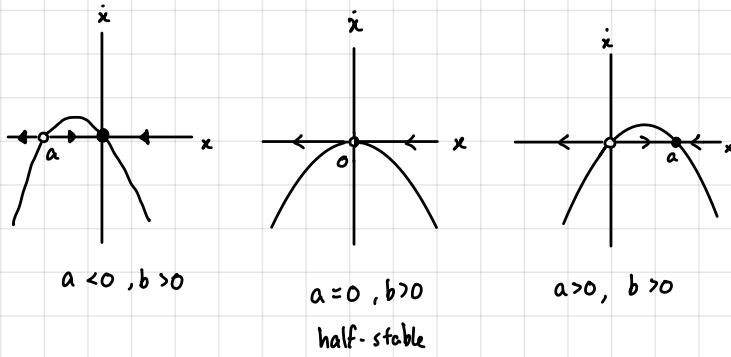
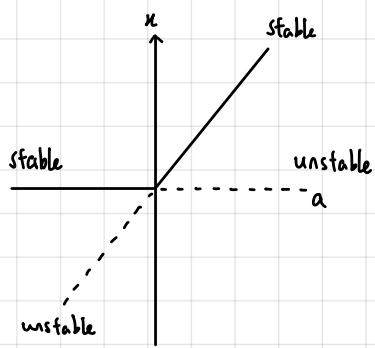
$x < 0 \Rightarrow$ stable fixed point

unstable

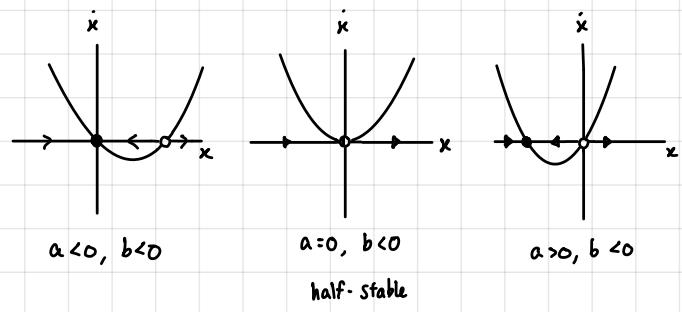
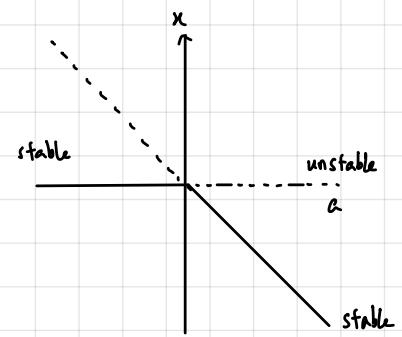


4. $\dot{x} = ax - bx^2$, type of bifurcation is transcritical bifurcation

for $b > 0$:



for $b < 0$:



Session 2

$$1. \text{ a) } \dot{x} = x - y - x(x^2 + 5y^2)$$

$$\dot{y} = x + y - y(x^2 + y^2)$$

for any system with fixed point at origin (0,0): x and y are deviations from fixed point.

We can linearize the equation by omitting nonlinear terms in x, y :

$$\begin{aligned} \therefore \dot{x} &= x - y \Rightarrow \text{Jacobian } A = \begin{pmatrix} \frac{\partial}{\partial x}(x-y) & \frac{\partial}{\partial y}(x-y) \\ \frac{\partial}{\partial x}(x+y) & \frac{\partial}{\partial y}(x+y) \end{pmatrix}_{(x^*, y^*)} \\ &= \begin{pmatrix} 1-y & x-1 \\ 1+y & x+1 \end{pmatrix}_{(x^*, y^*)} \\ &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

$$\det(A) = \det \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \text{Trace}(A) = 1+1 = 2$$

$$= 1 - (-1)$$

$$= 2$$

$$\begin{aligned} \text{eigenvalues: } \lambda_{1,2} &= \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} \\ &= \frac{2 \pm \sqrt{2^2 - 4(-4)}}{2} \\ &= \frac{2 \pm \sqrt{-4}}{2} \\ &= \frac{2 \pm 2i}{2} \\ &= 1 \pm i \end{aligned}$$

$\because \operatorname{Re}(\lambda) = 1 > 0$, and the eigenvalues are complex conjugates with $\det(A) > 0$, then the fixed point is a spiral with growing oscillations $\Rightarrow (0,0)$ is an unstable spiral.

$$\text{b) } x = r \cos \theta, \quad y = r \sin \theta$$

$$x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta$$

$$x^2 + y^2 = r^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2r \frac{dr}{dt}$$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = r \frac{dr}{dt}$$

$$x(x - y - x(x^2 + 5y^2)) + y(x + y - y(x^2 + y^2)) = r \frac{dr}{dt}$$

$$x^2 - xy - x^2(x^2 + 5y^2) + yx + y^2 - y^2(x^2 + y^2) = r \frac{dr}{dt}$$

$$x^2 + y^2 - x^2(x^2 + 5y^2) - y^2(x^2 + y^2) = r \frac{dr}{dt}$$

$$r^2 - x^4 - 5x^2y^2 - x^2y^4 - y^4 = r \frac{dr}{dt}$$

$$r^2 - 6x^2y^2 - x^4 - y^4 = r \frac{dr}{dt}$$

$$r^2 - (x^4 + 6x^2y^2 + y^4) = r \frac{dr}{dt}$$

$$r^2 - (r^4 \cos^4 \theta + r^4 \sin^4 \theta + 6(r^4 \sin^2 \theta \cos^2 \theta)) = r \frac{dr}{dt}$$

$$\frac{dr}{dt} = r(1 - r^2(\cos^4 \theta + \sin^4 \theta) - 6r^2 \sin^2 \theta \cos^2 \theta)$$

$$\frac{dr}{dt} = r(1 - r^2(1 - \frac{\sin^2 2\theta}{2}) - 6r^2(\sin \theta \cos \theta)^2)$$

$$\frac{dr}{dt} = r(1 - r^2 + r^2(\frac{\sin^2 2\theta}{2}) - 6r^2(\frac{\sin 2\theta}{2})^2)$$

$$\frac{dr}{dt} = r[1 - r^2 + r^2(\frac{\sin^2 2\theta}{2} - \frac{6}{4} \sin^2 2\theta)]$$

$$\boxed{\frac{dr}{dt} = r[1 - r^2 - r^2 \sin^2(2\theta)]}$$

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\frac{y}{x} = \tan \theta$$

$$\frac{x(\frac{dy}{dt}) - y(\frac{dx}{dt})}{x^2} = \sec^2 \theta \frac{d\theta}{dt}$$

$$x(\frac{dy}{dt}) \cdot y(\frac{dx}{dt}) = r^2 \cos^2 \theta (\sec^2 \theta) \frac{d\theta}{dt}$$

$$x(x + y - y(x^2 + y^2)) \cdot y(x - y - x(x^2 + 5y^2)) = r^2 \frac{d\theta}{dt}$$

$$x^2 + xy - xy(x^2 + y^2) - xy + y^2 + xy(x^2 + 5y^2) = r^2 \frac{d\theta}{dt}$$

$$x^2 + y^2 - x^2y - xy^2 + x^2y + 5xy^2 = r^2 \frac{d\theta}{dt}$$

$$x^2 + 4xy^2 = r^2 \frac{d\theta}{dt}$$

$$r^2 + 4(r \cos \theta)(r \sin \theta)^2 = r^2 \frac{d\theta}{dt}$$

$$r^2 + 4r^4 \cos \theta \sin^2 \theta = r^2 \frac{d\theta}{dt}$$

$$\boxed{\frac{d\theta}{dt} = 1 + 4r^2 \cos \theta \sin^2 \theta}$$

$$1.c) \frac{dr}{dt} = r \left[1 - r^2 - r^2 \sin^2(2\theta) \right], \quad \frac{d\theta}{dt} = 1 + 4r^2 \cos\theta \sin^2\theta$$

i) $\forall r < r_1 = \frac{1}{\sqrt{2}}$:

$$\frac{dr}{dt} = r^3 \left[\frac{1}{r^2} - 1 - \sin^2(2\theta) \right]$$

$$= r - r^3 - r^3 \sin^2(2\theta)$$

$\because 0 < r < 1$ and $r^3 \ll r$ $\forall r = r_1$,

$0 \leq \sin^2(2\theta) \leq 1$, then $r - r^3 - r^3 \sin^2(2\theta) > 0$

$$\Rightarrow \frac{dr}{dt} > 0 \quad \forall r < r_1 = \frac{1}{\sqrt{2}}$$

$$\text{at } r = \frac{1}{\sqrt{2}}, \quad \frac{dr}{dt} = \frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}}\right)^3 - \left(\frac{1}{\sqrt{2}}\right)^3 \sin^2(2\theta)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{8}} - \frac{1}{\sqrt{8}} \sin^2(2\theta)$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} \sin^2(2\theta)$$

$$= \frac{\sqrt{2}}{4} (1 - \sin^2(2\theta))$$

$\because 0 \leq \sin^2(2\theta) \leq 1$, then $\frac{dr}{dt} > 0$ for $r = \frac{1}{\sqrt{2}}$

$$\Rightarrow \frac{dr}{dt} > 0 \quad \forall r < r_1 = \frac{1}{\sqrt{2}}$$

ii) $\forall r \geq r_2 = 1$:

$$\frac{dr}{dt} = r - r^3 - r^3 \sin^2(2\theta)$$

$$r = 1 \quad , \quad \text{for } r > 1 :$$

$$\frac{dr}{dt} = 1 - 1 - \sin^2(2\theta)$$

$$r^3 \gg r,$$

$$= -\sin^2(2\theta)$$

$$\frac{dr}{dt} = r - r^3 (1 + \sin^2(2\theta))$$

$\because 0 \leq \sin^2(2\theta) \leq 1$, then

$$\frac{dr}{dt} < 0 \quad \text{for } r = 1$$

$$\text{then } r - r^3 (1 + \sin^2(2\theta)) < 0$$

$$\Rightarrow \frac{dr}{dt} < 0 \quad \forall r > 1$$

$$\therefore \frac{dr}{dt} < 0 \quad \forall r \geq r_2 = 1$$

iii) $\frac{\sqrt{2}}{2} \leq r \leq 1$: Consider two concentric circles $r_1 = \frac{1}{\sqrt{2}}$, $r_2 = 1$,

From part i and ii, we know that $\frac{dr}{dt} > 0$

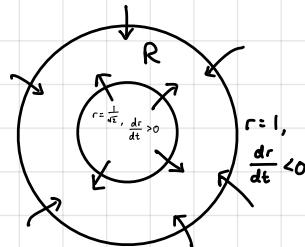
$$\text{for } r \leq r_1 = \frac{1}{\sqrt{2}}, \quad \text{and} \quad \frac{dr}{dt} < 0 \quad \text{for } r \geq r_2 = 1,$$

$$\Rightarrow \frac{dr}{dt} > 0 \text{ on inner circle and } \frac{dr}{dt} < 0 \text{ for outer circle.}$$

\therefore We have the annulus $0 < \frac{1}{\sqrt{2}} \leq r \leq 1$ which is our bounded region R .

\Rightarrow in this trapping region R , we have trajectories outside the region pointing inwards (from part ii), and trajectories inside the region on the inner circle pointing outwards towards R (from part i), thus all trajectories in R are confined.

(satisfies condition (1) and (4) of Poincaré-Bendixson theorem).



$\therefore \frac{dr}{dt} = r[1 - r^2 - r^2 \sin^2(2\theta)]$ is continuously differentiable

and the only fixed point is at the origin and unstable, which is outside of our trapping region R

$\frac{1}{\sqrt{2}} \leq r \leq 1$, (satisfies (2) and (3) of Poincaré-Bendixson theorem)

\Rightarrow By the Poincaré-Bendixson theorem, there exists a closed limiting cycle for $\frac{1}{\sqrt{2}} \leq r \leq 1$ which does not contain any fixed points.

$$2. \dot{x} = x(1-x) - h, \quad h \geq 0$$

To find bifurcation values, $\dot{x}=0$:

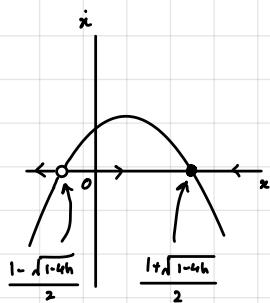
$$x(1-x) - h = 0$$

$$x - x^2 - h = 0$$

$$x^2 - x + h = 0$$

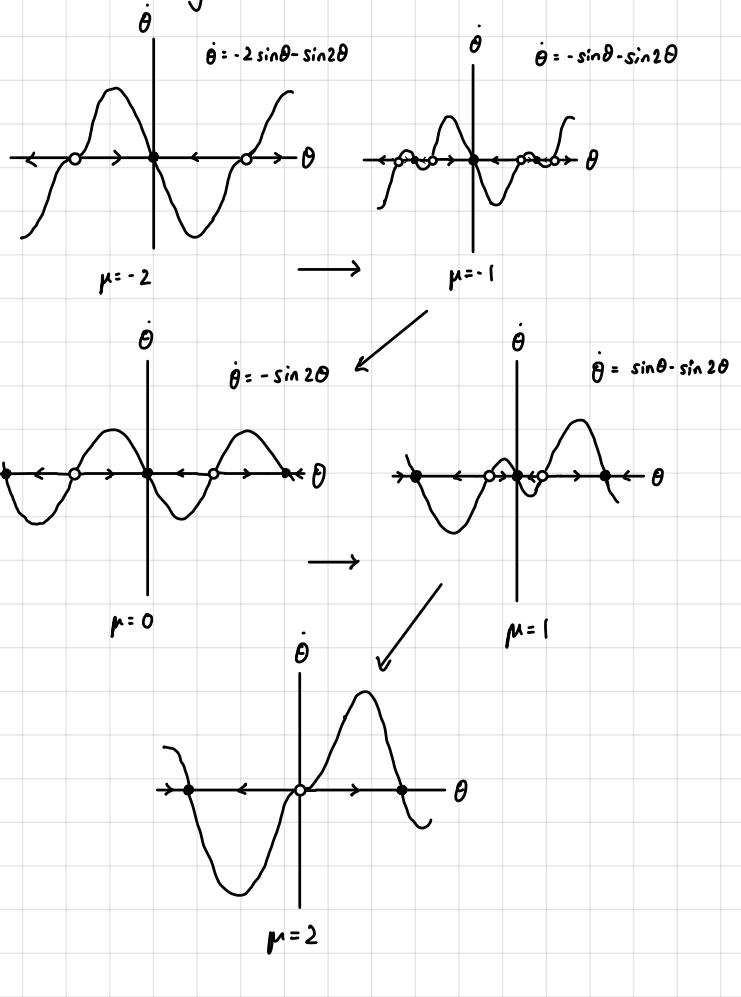
$$x = \frac{1 \pm \sqrt{1^2 - 4(1)(h)}}{2}$$

$$x^* = \frac{1 \pm \sqrt{1-4h}}{2}$$



$$3. \dot{\theta} = \mu \sin \theta - \sin 2\theta$$

Plotting $\dot{\theta}$ against θ , we have:



\Rightarrow saddle-node bifurcation, as if we vary h , then the graph translates vertically and the fixed point changes from one stable and one unstable, for $h \geq 0$ to one half stable point at $h = \frac{1}{4}$, and then no fixed points for $h < 0$.

∴ bifurcation occurs at $h = \frac{1}{4}$ and

$$x = \frac{1 \pm \sqrt{1-4(\frac{1}{4})}}{2}$$

$$x = \frac{1}{2}$$

\therefore The origin and the nearest two other fixed points switch stability from $\mu = -2$ to $\mu = 2$, then the bifurcation that occurs is a subcritical pitchfork bifurcation.

$$4. \frac{dR}{dt} = aR + bJ, \quad \frac{dJ}{dt} = bR + aJ$$

Let matrix A be $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

$$\text{Trace}(A) = \tau, \quad \det(A) = \Delta, \quad \tau^2 - 4\Delta = 4b^2$$

$$= 2a \quad = a^2 - b^2$$

$$\det(A - \lambda I) = \det \begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix}$$

$$= (a-\lambda)^2 - b^2$$

$$= a^2 - 2a\lambda + \lambda^2 - b^2$$

$$= \lambda^2 - 2a\lambda + a^2 - b^2$$

$$\Rightarrow \lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2 - b^2)}}{2}$$

$$\lambda = a \pm b$$

$$\lambda = a \pm b \quad (\text{eigenvalues})$$

$\begin{pmatrix} R \\ J \end{pmatrix} = A \begin{pmatrix} R \\ J \end{pmatrix}$ - To find eigenvectors:

$$\begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\lambda = \lambda_1 = a+b$

$$\Rightarrow \begin{pmatrix} a-a-b & b \\ b & a-a-b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -b & b \\ b & -b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left. \begin{array}{l} -b v_1 + b v_2 = 0 \\ b v_1 - b v_2 = 0 \end{array} \right\} \quad \left. \begin{array}{l} b(-v_1 + v_2) = 0 \\ b(v_1 - v_2) = 0 \end{array} \right\}$$

\therefore the eigenvector corresponding to $\lambda_1 = a+b$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda = \lambda_2 = a-b$:

$$\begin{pmatrix} a-a+b & b \\ b & a-a+b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} b & b \\ b & b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left. \begin{array}{l} b v_1 + b v_2 = 0 \\ b v_1 + b v_2 = 0 \end{array} \right\} \quad \left. \begin{array}{l} b(v_1 + v_2) = 0 \\ b(v_1 + v_2) = 0 \end{array} \right\}$$

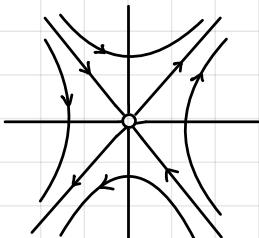
\therefore the eigenvector corresponding to $\lambda_2 = a-b$ is: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

1. For $a < b$,

$$\lambda_1 = a+b > 0$$

$$\lambda_2 = a-b < 0$$

$$\therefore \lambda_1 > 0 > \lambda_2$$



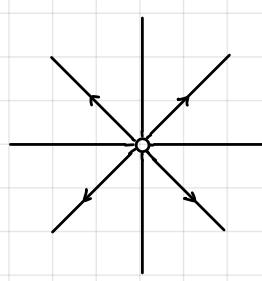
All trajectories approach the line $R = J$, and the fixed point $(0,0)$ is a saddle point

2. For $a < b < 0$,

$$\lambda_1 = a+b < 0$$

$$\lambda_2 = a-b < 0$$

$$\therefore \lambda_1 < \lambda_2 < 0$$



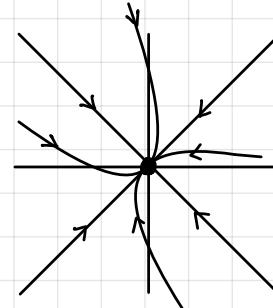
All trajectories lead out to infinity from the unstable node $(0,0)$. \Rightarrow source node

3. For $0 < b < a$

$$\lambda_1 = a+b > 0$$

$$\lambda_2 = a-b > 0$$

$$\lambda_1 > \lambda_2 > 0$$



The fixed point $(0,0)$ is a stable node, all trajectories lead to $(0,0)$